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ORDERED LOCATION PROBLEMS

JERZY GRZYBOWSKI, DIETHARD PALLASCHKE, AND RYSZARD URBAŃSKI

Abstract. For finitely many points in a metric space, called facilities, Endre Weiszfeld studied the differentiable convex optimization problem of finding a point such that the sum of all distances from this point to the given facilities is minimal. In this paper we consider a modified problem: Namely, after ordering the distances to all facilities by value, determine a point so, that in the order by value the sum of those distances which belong to a given connected ascending subsequence of indexes is minimal. This is a quasi-differentiable optimization problem with a goal function which is a difference of convex functions. Necessary optimality conditions are derived which are similar to those of the classical cases studied by Weiszfeld.

1. Introduction

We describe first our problem in more details: For a metric space (X,d) and given points $v_1, v_2, \ldots, v_n \in X$, E. Weiszfeld (see [9] and [5]) studied the problem of finding a point $\bar{x} \in X$ such that the sum of distances $\sum_{i=1}^{n} d(\bar{x}, v_i)$ is minimal. In this paper we consider the modified problem of finding a point $\bar{x} \in X$ such that for $l, k > 0$ with $l + k < n$ there exists a permutation π of the indexes such that the sum $\sum_{i=l+1}^{l+k} d(\bar{x}, v_{\pi(i)})$ is minimal under the condition that $d(\bar{x}, v_{\pi(1)}) \leq d(\bar{x}, v_{\pi(2)}) \leq \cdots \leq d(\bar{x}, v_{\pi(n)})$ holds. This problem will be called the *ordered location problem* and was first studied from the algorithmic point of view for an arbitrary metric space in [4]. It turns out, that in the unrestricted case for $X = \mathbb{R}^n$ endowed with the Euclidean metric it is a quasi-differentiable optimization problem with a goal function which is a difference of convex functions.

An access to this problem is possible by the class of *ordered median functions,* which were introduced by S. Nickel and J. Puerto [6] for the treatment of continuous location problems. Roughly speaking these functions are weighted averages of ordered elements and can be defined as follows:

For a (row)-vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let us denote by $x_{\leq} = \big(x_{(1)}, \ldots, x_{(n)}\big)$ the rearrangement of the components of x, sorted by value, i.e. $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and consider the mapping

$$\mathrm{sort}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{given by} \quad x \mapsto x_{\leq}.$$

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Definition 1.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be given. Then the function $f_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called an ordered median function if for every $x \in \mathbb{R}^n$ holds $f_\lambda(x) = \langle \lambda, \mathrm{sort}_n(x) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Obviously every ordered median function is a continuous piecewise linear function on $\mathbb{R}^n$ and therefore a DCH-function (see [7]) , i.e. a function which can be represented as a difference of two continuous sublinear functions.

Throughout this paper we will mainly use the notation $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ instead of $x_{\pi(1)} \leq x_{\pi(2)} \cdots \leq x_{\pi(n)}$ for the ordered components of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

In [3] a simple explicit representation of an ordered median function as a difference of two continuous sublinear functions is given as follows:

Consider the following sublinear functions $\theta_1, \theta_2, \ldots, \theta_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by:

$$\begin{aligned}
\theta_1(x) &= x_1 + x_2 + \cdots + x_n, \\
\theta_2(x) &= \max \left\{ x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}} \ : \begin{array}{c} {\scriptstyle i_1, i_2, \ldots, i_{n-1} \in \{1,2,..,n\}} \\ i_1 < i_2 < \cdots < i_{n-1} \end{array} \right\}, \\
&\ \vdots \\
\theta_r(x) &= \max \left\{ x_{i_1} + x_{i_2} + \cdots + x_{i_{n+1-r}} \ : \begin{array}{c} {\scriptstyle i_1, i_2, \ldots, i_{n+1-r} \in \{1,2,..,n\}} \\ i_1 < i_2 < \cdots < i_{n+1-r} \end{array} \right\}, \\
&\ \vdots \\
\theta_{n-1}(x) &= \max \left\{ x_{i_1} + x_{i_2} \ : \begin{array}{c} {\scriptstyle i_1, i_2 \in \{1,2,..,n\}} \\ i_1 < i_2 \end{array} \right\}, \\
\theta_n(x) &= \max \{x_1, \ldots, x_n\}.
\end{aligned}$$

Let us remark that the sublinear functions $\theta_1, \theta_2, \ldots, \theta_n$ are also ordered median functions, known as *centrum functions* (see [6]) , where θ_r and generated by the vector $\lambda = (\underbrace{0, \ldots, 0}_{r-1 \text{ elements}}, \underbrace{1, \ldots, 1, 1, \ldots, 1}_{n-r+1 \text{ elements}}) \in \mathbb{R}^n$.

In [3], Remark 1 the following statement was proved:

Proposition 1.2. *Every ordered median function $f_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f_\lambda(x) = \langle \lambda, \mathrm{sort}_n(x) \rangle$ with $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ has the following DCH-representation:*

$$f_\lambda(x) = \langle \lambda, \mathrm{sort}_n(x) \rangle = \sum_{j=1}^n \rho_j \theta_j$$

with $\rho_1 = \lambda_1$ and $\rho_i = \lambda_i - \lambda_{i-1}$ for $i \in \{2, \ldots, n\}$.

Although the following Corollary is an immediate consequence of Proposition 1.2, we will give a direct proof of it.

Corollary 1.3. *For a binary vector* $\lambda = \big(\underbrace{0,\dots,0}_{l_1 \text{ elements}}, \underbrace{1,\dots,1}_{k \text{ elements}}, \underbrace{0,\dots,0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ *with block size* k, *left margin* l_1 *and right margin* l_2 *holds*

$$f_\lambda(x) = \langle \lambda, \mathrm{sort}_n(x)\rangle = \theta_{l_1+1} - \theta_{l_1+k+1}.$$

Proof. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ let $x_\leq = \big(x_{(1)}, \dots, x_{(n)}\big)$ be the rearrangement of the components of x, sorted by value, i.e. $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. Then $f_\lambda(x) = \sum_{i=l_1+1}^{l_1+k} x_{(i)} = \sum_{i=l_1+1}^{n} x_{(i)} - \sum_{i=l_1+k+1}^{n} x_{(i)} = \theta_{l_1+1}(x) - \theta_{l_1+k+1}(x)$, which finishes the proof. □

2. The ordered location problem

Ordered median functions have been used for the formulation of objectives for optimization problems in location theory (see [4] and [2]). Here we assume throughout the paper that the metric space (X, d) is given by $X = \mathbb{R}^m$ endowed with the Euclidean distance

$$d(x, y) = \|x - y\| = \sqrt{(\langle x - y, x - y\rangle)},$$

and request that the ordered median functions belong to the binary vector of the form $\lambda = \big(\underbrace{0,\dots,0}_{l_1 \text{ elements}}, \underbrace{1,\dots,1}_{k \text{ elements}}, \underbrace{0,\dots,0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ with block size $k > 0$, left margin $l_1 > 0$ and right margin $l_2 > 0$. Moreover let $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ a finite subset of pairwise disjoint points and for the Euclidean metric d on $\mathbb{R}^m$ define

$$D(x) = (d_1(x), \dots, d_n(x)) \in \mathbb{R}^n \text{ by } \ d_i(x) = d(x, v_i), \ x \in \mathbb{R}^m \text{ and } \ i \in \{1, 2, \dots, n\}.$$

Then the optimization problem

$$(\mathbf{OLP}) \qquad \min f_\lambda(D(x)) \text{ under } \ x \in \mathbb{R}^n$$

is called the *ordered location problem.* Note, that the ordered location problem is a quasi-differentiable optimization problem.

General Assumption:
The metric space (X, d) is given by $X = \mathbb{R}^m$ endowed with the Euclidean distance $d(x, y)$ and for every point $x_0 \in \mathbb{R}^m$ in which the goal function $(f_\lambda \circ D)$ has a local minimum the condition $x_0 \notin V = \{v_1, \dots, v_n\}$ holds. $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ a finite subset of pairwise disjoint points

Proposition 2.1. *For an arbitrary vector* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ *let* $f_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be the corresponding ordered median function. Moreover let* $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ *a finite subset of pairwise disjoint points and assume that* $x_0 \notin V = \{v_1, \dots, v_n\}$. *If additionally* $D(x_0) = (d_1(x_0), \dots, d_n(x_0)) \in \mathbb{R}^n$ *with* $d_i(x_0) = d(x_0, v_i)$ *consists of pairwise different components, then the function*

$$f_\lambda \circ D : \mathbb{R}^m \longrightarrow \mathbb{R} \text{ with } \ x \mapsto f_\lambda\,(D(x))$$

is continuously differentiable in a suitable neighborhood of $x_0 \in \mathbb{R}^m \setminus V$.

Proof. Let us choose a different representation of $f_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $f_\lambda(x) = \langle \lambda, \mathrm{sort}_n(x) \rangle = \sum_{i=1}^n \lambda_i \varphi_i(x)$ where the order by value is given by the arrangement functions $\varphi_1, \varphi_2, \ldots, \varphi_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ with:

$$\begin{aligned}
\varphi_1(x) &= \min\{x_1, \ldots, x_n\} \\
\varphi_2(x) &= \min\{\max\{x_i, x_j\} \;\mid\; i < j \text{ and } \; i, j \in \{1, 2, \ldots, n\}\} \\
\varphi_3(x) &= \min\{\max\{x_i, x_j, x_l\} \;\mid\; i < j < l \text{ and } \; i, j, l \in \{1, 2, .., n\}\} \\
&\;\;. \\
\varphi_r(x) &= \min\{\max\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \mid i_1 < i_2 < \cdots < i_r \text{ and } \; i_1, i_2, \cdots_r \in \{1, 2, \ldots, n\}\} \\
&\;\;. \\
\varphi_n(x) &= \max\{x_1, \ldots, x_n\}\,.
\end{aligned}$$

Since $x_0 \notin V$ the function $x \mapsto d_i(x)$ is smooth on $R^n \setminus \{v_i\}$ for the Euclidean metric. Since the vector $D(x_0) = (d_1(x_0), \ldots, d_n(x_0)) \in \mathbb{R}^n$ consists of pairwise different components the sequence $(\varphi_1(D(x_0)), \varphi_2(D(x_0)), \ldots, \varphi_n(D(x_0)))$ is strictly monotone. This holds also for $(\varphi_1(D(x)), \varphi_2(D(x)), \ldots, \varphi_n(D(x)))$ for every $x \notin V$ which is close to x_0. So every summand in this representation is continuously differentiable around x_0 which proves the assertion. □

Necessary Optimality Conditions:

Now we will turn our attention to necessary optimality conditions for the ordered location problem (OLP)

$$\min f_\lambda(D(x)) \text{ under } \; x \in \mathbb{R}^m$$

$$f_\lambda \circ D : \mathbb{R}^m \longrightarrow \mathbb{R} \text{ is given by } \; f_\lambda\left(D(x)\right) = \theta_{l_1+1}\left(D(x)\right) - \theta_{l_1+k+1}\left(D(x)\right).$$

Since f_λ is a DCH-function and the Euclidean distance is convex the composed function $f_\lambda \circ D$ is quasi-differentiable in the sense of V.F. Demyanov and A.M. Rubinov (see [1]). By quasi-differential calculus (see [1]) we get for the quasi-differential at $x_0 \in \mathbb{R}^n$ the following pair of compact convex sets

$$\textbf{(QD)} \qquad \mathcal{D}\left(f_\lambda \circ D\right)\Big|_{x_0} = \left(\partial\left(\theta_{l_1+1} \circ D\right)\Big|_{x_0},\; \partial\left(\theta_{l_1+k+1} \circ D\right)\Big|_{x_0}\right)$$

where "∂" denotes the convex subdifferential in $\mathbb{R}^m$ (see [8]). A necessary optimality condition for a local minimum in $x_0 \in \mathbb{R}^m$ (see [1] Theorem 16.4) is:

$$\textbf{(NOC)} \qquad \partial\left(\theta_{l_1+k+1} \circ D\right)\Big|_{x_0} \subseteq\; \partial\left(\theta_{l_1+1} \circ D\right)\Big|_{x_0}.$$

Similarly a sufficient optimality condition for $x_0 \in \mathbb{R}^m$ (see [1] Theorem 16.7) to be a strict local minimum is

$$\textbf{(SOC)} \qquad \partial\left(\theta_{l_1+k+1} \circ D\right)\Big|_{x_0} \subseteq\; \mathrm{int}\;\partial\left(\theta_{l_1+1} \circ D\right)\Big|_{x_0},$$

where $''\mathrm{int}\;(A)''$ denotes the interior of a set A.

Note furthermore that a point $x \in \mathbb{R}^m$ is in the interior of a convex set $A \subset \mathbb{R}^m$

of dimension $k \le m$ if an only if it is a convex combination with strict positive coefficients of $(k+1)$ affine independent elements of A.

Let us finally remark that the subdifferential of a finite maxima of convex functions is the convex hull of the gradients of the active functions in this point, i.e. for $r \in \{2, .., n\}$ one has:

(SD)

$$\partial\left(\theta_r \circ D\right)\big|_{x_0} = \operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1} + d_{i_2} + \cdots + d_{i_{n+1-r}}\right) \;\middle|\; \left(\theta_r \circ D\right)(x_0) = \left(d_{i_1} + d_{i_2} + \cdots + d_{i_{n+1-r}}\right)(x_0) \quad \text{and} \quad \begin{matrix} i_1, i_2, \ldots, i_{n+1-r} \in \{1, 2, .., n\} \\ i_1 < i_2 < \ldots. < i_{n+1-r} \end{matrix}\right\} \subset \mathbb{R}^n.$$

Proposition 2.2. *In the notation of an ordered location problem (OLP) let* $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ *be a finite set of pairwise disjoint points and* $\lambda = (\underbrace{0, \ldots, 0}_{l_1 \text{ elements}}, \underbrace{1, \ldots, 1}_{k \text{ elements}}, \underbrace{0, \ldots, 0}_{l_2 \text{ elements}}) \in \mathbb{R}^n$ *be given. Assume that* $f_\lambda \circ D$ *has a local minimum in in* $x_0 \in \mathbb{R}^m$ *and* $x_0 \notin V$. *Let us write for the vector* $D(x)$ *in the ordered form*

$$\operatorname{sort}_n(D(x_0)) = D(x_0)_{\le} = \left(d_{(1)}(x_0), d_{(2)}(x_0), \ldots, d_{(n)}(x_0)\right)$$

and assume that $d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ *holds. If* $f_\lambda \circ D$ *is not Gâteaux-differentiable in* $x_0 \in \mathbb{R}^m$ *then*

$$d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0).$$

Proof. First note that $f_\lambda(D(x)) : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $f_\lambda\left(D(x)\right) = \theta_{l_1+1}\left(D(x)\right) - \theta_{l_1+k+1}\left(D(x)\right)$ is quasi-differentiable at $x_0 \in \mathbb{R}^m$ and hence directional differentiable with quasi-differential

$$\mathcal{D}\left(f_\lambda \circ D\right)\big|_{x_0} = \left(\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}, \partial\left(\theta_{l_1+k+1} \circ D\right)\Big|_{x_0}\right).$$

Next observe that by formula (SD) and the assumption $d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ the set $\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0}$ is a singleton. Since $f_\lambda \circ D$ is not Gâteaux-differentiable in $x_0 \in \mathbb{R}^m$ the set $\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$ has at least two extremal points ([1], 10.1). Hence the ordered sequence $\operatorname{sort}_n(D(x_0)) = D(x_0)_{\le} = \left(d_{(1)}(x_0), d_{(2)}(x_0), \ldots, d_{(n)}(x_0)\right)$ must be of the form:

$$d_{(1)}(x_0) \le \cdots \le d_{(l_1)}(x_0) = \underbrace{d_{(l_1+1)}(x_0) \le d_{(l_1+2)}(x_0) \le \cdots \le d_{(n)}(x_0)}_{(n-l_1) \text{ elements}}$$

with the equality $d_{(l_1+1)}(x_0) = d_{(l_1)}(x_0)$. Now we show that $\nabla\big|_{x_0} d_{(l_1+1)} \neq \nabla\big|_{x_0} d_{(l_1)}$, because otherwise

$$\frac{x_0 - v_{(l_1+1)}}{d_{(l_1+1)}(x_0)} = \nabla\big|_{x_0} d_{(l_1+1)} = \nabla\big|_{x_0} d_{(l_1)} = \frac{x_0 - v_{(l_1)}}{d_{(l_1)}(x_0)}$$

which implies $v_{(l_1)} = v_{(l_1+1)}$ in contradiction to the definition of V. $\square$

Proposition 2.3. *In the notation of an ordered location problem (OLP) let* $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ *be a finite set of pairwise disjoint points and* $\lambda = \big(\underbrace{0,\dots,0}_{l_1 \text{ elements}}, \underbrace{1,\dots,1}_{k \text{ elements}}, \underbrace{0,\dots,0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ *be given. If for an* $x_0 \in \mathbb{R}^m \setminus V$ *in the ordered sequence of distances holds that* $d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)$ *and* $d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)$ *then* $x_0 \in \mathbb{R}^m \setminus V$ *is no solution of the ordered location problem*

$$\min f_\lambda(D(x)) \text{ under } \ x \in \mathbb{R}^m.$$

Proof. As seen in the above proof the set $\partial\,(\,\theta_{l_1+1} \circ D)\,\big|_{x_0}$ consists only of one element, whereas $\partial\,(\theta_{l_1+k+1} \circ D)\,\big|_{x_0}$ contains the line segment between two distinct elements. Hence the necessary optimality condition (NOC) is not satisfied for $x_0 \in \mathbb{R}^m$ and therefore x_0 is no local minimum point of $f_\lambda \circ D$ which implies that x_0 is no solution of the ordered location problem. $\square$

3. Types of local minima

By the necessary optimality condition (NOC) we have to consider for the optimization problem:

$$\min f_\lambda(D(x)) \text{ under } \ x \in \mathbb{R}^m$$

$f_\lambda \circ D : \mathbb{R}^m \longrightarrow \mathbb{R}$ given by

$$f_\lambda\,(D(x)) = \langle \lambda, \mathrm{sort}_n(D(x))\rangle = \theta_{l_1+1}\,(D(x)) - \theta_{l_1+k+1}\,(D(x))\,.$$

the two subdifferentials $\partial\,(\theta_{l_1+1} \circ D)\big|_{x_0}$ and $\partial\,(\theta_{l_1+k+1} \circ D)\big|_{x_0}$, where we assume that $f_\lambda \circ D$ has in $x_0 \in \mathbb{R}^m$ a local minimum. If we assume that the ordered sequence $\mathrm{sort}_n(D(x_0)) = D(x_0)_\leq = \big(d_{(1)}(x_0), d_{(2)}(x_0), \dots, d_{(n)}(x_0)\big)$ is of the form:

$$d_{(1)}(x_0) \leq \cdots \leq d_{(l_1)}(x_0) = \underbrace{d_{(l_1+1)}(x_0) \leq d_{(l_1+2)}(x_0) \leq \cdots \leq d_{(n)}(x_0)}_{(n-l_1) \text{ elements}}$$

then it follows from formula (SD) that in the case $d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)$ the subdifferential $\partial\,(\theta_{l_1+1} \circ D)\big|_{x_0}$ consists only of one element, whereas in the case $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$ it has at least two different extremal points as shown in the proof of Proposition 2.2.

Hence the structure of the subdifferential depends only on the condition whether $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$ or $d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)$ holds, which we take as a basis for classifying local minima. To get a better overview about all this cases we first give the following list of all formal logical combinations:

Nr.	List of all formal logical combinations
1	$\big(d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) = d_{(l_1+k+1)}(x_0)\big)$
2	$\big(d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big)$
3	$\big(d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) = d_{(l_1+k+1)}(x_0)\big)$
4	$\big(d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big)$
5	$\big(d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) = d_{(l_1+k+1)}(x_0)\big)$
6	$\big(d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big)$
7	$\big(d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) = d_{(l_1+k+1)}(x_0)\big)$
8	$\big(d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big) \;\wedge\; \big(d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)\big)$

The cases 3) and 7) are not possible, because the sequence of distances is ordered by value and the last condition contradicts the two previous ones. The cases 1) and 2) are treated in Proposition 2.3, where we proved that the first two conditions $\big(d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)\big)$ and $\big(d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)\big)$ already imply that $x_0 \in \mathbb{R}^m \setminus V$ is no solution of the ordered location problem. Weiszfeld conditions for the remaining cases will be given in the next section: for case 4) in Theorem 4.1 a), for case 5) in Theorem 4.2 b), for case 6) in Theorem 4.2 a), and finally for the case 8) in Theorem 4.1 b).

4. Necessary optimality conditions in Weiszfeld terms

In this section we derive Weiszfeld formulations for the ordered location problem. For a finite subset $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ of pairwise disjoint points and a binary vector $\lambda = \big(\underbrace{0,\ldots,0}_{l_1 \text{ elements}}, \underbrace{1,\ldots,1}_{k \text{ elements}}, \underbrace{0,\ldots,0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ with block size k, left margin l_1 and right margin l_2 we consider the optimization problem:

$$\min f_\lambda(D(x)) \text{ under } \; x \in \mathbb{R}^m$$

$f_\lambda \circ D : \mathbb{R}^m \longrightarrow \mathbb{R}$ given by

$$f_\lambda\left(D(x)\right) = \langle \lambda, \mathrm{sort}_n(D(x))\rangle = \theta_{l_1+1}\left(D(x)\right) - \theta_{l_1+k+1}\left(D(x)\right).$$

First we will fix some notation: For $x_0 \in \mathbb{R}^m$ and $l \in \{l_1+1, l_1+k+1\}$ we call:

$$\begin{aligned}\mathcal{I}(x_0, l) &= \{i \in \{1, \ldots, n\} \mid d(x_0, v_i) = \varphi_l\left(D(x_0)\right)\} \\ &= \left\{j \in \{1, \ldots, n\} \mid d_j(x_0) = d_{(l)}(x_0)\right\}\end{aligned}$$

the set of *"active indexes for $x_0 \in \mathbb{R}^m$ at the order-position l."*

If we assume, that

$$\mathrm{sort}_n(D(x_0) = D(x_0)_{\leq} = \left(d_{(1)}(x_0), d_{(2)}(x_0), \ldots, d_{(n)}(x_0)\right)$$

then one has for the ordered sequence (for instance at $l = l_1 + 1$) :

$$\begin{aligned} d_{(1)}(x_0) \leq d_{(2)}(x_0) \leq \cdots \leq d_{(l_1-r-1)}(x_0) &< d_{(l_1-r)}(x_0) = d_{(l_1-r+1)}(x_0) \\ &= d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0) = \cdots = d_{(l_1+s)}(x_0) \\ &< d_{(l_1+s+1)}(x_0) \leq \cdots \leq d_{(n)}(x_0)\end{aligned}$$

$(r+s)$ permuted indexes for equal distances in the positions $l_1 - r + 1, l_1 - r + 2, \ldots, l_1 - 1, l_1$ left of index $l_1 + 1$ and $l_1 + 2, l_1 + 3, \ldots, l_1 + s$ on the right hand side. So, on the left hand side of $l_1 + 1$ are r positions for equal distances and on the right hand side of $l_1 + 1$ are $(s - 1)$ positions for equal distances. Including $l_1 + 1$ to the right hand side index set, we will call the number s the *degree of summation* for $\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$. Hence $\mathcal{I}(x_0, l_1 + 1) = \{(l_1 - r + 1) \ldots, (l_1 - 1), (l_1), (l_1 + 1), \ldots, (l_1 + s)\}$ in the notation of the permuted indexes of the ordered sequence of distances.

Theorem 4.1. *Let* $\lambda = \big(\underbrace{0, \ldots, 0}_{l_1 \text{ elements}}, \underbrace{1, \ldots, 1}_{k \text{ elements}}, \underbrace{0, \ldots, 0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ *be a binary vector with block size k, left margin l_1 and right margin l_2 and f_λ the corresponding ordered median function. Moreover let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ be a finite subset of pairwise disjoint points and for the Euclidean metric d on $\mathbb{R}^m$, put $D(x) = (d_1(x), \ldots, d_n(x)) \in \mathbb{R}^n$ by $d_i(x) = d(x, v_i)$ for $x \in X$ and $i \in \{1, 2, \ldots, n\}$. Assume that $x_0 \notin V$ and that $f_\lambda \circ D$ has in $x_0 \in \mathbb{R}^m$ a local minimum and that* $\mathrm{sort}_n(D(x_0)) = D(x_0)_{\leq} = \left(d_{(1)}(x_0), d_{(2)}(x_0), \ldots, d_{(n)}(x_0)\right)$ *and $d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ hold.*

a) *If $d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)$ then*

$$x_0 = \frac{\sum_{i=l_1+1}^{l_1+k} \frac{v_{(i)}}{d_{(i)}(x_0)}}{\sum_{i=l_1+1}^{l_1+k} \frac{1}{d_{(i)}(x_0)}}.$$

b) *If $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$ and (in the notation of the permuted indexes of the ordered sequence of distances) the active index set at $x_0 \in \mathbb{R}^m$ on level $l_1 + 1$ with degree of summation s for $\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$ is*

$$\mathcal{I}(x_0, l_1 + 1) = \{(l_1 - r + 1), \ldots, (l_1 - 1), (l_1), (l_1 + 1), \ldots, (l_1 + s)\}$$

then there exists a point

$$p \in \mathrm{conv}\left\{v_{(l_1-r+1)}, v_{(l_1-r+2)}, \ldots, v_{(l_1)}, v_{(l_1+1)}, \ldots, v_{(l_1+s)}\right\}$$

such that

$$x_0 = \frac{s \cdot \left(\frac{p}{d*}\right) + \sum_{i=l_1+s+1}^{l_1+k} \frac{v_{(i)}}{d_{(i)}(x_0)}}{\frac{s}{d^*} + \sum_{i=l_1+s+1}^{l_1+k} \frac{1}{d_{(i)}(x_0)}}$$

holds with $d^* = d_{(l_1+1)}(x_0)$.

Proof. **a)** Since by assumption $d_{(l_1)}(x_0) \neq d_{(l_1+1)}(x_0)$ and $d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ we have

$$\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0} = \left\{\nabla\big|_{x_0}\left(d_{(l_1+1)} + d_{(l_1+2)} + \cdots + d_{(n)}\right)\right\} \qquad \text{and}$$

$$\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0} = \left\{\nabla\big|_{x_0}\left(d_{(l_1+k+1)} + d_{(l_1+k+2)} + \cdots + d_{(n)}\right)\right\}$$

and by the necessary optimality condition (NOC) $\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0} \subseteq \partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$ we get

$$\nabla\big|_{x_0}\left(d_{(l_1+1)} + d_{(l_1-1+2)} + \cdots + d_{(l_1+k)}\right) = 0.$$

Now $\nabla\big|_{x_0} d_i = \frac{x_0 - v_i}{d_i(x_0)}$, which gives:

$$\sum_{i=l_1+1}^{l_1+k} \frac{x_0 - v_{(i)}}{d_{(i)}(x_0)} = 0,$$

and finally

$$x_0 = \frac{\sum_{i=l_1+1}^{l_1+k} \frac{v_{(i)}}{d_{(i)}(x_0)}}{\sum_{i=l_1+1}^{l_1+k} \frac{1}{d_{(i)}(x_0)}}.$$

b) Now we consider the case where $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$ and assume that $d_{(l_1+k)}(x_0) \neq d_{(l_1+k+1)}(x_0)$. Then the ordered sequence of distances looks like:

$$\begin{aligned} &d_{(1)}(x_0) \leq d_{(2)}(x_0) \cdots \leq d_{(l_1-r)}(x_0) < d_{(l_1-r+1)}(x_0) = d_{(l_1-r+2)}(x_0) = \cdots \\ &\qquad = d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0), \cdots = d_{(l_1+s)}(x_0) < d_{(l_1+s+1)}(x_0) \leq \cdots \leq d_{(n)}(x_0) \end{aligned}$$

By subdifferential calculus (see [8]) one has:

$$\begin{aligned} \partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0} = \operatorname{conv}&\left\{\nabla\big|_{x_0}\left(d_{i_1} + d_{i_2} + \cdots + d_{i_s}\right) \;\middle|\; \begin{smallmatrix} i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s \end{smallmatrix}\right\} \\ &+\nabla\big|_{x_0}\left(d_{(l_1+s+1)} + d_{(l_1+s+2)} + \cdots + d_{(n)}\right) \end{aligned}$$

and

$$\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0} = \left\{\nabla\big|_{x_0}\left(d_{(l_1+k+1)} + d_{(l_1+k+2)} + \cdots + d_{(n)}\right)\right\}$$

By the necessary optimality condition (NOC) $\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0} \subseteq \partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$ we get

$$\begin{aligned} \{0\} \in \operatorname{conv}&\left\{\nabla\big|_{x_0}\left(d_{i_1} + d_{i_2} + \cdots + d_{i_s}\right) \;\middle|\; \begin{smallmatrix} i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s \end{smallmatrix}\right\} \\ &+\nabla\big|_{x_0}\left(d_{(l_1+s+1)} + d_{(l_1+s+2)} + \cdots + d_{(l_1+k)}\right). \qquad (\mathbf{NOC'}) \end{aligned}$$

Hence there exist real numbers $t_{i_1 i_2 < \cdots < i_s} \geq 0$ with $\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s}} t_{i_1 i_2 \cdots s} = 1$ such that

$$\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s}} t_{i_1 i_2 \cdots s} \left[\nabla\big|_{x_0} (d_{i_1} + d_{i_2} + \cdots + d_{i_s}) \right] + \nabla\big|_{x_0} \left(d_{(l_1+s+1)} + \cdots + d_{(l_1+k)} \right) = 0.$$

Since $\nabla\big|_{x_0} (d_i) = \frac{x_0 - v_i}{d_i(x_0)}$, this gives:

$$\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s}} t_{i_1 i_2 \cdots s} \left[\frac{s \cdot x_0 - (v_{i_1} + v_{i_2} + \cdots + v_{i_s})}{d^*} \right] + \left(\frac{x_0}{d_{(l_1+s+1)}} + \cdots + \frac{x_0}{d_{(l_1+k)}} \right)$$
$$- \left(\frac{v_{(l_1+s+1)}}{d_{(l_1+s+1)}} + \cdots + \frac{v_{(l_1+k)}}{d_{(l_1+k)}} \right) = 0$$

and finally:

$$\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s}} -t_{i_1 i_2 \cdots s} \left[\frac{(v_{i_1} + v_{i_2} + \cdots + v_{i_s})}{d^*} \right] + x_0 \cdot \left(\frac{s}{d^*} + \frac{1}{d_{(l_1+s+1)}} + \cdots + \frac{1}{d_{(l_1+k)}} \right)$$
$$- \left(\frac{v_{(l_1+s+1)}}{d_{(l_1+s+1)}} + \cdots + \frac{v_{(l_1+k)}}{d_{(l_1+k)}} \right) = 0$$

and

$$\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i_1 < i_2 < \cdots < i_s}} t_{i_1 i_2 \cdots s} \left[\frac{(v_{i_1} + v_{i_2} + \cdots + v_{i_s})}{d^*} \right] + \left(\frac{v_{(l_1+s+1)}}{d_{(l_1+s+1)}} + \cdots + \frac{v_{(l_1+k)}}{d_{(l_1+k)}} \right)$$
$$= x_0 \cdot \left(\frac{s}{d^*} + \frac{1}{d_{(l_1+s+1)}} + \cdots + \frac{1}{d_{(l_1+k)}} \right).$$

Put for $i \in \{(l_1 - r), (l_1 - r + 1), \ldots, (l_1), (l_1 + 1), \ldots, (l_1 + s)\ \} = \mathcal{I}(x_0, l_1 + 1)$,

$$\tau_i = \frac{1}{s} \left(\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1+1) \\ i \in \{i_1, .., i_s\}}} t_{i_1 i_2 \cdots s} \right) \qquad \text{and} \qquad p = \sum_{i \in \mathcal{I}(x_0, l_1)+1} \tau_i v_i$$

then

$$\sum_{\substack{i_1, i_2, \ldots, i_s \in \mathcal{I}(x_0, l_1) \\ i_1 < i_2 < \cdots < i_s}} t_{i_1 i_2 \cdots s} \left[\frac{(v_{i_1} + v_{i_2} + \cdots + v_{i_s})}{d^*} \right] = \frac{s}{d^*} \cdot \left(\sum_{i \in \mathcal{I}(x_0, l_1)} \tau_i v_i \right) = \frac{s}{d^*} \cdot p,$$

which gives

$$\left(s \cdot \frac{p}{d^*} + \frac{v_{(l_1+s+1)}}{d_{(l_1+s+1)}} + \cdots + \frac{v_{(l_1+k)}}{d_{(l_1+k)}} \right) = x_0 \cdot \left(\frac{s}{d^*} + \frac{1}{d_{(l_1+s+1)}} + \cdots + \frac{1}{d_{(l_1+k)}} \right)$$

and finishes the proof. $\square$

Theorem 4.2. *Let* $\lambda = \big(\underbrace{0,\dots,0}_{l_1 \text{ elements}}, \underbrace{1,\dots,1}_{k \text{ elements}}, \underbrace{0,\dots,0}_{l_2 \text{ elements}}\big) \in \mathbb{R}^n$ *be a binary vector with block size k, left margin l_1 and right margin l_2 and f_λ the corresponding ordered median function. Moreover let $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ be a finite subset of pairwise disjoint points and for the Euclidean metric d on $\mathbb{R}^m$, put $D(x) = (d_1(x), \dots, d_n(x)) \in \mathbb{R}^n$ by $d_i(x) = d(x, v_i)$ for $x \in X$, $i \in \{1, 2, \dots, n\}$. Assume that $x_0 \notin V$ and that $f_\lambda \circ D$ has in $x_0 \in \mathbb{R}^m$ a local minimum and that $\mathrm{sort}_n(D(x_0)) = D(x_0)_\leq = \big(d_{(1)}(x_0), d_{(2)}(x_0), \dots, d_{(n)}(x_0)\big)$ and $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$ and $d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)$ hold. Furthermore let us write in the notation of the permuted indexes of the ordered sequence of distances on level l_1+1 and level l_1+k+1*

$$\mathcal{I}(x_0, l_1+1) = \{(l_1 - r + 1), \dots, (l_1 - 1), (l_1), (l_1+1), \dots, (l_1+s)\}$$

and

$$\mathcal{I}(x_0, l_1+k+1) = \{(l_1 + k - \bar{r} + 1), \dots, (l_1+k), (l_1+k+1), \dots, (l_1+k+\bar{s})\}.$$

Then:

a) *If $d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ holds then for every set $S = \{i_1, \dots, i_{\bar{s}}\} \subset \mathcal{I}(x_0, l_1+k+1)$ of pairwise different indexes with $\bar{s}$ elements, there exists a point*

$$p \in \operatorname{conv}\left\{v_{(l_1-r+1)}, v_{(l_1-r+2)}, \dots, v_{(l_1)}, v_{(l_1+1)}, \dots, v_{(l_1+s)}\ \right\}$$

such that

$$x_0 = \frac{s \cdot \left(\frac{p}{d*}\right) + \sum_{i \in R \setminus S} \frac{v_i}{d_i(x_0)}}{\frac{s}{d^*} + \sum_{i \in R \setminus S} \frac{1}{d_i(x_0)}}$$

with $R = \{(l_1+s+1), .., (l_1+k), (l_1+k+1), \dots, (l_1+k+\bar{s})\}$ and $d^* = d_{(l_1+1)}(x_0)$ holds. Here the number s is the degree of summation for $\partial\left(\theta_{l_1+1} \circ D\right)\big|_{x_0}$.

b) *If $d_{(l_1+1)}(x_0) = d_{(l_1+k+1)}(x_0)$ or equivalently $\mathcal{I}(x_0, l_1+1) = \mathcal{I}(x_0, l_1+k+1)$ then for every $S = \{j_1, \dots, j_{\bar{s}}\} \subset \mathcal{I}(x_0, l_1+k+1)$ of pairwise different indexes with $\bar{s}$ elements there exists a point*

$$p \in \operatorname{conv}\left\{\mathcal{I}(x_0, l_1+k+1)\right\}$$

such that such that

$$x_0 = \frac{1}{k}\left(\sum_{i_t \in \mathcal{I}(x_0, l_1+k+1) \setminus S} v_{i_t} \ - \ r \cdot p\right),$$

with r taken from $\mathcal{I}(x_0, l_1+1)$. Note that r is also the degree of summation at the instance (l_1+k+1) for $\partial\left(\theta_{l_1+k+1} \circ D\right)\big|_{x_0}$.

Proof. a) This part of the proof is similar to the proof of Theorem 4.1 b). Now by assumption $d_{(l_1)}(x_0) = d_{(l_1+1)}(x_0)$, $d_{(l_1+k)}(x_0) = d_{(l_1+k+1)}(x_0)$ and $d_{(l_1+1}(x_0) \neq d_{(l_1+k+1)}(x_0)$. Hence we have

$$\partial\left(\theta_{l_1+1} \circ D\right)\Big|_{x_0} = \operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1} + d_{i_2} + \dots + d_{i_s}\right) \ \Big| \ \begin{matrix} {\scriptstyle i_1, i_2, \dots, i_s \in \mathcal{I}(x_0, l_1+1)} \\ i_1 < i_2 < \dots < i_s \end{matrix}\right\}$$
$$+\nabla\big|_{x_0}\left(d_{(l_1+s+1)} + d_{(l_1+s+2)} + \dots + d_{(n)}\right)$$

and

$$\partial\left(\theta_{l_1+k+1}\circ D\right)\Big|_{x_0} = \operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_{\bar{s}}}\right) \;\Big|\; \begin{matrix} {\scriptstyle i_1,i_2,\ldots,i_{\bar{s}}\in\mathcal{I}(x_0,l_1+k+1)} \\ i_1<i_2<\cdots<i_{\bar{s}} \end{matrix}\right\}$$
$$+\nabla\big|_{x_0}\left(d_{(l_1+\bar{s}+1)}+d_{(l_1+\bar{s}+2)}+\cdots+d_{(n)}\right).$$

Since $d_{(l_1+1)}(x_0) \neq d_{(l_1+k+1)}(x_0)$ the necessary optimality condition (NOC) $\partial\left(\theta_{l_1+k+1}\circ D\right)\big|_{x_0} \subseteq \partial\left(\theta_{l_1+1}\circ D\right)\big|_{x_0}$ is equivalent to the condition, that for every $S=\{i_1,\ldots,i_{\bar{s}}\}\subset\mathcal{I}(x_0,l_1+k+1)$ of pairwise different indexes with $\bar{s}$ elements

$$\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_{\bar{s}}}\right) + \nabla\big|_{x_0}\left(d_{(l_1+\bar{s}+1)}+d_{(l_1+\bar{s}+2)}+\cdots+d_{(n)}\right)$$
$$\in \partial\left(\theta_{l_1+1}\circ D\right)\big|_{x_0}$$

which gives that:

$$0\in\operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_s}\right) \;\Big|\; \begin{matrix} {\scriptstyle i_1,i_2,\ldots,i_s\in\mathcal{I}(x_0,l_1+1)} \\ i_1<i_2<\cdots<i_s \end{matrix}\right\}$$
$$+\nabla\big|_{x_0}\left(\sum_{j\in R\setminus S} d_j\right)$$

where $R=\{(l_1+s+1),\ldots,(l_1+k),(l_1+k+1),\ldots,(l_1+k+\bar{s})\}$ is the set of remaining indexes. Now we can proceed exactly as in the proof of the above Theorem 4.1 after equation (NOC') to derive the claimed result.

b) By assumption we have

$$\mathcal{I}(x_0,l_1+1)=\mathcal{I}(x_0,l_1+k+1)$$
$$=\{(l_1-r+1),..,(l_1),(l_1+1),..,(l_1+k),(l_1+k+1),\ldots,(l_1+k+\bar{s})\}$$

with $r+\bar{s}+k$ elements. Since $k+\bar{s}=s$ we have

$$\partial\left(\theta_{l_1+1}\circ D\right)\Big|_{x_0} = \operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_{k+\bar{s}}}\right) \;\Big|\; \begin{matrix} {\scriptstyle i_1,i_2,\ldots,i_s\in\mathcal{I}(x_0,l_1+1)} \\ i_1<i_2<\cdots<i_{k+\bar{s}} \end{matrix}\right\}$$
$$+\nabla\big|_{x_0}\left(d_{(l_1+k+\bar{s}+1)}+d_{(l_1+k+\bar{s}+2)}+\cdots+d_{(n)}\right)$$

and

$$\partial\left(\theta_{l_1+k+1}\circ D\right)\Big|_{x_0} = \operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_{\bar{s}}}\right) \;\Big|\; \begin{matrix} {\scriptstyle i_1,i_2,\ldots,i_{k+\bar{s}}\in\mathcal{I}(x_0,l_1+k+1)} \\ i_1<i_2<\cdots<i_{\bar{s}} \end{matrix}\right\}$$
$$+\nabla\big|_{x_0}\left(d_{(l_1+k+\bar{s}+1)}+d_{(l_1+k+\bar{s}+2)}+\cdots+d_{(n)}\right).$$

Now the necessary optimality condition (NOC) $\partial\left(\theta_{l_1+k+1}\circ D\right)\big|_{x_0} \subseteq \partial\left(\theta_{l_1+1}\circ D\right)\big|_{x_0}$ is equivalent to the condition, that for every $S=\{j_1,\ldots,j_{\bar{s}}\}\subset\mathcal{I}(x_0,l_1+k+1)$ of pairwise different indexes with $\bar{s}$ elements which gives that:

$$\nabla\big|_{x_0}\left(d_{j_1}+d_{j_2}+\cdots+d_{j_{\bar{s}}}\right)$$
$$\in\operatorname{conv}\left\{\nabla\big|_{x_0}\left(d_{i_1}+d_{i_2}+\cdots+d_{i_{k+\bar{s}}}\right) \;\Big|\; \begin{matrix} {\scriptstyle i_1,i_2,\ldots,i_{k+\bar{s}}\in\mathcal{I}(x_0,l_1+1)} \\ i_1<i_2<\cdots<i_{k+\bar{s}} \end{matrix}\right\}.$$

Hence there exist real numbers $t_{i_1 i_2 \cdots k+\bar{s}} \geq 0$ with $\sum_{\substack{i_1,i_2,\ldots,i_{k+\bar{s}} \in \mathcal{I}(x_0,l_1+k+1) \\ i_1<i_2<\cdots<i_{k+\bar{s}}}} t_{i_1 i_2 \cdots k+\bar{s}} = 1$ such that

$$\sum_{\substack{i_1,i_2,\ldots,i_{k+\bar{s}} \in \mathcal{I}(x_0,l_1+k+1) \\ i_1<i_2<\cdots<i_{k+\bar{s}}}} t_{i_1 i_2 \cdots k+\bar{s}} \left[\nabla\big|_{x_0} \left(d_{i_1} + d_{i_2} + \cdots + d_{i_{k+\bar{s}}}\right) - \nabla\big|_{x_0} \left(d_{j_1} + d_{j_2} + \cdots + d_{j_{\bar{s}}}\right) \right]$$

$$(*) \qquad = 0.$$

Rewriting this formula by ordering with respect to the terms $\nabla\big|_{x_0} d_{i_i}$ gives for the coefficients c_{i_t} of $\nabla\big|_{x_0} d_{i_t}$ for the indexes $i_t \in \mathcal{I}(x_0, l_1+k+1) \setminus S$ and for $i_{\bar{t}} \in S$ resp.

$$c_{i_t} = \left(\sum_{\substack{i_1,i_2,\ldots,i_{k+\bar{s}} \in \mathcal{I}(x_0,l_1+k+1) \\ i_t \in \{i_1,i_2,\ldots,i_{k+\bar{s}}\} \\ i_1<i_2<\cdots<i_{k+\bar{s}}}} t_{i_1 i_2 \cdots k+\bar{s}} \right) \text{ resp. } \bar{c}_{i_{\bar{t}}} = \left(\sum_{\substack{i_1,i_2,\ldots,i_{k+\bar{s}} \in \mathcal{I}(x_0,l_1+k+1) \\ i_{\bar{t}} \notin \{i_1,i_2,\ldots,i_{k+\bar{s}}\} \\ i_1<i_2<\cdots<i_{k+\bar{s}}}} t_{i_1 i_2 \cdots k+\bar{s}} \right).$$

Hence formula $(*)$ can be written as:

$$\sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)\setminus S} c_{i_t} \nabla\big|_{x_0} d_{i_t} \; - \; \sum_{i_t \in S} \bar{c}_{i_t} \nabla\big|_{x_0} d_{i_t} = 0.$$

Since $\sum_{\substack{i_1,i_2,\ldots,i_{k+\bar{s}} \in \mathcal{I}(x_0,l_1+k+1) \\ i_1<i_2<\cdots<i_{k+\bar{s}}}} t_{i_1 i_2 \cdots k+\bar{s}} = 1$ which implies $c_{i_t} = (1 - \bar{c}_{i_t})$ the above formula can be written as:

$$\sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)\setminus S} \nabla\big|_{x_0} d_{i_t} \; - \sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)} \bar{c}_{i_t} \nabla\big|_{x_0} d_{i_t} = 0.$$

Similar as at the end of the proof of Theorem 4.1 b) one can prove that every $t_{i_1 i_2 \cdots k+\bar{s}}$ appears in the collection of all sums $\bar{c}_{i_t}$ exactly r times (which is the degree of summation for $\partial \left(\theta_{l_1+k+1} \circ D\right)\Big|_{x_0}$), which gives finally that formula $(*)$ can be written as:

$$\sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)\setminus S} \nabla\big|_{x_0} d_{i_t} \; - \; r \cdot \sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)} \bar{\tau}_{i_t} \nabla\big|_{x_0} d_{i_t} = 0$$

with $\bar{\tau}_{i_t} = \frac{1}{r} \bar{c}_{i_t}$ and $\sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)} \bar{\tau}_{i_t} = 1$.

Since $\nabla\big|_{x_0} d_i = \frac{x_0 - v_i}{d_i(x_0)}$, and all distances are equal this gives:

$$(k+r) \cdot x_0 - \sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)\setminus S} v_{i_t} \; - \; r \cdot x_0 \; + \; r \cdot \sum_{i_t \in \mathcal{I}(x_0,l_1+k+1)} \bar{\tau}_{i_t} v_{i_t} = 0$$

which finishes the proof. $\square$

5. Examples

In this section we will discuss four typical examples of an ordered location problem for which in the first case the goal function is smooth in the solution point and in all other cases nonsmooth.

The smooth case

We begin with the case where the goal function is smooth in the solution point $x_0 \in \mathbb{R}^2$ and recall that a necessary optimality condition for this case is given in Theorem 4.1 a).

Therefore take $n = 6$, $l_1 = 1$, $k = 3$ and $l_2 = 2$, and choose for the points $V = \{v_1, v_2, \dots, v_6\} \subset \mathbb{R}^2$ (see Figure 1)

$$v_1 = (0, -\frac{1}{2}), \quad v_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}), \quad v_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$$

and

$$v_4 = (0, 1), \quad v_5 = (\sqrt{2}, 1), \quad v_6 = (0, 2).$$

Then for $\lambda = (0, 1, 1, 1, 0, 0) \in \mathbb{R}^6$ the goal function

$$f_\lambda(D(x)) = \langle \lambda, \mathrm{sort}_n(D(x)) \rangle = \theta_2(D(x)) - \theta_5(D(x))$$

has to be minimized for $x \in \mathbb{R}^2$. Here θ_2, θ_5 are centrum functions and $D(x) = (d_1(x), \dots, d_6(x))$ is the vector of the Euclidean distances $d_i(x) = d(x, v_i)$.

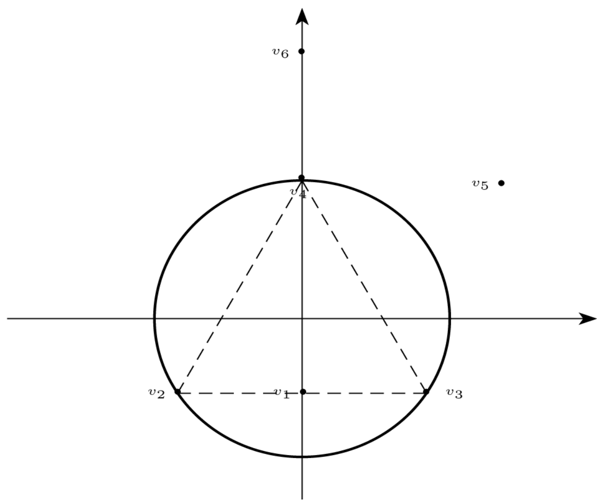

Figure 1. Ordered location problem for $\lambda = (0, 1, 1, 1, 0, 0)$

It follows from [5] Theorem 4.33 that the solution of the above ordered location problem is $x_0 = (0, 0)$. This is also the solution of the location problem minimize $(d_2(x) + d_3(x) + d_4(x))$ for $x \in \mathbb{R}^2$, i.e. for the minimal distance problem of the points $V = \{v_2, v_3, v_4\} \subset \mathbb{R}^2$, which is a smooth convex optimization problem. In this solution point $x_0 \in \mathbb{R}^2$ the goal function is differentiable and for the vector of distances $D(x) = (d_1(x), \ldots, d_6(x))$ with $d_i(x) = d(x, v_i)$, in the Euclidean distances metric holds:

$$d_1(x_0) \ < \ d_2(x_0) \ = \ d_3(x_0) \ = \ d_4(x_0) \ < \ d_5(x_0) \ < \ d_6(x_0).$$

The nonsmooth case

Now we consider an example where the goal function is nonsmooth in the solution point $x_0 \in \mathbb{R}^2$ and recall that a necessary optimality condition for this case is given in Theorem 4.1 b).

Therefore take $n = 8$, $l_1 = 2$, $k = 4$ and $l_2 = 2$, and choose the points $V = \{v_1, v_2, \ldots, v_8\} \subset \mathbb{R}^2$ as follows:

$$v_1 = (0.0, 2.5), \quad v_2 = (2.0, 1.5), \quad v_3 = (3.0, 0.0), \quad v_4 = (2.0, -2.5)$$
$$v_5 = (0.0, -3.0), \quad v_6 = (-3.0, -2.5), \quad v_7 = (-3.5, 0.0) \quad v_8 = (-2.0, 2.0).$$

The polytope is depicted in Figure 2. By evaluating condition (NOC') numerically,

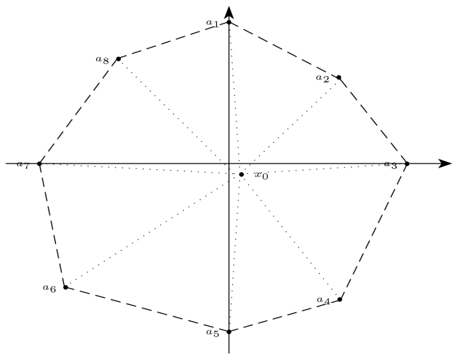

FIGURE 2. Ordered location problem for $\lambda = (0, 0, 1, 1, 1, 1, 0, 0)$

we get for the optimal solution x_0 of the corresponding optimization problem with the goal function

$$f_\lambda(D(x)) = \langle \lambda, \mathrm{sort}_n(D(x)) \rangle = \theta_3(D(x)) - \theta_7(D(x))$$

$x_0 = (0.25, -0.25)$. In this point the goal function is not differentiable and for the vector of distances $D(x) = (d_1(x), \ldots, d_8(x))$ with $d_i(x) = d(x, v_i)$ in the Euclidean metric holds:

$$d_2(x_0) \ < \ d_1(x_0) \ = \ d_3(x_0) \ = \ d_5(x_0) \ < \ d_4(x_0) \ < \ d_8(x_0) \ < \ d_7(x_0) \ < \ d_6(x_0).$$

It follows from the numerical evaluation of (NOC') that the artificial point p from the necessary optimality condition in Theorem 4.1 b) is $p = (0.3786, -0.13644) \in \mathrm{conv}\,\{v_1, v_3, v_5\}$ and the degree of summation is 2.

The case of equal distances

The next two examples concern both parts of Theorem 4.2. We consider two subcases for the following $n = 8$ points, namely $V = \{v_1, v_2, \dots, v_8\} \subset \mathbb{R}^2$

$$v_1 = (1.0, 1.0), \quad v_2 = (1.0, -1.0), \quad v_3 = (-1.0, 1.0), \quad v_4 = (-1.0, -1.0)$$

$$v_5 = (\sqrt{2}, 0.0), \quad v_6 = (0.0, \sqrt{2}), \quad v_7 = (0.0, -\sqrt{2}), \quad v_8 = (-\sqrt{2}, 0.0)$$

and take $x_0 = (0.0, 0.0)$. All distances $d(x_0, v_i) = \sqrt{2},\ i = 1, \dots, 8$ are equal.

a) Now take $n = 8,\ l_1 = 2,\ k = 4$ and $l_2 = 2$, Then $\lambda = (0, 0, 1, 1, 1, 1, 0, 0) \in \mathbb{R}^8$ and the goal function

$$f_\lambda(D(x)) = \langle \lambda, \mathrm{sort}_n(D(x)) \rangle = \theta_3(D(x)) - \theta_7(D(x))$$

has to be minimized for $x \in \mathbb{R}^2$. In this case $x_0 = (0.0, 0.0)$ is a solution of the corresponding ordered location problem, we have $\partial\,(\theta_3 \circ D)\big|_{x_0} = \partial\,(\theta_7 \circ D)\big|_{x_0} = \frac{1}{2}\sqrt{2}A$, where $A \subset \mathbb{R}^2$ is depicted in Figure 4. The function $x \mapsto \theta_3(D(x)) - \theta_7(D(x))$ and the level lines around x_0 are depicted in Figure 3.

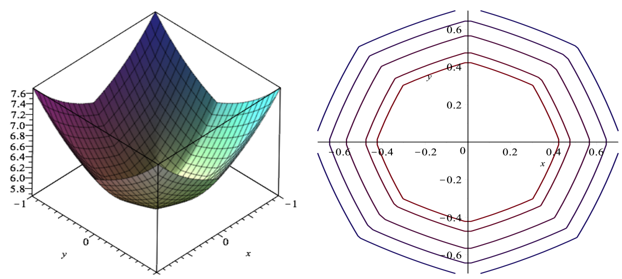

FIGURE 3. Plot of $x \mapsto \theta_3(D(x)) - \theta_7(D(x))$ with level lines around x_0

b) Now take $n = 8,\ l_1 = 1,\ k = 5$ and $l_2 = 2$, Then $\lambda = (0, 1, 1, 1, 1, 1, 0, 0) \in \mathbb{R}^8$. Now the goal function

$$f_\lambda(D(x)) = \langle \lambda, \mathrm{sort}_n(D(x)) \rangle = \theta_2(D(x)) - \theta_7(D(x))$$

has to be minimized for $x \in \mathbb{R}^2$. But it turns out that $x_0 = (0.0, 0.0)$ is now not a local minimum, but even a local maximum of the goal function and satisfies the sufficient optimality condition (SOC). We have $\partial\,(\theta_2 \circ D)\big|_{x_0} = \frac{1}{2}\sqrt{2}B$ and $\partial\,(\theta_7 \circ D)\big|_{x_0} = \frac{1}{2}\sqrt{2}A$, where the subsets $A, B \subset \mathbb{R}^2$ are depicted in the Figure 4.

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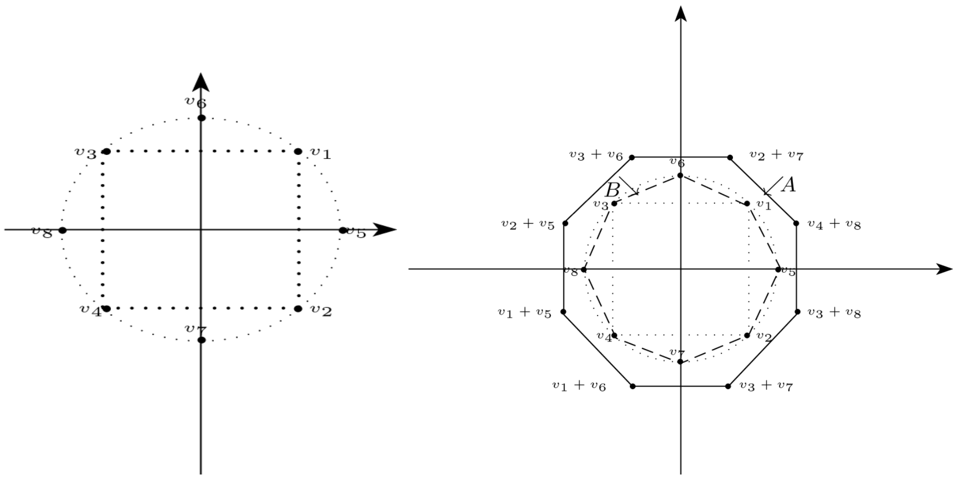

FIGURE 4. The sets V, A and B

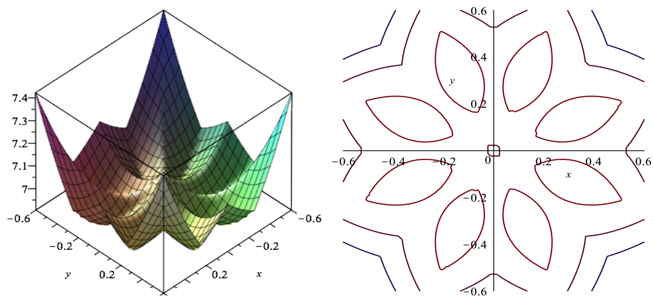

FIGURE 5. Plot of $x \mapsto \theta_2(D(x)) - \theta_7(D(x))$ with level lines around x_0

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J. Grzybowski
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, PL-61-614 Poznań, Poland
E-mail address: jgrz@amu.edu.pl

D. Pallaschke
Institute of Operations Research, University of Karlsruhe, Karlsruhe Institute of Technology (KIT), D-76131 Karlsruhe, Germany
E-mail address: diethard.pallaschke@kit.edu

R. Urbański
Faculty of Mathematics and Computer Science, Adam Mickiewicz University , PL-61-614 Poznań, Poland
E-mail address: rich@amu.edu.pl