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AN ALGEBRAIC PROPERTY OF THE BOUNDARIES OF BANACH SPACES

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Abstract. Some connections between the concepts of boundary and of norming set of a Banach space and the linear structure are investigated. In particular we prove that, if X is a Banach space which does not contain an isomorphic copy of c_0, $B \subset S_{X^*}$ is a boundary of X and H is a maximal linearly independent subset of B, then H is norming.

Let $(X, \|\cdot\|)$ be a Banach space. A subset $B \subset S_{X^*}$ of the unit sphere S_{X^*} of X is called a boundary of X if for any $x \in X$ there exists $f \in B$ with $f(x) = \|x\|$. From the Krein-Milman theorem it follows that the set $\mathrm{ext}B_{X^*}$ of all the extreme points of the unit ball B_{X^*} of X^* is a boundary. Easy examples show that a boundary may be a proper subset of $\mathrm{ext}B_{X^*}$. The separation theorem shows that, for any boundary B, we have $w^* - \mathrm{cl}\,\mathrm{co}B = B_{X^*}$. In particular it follows that, when X is infinite-dimensional, any of its boundaries must be infinite, so at least countable. This is the case of separable polyhedral spaces (see [1, 2]). Therefore, if X is not polyhedral, then any of its boundaries is uncountable, i.e. is massive in the sense of cardinality.

Recall that, for $r > 0$, a subset $C \subset B_{X^*}$ is called $r-$norming if

$$\sup_{f \in C} |f(x)| \geq r\|x\|$$

for every $x \in X$. C is called norming if it is $r-$norming for some $r > 0$. It easily follows from the separation theorem that a set C is $r-$norming if and only if

$$w^* - \mathrm{cl}\ \mathrm{co}(\pm C) \supset rB_{X^*}.$$

Our first result results provides a connection between Hamel bases and norming sets in the dual space.

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Proposition 1. *Let X be a Banach space and $H \subset S_{X^*}$ be a Hamel basis of X^*. Then H is a norming set.*

Proof. Any $f \in B_{X^*}$ can be represented in exactly one way as

$$f = \sum_{i\in\sigma_f,\ |\sigma_f|<\infty,\ f_i\in H} a_i(f)f_i.$$

For $n = 1, 2, ...$ put

$$B_n = \{f \in B_{X^*} : f = \sum a_i(f)f_i,\ \sum |a_i(f)| < n\}.$$

Clearly, any B_n is convex and symmetric, the sequence $\{B_n\}$ is increasing and $B_{X^*} = \cup_{n=1}^{\infty} B_n$. By the Baire category an index m and $\delta > 0$ exist such that

$$w^* - \mathrm{cl}B_m \supset \delta B_{X^*}.$$

It follows that

$$w^* - \mathrm{cl}\,\mathrm{co}(\pm H) \supset \frac{1}{m} w^* - \mathrm{cl}B_m \supset \frac{\delta}{m} B_{X^*}$$

so H turns out to be $\delta/m-$norming. The proof is complete. □

Clearly not any norming set contains a Hamel basis.
It is natural to ask whether any boundary of X contains a Hamel basis of X^*. The answer is in the negative (think of separable polyhedral spaces, e.g. c_0). However, as the following two examples show, the answer is in the negative even for reflexive spaces.

Example A. In [6] an example of an incomplete normed space L is given such that every functional $f \in L^*$ attains its norm on B_L. By the James theorem the completion $\tilde{L}$ of L is reflexive. If we consider L as a subspace of L^{**}, then the set $B = B_L$ is a boundary for L^*. Clearly, span $B = L \neq \tilde{L} = L^{**}$.

Notice that R.C. James constructed his example as an attempt to answer the following question by F. Deutsch (see [6]): "Is it true that a normed space X is complete if, for each convex closed subset $K \subset X$, each point $x \in X$ has a closest point x_0 in K ?" Since (for $x \notin K$) any such point x_0 must be a support point of K (Hahn-Banach separation theorem), the answer (for separable X) follows from the following Theorem.

Theorem 2 ([4])**.** *A separable normed space X is complete if and only if any convex closed bounded subset of X has a support point.*

Example B. We use some ideas from [7]. For any subset A of a linear space and any positive integer k, denote by $\mathrm{span}_k A$ the set $\{\sum_{i\in\sigma\ |\sigma|=k} a_i x_i : a_i \in \mathbb{R},\ x_i \in A\}$. Let $\{E_n\}$ be a sequence of finite-dimensional spaces such that $\dim E_n = n$ and B_{E_n} is a polytope for any $n = 1, 2, ...$ The space $X = \oplus_{l_2} \sum_{n=1}^{\infty} E_n$ is reflexive and

$$\mathrm{ext}B_{X^*} = \{\{f_i\}_{i=1}^{\infty} : \sum \|f_i\|^2 = 1, f_i/\|f_i\| \in \mathrm{ext}B_{E_i^*}, i = 1, 2, ...\}.$$

We claim that

$$\text{span ext}B_{X^*} \neq X^*.$$

In fact two complementary cases can be considered.

(a) There exists an integer k such that, for every $n > k$ and every $f \in E_n^*$, it happens that $f \in \text{span}_k\text{ext}B_{E_n^*}$.

(b) For every integer k there exists an index $n_k > k$ such that, for some $h_{n_k} \in E_{n_k}^*$, it happens that $h_{n_k} \notin \text{span}_k\text{ext}B_{E_{n_k}^*}$, where without loss of generality we can assume $\|h_{n_k}\| < \frac{1}{n_k^2}$.

We investigate both cases.

Case (a). Consider the space E_{k+1}^* and put

$$\mathcal{A} = \{\{g_i\}_{i=1}^k : g_i \in \text{ext}B_{E_{k+1}^*}, i = 1, ..., k\}.$$

Since E_{k+1} is polyhedral, it follows that $\mathcal{A}$ is a finite set. We are in case (a), so we have

$$\cup_{\{g_i\}_1^k \in \mathcal{A}} \text{span}\{g_i\}_1^k = E_{k+1}^*$$

that contradicts the Baire category theorem.

Case (b). Put $f = (f_j)_{j=1}^\infty$ where

$$f_{n_k} = h_{n_k}, \quad f_j = 0 \; j \neq n_k, \quad k = 1, 2, ... \, .$$

It is not difficult to see that $f \in X^*$, however $f \notin \text{span ext}B_{X^*}$.

We are done. $\square$

The following property of boundaries was established in [3]. It has many applications in the Geometry of Banach spaces as well as in the Function Theory (see e.g. [3, 5]).

Theorem 3. *Assume that X is a Banach space which does not contain an isomorphic copy of c_o. Let B be a boundary of X, $B = \cup_{n=1}^\infty B_n$ with $\{B_n\}_{n=1}^\infty$ an increasing sequence. Then there is an index m such that the set B_m is norming.*

The main purpose of this paper is to show that, if a Banach space X does not contain an isomorphic copy of c_0, then any boundary of X has a sort of "algebraic - linear - topological" massiveness property. That will be done with the aid of Theorem 3.

Theorem 4. *Let X be a Banach space which does not contain an isomorphic copy of c_0, and B be a boundary of X. Assume that $K \subset B_{X^*}$ is a w^*-compact set such that*

(M) *for any $f \in B$ there is a finite Borel positive measure $\mu = \mu_f$ on K representing f, i.e.*

$$f(x) = \int_K t(x) d\mu(t) \quad \forall x \in X.$$

Then the set K is norming. Conversely, if X is separable and contains an isomorphic copy of c_0, then there are an equivalent norm $|||\cdot|||$ on X and a w^*-compact non-norming subset $K \subset B_{(X,|||\cdot|||)^*}$ such that for any $f \in B = \mathrm{ext}B_{(X,|||\cdot|||)^*}$ there is a Borel positive measure on K representing f.

Proof. We start with the proof of the first implication in the statement. Here the proof runs along the lines of the proof of Proposition 1 but, instead of using the Baire category theorem, we use Theorem 3.
Put $B_n = \{f \in B : \mu_f(K) \leq n\},\ n = 1, 2, \ldots$. Clearly, $\{B_n\}$ is increasing and $B = \cup_{n=1}^{\infty} B_n$. By Theorem 3 there is an index m such that B_m is $r-$norming for some $r > 0$. Take $x \in S_X$ and find $f \in B_m$ with $|f(x)| \geq r.$. We have

$$r \leq |f(x)| = |\int_K t(x)d\mu_f(t)| \leq \int_K |t(x)|d\mu(t) \leq \sup_{t\in K} |t(x)|\mu_f(K) \leq m \sup_{t\in K} |t(x)|,$$

hence $\sup_{t\in K} |t(x)| \geq r/m$. Therefore the set K is $r/m-$norming.

Now assume that X is separable and contains a subspace Y isomorphic to c_0. By the Sobchyk theorem Y is complemented in X, i.e. $X = Y \oplus Z$. For any $n \in \mathbb{N}$ denote by M_n the 2-dimensional Euclidean space, i.e. $M_n = \mathbb{R}^2$. Consider the Banach space $L = \oplus_{c_0} \sum_{n=1}^{\infty} M_n$. It is not difficult to see that L is isomorphic to c_0. Moreover, $L^* = \oplus_{l_1} \sum_{n=1}^{\infty} M_n^*$ and $\mathrm{ext}B_{L^*} = \cup_{n=1}^{\infty} S_{M_n^*}$. Put $X_1 = L \oplus_\infty Z$. Clearly X_1 is isomorphic to X, hence we can consider X_1 as X in that equivalent norm $|||\cdot|||$. Clearly, $X_1^* = L^* \oplus_{l_1} Z^*$ and $\mathrm{ext}B_{X_1^*} = \mathrm{ext}B_{L^*} \cup \mathrm{ext}B_{Z^*} = \cup_{n=1}^{\infty} S_{M_n^*} \cup \mathrm{ext}B_{Z^*}$. Next we choose in each M_n a basis $\{u_n, v_n\}$ such that the associated linear functionals $\{u_n^*, v_n^*\} \subset M_n^*$ have the following properties: $|||u_n^*||| = |||v_n^*||| = 1,\ |||u_n^* - v_n^*||| < 1/n$. Note that subspaces $[u_n^*]_{n=1}^{\infty}$ and $[v_n^*]_{n=1}^{\infty}$ (the $|||\cdot|||-$closed linear spans of $\{u_n^*\}_{n=1}^{\infty}$ and $\{v_n^*\}_{n=1}^{\infty}$) are w^*-closed. In fact both sequences $\{u_n^*\}_{n=1}^{\infty}$ and $\{v_n^*\}_{n=1}^{\infty}$ are just l_1-bases and w^*-convergence in subspaces they generate is just a "coordinate$-$wise" convergence (note that the sequences of the functionals associated to these bases are respectively $\{u_n\} \subset X$ and $\{v_n\} \subset X$). Moreover, since $|||u_n^* - v_n^*||| < 1/n$, it follows that the direct sum $[u_n^*]_{n=1}^{\infty} + [v_n^*]_{n=1}^{\infty}$ is not $|||\cdot|||-$closed.

Put $K = w^* - \mathrm{cl}\{\pm\{u_n^*, v_n^*\}_{n=1}^{\infty} \cup \mathrm{ext}B_{Z^*}\}$: an easy consideration shows that K satisfies condition (M) with $B = \mathrm{ext}B_{X_1^*}$. However, K is not norming.

The proof is complete. □

Remark. Let X be a Banach space such that X^* is strictly convex. Put $B = \mathrm{ext}B_{X^*} = S_{X^*}$. If $K \subset B_{X^*}$ satisfies condition (M), then a category argument shows that K is norming. This explains why referring to an equivalent norm in the second part of the statement of Theorem 4 is necessary.

One, in our opinion significant, application of Theorem 4 is the next Theorem.

Theorem 5. *Let X be a Banach space which does not contain an isomorphic copy of c_0. Assume that $B \subset S_{X^*}$ is a boundary and H is a maximal linearly independent subset of B. Then H is norming. Hence $w^* - \mathrm{cl}\ \mathrm{co}H \supset rB_{X^*}$ for some $r > 0$.*

Proof. First note that H is norming if and only if $w^* - \mathrm{cl}H$ is norming. So without loss of generality we can assume that H is $w*-$ compact. Clearly, each $f \in B$ is a (finite) linear combination of elements from H. Hence H satisfies condition (M) (with $K = H$), and Theorem 4 finishes the proof. □

Remark. Theorem 5 can obviously be stated, keeping the same proof, in a more general form, just asking that $H \subset B$ has the property that any $f \in B$ can be represented as

$$f = \sum_{i=1}^{\infty} a_i h_i, \; h_i \in H, \quad i = 1, 2, ..., \quad \sum_{i=1}^{\infty} |a_i| < \infty.$$

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