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OPTIMALITY CONDITIONS IN NONCONVEX AND NONSMOOTH OPTIMIZATION REVISITED

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Abstract. This work is a continuation of the paper [15], in which the well-known necessary and sufficient optimality condition for nonsmooth convex optimization, given in the form of a variational inequality, is generalized to the nonconvex case by using the notion of weak subdifferentials. In this paper, we show that the same optimality conditions can be obtained under weaker conditions.

1. Introduction

The well-known optimality condition in nonsmooth convex analysis states that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then vector $\overline{x}$ minimizes f over a convex set $S \subset \mathbb{R}^n$ if and only if there exists a subgradient $x^* \in \partial f(\overline{x})$ such that

$$\langle x^*, x - \overline{x}\rangle \geq 0, \quad \forall x \in S \tag{1.1}$$

where

$$\partial f(\overline{x}) = \{x^* \in \mathbb{R}^n : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle, \text{ for all } x \in \mathbb{R}^n\} \tag{1.2}$$

is the subdifferential of f at $\overline{x}$ (see [21, Theorem 8.15, page 310] or [4, Proposition 1.8.1, page 168]). Equivalently, $\overline{x}$ minimizes f over a convex set $S \subset \mathbb{R}^n$ if and only if

$$0 \in \partial f(\overline{x}) + N_S(\overline{x}), \tag{1.3}$$

where

$$N_S(\overline{x}) = \{x^* \in \mathbb{R}^n : \langle x^*, x - \overline{x}\rangle \leq 0, \text{ for all } x \in S\} \tag{1.4}$$

denotes the normal cone to the set S at the point $\overline{x}$.

These relations were generalized by Kasimbeyli and Mammadov in [15] for the case of nonconvex and nonsmooth problems. Probably it was the first generalization in the form of a necessary and sufficient condition in the nonconvex case. They used the notions of a weak subdifferential and the augmented normal cone [2, 3, 15]. One of the advantages of weak subgradients is that they generate conical supporting surfaces to the epigraph of a function under investigation, instead of hyperplanes used by the classical subgradient of convex analysis. The other advantage of weak subgradients is due to its relation with directional derivatives.

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It is well-known that every convex function $f : X \to (-\infty, +\infty]$ on a Banach space admits the directional derivative

$$f'(\overline{x}; h) = \lim_{t \downarrow 0} \frac{f(\overline{x} + th) - f(\overline{x})}{t} \tag{1.5}$$

in any direction $h \in X$ at any point of its efficient domain $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$.

There are many generalizations of the directional derivative: generalized derivative introduced by Clarke [6], generalized directional derivative (upper subderivative) introduced by Rockafellar [19, 20], lower semiderivative introduced by Penot [18], contingent derivative/epiderivative introduced by Aubin [1], lower Dini (or Dini-Hadamard) directional derivative introduced by Ioffe [10] and so on.

By using a general notation f^g for the generalized directional derivatives mentioned above, the corresponding generalized subdifferential of f at $\overline{x}$ is defined by

$$\partial^g f(\overline{x}) = \{x^* \in \mathbb{R}^n : f^g(\overline{x}; h) \geq \langle x^*, h \rangle, \text{ for all } h \in \mathbb{R}^n\}. \tag{1.6}$$

This is a standard way to introduce subgradients via directional derivatives. For convex functions it is equivalent to the classical subdifferential (1.2) of convex analysis:

$$\partial f(\overline{x}) = \{x^* \in \mathbb{R}^n : f'(\overline{x}; h) \geq \langle x^*, h \rangle, \text{ for all } h \in \mathbb{R}^n\}. \tag{1.7}$$

The inequality relation given in (1.6) and (1.7) is strengthened in convex analysis by showing that the directional derivative of a convex function can be represented as a pointwise maximum of its subgradients.

By using a special class of superlinear functions, Azimov and Gasimov [2, 3] introduced the concept of weak subgradient – one of the natural generalizations of the classic subgradient of convex analysis. The definition of weak subgradients does not use directional derivatives. Rather, it is based on the (conical) supporting philosophy directly for the epigraph of the function under consideration.

This approach was used to develop a nonlinear separation methodology without the convexity assumption (see, e.g., [8,12,13]) and to derive a collection of optimality conditions, duality relations and solution algorithms for a wide class of nonconvex optimization problems [5, 7, 9, 16, 17, 22, 23].

Kasimbeyli [11] introduced the notion of radial epiderivative for nonconvex single-valued and set-valued maps and established relationships between the radial epiderivatives, the weak subdifferentials and the directional derivatives. Kasimbeyli and Mammadov [14] established conditions that guarantee a representation of the directional derivative as a pointwise supremum of weak subgradients of a nonconvex real-valued function. A similar representation is also established for the radial epiderivative of a nonconvex function. Because of these reasons, the weak subgradient becomes a powerful tool in nonconvex analysis.

The optimality condition in [15] was proved under the following assumption:

Assumption 1.1. Let the directional derivative $f'(\overline{x}; \cdot)$ of f at $\overline{x}$ be lower semicontinuous on $K = \operatorname{cone}(S - \overline{x})$ and the following two conditions hold:

(i) there exists $\sigma > 0$ such that

$$f(x) - f(\overline{x}) \geq \sigma f'(\overline{x}; x - \overline{x}) \ \text{ for all } x \in S; \tag{1.8}$$

(ii)

$$\beta(\overline{x}) \doteq \inf\{f'(\overline{x}; h) : \ h \in K \cap U\} > 0, \tag{1.9}$$

where U denotes the unit sphere of $\mathbb{R}^n$.

Kasimbeyli and Mammadov [15] proved that, under Assumption 1.1, there exists a weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ such that the relation

$$\langle x^*, x - \overline{x}\rangle + \alpha \|x - \overline{x}\| \geq 0, \quad \text{for all } x \in S \tag{1.10}$$

is necessary and sufficient for the optimality of $\overline{x} \in S$ to the problem of minimization of f over S, where

$$\partial_S^w f(\overline{x}) = \{(x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle + \alpha\|x - \overline{x}\| \ \text{ for all } x \in S\} \tag{1.11}$$

denotes the weak subdifferential of f at $\overline{x}$ on S.

The aim of this paper is to generalize and weaken this assumption. We first show that the claim remains true if there exists a homogeneous function (instead of the directional derivative) satisfying the conditions in the above assumption. Then, this condition is used to formulate a weaker condition than (1.8).

The paper is organized as follows. The definition of weak subdifferentials and some preliminary results are presented in the next section. Necessary and sufficient conditions for optimality are given in Section 2. In Section 3, we formulate necessary and sufficient optimality conditions in terms of augmented normal cones. Finally, Section 4 concludes the paper.

2. Necessary and sufficient optimality conditions

In this section, we present a generalization of the optimality condition given in (1.1) to the nonconvex case under some weaker conditions than those presented in Assumption 1.1. The condition given by (1.1) guarantees the existence of a supporting hyperplane to the given convex set of feasible solutions at the optimal point. The normal vector of this hyperplane is a subgradient of the function being optimized and the corresponding theorem is based on a well-known separation theorem of convex analysis. The optimality condition formulated in this section guarantees the existence of a weak subgradient, that is a pair consisting of some linear functional and some real number such that the graph of the homogeneous function defined by this pair, is a conical supporting surface to the (possibly nonconvex) set of feasible solutions at the optimal point (see (2.4)).

We begin with the definition of the weak subgradient. Let $(X, \|\cdot\|_X)$ be a real normed space and let X^* be the topological dual of X.

Definition 2.1. Let $f : X \to \mathbb{R}$ be a real-valued function and let $\overline{x} \in X$ be a given point where $f(\overline{x})$ is finite. A pair $(x^*, \alpha) \in X^* \times \mathbb{R}$ is called the weak subgradient of f at $\overline{x}$ if

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle + \alpha\|x - \overline{x}\|, \text{ for all } x \in X. \tag{2.1}$$

The set

$$\partial^w f(\overline{x}) = \{(x^*, \alpha) \in X^* \times \mathbb{R} : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle + \alpha\|x - \overline{x}\|, \text{ for all } x \in X\} \tag{2.2}$$

of all weak subgradients of f at $\overline{x}$ is called the weak subdifferential of f at $\overline{x}$. If $\partial^w f(\overline{x}) \neq \emptyset$, then f is said to be weakly subdifferentiable at $\overline{x}$.

Remark 2.2. It is obvious that, when f is subdifferentiable at $\overline{x}$ (in the convex analysis sense), then f is also weakly subdifferentiable at $\overline{x}$, that is, if $x^* \in \partial f(\overline{x})$ then by definition, $(x^*, \alpha) \in \partial^w f(\overline{x})$ for every $\alpha \leq 0$. It follows from Definition 2.1 that the pair $(x^*, \alpha) \in X^* \times \mathbb{R}$ is a weak subgradient of f at $\overline{x} \in X$ if there exists a continuous function

$$g(x) = \langle x^*, x - \overline{x} \rangle + f(\overline{x}) + \alpha \|x - \overline{x}\| \tag{2.3}$$

such that $g(x) \leq f(x)$ for all $x \in X$. Note that we have $g(\overline{x}) = f(\overline{x})$. The set

$$\text{hypo}(g) = \{(x, \gamma) \in X \times \mathbb{R} : g(x) \geq \gamma\}$$

is a closed cone in $X \times \mathbb{R}$ with vertex at $(\overline{x}, f(\overline{x}))$ Indeed,

$$\begin{aligned} & \mathit{hypo}(g) - (\overline{x}, f(\overline{x})) \\ = \; & \{(x - \overline{x}, \alpha - f(\overline{x})) \in X \times \mathbb{R} : \langle x^*, x - \overline{x} \rangle - c \|x - \overline{x}\| \geq \alpha - f(\overline{x})\} \\ = \; & \{(u, \beta) \in X \times \mathbb{R} : \langle x^*, u \rangle - c \|u\| \geq \beta\}. \end{aligned}$$

Thus, it follows from (2.1) and (2.3) that

$$\text{graph}(g) = \{(x, \gamma) \in X \times \mathbb{R} : g(x) = \gamma\}$$

is a conic surface which is supporting the set

$$\text{epi}(f) = \{(x, \gamma) \in X \times \mathbb{R} : f(x) \leq \gamma\}$$

at the point $(\overline{x}, f(\overline{x}))$ in the sense that

$$\text{epi}(f) \subset \text{epi}(g), \text{ and } \text{cl}(\text{epi}(f)) \cap \text{graph}(g) \neq \emptyset.$$

The function g from (2.3) is superlinear concave if $\alpha < 0$, and in this case hypo(g) is a closed convex cone. Obviously, when $\alpha \geq 0$, the function g becomes convex.

It follows from this remark and from Definition 2.1 that the class of weakly subdifferentiable functions is essentially larger than the class of subdifferentiable functions (see [2, 3, 11, 14]).

We consider the following problem: under which conditions there exists a weak subgradient $(x^*, \alpha) \in \partial^w_S f(\overline{x})$ such that the relation

$$\langle x^*, x - \overline{x} \rangle + \alpha \|x - \overline{x}\| \geq 0, \quad \text{for all } x \in S \tag{2.4}$$

is necessary and sufficient for the optimality of $\overline{x} \in S$ for the problem of minimization of f over S, which does not have to be convex?

This problem is considered in [15]. Under the conditions given in Assumption 1.1, it was proved that the existence of such a subgradient is necessary and sufficient for the optimality of $\overline{x} \in S$ for the problem of minimization of f over S.

The aim of this paper is to weaken this assumption. Consider the following assumption.

Assumption 2.3. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function. Let $K =$ cone$(S - \overline{x})$ and suppose that the following two conditions hold:

(i)

$$f(x) - f(\overline{x}) \geq g(x - \overline{x}) \text{ for all } x \in S; \tag{2.5}$$

(ii)

$$\beta(\overline{x}) \doteq \inf\{g(h): \ h \in K \cap U\} > 0. \tag{2.6}$$

It should be noted that, if $\overline{x} \in S$ is a minimizer of the function f and if f is differentiable at that point with $\nabla f(\overline{x}) = \mathbf{0}$, then the condition (2.4) is satisfied for the zero weak subgradient $(x^*, \alpha) = (\mathbf{0}, 0) \in \partial_S^w f(\overline{x})$. In addition, if $\overline{x} \in \text{int } S$, then this zero weak subgradient is the only weak subgradient in $\partial_S^w f(\overline{x})$ satisfying the condition $\alpha \geq 0$. Therefore, in the case $\nabla f(\overline{x}) = \mathbf{0}$, the necessity of the condition (2.4) is trivial.

Before formulating the necessary and sufficient optimality conditions, we first present the following definition from [12] that provides a certain kind of separation property.

Definition 2.4. [12, Definition 4.1] Let C and K be closed cones in $\mathbb{R}^n$, let U denote the unit sphere of $\mathbb{R}^n$, and let $C_U = C \cap U$ and $K_U = K \cap U$. Denote $\widetilde{C} = \text{co}(C_U)$, $K_U^\partial = K_U \cap \text{bd}(K)$ and $\widetilde{K}^\partial = \text{co}(K_U^\partial \cup \{\mathbf{0}\})$. The cones C and K are said to satisfy the separation property with respect to the given norm $\|\cdot\|$ (the same norm that defines the unit sphere), if

$$\widetilde{C} \cap \widetilde{K}^\partial = \emptyset, \tag{2.7}$$

where co and bd denote the convex hull and the boundary of a set, respectively.

The following lemma is proved in [15, Lemma 3]. We will use it in the proof of the main theorem.

Lemma 2.5. *Let $a \in U$ and let $\varepsilon \in (0, \sqrt{2})$. Let*

$$C = \text{cone}(\{x \in U: \ ||x - a|| \leq \varepsilon\}), \tag{2.8}$$

where the Euclidean norm is used, and K be a closed cone (which does not have to be convex) in $\mathbb{R}^n$ such that $C \cap K = \{\mathbf{0}\}$. Then, C is a closed convex and pointed cone, and the cones C and K satisfy the separation property given in Definition 2.4 with respect to the Euclidean norm.

The following theorem quoted from [15] gives a sufficient condition for a point to be a minimizer of some function over given set, in terms of weak subgradients.

Theorem 2.6. *([15, Theorem 3]) Let $f: S \to R$ be a given function. If there exists a weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ such that the relation (2.4) is satisfied, then $\overline{x} \in S$ minimizes f on S.*

Proof. If, for some weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$, the relation (2.4) is satisfied, then by definition (1.11) of $\partial_S^w f(\overline{x})$, we obtain $f(x) - f(\overline{x}) \geq 0$ for all $x \in S$, which means that $\overline{x}$ minimizes f over S. □

Now we consider the problem when the relation (2.4) becomes also a necessary condition for being a minimizer. We are interested in the existence of non-zero weak subgradients satisfying (2.4). This is true for the trivial case when $S = \{\overline{x}\}$. Below it is assumed that set $S \setminus \{\overline{x}\}$ is nonempty.

Definition 2.7. A nonempty subset S of $\mathbb{R}^n$ is called cone-shaped at $\overline{x} \in S$ if $\text{cl}(\text{cone}(S - \overline{x})) \neq \mathbb{R}^n$.

Theorem 2.8. *Let $S \subseteq \mathbb{R}^n$ be cone-shaped at $\overline{x}$. Suppose that $S \setminus \{\overline{x}\} \neq \emptyset$ and let $f : S \to \mathbb{R}$ be a given function. Assume that $\overline{x} \in S$ is a minimizer of f over S, and Assumption 2.3 holds. Then, there exists a weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ with $x^* \neq \mathbf{0}$ and $\alpha \geq 0$ such that*

$$\langle x^*, x - \overline{x} \rangle + \alpha \, \|x - \overline{x}\| \geq 0 \text{ for all } x \in S,$$

and

$$\langle x^*, z - \overline{x} \rangle + \alpha \, \|z - \overline{x}\| < 0 \text{ for some } z \notin S. \tag{2.9}$$

Proof. Suppose that $\overline{x}$ minimizes f over S. Denote $\bar{K} = \text{cl}(\text{cone}(S - \overline{x}))$. By the hypotheses, S is cone-shaped at $\overline{x}$, which means that $\bar{K} \neq \mathbb{R}^n$. Then, there exists a point $a \in U$ such that $a \in \mathbb{R}^n \setminus \bar{K}$, where U is the unit sphere. Take any $\varepsilon > 0$ and define the cone

$$C = \text{cone}\{x \in U : \ ||a - x|| \leq \varepsilon\}.$$

Since $\mathbb{R}^n \setminus \bar{K}$ is an open set, there exists a positive number ε such that $C \setminus \{\mathbf{0}\} \subset \text{int}(\mathbb{R}^n \setminus \bar{K})$; that is, $C \cap \bar{K} = \{\mathbf{0}\}$. Then, by Lemma 2.5, it follows that C and $\bar{K}$ satisfy the separation property given in Definition 2.4 with respect to the Euclidean norm in $\mathbb{R}^n$. Then, by [12, Theorem 4.3], there exists a pair (y^*, γ) with $y^* \neq \mathbf{0}$ and $\gamma \geq 0$ such that the function $\varphi(y) = \langle y^*, y \rangle + \gamma \|y\|$ is strongly monotonically increasing (with respect to the cone C) on $\mathbb{R}^n$ and the sublevel set of this function separates the cones C and $\text{bd}(\bar{K})$ in the following sense:

$$\langle y^*, y \rangle + \gamma \|y\| < 0 \leq \langle y^*, x \rangle + \gamma \|x\| \tag{2.10}$$

for all $y \in C \setminus \{\mathbf{0}\}$ and for all $x \in \text{bd}(\bar{K})$. Clearly, this relation holds for all $y \in C \setminus \{\mathbf{0}\}$ and for all $x \in \bar{K}$, and in particular for all $x \in S - \overline{x}$. Therefore, the right-hand side of the inequality in the last relation can also be written as

$$\langle y^*, x - \overline{x} \rangle + \gamma \|x - \overline{x}\| \geq 0 \ \text{ for all } x \in S. \tag{2.11}$$

Take any $\lambda > 0$ and denote $x_\lambda^* = \lambda y^*$, $\alpha_\lambda = \lambda\gamma$. Consider

$$\eta(\lambda) \doteq \sup \left\{ \left\langle x_\lambda^*, \frac{x - \overline{x}}{||x - \overline{x}||} \right\rangle + \alpha_\lambda : \ x \in S, \ x \neq \overline{x} \right\}. \tag{2.12}$$

It is not difficult to observe that

$$\begin{aligned} \eta(\lambda) &\leq \sup \left\{ \left\langle x_\lambda^*, \frac{x - \overline{x}}{||x - \overline{x}||} \right\rangle + \alpha_\lambda : \ x \in \mathbb{R}^n, \ x \neq \overline{x} \right\} \\ &= \left\langle x_\lambda^*, \frac{x_\lambda^*}{||x_\lambda^*||} \right\rangle + \alpha_\lambda \\ &= \lambda(||y^*|| + \gamma). \end{aligned}$$

This implies $\eta(\lambda) \to 0$ as $\lambda \to 0$. Then, from (2.6) there exists a number $\overline{\lambda} > 0$ such that $\eta(\overline{\lambda}) \leq \beta(\overline{x})$. By denoting $x^* = x_{\overline{\lambda}}^*$ and $\alpha = \alpha_{\overline{\lambda}}$, we have

$$\sup \left\{ \left\langle x^*, \frac{x - \overline{x}}{||x - \overline{x}||} \right\rangle + \alpha : \ x \in S, \ x \neq \overline{x} \right\} \leq \inf \left\{ g\left(\frac{x - \overline{x}}{||x - \overline{x}||} \right) : \ x \in S, \ x \neq \overline{x} \right\}.$$

Hence, for all $x \in S,\ x \neq \overline{x}$, we have

$$\left\langle x^*, \frac{x-\overline{x}}{||x-\overline{x}||} \right\rangle + \alpha \leq g\left(\frac{x-\overline{x}}{||x-\overline{x}||}\right),$$

or

$$\langle x^*, x-\overline{x}\rangle + \alpha \parallel x-\overline{x} \parallel \leq g(x-\overline{x}).$$

Together with condition (2.5), we obtain that

$$f(x) - f(\overline{x}) \geq \langle x^*, x-\overline{x}\rangle + \alpha||x-\overline{x}||,\ \forall x \in S,$$

which means $(x^*, \alpha) \in \partial^w_S f(\overline{x})$. Now, recalling that $x^* = \overline{\lambda} y^*$ and $\alpha = \overline{\lambda}\gamma$, the desired relation follows from (2.11):

$$\langle x^*, x-\overline{x}\rangle + \alpha||x-\overline{x}|| = \overline{\lambda}(\langle y^*, x-\overline{x}\rangle + \gamma||x-\overline{x}||) \geq 0,\ \forall x \in S.$$

Now we prove (2.9). From (2.10), there exists $\widetilde{z} \in C \setminus \{\mathbf{0}\}$ such that

$$\langle y^*, \widetilde{z}\rangle + \gamma||\widetilde{z}|| < 0.$$

Denote $z = \widetilde{z} + \overline{x}$. Then,

$$\langle y^*, z-\overline{x}\rangle + \gamma||z-\overline{x}|| < 0. \tag{2.13}$$

On the other hand, since $(C \setminus \{\mathbf{0}\}) \cap (S - \overline{x}) = \emptyset$, we have $z - \overline{x} = \widetilde{z} \notin S - \overline{x}$, which means that $z \notin S$. Therefore, multiplying (2.13) by $\overline{\lambda} > 0$ and noting that $x^* = \overline{\lambda} y^*$ and $\alpha = \overline{\lambda}\gamma$, we obtain (2.9). □

The following corollary is a straightforward consequence of the previous theorem.

Corollary 2.9. *Let $S \subseteq \mathbb{R}^n$ be cone-shaped at $\overline{x}$. Suppose that $S \setminus \{\overline{x}\} \neq \emptyset$ and let $f : S \to R$ be a given function. Assume that $\overline{x} \in S$ minimize f over S. Then, there exists a weak subgradient $(x^*, \alpha) \in \partial^w_S f(\overline{x})$ with $x^* \neq \mathbf{0}$ and $\alpha \geq 0$ such that*

$$\langle x^*, x-\overline{x}\rangle + \alpha \|x-\overline{x}\| \geq 0 \text{ for all } x \in S,$$

and

$$\langle x^*, z-\overline{x}\rangle + \alpha \|z-\overline{x}\| < 0 \text{ for some } z \notin S$$

if either one of the following statements holds:

(i) *f is a weakly subdifferentiable function at $\bar{x}$, and there exists a weak subgradient (u^*, δ) such that the function $g(h) = \langle u^*, h\rangle + \delta\|h\|$ satisfies conditions (2.5) and (2.6) of Assumption 2.3.*

(ii) *f is a directionally differentiable function at $\bar{x}$, and there exists a real number δ such that the function $g(h) = f'(\overline{x}; h) + \delta||h||$ satisfies conditions (2.5) and (2.6) of Assumption 2.3.*

3. Optimality conditions via augmented normal cones

Let $S \subset \mathbb{R}^n$ and $\overline{x} \in S$. As mentioned above (see (1.4)), the normal cone to S at $\overline{x}$ is defined as follows:

$$N_S(\overline{x}) \doteq \{x^* \in \mathbb{R}^n : \ \langle x^*, x - \overline{x}\rangle \leq 0, \ \forall x \in S\}. \tag{3.1}$$

Clearly, if $\overline{x} \in \text{int } S$, then the set $N_S(\overline{x})$ contains only the element $\mathbf{0} \in \mathbb{R}^n$. Such a normal cone is called trivial. If $\overline{x} \notin \text{int } S$ and S is convex, then the normal cone $N_S(\overline{x})$ contains non-zero elements. However, if S is not convex, $\mathbf{0} \in \mathbb{R}^n$ may be the only element in this cone. A normal cone is called nontrivial if it contains non-zero elements.

Below, we recall the definition of the augmented normal cone introduced by Kasimbeyli and Mammadov [15].

Definition 3.1. Let $S \subset \mathbb{R}^n$ and $\overline{x} \in S$ with $S \setminus \{\overline{x}\} \neq \emptyset$. The set

$$N_S^a(\overline{x}) \doteq \{(x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \langle x^*, x - \overline{x}\rangle + \alpha||x - \overline{x}|| \leq 0 \text{ for all } x \in S\},$$

is called an augmented normal cone to S at $\overline{x}$.

Since, for pairs (x^*, α) with $\alpha \leq -\|x^*\|$, the inequality $\langle x^*, x - \overline{x}\rangle + \alpha||x - \overline{x}|| \leq 0$ is obviously satisfied for all $x \in \mathbb{R}^n$, an augmented normal cone consisting of only such elements is called trivial. The trivial augmented normal cone is denoted by $N_S^{triv}(\overline{x})$ and defined as

$$N_S^{triv}(\overline{x}) = \{(x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \leq -\|x^*\|\}.$$

It follows from the definitions of normal and augmented normal cones that, if the normal cone $N_S(\overline{x})$ is not empty for a given set $S \subset \mathbb{R}^n$, then for every $x^* \in N_S(\overline{x})$, the pair $(x^*, 0)$ belongs to the augmented normal cone $N_S^a(\overline{x})$. Conversely, if $(x^*, \alpha) \in N_S^a(\overline{x})$ with $\alpha \geq 0$, then $x^* \in N_S(\overline{x})$.

Definition 3.2. $\overline{x} \in S$ is called an *n-interior* point of $S \subset \mathbb{R}^n$ if the augmented normal cone at this point contains only trivial elements: $N_S^a(\overline{x}) = N_S^{triv}(\overline{x})$. By n-int$(S)$, we denote the set of all *n-interior* points of S.

The following assertions, which characterize the trivial augmented normal cones and interior points, are proved in [15].

Proposition 3.3. (i) ([15, Lemma 4]) *If $\overline{x} \in int\ S$, then $N_S^a(\overline{x}) = N_S^{triv}(\overline{x})$.*
(ii) ([15, Lemma 5]) *S is cone-shaped at $\overline{x}$ if and only if $\overline{x} \notin$ n-int(S) (i.e. $N_S^a(\overline{x}) \neq N_S^{triv}(\overline{x})$).*

Now we present a generalization of the optimality condition given in (1.3) to optimization problems without any convexity assumption. The new condition is formulated in terms of weak subdifferentials and augmented normal cones. We prove that the optimality condition given in the form of the existence of nonzero solutions to the variational inequality (2.4) can equivalently be formulated as the existence of nontrivial solutions to the following problem:

$$(\mathbf{0}, 0) \in \partial_S^w f(\overline{x}) + N_S^a(\overline{x}), \tag{3.2}$$

where $(\mathbf{0}, 0)$ denotes the zero of $\mathbb{R}^n \times \mathbb{R}$.

Note that such a generalization was firstly given in [15], where the authors presented nice illustrative examples. Here, we prove the same assertion under the weakened conditions.

Theorem 3.4. *Let $S \subset \mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a given function. Assume that $\overline{x} \in S$ minimizes f over S, S is cone-shaped at $\overline{x}$, $S \setminus \{\overline{x}\} \neq \emptyset$, and Assumption 2.3 holds. Then, there exists a nontrivial solution to (3.2):*

$$(\mathbf{0}, 0) \in \partial_S^w f(\overline{x}) + N_S^a(\overline{x});$$

that is, there exists a weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ such that $-(x^*, \alpha) \in N_S^a(\overline{x})$, $x^* \neq \mathbf{0}$ and $\alpha > -||x^*||$.

Proof. Note that all conditions of Theorem 2.8 are satisfied. By this theorem, there exists a weak subgradient $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ such that $x^* \neq \mathbf{0}$, $\alpha \geq 0$ and the variational inequality (2.4) is satisfied, that is:

$$\langle x^*, x - \overline{x} \rangle + \alpha \, \|x - \overline{x}\| \geq 0 \text{ for all } x \in S.$$

By multiplying both sides of this inequality by -1, we obtain:

$$\langle -x^*, x - \overline{x} \rangle + (-\alpha) \, \|x - \overline{x}\| \leq 0 \text{ for all } x \in S,$$

which means that $(-x^*, -\alpha) \in N_S^a(\overline{x})$, and since $x^* \neq \mathbf{0}$ and $\alpha \geq 0$, we obtain $\alpha \geq 0 > -||x^*||$ and the proof is complete. □

It is not difficult to observe that the proof of Theorem 3.4 establishes a much stronger fact, which states that a nontrivial solution (x^*, α) to (3.2) could be chosen so that not only $\alpha \geq -||x^*||$ but also $\alpha \geq 0$.

Theorem 3.5. *If (3.2) has a solution, then $\overline{x} \in S$ is a minimizer of function f on S.*

Proof. Let (x^*, α) be a solution to (3.2). That is, let $(x^*, \alpha) \in \partial_S^w f(\overline{x})$ and let $-(x^*, \alpha) \in N_S^a(\overline{x})$. Then, from the last inclusion we have:

$$\langle -x^*, x - \overline{x} \rangle + (-\alpha) \, \|x - \overline{x}\| \leq 0 \text{ for all } x \in S,$$

or

$$\langle x^*, x - \overline{x} \rangle + \alpha \, \|x - \overline{x}\| \geq 0 \text{ for all } x \in S.$$

Now, by taking into account the inclusion $(x^*, \alpha) \in \partial_S^w f(\overline{x})$, the last inequality yields:

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle + \alpha \, \|x - \overline{x}\| \geq 0 \text{ for all } x \in S,$$

which completes the proof. □

Corollary 3.6. *Let $S \subseteq \mathbb{R}^n$ be cone-shaped at $\overline{x}$. Suppose that $S \setminus \{\overline{x}\} \neq \emptyset$ and let $f : S \to \mathbb{R}$ be a given function. Assume that $\overline{x} \in S$ is a minimizer of f over S. Then, there exists a nontrivial solution to (3.2) if either one of the following statements holds:*

(i) *f is a weakly subdifferentiable function at $\bar{x}$, and there exists a weak subgradient (u^*, δ) such that the function $g(h) = \langle u^*, h \rangle + \delta \|h\|$ satisfies the conditions (2.5) and (2.6) of Assumption 2.3.*

(ii) *f is a directionally differentiable function at $\bar{x}$, and there exists a real number δ such that the function $g(h) = f'(\overline{x}; h) + \delta||h||$ satisfies the conditions (2.5) and (2.6) of Assumption 2.3.*

Proof. The proof is obvious and is therefore omitted. □

4. Conclusions

In this paper, necessary and sufficient optimality conditions for optimization problems without any convexity assumption are studied. The class of problems considered is described by a special class of directionally differentiable and weakly subdifferentiable functions. The necessary and sufficient optimality condition of nonsmooth convex optimization, given in the form of variational inequality (1.1), is generalized to the nonconvex case by using the notion of weak subdifferential. This is a generalization of the theorem obtained by Kasimbeyli and Mammadov [15] in nonconvex and nonsmooth optimization. The condition (1.1) has an equivalent formulation (1.3) in terms of subgradients and normal cones in convex optimization. In this paper, a similar condition is obtained in terms of the weak subdifferentials and the augmented normal cones.

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