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BOLZA OPTIMAL CONTROL PROBLEMS WITH LINEAR EQUATIONS AND NONCONVEX INTEGRANDS ON LARGE INTERVALS

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Abstract. We study the structure of approximate solutions of Bolza optimal control problems, governed by linear equations, with autonomous nonconvex integrands, on large intervals. It is established that these optimal control problems have the turnpike property and that this property is stable under small perturbations of integrands. We also describe the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals. It is established that in these regions optimal trajectories converge to solutions of the corresponding infinite horizon optimal control problem which depend only on the integrand. This convergence is stable under small perturbations of integrands.

1. Introduction

In our recent research [49] we analyzed the structure of approximate solutions of Lagrange optimal control problems, governed by linear equations, with autonomous nonconvex integrands, on large intervals. It should be mentioned that the growing significance of the study of (approximate) solutions of variational and optimal control problems defined on infinite intervals and on sufficiently large intervals has been realized in the recent years [2, 4–10, 13–15, 17–20, 23, 28, 30, 31, 34, 35, 37, 38, 42, 43, 46, 48]. This is due not only to theoretical achievements in this area, but also because of numerous applications to engineering [1, 12, 26, 43], models of economic dynamics [11, 12, 16, 21, 25, 29, 33, 36, 39–41, 43, 45, 48], the game theory [22, 24, 43, 44, 48], models of solid-state physics [3] and the theory of thermodynamical equilibrium for materials [27, 32]. In [49], for the Lagrange optimal control problems, governed by linear equations, with autonomous nonconvex integrands, on large intervals we proved that the turnpike phenomenon holds and described the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals. It is established that in these regions optimal trajectories converge to solutions of the corresponding infinite horizon optimal control problem which depend only on the integrand. In the present paper we generalize these results for Bolza problems.

We study the structure of approximate optimal trajectories of linear control systems described by a differential equation

$$x'(t) = Ax(t) + Bu(t) \text{ for almost every (a. e.) } t \in \mathcal{I}, \tag{1.1}$$

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where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ (here $-\infty < T_1 < T_2 < \infty$), n, m are natural numbers, $x : \mathcal{I} \to R^n$ is an absolutely continuous (a. c.) function and the control function $u : \mathcal{I} \to R^m$ is Lebesgue measurable, and A and B are given matrices of dimensions $n \times n$ and $n \times m$ with integrands $f : R^n \times R^m \to R^1$.

Note that if $\mathcal{I}$ is an unbounded interval, then $x : \mathcal{I} \to R^n$ is an absolutely continuous function if and only if it is an absolutely continuous function on any bounded subinterval of $\mathcal{I}$.

We assume that the linear system (1.1) is controllable and that the integrand f is a continuous function.

We denote by $|\cdot|$ the Euclidean norm and by $\langle \cdot, \cdot \rangle$ the inner product in the k-dimensional Euclidean space R^k. For every $s \in R^1$ set $s_+ = \max\{s, 0\}$. For every nonempty set X and every function $h : X \to R^1 \cup \{\infty\}$ set

$$\inf(h) = \inf\{h(x) : \ x \in X\}.$$

Let a_0 be a positive number and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t \to \infty} \psi(t) = \infty. \tag{1.2}$$

Suppose that $f : R^n \times R^m \to R^1$ is a continuous function such that the following assumption holds:

(A1)

(i) for every point $(x, u) \in R^n \times R^m$,

$$f(x, u) \geq \max\{\psi(|x|), \ \psi(|u|),$$

$$\psi([|Ax + Bu| - a_0|x|]_+)[|Ax + Bu| - a_0|x|]_+\} - a_0; \tag{1.3}$$

(ii) for every point $x \in R^n$ the function $f(x, \cdot) : \ R^m \to R^1$ is convex;

(iii) for every pair of positive numbers M, ϵ there exist positive numbers Γ, δ such that

$$|f(x_1, u_1) - f(x_2, u_2)| \leq \epsilon \max\{f(x_1, u_1), f(x_2, u_2)\}$$

for each $u_1, u_2 \in R^m$ and each $x_1, x_2 \in R^n$ which satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma, \ i = 1, 2,$$

$$\max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta;$$

(iv) for every positive number K there exists a positive constant a_K and an increasing function

$$\psi_K : [0, \infty) \to [0, \infty)$$

such that

$$\psi_K(t) \to \infty \text{ as } t \to \infty$$

and

$$f(x, u) \geq \psi_K(|u|)|u| - a_K$$

for every point $u \in R^m$ and every point $x \in R^n$ satisfying $|x| \leq K$.

Let $T_1 \in R^1$ and $T_2 > T_1$. A pair of an absolutely continuous function $x : [T_1, T_2] \to R^n$ and a Lebesgue measurable function $u : [T_1, T_2] \to R^m$ is called an (A, B)-trajectory-control pair if (1.1) holds with $\mathcal{I} = [T_1, T_2]$. Denote by

$X(A, B, T_1, T_2)$ the set of all (A, B)-trajectory-control pairs $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$.

Let $T \in R^1$ and $\mathcal{I} = [T, \infty)$ be an infinite closed subinterval of R^1. Denote by $X(A, B, T, \infty)$ the set of all pairs of a.c. functions $x : [T, \infty) \to R^n$ and Lebesgue measurable functions $u : [T, \infty) \to R^m$ satisfying (1.1).

Note that a function h satisfies (A1) if $h \in C^1(R^n \times R^m)$, (A1)(i), (A1)(ii), (A1)(iv) hold, and for each $K > 0$ there exists an increasing function $\tilde{\psi} : [0, \infty) \to [0, \infty)$ such that for each $x \in R^n$ satisfying $|x| \le K$ and each $u \in R^m$,

$$\max\{|\partial h/\partial x(x, u)|,\ |\partial h/\partial u(x, u)|\} \le \tilde{\psi}(|x|)(1 + \psi_K(|u|)|u|).$$

The performance of the above control system is measured on any finite interval $[T_1, T_2] \subset [0, \infty)$ and for any $(x, u) \in X(A, B, T_1, T_2)$ by the integral functional

$$I^f(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(x(t), u(t))dt. \tag{1.4}$$

We consider the following optimal control problems

$$(P_1) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,\ x(T) = z,$$

$$(P_2) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,$$

$$(P_3) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T),$$

$$(P_4) \qquad I^f(0, T, x, u) + g(x(0), x(T)) \to \min,$$
$$(x, u) \in X(A, B, 0, T),$$

where $y, z \in R^n$, $T > 0$ and $g : R^n \times R^n \to R^1$ is a lower semicontinuous function which is bounded on bounded sets. The study of these problems is based on the properties of solutions of the corresponding infinite horizon optimal control problem associated with the control system (1.1) and the integrand f. Problems $(P_1) - (P_3)$ where analyzed in [49] while in this paper we study problems (P_4).

We establish the turnpike property for the approximate solutions of problems (P_4) and show that in regions close to the endpoints of the time interval their approximate solutions are determined only by the pair (f, g) and are essentially independent of the choice of the interval.

A number

$$\mu(f) := \inf\{\liminf_{T\to\infty} T^{-1} I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, \infty)\} \tag{1.5}$$

is called the minimal long-run average cost growth rate of f. In view of (A1)(i), we have $-\infty < \mu(f)$.

We say that a pair $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$ is (f, A, B)-overtaking optimal [43, 48] if for every pair $(x, u) \in X(A, B, 0, \infty)$ such that $x(0) = \tilde{x}(0)$ the inequality

$$\limsup_{T\to\infty} [I^f(0, T, \tilde{x}, \tilde{u}) - I^f(0, T, x, u)] \le 0$$

holds.

We say that a pair $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-minimal [43,48] if for every positive number T,

$$I^f(0, T, x, u) \le I^f(0, T, y, v)$$

for every pair $(y, v) \in X(A, B, 0, T)$ such that $y(0) = x(0)$, $y(T) = x(T)$.

Let $(x_f, u_f) \in R^n \times R^m$ satisfy

$$Ax_f + Bu_f = 0. \tag{1.6}$$

It is clear that $\mu(f) \le f(x_f, u_f)$. It is not difficult to see that the following result holds.

Proposition 1.1 (Proposition 3.1 of [49]). *Assume that $\mu(f) = f(x_f, u_f)$ and let $x(t) = x_f$, $u(t) = u_f$ for all $t \in [0, \infty)$. Then the pair $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-minimal.*

We suppose that the following assumption holds.

(A2) $\mu(f) = f(x_f, u_f)$ and if $(x, u) \in R^n \times R^m$ satisfies

$$Ax + Bu = 0, \ \mu(f) = f(x, u),$$

then $x = x_f$.

In [49] we proved the following result.

Proposition 1.2 (Proposition 3.4 of [49]). *For every trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ either*

$$I^f(0, T, x, u) - T\mu(f) \to \infty \text{ as } T \to \infty$$

or $\sup\{|I^f(0, T, x, u) - T\mu(f)| : \ T > 0\} < \infty$.

A trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ is called (f, A, B)-good [43,48] if

$$\sup\{|I^f(0, T, x, u) - T\mu(f)| : \ T > 0\} < \infty.$$

We have the following result.

Proposition 1.3 (Proposition 3.5 of [49]). *For any (f, A, B)-good pair*

$$(x, u) \in X(A, B, 0, \infty)$$

the inequality

$$\sup\{|x(t)| : \ t \in [0, \infty)\} < \infty$$

holds.

We suppose that the following assumption holds.

(A3) For every (f, A, B)-good trajectory-control pair

$$(x, u) \in X(A, B, 0, \infty)$$

the equality $\lim_{t\to\infty} x(t) = x_f$ holds.

Several examples of integrands satisfying assumptions (A1)-(A3) are considered in Section 3.1 of [49].

2. Turnpike results for problems (P_1), (P_2) and (P_3)

We use the notation, definitions and assumptions introduced in Section 1.

Let $T > 0$ and $y, z \in R^n$. Set

$$\sigma(f, y, z, T) = \inf\{I^f(0, T, x, u) :$$

$$(x, u) \in X(A, B, 0, T) \text{ and } x(0) = y,\ x(T) = z\}, \tag{2.1}$$

$$\sigma(f, y, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T) \text{ and } x(0) = y\}, \tag{2.2}$$

$$\widehat{\sigma}(f, z, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T) \text{ and } x(T) = z\}, \tag{2.3}$$

$$\sigma(f, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T)\}. \tag{2.4}$$

The results of this section were obtained in [49]. The following theorem establishes the turnpike property of approximate solutions of problems (P_1), (P_2) and (P_3).

Theorem 2.1 (Theorem 3.7 of [49])**.** *Let ϵ, $M_0, M_1 > 0$. Then there exist $L > 0$, $\delta \in (0, \epsilon)$ such that for each $T > 2L$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L]$,*

$$I^f(S, S + L, x, u) \le \sigma(f, x(S), x(S + L), L) + \delta$$

and satisfies at least one of the following conditions below

(a) $|x(0)|,\ |x(T)| \le M_0,\ I^f(0, T, x, u) \le \sigma(f, x(0), x(T), T) + M_1$;
(b) $|x(0)| \le M_0,\ I^f(0, T, x, u) \le \sigma(f, x(0), T) + M_1$;
(c) $I^f(0, T, x, u) \le \sigma(f, T) + M_1$

there exist $p_1 \in [0, L]$, $p_2 \in [T - L, T]$ such that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \le \delta$, then $p_2 = T$.

It should be mentioned that turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [41]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path and a turnpike). This property was further investigated for optimal trajectories of models of economic dynamics. See, for example, Makarov and Rubinov [29], McKenzie [33], Rubinov [39, 40] and the references mentioned there.

Theorem 2.2 (Theorem 3.8 of [49])**.** *Let $x_0 \in R^n$. Then there exists an (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ satisfying $x(0) = x_0$.*

Proposition 2.3 (Proposition 3.36 of [49])**.** *Any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-good.*

The next result describes the limit behavior of overtaking optimal trajectories.

Theorem 2.4 (Theorem 3.9 of [49]). *Let $M, \epsilon > 0$. Then there exists $L > 0$ such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ which satisfies $|x(0)| \leq M$ the inequality*

$$|x(t) - x_f| \leq \epsilon$$

holds for all numbers $t \geq L$. Moreover, there exists $\delta > 0$ such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ satisfying $|x(0) - x_f| \leq \delta$, the inequality

$$|x(t) - x_f| \leq \epsilon$$

holds for all numbers $t \geq 0$.

The next result shows the equivalence of the optimality criterions introduced above.

Theorem 2.5 (Theorem 3.10 of [49]). *Assume that $(x, u) \in X(A, B, 0, \infty)$. Then the following conditions are equivalent:*

(i) *(x, u) is (f, A, B)-overtaking optimal;* (ii) *(x, u) is (f, A, B)-minimal and (f, A, B)-good;*

(iii) *(x, u) is (f, A, B)-minimal and*

$$\lim_{t \to \infty} x(t) = x_f;$$

(iv) *(x, u) is (f, A, B)-minimal and $\liminf_{t \to \infty} |x(t)| < \infty$.*

3. Structure of solutions in the regions close to the end points

We use the notation, definitions and assumptions introduced in Sections 1 and 2.

For every point $z \in R^n$ denote by $\Lambda(z)$ the collection of all (f, A, B)-overtaking optimal pairs $(x, u) \in X(A, B, 0, \infty)$ such that $x(0) = z$, which is nonempty in view of Theorem 2.2.

Let $z \in R^n$. Define

$$\pi^f(z) = \liminf_{T \to \infty} [I^f(0, T, x, u) - T\mu(f)], \tag{3.1}$$

where $(x, u) \in \Lambda(z)$. By Propositions 1.2 and 2.3, $\pi^f(z)$ is finite, well defined and does not depend on the choice of $(x, u) \in \Lambda(z)$. The following results were obtained in Section 3.3 of [49].

Proposition 3.1 (Proposition 3.11 of [49]). *1. Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-good. Then*

$$\pi^f(x(0)) \leq \liminf_{T \to \infty} [I^f(0, T, x, u) - T\mu(f)]$$

and for each pair of numbers $S > T \geq 0$,

$$\pi^f(x(T)) \leq I^f(T, S, x, u) - (S - T)\mu(f) + \pi^f(x(S)). \tag{3.2}$$

2. Let $S > T \geq 0$ and $(x, u) \in X(A, B, T, S)$. Then (3.2) holds.

Proposition 3.2 (Proposition 3.12 of [49]). *Let $(x, u) \in X(A, B, 0, \infty)$ be an (f, A, B)-overtaking optimal pair. Then for each pair of numbers $S > T \geq 0$,*

$$\pi^f(x(T)) = I^f(T, S, x, u) - (S - T)\mu(f) + \pi^f(x(S)).$$

Proposition 3.3 (Propositions 3.13, 3.14, 3.16 and 3.17 of [49]). *$\pi^f(x_f) = 0$, the function π^f is continuous at x_f, the function π^f is lower semicontinuous and for each $M > 0$ the set $\{x \in R^n : \pi^f(x) \le M\}$ is bounded.*

Proposition 3.4 (Proposition 3.15 of [49]). *Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-overtaking optimal. Then*

$$\pi^f(x(0)) = \lim_{T\to\infty} [I^f(0, T, x, u) - T\mu(f)].$$

Proposition 3.5 (Proposition 3.18 of [49]). *Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-good pair such that for all $T > 0$,*

$$I^f(0, T, x, u) - T\mu(f) = \pi^f(x(0)) - \pi^f(x(T)).$$

Then $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-overtaking optimal.

Consider a linear control system

$$x'(t) = -Ax(t) - Bu(t), \tag{3.3}$$

$$x(0) = x_0$$

which is also controllable. For the triplet $(f, -A, -B)$ we use all the notation and definitions introduced for the triplet (f, A, B). It is not difficult to see that assumption (A1) holds for the triplet $(f, -A, -B)$.

Let $T_1 \in R^1$, $T_2 > T_1$. A pair of an absolutely continuous function $x : [T_1, T_2] \to R^n$ and a Lebesgue measurable function $u : [T_1, T_2] \to R^m$ is called an $(-A, -B)$-trajectory-control pair if (3.3) holds for a. e. $t \in [T_1, T_2]$. Denote by $X(-A, -B, T_1, T_2)$ the set of all $(-A-, B)$-trajectory-control pairs $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$.

Let $T \in R^1$. Denote by $X(-A, -B, T, \infty)$ the set of all pairs of a. c. functions $x : [T, \infty) \to R^n$ and Lebesgue measurable functions $u : [T, \infty) \to R^m$ satisfying (3.3) for a. e. $t \ge T$, which are called $(-A, -B)$-trajectory-control pairs.

Assume that $S_1 \in R^1$, $S_2 > S_1$ and that $(x, u) \in X(A, B, S_1, S_2)$. For all $t \in [S_1, S_2]$ set

$$\bar{x}(t) = x(S_2 - t + S_1),\ \bar{u}(t) = u(S_2 - t + S_1). \tag{3.4}$$

By (1.1) and (3.4) for a. e. $t \in [S_1, S_2]$,

$$\begin{aligned} \bar{x}'(t) &= -x'(S_2 - t + S_1) \\ &= -Ax(S_2 - t + S_1) - Bu(S_2 - t + S_1) \\ &= -A\bar{x}(t) - B\bar{u}(t), \\ (\bar{x}, \bar{u}) &\in X(-A, -B, S_1, S_2). \end{aligned} \tag{3.5}$$

In view of (3.4),

$$\begin{aligned} \int_{S_1}^{S_2} f(\bar{x}(t), \bar{u}(t))dt &= \int_{S_1}^{S_2} f(x(S_2 - t + S_1), u(S_2 - t + S_1))dt \\ &= \int_{S_1}^{S_2} f(x(t), u(t))dt. \end{aligned} \tag{3.6}$$

For every pair of numbers $T_2 > T_1$ and every trajectory-control pair $(x,u) \in X(-A,-B,T_1,T_2)$ define

$$I^f(T_1,T_2,x,u) = \int_{T_1}^{T_2} f(x(t),u(t))dt. \tag{3.7}$$

For every pair of points $y,z \in R^n$ and every positive number T define

$$\sigma_-(f,y,z,T) = \inf\{I^f(0,T,x,u) :$$
$$(x,u) \in X(-A,-B,0,T) \text{ and } x(0)=y,\ x(T)=z\}, \tag{3.8}$$
$$\sigma_-(f,y,T) = \inf\{I^f(0,T,x,u) :$$
$$(x,u) \in X(-A,-B,0,T) \text{ and } x(0)=y\}, \tag{3.9}$$
$$\widehat{\sigma}_-(f,z,T) = \inf\{I^f(0,T,x,u) :$$
$$(x,u) \in X(-A,-B,0,T) \text{ and } x(T)=z\}, \tag{3.10}$$

$$\sigma_-(f,T) = \inf\{I^f(0,T,x,u) :\ (x,u) \in X(-A,-B,0,T)\}. \tag{3.11}$$

The following auxiliary results were proved in Section 3.3 of [49].

Proposition 3.6. *Let $S_2 > S_1$ be real numbers, $M \geq 0$ and that $(x_i,u_i) \in X(A,B,S_1,S_2)$, $i=1,2$. Then*

$$I^f(S_1,S_2,x_1,u_1) \geq I^f(S_1,S_2,x_2,u_2) - M$$

if and only if $I^f(S_1,S_2,\bar{x}_1,\bar{u}_1) \geq I^f(S_1,S_2,\bar{x}_2,\bar{u}_2) - M$.

Proposition 3.7. *Let $S_2 > S_1$ be real numbers and*

$$(x,u) \in X(A,B,S_1,S_2).$$

Then the following assertions hold:

$$I^f(S_1,S_2,x,u) \leq \sigma(f,S_2-S_1)+M$$
if and only if $I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,S_2-S_1)+M$;
$$I^f(S_1,S_2,x,u) \leq \sigma(f,x(S_1),x(S_2),S_2-S_1)+M$$
if and only if $I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,\bar{x}(S_1),\bar{x}(S_2),S_2-S_1)+M$;
$$I^f(S_1,S_2,x,u) \leq \sigma(f,x(S_1),S_2-S_1)+M$$
if and only if $I^f(S_1,S_2,\bar{x},\bar{u}) \leq \widehat{\sigma}_-(f,\bar{x}(S_2),S_2-S_1)+M$;
$$I^f(S_1,S_2,x,u) \leq \widehat{\sigma}(f,x(S_2),S_2-S_1)+M$$
if and only if $I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,\bar{x}(S_1),S_2-S_1)+M$.

Define

$$\mu_-(f) = \inf\{\liminf_{T\to\infty} T^{-1}I^f(0,T,x,u) :\ (x,u) \in X(-A,-B,0,\infty)\}. \tag{3.12}$$

Proposition 3.8. $\mu_-(f) = \mu(f) = f(x_f,u_f)$.

Proposition 3.9. *For any $(f,-A,-B)$-good trajectory-control pair $(x,u) \in X(-A,-B,0,\infty)$,*

$$\lim_{t\to\infty} x(t) = x_f.$$

Therefore $(f, -A, -B)$ satisfies all the assumptions posed for the triplet (f, A, B) and all the results stated above for the triplet (f, A, B) are also true for $(f, -A, -B)$.

For every point $z \in R^n$, define

$$\pi_-^f(z) = \liminf_{T \to \infty} [I^f(0, T, x, u) - T\mu(f)],$$

where $(x, u) \in X(-A, -B, 0, \infty)$ is an $(f, -A, -B)$-overtaking optimal pair such that $x(0) = z$.

In Chapter 3 of [49] we prove the following two theorems which describe the structure of solutions of problems (P_2) and (P_3) in the regions closed to the end points.

Theorem 3.10. *Let $L_0 > 0$, $\epsilon \in (0, 1)$, $M > 0$. Then there exist $\delta > 0$ and $L_1 > L_0$ such that for each $T \geq L_1$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$|x(0)| \leq M, \ I^f(0, T, x, u) \leq \sigma(f, x(0), T) + \delta$$

there exists an $(f, -A, -B)$-overtaking optimal pair

$$(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$$

such that

$$\pi_-^f(\bar{x}_*(0)) = \inf(\pi_-^f),$$
$$|x(T - t) - \bar{x}_*(t)| \leq \epsilon \text{ for all } t \in [0, L_0].$$

Theorem 3.11. *Let $L_0 > 0$ and $\epsilon > 0$. Then there exist $\delta > 0$ and $L_1 > L_0$ such that for each $T \geq L_1$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$I^f(0, T, x, u) \leq \sigma(f, 0, T) + \delta$$

there exist an (f, A, B)-overtaking optimal pair $(x_*, u_*) \in X(A, B, 0, \infty)$ and an $(f, -A, -B)$-overtaking optimal pair $(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$ such that

$$\pi^f(x_*(0)) = \inf(\pi^f),$$
$$\pi_-^f(\bar{x}_*(0)) = \inf(\pi_-^f)$$

and for all $t \in [0, L_0]$,

$$|x(t) - x_*(t)| \leq \epsilon, \ |x(T - t) - \bar{x}_*(t)| \leq \epsilon.$$

4. Spaces of integrands

We use the notation, definitions and assumptions introduced in Sections 1-3. Recall that $a_0 > 0$ and $\psi : [0, \infty) \to [0, \infty)$ is an increasing function such that

$$\lim_{t \to \infty} \psi(t) = \infty.$$

We continue to study the structure of optimal trajectories of the controllable linear control system

$$x' = Ax + Bu,$$

where A and B are given matrices of dimensions $n \times n$ and $n \times m$, with the continuous integrand $f : R^n \times R^m \to R^1$ which satisfies assumptions (A1)-(A3) and (1.6).

Denote by $\mathfrak{M}$ the set of all borelian functions $g : R^{n+m+1} \to R^1$ which satisfy

$$g(t, x, u) \geq \max\{\psi(|x|), \ \psi(|u|),$$

$$\psi([|Ax+Bu|-a_0|x|]_+)[|Ax+Bu|-a_0|x|]_+\}-a_0 \tag{4.1}$$

for each $(t,x,u)\in R^{n+m+1}$.

We equip the set $\mathfrak{M}$ with the uniformity which is determined by the following base:

$$E(N,\epsilon,\lambda)=\{(f,g)\in\mathfrak{M}\times\mathfrak{M}:\ |f(t,x,u)-g(t,x,u)|\le\epsilon$$
$$\text{for each } (t,x,u)\in R^{n+m+1} \text{ satisfying } |x|,|u|\le N\}$$
$$\cap\{(f,g)\in\mathfrak{M}\times\mathfrak{M}:\ (|f(t,x,u)|+1)(|g(t,x,u)|+1)^{-1}\in[\lambda^{-1},\lambda]$$
$$\text{for each } (t,x,u)\in R^{n+m+1} \text{ satisfying } |x|\le N\}, \tag{4.2}$$

where $N>0$, $\epsilon>0$ and $\lambda>1$.

It is clear that the uniform space $\mathfrak{M}$ is Hausdorff and has a countable base. Therefore $\mathfrak{M}$ is metrizable. It is not difficult to show that the uniform space $\mathfrak{M}$ is complete.

Denote by $\mathfrak{M}_b$ the set of all functions $g\in\mathfrak{M}$ which are bounded on bounded subsets of R^{n+m+1}. Clearly, $\mathfrak{M}_b$ is a closed subset of $\mathfrak{M}$. We consider the topological subspace $\mathfrak{M}_b\subset\mathfrak{M}$ equipped with the relative topology.

For each a pair of numbers $T_1\in R^1$, $T_2>T_1$, each $(x,u)\in X(A,B,T_1,T_2)$ and each borelian bounded from below function $g:[T_1,T_2]\times R^n\times R^m\to R^1$ set

$$I^g(T_1,T_2,x,u)=\int_{T_1}^{T_2} g(t,x(t),u(t))dt.$$

We consider the following optimal control problems

$$I^g(T_1,T_2,x,u)\to\min,$$
$$(x,u)\in X(A,B,T_1,T_2) \text{ such that } x(T_1)=y,\ x(T_2)=z,$$
$$I^g(T_1,T_2,x,u)\to\min,$$
$$(x,u)\in X(A,B,T_1,T_2) \text{ such that } x(T_1)=y,$$
$$I^g(T_1,T_2,x,u)\to\min,$$
$$(x,u)\in X(A,B,T_1,T_2),$$

where $y,z\in R^n$, $\infty>T_2>T_1>-\infty$ and $g\in\mathfrak{M}$.

Let $y,z\in R^n$, $T_1\in R^1$, $T_2>T_1$ and $g:[T_1,T_2]\times R^n\times R^m\to R^1$ be a borelian bounded from below function. Set

$$\sigma(g,y,z,T_1,T_2)=\inf\{I^g(T_1,T_2,x,u):$$
$$(x,u)\in X(A,B,T_1,T_2) \text{ and } x(T_1)=y,\ x(T_2)=z\}, \tag{4.3}$$

$$\sigma(g,y,T_1,T_2)=\inf\{I^g(T_1,T_2,x,u):$$
$$(x,u)\in X(A,B,T_1,T_2) \text{ and } x(T_1)=y\}, \tag{4.4}$$

$$\widehat{\sigma}(g,z,T_1,T_2)=\inf\{I^g(T_1,T_2,x,u):$$
$$(x,u)\in X(A,B,T_1,T_2) \text{ and } x(T_2)=z\}, \tag{4.5}$$

$$\sigma(g,T_1,T_2)=\inf\{I^g(T_1,T_2,x,u):\ (x,u)\in X(A,B,T_1,T_2)\}. \tag{4.6}$$

Recall that $f : R^n \times R^m \to R^1$ is a continuous function which satisfies (1.6) and assumptions (A1), (A2) and (A3). For each $(t, x, u) \in R^{n+m+1}$ set

$$F(t, x, u) = f(x, u). \tag{4.7}$$

The following stability results were obtained in Chapter 4 of [49]. They show that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective function (integrand) f.

Theorem 4.1. *Let $\epsilon, M > 0$. Then there exist $L_0 \ge 1$ and $\delta_0 > 0$ such that for each $L_1 \ge L_0$ there exists a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that the following assertion holds.*

Assume that $T > 2L_1$, $g \in \mathcal{U}$, $(x, u) \in X(A, B, 0, T)$ and that a finite sequence of numbers $\{S_i\}_{i=0}^q$ satisfies

$$S_0 = 0,\ S_{i+1} - S_i \in [L_0, L_1],\ i = 0, \dots, q-1,\ S_q \in (T - L_1, T],$$

$$I^g(S_i, S_{i+1}, x, u) \le (S_{i+1} - S_i)\mu(f) + M$$

for each integer $i \in [0, q-1]$,

$$I^g(S_i, S_{i+2}, x, u) \le \sigma(g, x(S_i), x(S_{i+2}), S_i, S_{i+2}) + \delta_0$$

for each nonnegative integer $i \le q - 2$ and

$$I^g(S_{q-2}, T, x, u) \le \sigma(g, x(S_{q-2}), x(T), S_{q-2}, T) + \delta_0.$$

Then there exist $p_1, p_2 \in [0, T]$ such that $p_1 \le p_2$, $p_1 \le 2L_0$, $p_2 > T - 2L_1$ and that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \le \delta$, then $p_2 = T$.

Theorem 4.2. *Let $\epsilon \in (0, 1)$, $M_0, M_1 > 0$. Then there exist $L > 0$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > 2L$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L]$,*

$$I^g(S, S + L, x, u) \le \sigma(g, x(S), x(S + L), S, S + L) + \delta$$

and satisfies at least one of the following conditions below

(a) $|x(0)|,\ |x(T)| \le M_0$,

$$I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$$

(b) $|x(0)| \le M_0$,

$$I^g(0, T, x, u) \le \sigma(g, x(0), 0, T) + M_1;$$

(c) $I^g(0, T, x, u) \le \sigma(g, 0, T) + M_1$
there exist $p_1 \in [0, L]$, $p_2 \in [T - L, T]$ such that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \le \delta$, then $p_2 = T$.

Theorem 4.3. *Let $\epsilon \in (0, 1)$, $M_0, M_1 > 0$. Then there exist $l > 0$, an integer $Q \ge 1$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > lQ$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies at least one of the following conditions below*

(a) $|x(0)|,\ |x(T)| \le M_0$,

$$I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$$

(b) $|x(0)| \le M_0$,

$$I^g(0, T, x, u) \le \sigma(g, x(0), 0, T) + M_1;$$

(c) $I^g(0, T, x, u) \le \sigma(g, 0, T) + M_1$
there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ *such that* $q \le Q$, *for all* $i = 1, \dots, q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ *for all integers* i *satisfying* $1 \le i < q$ *and that*

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^q [a_i, b_i].$$

In Chapter 4 of [49] we also prove the following two stability results. They show that the convergence of approximate solutions on large intervals, in the regions close to the end points, is stable under small perturbations of the objective function (integrand) f.

Theorem 4.4. *Let* $L_0 > 0$, $\epsilon \in (0, 1)$, $M > 0$. *Then there exist* $\delta > 0$, *a neighborhood* $\mathcal{U}$ *of* F *in* $\mathfrak{M}_b$ *and* $L_1 > L_0$ *such that for each* $T \ge L_1$, *each* $g \in \mathcal{U}$ *and each* $(x, u) \in X(A, B, 0, T)$ *which satisfies*

$$|x(0)| \le M,\ I^g(0, T, x, u) \le \sigma(g, x(0), 0, T) + \delta$$

there exists an $(f, -A, -B)$*-overtaking optimal pair*

$$(x_*, u_*) \in X(-A, -B, 0, \infty)$$

such that

$$\pi_-^f(x_*(0)) = \inf(\pi_-^f),$$

$$|x(T - t) - x_*(t)| \le \epsilon \text{ for all } t \in [0, L_0].$$

Theorem 4.5. *Let* $L_0 > 0$, $\epsilon \in (0, 1)$. *Then there exist* $\delta > 0$, *a neighborhood* $\mathcal{U}$ *of* F *in* $\mathfrak{M}_b$ *and* $L_1 > L_0$ *such that for each* $T \ge L_1$, *each* $g \in \mathcal{U}$ *and each* $(x, u) \in X(A, B, 0, T)$ *which satisfies*

$$I^g(0, T, x, u) \le \sigma(g, 0, T) + \delta$$

there exist an (f, A, B)*-overtaking optimal pair* $(x_*, u_*) \in X(A, B, 0, \infty)$ *and an* $(f, -A, -B)$*-overtaking optimal pair* $(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$ *such that*

$$\pi^f(x_*(0)) = \inf(\pi^f),$$

$$\pi_-^f(\bar{x}_*(0)) = \inf(\pi_-^f)$$

and for all $t \in [0, L_0]$,

$$|x(t) - x_*(t)| \le \epsilon,\ |x(T - t) - \bar{x}_*(t)| \le \epsilon.$$

5. Bolza optimal control problems

We use the notation, definitions and assumptions introduced in Sections 1-4. Recall that $f : R^n \times R^m \to R^1$ is a continuous function which satisfies assumptions (A1)-(A3) and (1.6).

Let $a_1 > 0$. Denote by $\mathfrak{A}$ the set of all lower semicontinuous functions $h : R^n \times R^n \to R^1$ which are bounded on bounded subsets of $R^n \times R^n$ and satisfy

$$h(z_1, z_2) \ge -a_1 \text{ for all } z_1, z_2 \in R^n. \tag{5.1}$$

We equip the set $\mathfrak{A}$ with the uniformity which is determined by the following base:

$$E(N, \epsilon) = \{(h_1, h_2) \in \mathfrak{A} \times \mathfrak{A} : \ |h_1(z) - h_2(z)| \le \epsilon$$

$$\text{for each } z \in R^n \times R^n \text{ satisfying } |z| \le N\}, \tag{5.2}$$

where $N > 0$, $\epsilon > 0$. Clearly, the uniform space $\mathfrak{A}$ is metrizable and complete.

We consider the following optimal control problem

$$I^g(T_1, T_2, x, u) + h(x(T_1), x(T_2)) \to \min,$$

$$(x, u) \in X(A, B, T_1, T_2),$$

where $\infty > T_2 > T_1 > -\infty$, $g \in \mathfrak{M}$ and $h \in \mathfrak{A}$.

Let $T_1 \in R^1$, $T_2 > T_1$, $g : [T_1, T_2] \times R^n \times R^m \to R^1$ be a borelian bounded from below function and $h \in \mathfrak{A}$. Set

$$\sigma(g, h, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) + h(x(T_1), x(T_2)) :$$

$$(x, u) \in X(A, B, T_1, T_2)\}.$$

In view of Proposition 3.3, there exists $M_* > 0$ such that

$$\{x \in R^n : \ \pi^f(x) = \inf(\pi^f)\} \cup \{x \in R^n : \ \pi^f_-(x) = \inf(\pi^f_-)\}$$

$$\subset \{x \in R^n : \ |x| \le M_*\}. \tag{5.3}$$

We prove the following turnpike results for the Bolza optimal control problems which show that the turnpike phenomenon is stable under small perturbations of the objective functions.

Theorem 5.1. *Let $\epsilon > 0$, $M_1, M_2 > 0$. Then there exist $l > 0$, an integer $Q \ge 1$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > lQ$, each $g \in \mathcal{U}$, each $h \in \mathfrak{A}$ which satisfies*

$$|h(z_1, z_2)| \le M_1$$

for all $z_1, z_2 \in R^n$ satisfying $|z_1|, |z_2| \le M_* + 1$ and each

$$(x, u) \in X(A, B, 0, T)$$

which satisfies

$$I^g(0, T, x, u) + h(x(0), x(T)) \le \sigma(g, h, 0, T) + M_2$$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that $q \le Q$, for all $i = 1, \dots, q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ *for all integers* i *satisfying* $1 \le i < q$ *and that*

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0,T] \setminus \cup_{i=1}^{q} [a_i, b_i].$$

Theorem 5.2. *Let* $\epsilon \in (0,1)$, $M_1, M_2 > 0$. *Then there exist* $L > 0$, $\delta \in (0,\epsilon)$ *and a neighborhood* $\mathcal{U}$ *of* F *in* $\mathfrak{M}_b$ *such that for each* $T > 2L$, *each* $g \in \mathcal{U}$, *each* $h \in \mathfrak{A}$ *which satisfies*

$$|h(z_1, z_2)| \le M_1$$

for all $z_1, z_2 \in R^n$ *satisfying*

$$|z_1|, |z_2| \le M_* + 1$$

and each $(x,u) \in X(A,B,0,T)$ *which satisfies for each* $S \in [0, T-L]$,

$$I^g(S, S+L, x, u) \le \sigma(g, x(S), x(S+L), S, S+L) + \delta$$

and satisfies

$$I^g(0,T,x,u) + h(x(0), x(T)) \le \sigma(g,h,0,T) + M_2$$

there exist $p_1 \in [0, L]$, $p_2 \in [T-L, T]$ *such that*

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, *then* $p_1 = 0$ *and if* $|x(T) - x_f| \le \delta$, *then* $p_2 = T$.

In this paper we also prove the following stability result for our Bolza optimal control problems. This result shows that the convergence of approximate solutions on large intervals, in the regions close to the end points, is stable under small perturbations of the objective functions.

Theorem 5.3. *Let* $L_0 > 0$, $\epsilon \in (0,1)$ *and* $h \in \mathfrak{A}$. *Then there exist* $\delta > 0$, *a neighborhood* $\mathcal{U}$ *of* F *in* $\mathfrak{M}_b$, *a neighborhood* $\mathcal{V}$ *of* h *in* $\mathfrak{A}$ *and* $L_1 > L_0$ *such that for each* $T \ge L_1$, *each* $g \in \mathcal{U}$, *each* $\xi \in \mathcal{V}$ *and each* $(x,u) \in X(A,B,0,T)$ *which satisfies*

$$I^g(0,T,x,u) + \xi(x(0), x(T)) \le \sigma(g, \xi, 0, T) + \delta$$

there exist an (f,A,B)*-overtaking optimal pair* $(x_1^*, u_1^*) \in X(A,B,0,\infty)$ *and an* $(f,-A,-B)$*-overtaking optimal pair* $(x_2^*, u_2^*) \in X(-A,-B,0,\infty)$ *such that*

$$\pi^f(x_1^*(0)) + \pi_-^f(x_2^*(0)) + h(x_1^*(0), x_2^*(0))$$

$$\le \pi^f(y_1) + \pi_-^f(y_2) + h(y_1, y_2)$$

for all $y_1, y_2 \in R^n$ *and that for all* $t \in [0, L_0]$,

$$|x(t) - x_1^*(t)| \le \epsilon, \ |x(T-t) - x_2^*(t)| \le \epsilon.$$

6. Auxiliary results

In the sequel we use the following auxiliary results.

Proposition 6.1 (Proposition 3.27 of [49]). *Let* $T > 0$ *and* $y, z \in R^n$. *Then there exists* $(x,u) \in X(A,B,0,T)$ *such that*

$$x(0) = y, \ x(T) = z,$$

$$I^f(0,T,x,u) = \sigma(f,y,z,T).$$

Proposition 6.2 (Proposition 4.5 of [47], Proposition 3.28 of [49]). *Let $M, \tau > 0$. Then*

$$\sup\{|\sigma(f, y, z, \tau)| : \ y, z \in R^n, \ |y|, |z| \le M\} < \infty.$$

Proposition 6.3 (Proposition 2.9 of [47]). *Let $g \in \mathfrak{M}$, $0 < c_1 < c_2$ and $D, \epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathfrak{M}$ such that for each $h \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + c_1, T_1 + c_2]$ and each trajectory-control pair $(x, u) \in X(A, B, T_1, T_2)$ which satisfies*

$$\min\{I^g(T_1, T_2, x, u), \ I^h(T_1, T_2, x, u)\} \le D$$

the inequality

$$|I^g(T_1, T_2, x, u) - I^h(T_1, T_2, x, u)| \le \epsilon$$

holds.

Proposition 6.4 (Proposition 4.2 of [47]). *Let $T_2 > T_1$ be real numbers, $\{(x_j, u_j)\}_{j=1}^{\infty} \subset X(A, B, T_1, T_2)$ and let the sequence $\{I^f(T_1, T_2, x_j, u_j)\}_{j=1}^{\infty}$ be bounded. Then there exist a subsequence $\{(x_{j_k}, u_{j_k})\}_{k=1}^{\infty}$ and $(x, u) \in X(A, B, T_1, T_2)$ such that*

$$x_{j_k}(t) \to x(t) \text{ as } k \to \infty \text{ uniformly in } [T_1, T_2],$$
$$u_{j_k} \to u \text{ as } k \to \infty \text{ weakly in } L^1(R^m; (T_1, T_2)),$$
$$I^f(T_1, T_2, x, u) \le \liminf_{k \to \infty} I^f(T_1, T_2, x_{j_k}, u_{j_k}).$$

Proposition 6.5 (Proposition 4.6 of [47]). *Let $M, \tau, \epsilon > 0$. Then there exists a number $\delta > 0$ such that for each $y_1, y_2, z_1, z_2 \in R^n$ satisfying*

$$|y_i|, |z_i| \le M, \ i = 1, 2, \ |y_1 - y_2|, \ |z_1 - z_2| \le \delta$$

the following relation holds:

$$|\sigma(f, y_1, z_1, \tau) - \sigma(f, y_2, z_2, \tau)| \le \epsilon.$$

Proposition 6.6 (Proposition 2.7 of [47]). *Let $M_1 > 0$ and $0 < \tau_0 < \tau_1$. Then there exists a positive number M_2 such that for each $T_1 \in R^1$, each $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying*

$$I^f(T_1, T_2, x, u) \le M_1$$

the inequality $|x(t)| \le M_2$ holds for all $t \in [T_1, T_2]$.

7. Proof of Theorem 5.1

By Theorem 4.5, there exist $\delta \in (0, 1)$, $L_1 > 1$ and a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$, such that the following property holds:

(P1) for each $T \ge L_1$, each $g \in \mathcal{U}_1$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies

$$I^g(0, T, x, u) \le \sigma(g, 0, T) + \delta$$

there exist an (f, A, B)-overtaking optimal pair $(x_*, u_*) \in X(A, B, 0, \infty)$ and an $(f, -A, -B)$-overtaking optimal pair $(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$ such that

$$\pi^f(x_*(0)) = \inf(\pi^f), \ \pi_-^f(\bar{x}_*(0)) = \inf(\pi_-^f)$$

and that for all $t \in [0, 1]$,

$$|x(t) - x_*(t)| \le 1, \ |x(T - t) - \bar{x}_*(t)| \le 1.$$

By Theorem 4.3, there exist $l > L_1$, an integer $Q \geq 1$ and a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P2) for each $T > lQ$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies

$$I^g(0, T, x, u) \leq \sigma(g, 0, T) + M_1 + M_2 + a_1 + 1$$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that $q \leq Q$, for all $i = 1, \ldots, q$,

$$0 \leq b_i - a_i \leq l,$$

$b_i \leq a_{i+1}$ for all integers i satisfying $1 \leq i < q$ and that

$$|x(t) - x_f| \leq \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^q [a_i, b_i].$$

Assume that

$$T > lQ,\ g \in \mathcal{U},\ h \in \mathfrak{A}, \tag{7.1}$$

$$|h(z_1, z_2)| \leq M_1 \text{ for all } z_1, z_2 \in R^n \text{ satisfying } |z_1|, |z_2| \leq M_* + 1, \tag{7.2}$$

and $(x, u) \in X(A, B, 0, T)$ satisfies

$$I^g(0, T, x, u) + h(x(0), x(T)) \leq \sigma(g, h, 0, T) + M_2. \tag{7.3}$$

By (4.1), there exists $(y, v) \in X(A, B, 0, T)$ such that

$$I^g(0, T, y, v) \leq \sigma(g, 0, T) + \delta. \tag{7.4}$$

Property (P1), (7.1) and (7.4) imply that there exist $z_1, z_2 \in R^n$ such that

$$\pi^f(z_1) = \inf(\pi^f),\ \pi_-^f(z_2) = \inf(\pi_-^f), \tag{7.5}$$

$$|z_1 - y(0)|,\ |z_2 - y(T)| \leq 1. \tag{7.6}$$

In view of (5.3), (7.5) and (7.6),

$$|y(0)|,\ |y(T)| \leq M_* + 1. \tag{7.7}$$

It follows from (7.2) and (7.7) that

$$h(y(0), y(T)) \leq M_1. \tag{7.8}$$

It follows from (5.1), (7.3), (7.4) and (7.8)

$$\begin{aligned} I^g(0, T, x, u) - a_1 &\leq I^g(0, T, x, u) + h(x(0), x(T)) \\ &\leq I^g(0, T, y, v) + h(y(0), y(T)) + M_2 \\ &\leq I^g(0, T, y, v) + M_1 + M_2 \\ &\leq \sigma(g, 0, T) + 1 + M_1 + M_2, \end{aligned}$$

$$I^g(0, T, x, u) \leq \sigma(g, 0, T) + 1 + M_1 + M_2 + a_1. \tag{7.9}$$

In view of (7.1), (7.9) and property (P2), there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that $q \leq Q$, for all $i = 1, \ldots, q$,

$$0 \leq b_i - a_i \leq l,$$

$b_i \leq a_{i+1}$ for all integers i satisfying $1 \leq i < q$ and that

$$|x(t) - x_f| \leq \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^q [a_i, b_i].$$

This completes the proof of Theorem 5.1.

8. Proof of Theorem 5.2

By Theorem 4.2, there exist $L_1 > 0$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P3) For each $T > 2L_1$, each $g \in \mathcal{U}_1$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L_1]$,

$$I^g(S, S + L_1, x, u) \le \sigma(g, x(S), x(S + L_1), S, S + L_1) + \delta$$

and satisfies

$$I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_2,$$
$$|x(0) - x_f| \le \delta, \ |x(T) - x_f| \le \delta$$

we have

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0, T].$$

By Theorem 5.1, there exist $l > 0$, an integer $Q \ge 1$ and a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P4) for each $T > lQ$, each $g \in \mathcal{U}$, each $h \in \mathfrak{A}$ which satisfies

$$|h(z_1, z_2)| \le M_1 \text{ for all } z_1, z_2 \in R^n \text{ satisfying } |z_1|, |z_2| \le M_* + 1$$

and each $(x, u) \in X(A, B, 0, T)$ which satisfies

$$I^g(0, T, x, u) + h(x(0), x(T)) \le \sigma(g, h, 0, T) + M_2$$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that $q \le Q$, for all $i = 1, \dots, q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ for all integers i satisfying $1 \le i < q$ and that

$$|x(t) - x_f| \le \delta \text{ for all } t \in [0, T] \setminus \cup_{i=1}^q [a_i, b_i].$$

Fix

$$L > 2L_1 + 4lQ. \tag{8.1}$$

Assume that

$$T > 2L, \ g \in \mathcal{U}, \ h \in \mathfrak{A} \tag{8.2}$$

satisfy

$$|h(z_1, z_2)| \le M_1 \text{ for all } z_1, z_2 \in R^n \text{ satisfying } |z_1|, |z_2| \le M_* + 1 \tag{8.3}$$

and $(x, u) \in X(A, B, 0, T)$ satisfy for each $S \in [0, T - L]$,

$$I^g(S, S + L, x, u) \le \sigma(g, x(S), x(S + L), S, S + L) + \delta \tag{8.4}$$

and

$$I^g(0, T, x, u) + h(x(0), x(T)) \le \sigma(g, h, 0, T) + M_2. \tag{8.5}$$

By (8.1), (8.2), (8.3), (8.5) and property (P4), there exist $\tau_1, \tau_2 \in [0, T]$ such that

$$\tau_1 \in [0, Ql], \ \tau_2 \in [T - Ql, T], \tag{8.6}$$

$$|x(\tau_i) - x_f| \le \delta, \ i = 1, 2. \tag{8.7}$$

If $|x(0) - x_f| \le \delta$, we may assume that $\tau_1 = 0$ and if $|x(T) - x_f| \le \delta$, we may assume that $\tau_2 = T$. It follows from (8.1), (8.2), (8.4)-(8.7) and property (P3) that

$$|x(t) - x_f| \le \epsilon,\ t \in [\tau_1, \tau_2].$$

Theorem 5.2 is proved.

9. Auxiliary results for Theorem 5.3

The following result easily follows from Proposition 3.3.

Lemma 9.1. *Let $h \in \mathfrak{A}$. Then the function*

$$\pi^f(z_1) + \pi^f_-(z_2) + h(z_1, z_2),\ (z_1, z_2) \in R^n \times R^n$$

is lower semicontinuous and bounded from below, for every number M the set

$$\{(z_1, z_2) \in R^n \times R^n :\ \pi^f(z_1) + \pi^f_-(z_2) + h(z_1, z_2) \le M\}$$

is bounded and there exist $z_1^*, z_2^* \in R^n$ such that

$$\pi^f(z_1^*) + \pi^f_-(z_2^*) + h(z_1^*, z_2^*) \le \pi^f(z_1) + \pi^f_-(z_2) + h(z_1, z_2)$$

for all $(z_1, z_2) \in R^n \times R^n$.

For each $h \in \mathfrak{A}$ define

$$\phi_h(z_1, z_2) = \pi^f(z_1) + \pi^f_-(z_2) + h(z_1, z_2) \text{ for all } (z_1, z_2) \in R^n \times R^n. \tag{9.1}$$

Lemma 9.2. *Let $h \in \mathfrak{A}$, $S_0 > 0$, $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, \epsilon)$ such that for each $(x_1, u_1) \in X(A, B, 0, S_0)$ and each $(x_2, u_2) \in X(-A, -B, 0, S_0)$ which satisfy*

$$\phi_h(x_1(0), x_2(0)) \le \inf(\phi_h) + \delta, \tag{9.2}$$

$$I^f(0, S_0, x_1, u_1) - S_0\mu(f) - \pi^f(x_1(0)) + \pi^f(x_1(S_0)) \le \delta, \tag{9.3}$$

$$I^f(0, S_0, x_2, u_2) - S_0\mu(f) - \pi^f_-(x_2(0)) + \pi^f_-(x_2(S_0)) \le \delta \tag{9.4}$$

there exist an (f, A, B)-overtaking optimal pair $(x_1^*, u_1^*) \in X(A, B, 0, \infty)$ and an $(f, -A, -B)$-overtaking optimal pair $(x_2^*, u_2^*) \in X(-A, -B, 0, \infty)$ such that

$$\phi_h(x_1^*(0), x_2^*(0)) = \inf(\phi_h) \tag{9.5}$$

and that for all $t \in [0, S_0]$,

$$|x_1(t) - x_1^*(t)| \le \epsilon,\ |x_2(t) - x_2^*(t)| \le \epsilon.$$

Proof. Assume that the lemma does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^\infty \subset (0, 1]$ and sequences

$$\{(x_{k,1}, u_{k,1})\}_{k=1}^\infty \subset X(A, B, 0, S_0),\ \{(x_{k,2}, u_{k,2})\}_{k=1}^\infty \subset X(-A, -B, 0, S_0)$$

such that

$$\lim_{k\to\infty} \delta_k = 0, \tag{9.6}$$

for all integer $k \ge 1$,

$$\phi_h(x_{k,1}(0), x_{k,2}(0)) \le \inf(\phi_h) + \delta_k, \tag{9.7}$$

$$I^f(0, S_0, x_{k,1}, u_{k,1}) - S_0\mu(f) - \pi^f(x_{k,1}(0)) + \pi^f(x_{k,1}(S_0)) \le \delta_k, \tag{9.8}$$

$$I^f(0, S_0, x_{k,2}, u_{k,2}) - S_0\mu(f) - \pi_-^f(x_{k,2}(0)) + \pi_-^f(x_{k,2}(S_0)) \le \delta_k \tag{9.9}$$

and that for each integer $k \ge 1$, each (f, A, B)-overtaking optimal pair $(\xi_1, \eta_1) \in X(A, B, 0, \infty)$ and each $(f, -A, -B)$-overtaking optimal pair

$$(\xi_2, \eta_2) \in X(-A, -B, 0, \infty)$$

satisfying

$$\phi_h(\xi_1(0), \xi_2(0)) = \inf(\phi_h) \tag{9.10}$$

we have

$$\sup\{|\xi_1(t) - x_{k,1}(t)|,\ |\xi_2(t) - x_{k,2}(t)| :\ t \in [0, S_0]\} > \epsilon. \tag{9.11}$$

In view of (5.1), (9.1), (9.7), Proposition 3.3 and Lemma 9.1, the sequences

$$\{\pi^f(x_{k,1}(0))\}_{k=1}^\infty,$$

$$\{\pi_-^f(x_{k,2}(0))\}_{k=1}^\infty$$

and

$$\{h(x_{k,1}(0), x_{k,2}(0))\}_{k=1}^\infty$$

are bounded. Together with Proposition 3.3 this implies that the sequences $\{x_{k,1}(0)\}_{k=1}^\infty$, $\{x_{k,2}(0)\}_{k=1}^\infty$ are bounded. Proposition 3.3, (4.1), (9.6), (9.8) and (9.9) imply that the sequences

$$\{I^f(0, S_0, x_{k,1}, u_{k,1})\}_{k=1}^\infty,\ \{I^f(0, S_0, x_{k,2}, u_{k,2})\}_{k=1}^\infty$$

are bounded. By Proposition 6.4, extracting a subsequence and re-indexing if necessary, we may assume without loss of generality that there exist $(x_1, u_1) \in X(A, B, 0, S_0)$ and $(x_2, u_2) \in X(-A, -B, 0, S_0)$ such that for $i = 1, 2$,

$$x_{k,i}(t) \to x_i(t) \text{ as } k \to \infty \text{ uniformly on } [0, S_0], \tag{9.12}$$

$$I^f(0, S_0, x_i, u_i) \le \liminf_{k\to\infty} I^f(0, S_0, x_{k,i}, u_{k,i}). \tag{9.13}$$

It follows from (9.7), (9.12) and the lower semicontinuity of π^f, π_-^f, h, ϕ_h that

$$\pi^f(x_1(0)) \le \liminf_{k\to\infty} \pi^f(x_{k,1}(0)),$$

$$\pi_-^f(x_2(0)) \le \liminf_{k\to\infty} \pi_-^f(x_{k,2}(0)), \tag{9.14}$$

$$h(x_1(0), x_2(0)) \le \liminf_{k\to\infty} h(x_{k,1}(0), x_{k,2}(0)), \tag{9.15}$$

$$\phi_h(x_1(0), x_2(0)) \le \liminf_{k\to\infty} \phi_h(x_{k,1}(0), x_{k,2}(0)) = \inf(\phi_h). \tag{9.16}$$

In view of (9.14)-(9.16),

$$\pi^f(x_1(0)) = \lim_{k\to\infty} \pi^f(x_{k,1}(0)),$$

$$\pi_-^f(x_2(0)) = \lim_{k\to\infty} \pi_-^f(x_{k,2}(0)),$$

$$h(x_1(0), x_2(0)) = \lim_{k\to\infty} h(x_{k,1}(0), (x_{k,2}(0)). \tag{9.17}$$

By (9.12) and the lower semicontinuity of the functions π^f, π^f_-,

$$\pi^f(x_1(S_0)) \le \liminf_{k\to\infty} \pi^f(x_{k,1}(S_0)),$$

$$\pi^f_-(x_2(S_0)) \le \liminf_{k\to\infty} \pi^f_-(x_{k,2}(S_0)). \tag{9.18}$$

It follows from (9.6), (9.8), (9.13), (9.17) and (9.18) that

$$\begin{aligned} &I^f(0, S_0, x_1, u_1) - S_0\mu(f) - \pi^f(x_1(0)) + \pi^f(x_1(S_0)) \\ &\quad\le \liminf_{k\to\infty}[I^f(0, S_0, x_{k,1}, u_{k,1}) - S_0\mu(f)] \\ &\qquad - \lim_{k\to\infty} \pi^f(x_{k,1}(0)) + \liminf_{k\to\infty} \pi^f(x_{k,1}(S_0)) \\ &\quad\le \liminf_{k\to\infty}[I^f(0, S_0, x_{k,1}, u_{k,1}) - S_0\mu(f) - \pi^f(x_{k,1}(0)) + \pi^f(x_{k,1}(S_0))] \\ &\quad\le \lim_{k\to\infty} \delta_k = 0. \end{aligned} \tag{9.19}$$

It follows from (9.6), (9.9), (9.13), (9.17) and (9.18) that

$$\begin{aligned} &I^f(0, S_0, x_2, u_2) - S_0\mu(f) - \pi^f_-(x_2(0)) + \pi^f_-(x_2(S_0)) \\ &\quad\le \liminf_{k\to\infty}[I^f(0, S_0, x_{k,2}, u_{k,2}) - S_0\mu(f)] \\ &\qquad - \lim_{k\to\infty} \pi^f_-(x_{k,2}(0)) + \liminf_{k\to\infty} \pi^f_-(x_{k,2}(S_0)) \\ &\quad\le \liminf_{k\to\infty}[I^f(0, S_0, x_{k,2}, u_{k,2}) - S_0\mu(f) \\ &\qquad - \pi^f_-(x_{k,2}(0)) + \pi^f_-(x_{k,2}(S_0))] \\ &\quad\le \lim_{k\to\infty} \delta_k = 0. \end{aligned} \tag{9.20}$$

In view of (9.19), (9.20) and Proposition 3.1,

$$I^f(0, S_0, x_1, u_1) - S_0\mu(f) - \pi^f(x_1(0)) + \pi^f(x_1(S_0)) = 0, \tag{9.21}$$

$$I^f(0, S_0, x_2, u_2) - S_0\mu(f) - \pi^f_-(x_2(0)) + \pi^f_-(x_2(S_0)) = 0, \tag{9.22}$$

Theorem 2.2 implies that there exists an (f, A, B)-overtaking optimal pair $(\tilde{x}_1, \tilde{u}_1) \in X(A, B, 0, \infty)$ such that

$$\tilde{x}_1(0) = x_1(S_0) \tag{9.23}$$

and an $(f, -A, -B)$-overtaking optimal pair $(\tilde{x}_2, \tilde{u}_2) \in X(-A, -B, 0, \infty)$ such that

$$\tilde{x}_2(0) = x_2(S_0) \tag{9.24}$$

For all $t > S_0$ and $i = 1, 2$ set

$$x_i(t) = \tilde{x}_i(t - S_0),\ u_i(t) = \tilde{u}_i(t - S_0). \tag{9.25}$$

It is not difficult to see that the pair $(x_1, u_1) \in X(A, B, 0, \infty)$ is an (f, A, B)-good pair and that the pair $(x_2, u_2) \in X(-A, -B, 0, \infty)$ is an $(f, -A, -B)$-good pair. By (9.21)-(9.25) and Propositions 3.1 and 3.2, for all $S > 0$,

$$I^f(0, S, x_1, u_1) - S\mu(f) - \pi^f(x_1(0)) + \pi^f(x_1(S)) = 0,$$

$$I^f(0,S,x_2,u_2) - S\mu(f) - \pi_-^f(x_2(0)) + \pi_-^f(x_2(S)) = 0.$$

Combined with Proposition 3.5 and (9.16) this implies that

$$(x_1,u_1) \in X(A,B,0,\infty)$$

is an (f,A,B)-overtaking optimal pair and that

$$(x_2,u_2) \in X(-A,-B,0,\infty)$$

is an $(f,-A,-B)$-overtaking optimal pair such that

$$\phi_h(x_1(0),x_2(0)) = \inf(\phi_h).$$

By (9.12), for all sufficiently large natural numbers k and $i = 1,2$,

$$|x_{k,i}(t) - x_i(t)| \le \epsilon/2 \text{ for all } t \in [0,S_0].$$

This contradicts (9.11). The contradiction we have reached proves Lemma 9.2. □

10. Proof of Theorem 5.3

By Lemma 9.2, there exists $\delta_0 \in (0,\epsilon)$ such that the following property holds:

(P5) for each $(x_1,u_1) \in X(A,B,0,L_0)$ and each $(x_2,u_2) \in X(A,B,0,L_0)$ which satisfy

$$\phi_h(x_1(0),x_2(0)) \le \inf(\phi_h) + 4\delta_0,$$
$$I^f(0,L_0,x_1,u_1) - L_0\mu(f) - \pi^f(x_1(0)) + \pi^f(x_1(L_0)) \le 4\delta_0,$$
$$I^f(0,L_0,x_2,u_2) - L_0\mu(f) - \pi_-^f(x_2(0)) + \pi_-^f(x_2(L_0)) \le 4\delta_0$$

there exist an (f,A,B)-overtaking optimal pair $(x_1^*,u_1^*) \in X(A,B,0,\infty)$ and an $(f,-A,-B)$-overtaking optimal pair $(x_2^*,u_2^*) \in X(-A,-B,0,\infty)$ such that

$$\phi_h(x_1^*(0),x_2^*(0)) = \inf(\phi_h)$$

and that for all $t \in [0,L_0]$, $i = 1,2$,

$$|x_i(t) - x_i^*(t)| \le \epsilon.$$

In view of Propositions 3.3 and 6.5, there exists $\delta_1 \in (0,\delta_0/4)$ such that:

for each $z \in R^n$ satisfying $|z - x_f| \le 2\delta_1$,

$$|\pi_-^f(z)| = |\pi_-^f(z) - \pi_-^f(x_f)| \le \delta_0/8, \tag{10.1}$$

$$|\pi^f(z)| = |\pi^f(z) - \pi^f(x_f)| \le \delta_0/8; \tag{10.2}$$

for each $y,z \in R^n$ satisfying

$$|y - x_f| \le 2\delta_1,\ |z - x_f| \le 2\delta_1$$

we have

$$|\sigma(f,y,z,1) - \mu(f)| \le \delta_0/8. \tag{10.3}$$

By Theorem 5.2, there exist $l_0 > 0$, $\delta_2 \in (0,\delta_1/8)$, a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$ and a neighborhood $\mathcal{V}_1$ of h in $\mathfrak{A}$ such that the following property holds:

(P6) for each $T > 2l_0$, each $g \in \mathcal{U}_1$, each $\xi \in \mathcal{V}_1$ and each

$$(x,u) \in X(A,B,0,T)$$

such that

$$I^g(0,T,x,u) + \xi(x(0),x(T)) \le \sigma(g,\xi,0,T) + \delta_2$$

we have

$$|x(t) - x_f| \leq \delta_1 \text{ for all } t \in [l_0, T - l_0].$$

By Theorem 2.2 and Lemma 9.1, there exist an (f, A, B)-overtaking optimal pair

$$(x_*, u_*) \in X(A, B, 0, \infty)$$

and an $(f, -A, -B)$-overtaking optimal pair

$$(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$$

such that

$$\phi_h(x_*(0), \bar{x}_*(0)) = \inf(\phi_h). \tag{10.4}$$

Assumption (A3) implies that there exists $l_1 > 0$ such that for all $t \geq l_1$,

$$|\bar{x}_*(t) - x_f| \leq \delta_1, \ |x_*(t) - x_f| \leq \delta_1. \tag{10.5}$$

By Proposition 6.3, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P7) for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 1, T_1 + 2L_0 + 2l_0 + 2l_1 + 4]$ and each trajectory-control pair $(x, u) \in X(A, B, T_1, T_2)$ which satisfies

$$\begin{aligned} &\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \\ &\quad \leq (|\mu(f)| + 2)(2L_0 + 2l_0 + 2l_1 + 6) + 8 \\ &\qquad + |\pi_-^f(\bar{x}_*(0))| + |\pi^f(x_*(0))| + |h(x_*(0), \bar{x}_*(0))| \\ &\qquad + a_1 + a_0(L_0 + l_0 + l_1 + 3) \end{aligned}$$

the inequality

$$|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \leq \delta_2/8$$

holds.

By Proposition 6.6, there exists $\Delta_0 > 0$ such that the following property holds:

(P8) for each $T_1 \in R^1$, each $T_2 \in [T_1 + 1, T_1 + 2L_0 + 2l_0 + 2l_1 + 8]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying

$$\begin{aligned} I^f(T_1, T_2, x, u) &\leq (|\mu(f)| + 2)(2L_0 + 2l_0 + 2l_1 + 6) \\ &\quad + 2|\pi_-^f(\bar{x}_*(0))| + 2|\pi^f(x_*(0))| \\ &\quad + |h(x_*(0), \bar{x}_*(0))| + a_1 + a_0(L_0 + l_0 + l_1 + 3) + 46 \end{aligned}$$

we have $|x(t)| \leq \Delta_0$ for all $t \in [T_1, T_2]$.

Let

$$\begin{aligned} \mathcal{V} = \{\xi \in \mathcal{V}_1 : \ &|\xi(z_1, z_2) - h(z_1, z_2)| \leq \delta_1/16 \\ &\text{for all } z_1, z_2 \in R^n \text{ satisfying } |z_i| \leq 2 + \Delta_0 + |x_*(0)| + |\bar{x}_*(0)|, \ i = 1, 2\}. \end{aligned} \tag{10.6}$$

Choose $\delta > 0$ and $L_1 > 0$ such that

$$\delta \leq 4^{-1}\delta_2(L_0 + l_0 + l_1 + 8)^{-1}, \tag{10.7}$$

$$L_1 > 4L_0 + 4l_0 + 4l_1 + 8. \tag{10.8}$$

Assume that

$$T \geq L_1, \ g \in \mathcal{U}, \ \xi \in \mathcal{V}, \ (x, u) \in X(A, B, 0, T), \tag{10.9}$$

$$I^g(0,T,x,u)+\xi(x(0),x(T))\le\sigma(g,\xi,0,T)+\delta. \tag{10.10}$$

It follows from property (P6) and (10.7)-(10.10) that

$$|x(t)-x_f|\le\delta_1 \text{ for all } t\in[l_0,T-l_0]. \tag{10.11}$$

By Proposition 6.1, there exists a trajectory-control pair

$$(\tilde{x},\tilde{u})\in X(A,B,0,T)$$

such that

$$\begin{aligned}
&\tilde{x}(t)=x_*(t),\ \tilde{u}(t)=u_*(t),\ t\in[0,L_0+l_0+l_1+3],\\
&\tilde{x}(t)=x(t),\ \tilde{u}(t)=u(t),\ t\in[L_0+l_0+l_1+4,T-l_0-l_1-L_0-4],\\
&\tilde{x}(t)=\bar{x}_*(T-t),\ \tilde{u}(t)=\bar{u}_*(T-t),\ t\in[T-l_0-l_1-L_0-3,T],\\
&I^f(l_0+l_1+L_0+3,l_0+l_1+L_0+4,\tilde{x},\tilde{u})\\
&=\sigma(f,x_*(l_0+l_1+L_0+3),x(l_0+l_1+L_0+4),1),\\
&I^f(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,\tilde{x},\tilde{u})\\
&=\sigma(f,x(T-l_0-l_1-L_0-4),\bar{x}_*(l_0+l_1+L_0+3),1).
\end{aligned} \tag{10.12}$$

By (10.5), (10.8), (10.9), (10.11) and (10.12),

$$|\tilde{x}(L_0+l_0+l_1+3)-x_f|=|x_*(L_0+l_0+l_1+3)-x_f|\le\delta_1, \tag{10.13}$$

$$|\tilde{x}(L_0+l_0+l_1+4)-x_f|=|x(L_0+l_0+l_1+4)-x_f|\le\delta_1, \tag{10.14}$$

$$|\tilde{x}(T-L_0-l_0-l_1-4)-x_f|=|x(T-L_0-l_0-l_1-4)-x_f|\le\delta_1, \tag{10.15}$$

$$|\tilde{x}(T-L_0-l_0-l_1-3)-x_f|=|\bar{x}_*(L_0+l_0+l_1+3)-x_f|\le\delta_1. \tag{10.16}$$

In view of (10.3) and (10.12)-(10.16),

$$|I^f(L_0+l_0+l_1+3,L_0+l_0+l_1+4,\tilde{x},\tilde{u})-\mu(f)|\le\delta_0/8, \tag{10.17}$$

$$|I^f(T-L_0-l_0-l_1-4,T-L_0-l_0-l_1-3,\tilde{x},\tilde{u})-\mu(f)|\le\delta_0/8. \tag{10.18}$$

Property (P7), (10.9), (10.17) and (10.18) imply that

$$\begin{aligned}
&|I^g(L_0+l_0+l_1+3,L_0+l_0+l_1+4,\tilde{x},\tilde{u})-\mu(f)|\\
&\quad\le|I^g(L_0+l_0+l_1+3,L_0+l_0+l_1+4,\tilde{x},\tilde{u})\\
&\qquad-I^f(L_0+l_0+l_1+3,L_0+l_0+l_1+4,\tilde{x},\tilde{u})|\\
&\qquad+|I^f(L_0+l_0+l_1+3,L_0+l_0+l_1+4,\tilde{x},\tilde{u})-\mu(f)|\\
&\quad\le\delta_2/8+\delta_0/8
\end{aligned} \tag{10.19}$$

and

$$\begin{aligned}
&|I^g(T-L_0-l_0-l_1-4,T-L_0-l_0-l_1-3,\tilde{x},\tilde{u})-\mu(f)|\\
&\quad\le|I^g(T-L_0-l_0-l_1-4,T-L_0-l_0-l_1-3,\tilde{x},\tilde{u})\\
&\qquad-I^f(T-L_0-l_0-l_1-4,T-L_0-l_0-l_1-3,\tilde{x},\tilde{u})|\\
&\qquad+|I^f(T-L_0-l_0-l_1-4,T-L_0-l_0-l_1-3,\tilde{x},\tilde{u})-\mu(f)|\\
&\quad\le\delta_2/8+\delta_0/8.
\end{aligned} \tag{10.20}$$

By (10.3), (10.8), (10.9) and (10.11),

$$\begin{aligned} &I^f(L_0+l_0+l_1+3, L_0+l_0+l_1+4, x, u) \\ &\quad \geq \sigma(f, x(L_0+l_0+l_1+3), x(L_0+l_0+l_1+4), 1) \\ &\quad \geq \mu(f) - \delta_0/8 \end{aligned} \tag{10.21}$$

and

$$I^f(T-L_0-l_0-l_1-4, T-L_0-l_0-l_1-3, x, u)$$

$$\geq \sigma(f, x(T-L_0-l_0-l_1-4), x(T-L_0-l_0-l_1-3), 1) \geq \mu(f) - \delta_0/8. \tag{10.22}$$

We show that

$$I^g(L_0+l_0+l_1+3, L_0+l_0+l_1+4, x, u) \geq \mu(f) - \delta_0/2. \tag{10.23}$$

Assume the contrary. Then

$$I^g(L_0+l_0+l_1+3, L_0+l_0+l_1+4, x, u) < \mu(f) - \delta_0/2. \tag{10.24}$$

Property (P7), (10.9) and (10.24) imply that

$$\begin{aligned} &I^f(L_0+l_0+l_1+3, L_0+l_0+l_1+4, x, u) \\ &\quad \leq I^g(L_0+l_0+l_1+3, L_0+l_0+l_1+4, x, u) + \delta_2/8 \\ &\quad < \mu(f) - \delta_0/2 + \delta_2/8 < \mu(f) - \delta_0/2 + \delta_0/16. \end{aligned}$$

This contradicts (10.21). The contradiction we have reached proves (10.23).

We show that

$$I^g(T-L_0-l_0-l_1-4, T-L_0-l_0-l_1-3, x, u) \geq \mu(f) - \delta_0/2. \tag{10.25}$$

Assume the contrary. Then

$$I^g(T-L_0-l_0-l_1-4, T-L_0-l_0-l_1-3, x, u) < \mu(f) - \delta_0/2. \tag{10.26}$$

Property (P7), (10.9) and (10.26) imply that

$$\begin{aligned} &I^f(T-L_0-l_0-l_1-4, T-L_0-l_0-l_1-3, x, u) \\ &\quad \leq I^g(T-L_0-l_0-l_1-4, T-L_0-l_0-l_1-3, x, u) + \delta_2/8 \\ &\quad < \mu(f) - \delta_0/2 + \delta_2/8 < \mu(f) - \delta_0/2 + \delta_0/16. \end{aligned}$$

This contradicts (10.22). The contradiction we have reached proves (10.25).

Since (x_*, u_*) is an (f, A, B)-overtaking optimal pair, (10.12) and Proposition 3.2 imply

$$\begin{aligned} I^f(0, l_0+l_1+L_0+3, T, \tilde{x}, \tilde{u}) &= I^f(0, l_0+l_1+L_0+3, x_*, u_*) \\ &= \mu(f)(l_0+l_1+L_0+3) \\ &\quad + \pi^f(x_*(0)) - \pi^f(x_*(l_0+l_1+L_0+3)). \end{aligned} \tag{10.27}$$

Since $(\bar{x}_*, \bar{u}_*)$ is an $(\bar{f}, -A, -B)$-overtaking optimal pair it follows from (3.6), (10.12) and Proposition 3.2 that

$$\begin{aligned} &I^f(T - l_0 - l_1 - L_0 - 3, T, \tilde{x}, \tilde{u}) \\ &\quad = I^f(0, l_0 + l_1 + L_0 + 3, \bar{x}_*, \bar{u}_*) \\ &\quad = \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + \pi_-^f(\bar{x}_*(0)) - \pi_-^f(\bar{x}_*(l_0 + l_1 + L_0 + 3)). \end{aligned} \tag{10.28}$$

In view of the choice of δ_1 (see (10.1), (10.2)) and (10.5),

$$|\pi^f(x_*(L_0 + l_0 + l_1 + 3))| \le \delta_0/8,$$

$$|\pi_-^f(\bar{x}_*(L_0 + l_0 + l_1 + 3))| \le \delta_0/8. \tag{10.29}$$

Combined with (10.27) and (10.28) these inequalities imply that

$$I^f(0, L_0 + l_0 + l_1 + 3, \tilde{x}, \tilde{u}) \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi^f(x_*(0)) + \delta_0/8,$$

$$I^f(T - l_0 - l_1 - L_0 - 3, T, \tilde{x}, \tilde{u}) \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi_-^f(\bar{x}_*(0)) + \delta_0/8. \tag{10.30}$$

Property (P7), (10.9) and (10.30) imply that

$$\begin{aligned} &I^g(0, L_0 + l_0 + l_1 + 3, \tilde{x}, \tilde{u}) \\ &\quad \le I^f(0, L_0 + l_0 + l_1 + 3, \tilde{x}, \tilde{u}) + \delta_2/8 \\ &\quad \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi^f(x_*(0)) + \delta_0/8 + \delta_2/8 \end{aligned} \tag{10.31}$$

and

$$\begin{aligned} &I^g(T - l_0 - l_1 - L_0 - 3, T, \tilde{x}, \tilde{u}) \\ &\quad \le I^f(T - l_0 - l_1 - L_0 - 3, T, \tilde{x}, \tilde{u}) + \delta_2/8 \\ &\quad \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi_-^f(\bar{x}_*(0)) + \delta_0/8 + \delta_2/8. \end{aligned} \tag{10.32}$$

By (10.10), (10.12), (10.19), (10.23), (10.26), (10.31) and (10.32),

$$\begin{aligned} \delta &\ge I^g(0, T, x, u) + \xi(x(0), x(T)) - (I^g(0, T, \tilde{x}, \tilde{u}) + \xi(\tilde{x}(0), \tilde{x}(T))) \\ &= (\xi(x(0), x(T)) - \xi(x_*(0), \bar{x}_*(0))) \\ &\quad + I^g(0, L_0 + l_0 + l_1 + 3, x, u) \\ &\quad - I^g(0, L_0 + l_0 + l_1 + 3, \tilde{x}, \tilde{u}) \\ &\quad + I^g(L_0 + l_0 + l_1 + 3, L_0 + l_0 + l_1 + 4, x, u) \\ &\quad - I^g(L_0 + l_0 + l_1 + 3, L_0 + l_0 + l_1 + 4, \tilde{x}, \tilde{u}) \\ &\quad + I^g(T - L_0 - l_0 - l_1 - 4, T - L_0 - l_0 - l_1 - 3, x, u) \\ &\quad - I^g(T - L_0 - l_0 - l_1 - 4, T - L_0 - l_0 - l_1 - 3, \tilde{x}, \tilde{u}) \\ &\quad + I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) \\ &\quad - I^g(T - l_0 - l_1 - L_0 - 3, T, \tilde{x}, \tilde{u}) \\ &\ge \xi(x(0), x(T)) - \xi(x_*(0), \bar{x}_*(0)) \\ &\quad + I^g(0, L_0 + l_0 + l_1 + 3, x, u) \\ &\quad - \mu(f)(l_0 + l_1 + L_0 + 3) - \pi^f(x_*(0)) - \delta_0/8 - \delta_2/8 \end{aligned} \tag{10.33}$$

$$
\begin{aligned}
&+ (\mu(f) - \delta_0/2 - \mu(f) - \delta_0/8 - \delta_2/8)\\
&+ (\mu(f) - \delta_0/2 - \mu(f) - \delta_0/8 - \delta_2/8)\\
&+ I^g(T - l_0 - l_1 - L_0 - 3, T, x, u)\\
&- \mu(f)(l_0 + l_1 + L_0 + 3) - \pi_-^f(\bar{x}_*(0)) - \delta_0/8 - \delta_2/8\\
\geq\ & \xi(x(0), x(T)) - \xi(x_*(0), \bar{x}_*(0))\\
&+ I^g(0, L_0 + l_0 + l_1 + 3, x, u) + I^g(T - l_0 - l_1 - L_0 - 3, T, x, u)\\
&- 2\mu(f)(L_0 + l_0 + l_1 + 3) - \pi^f(x_*(0)) - \pi_-^f(\bar{x}_*(0)) - 3\delta_0/2 - \delta_2/2.
\end{aligned}
$$

In view of (10.6) and (10.9),

$$|\xi(x_*(0), \bar{x}_*(0)) - h(x_*(0), \bar{x}_*(0))| \leq \delta_1/16. \tag{10.34}$$

It follows from (4.1), (5.1) (10.33) and (10.34) that

$$
\begin{aligned}
&I^g(0, L_0 + l_0 + l_1 + 3, x, u),\ I^g(T - l_0 - l_1 - L_0 - 3, T, x, u)\\
&\leq \delta + \delta_2/2 + 3\delta_0/2 + |h(x_*(0), \bar{x}_*(0))| + \delta_1/16 + a_1\\
&+2\mu(f)(L_0 + l_0 + l_1 + 3) + \pi^f(x_*(0)) + \pi_-^f(\bar{x}_*(0))\\
&+a_0(L_0 + l_0 + l_1 + 3).
\end{aligned} \tag{10.35}
$$

Property (P7), (10.9) and (10.35) imply that

$$|I^g(0, L_0 + l_0 + l_1 + 3, x, u) - I^f(0, L_0 + l_0 + l_1 + 3, x, u)| \leq \delta_2/8,$$

$$|I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) - I^f(T - l_0 - l_1 - L_0 - 3, T, x, u)| \leq \delta_2/8. \tag{10.36}$$

Property (P8), (10.35) and (10.36) imply that

$$
\begin{aligned}
&I^f(0, L_0 + l_0 + l_1 + 3, x, u),\ I^f(T - l_0 - l_1 - L_0 - 3, T, x, u)\\
&\leq \delta + \delta_2/8 + \delta_2/2 + 3\delta_0/2 + |h(x_*(0), \bar{x}_*(0))| + \delta_1/16 + a_1\\
&+2\mu(f)(L_0 + l_0 + l_1 + 3) + \pi^f(x_*(0)) + \pi_-^f(\bar{x}_*(0))\\
&+a_0(L_0 + l_0 + l_1 + 3).
\end{aligned}
$$

In view of the inequality above and property (P8),

$$|x(t)| \leq \Delta_0,\ t \in [0, L_0 + l_0 + l_1 + 3] \cup [T - l_0 - l_1 - L_0 - 3, T]. \tag{10.37}$$

By (10.6), (10.9) and (10.37),

$$|\xi(x(0), x(T)) - h(x(0), x(T))| \leq \delta_1/16. \tag{10.38}$$

It follows from (10.33)-(10.36) and (10.38) that

$$
\begin{aligned}
\delta + 3\delta_0/2 + \delta_2/2 \geq\ & h(x(0), x(T)) - h(x_*(0), \bar{x}_*(0)) - \delta_1/8\\
&+ I^f(0, L_0 + l_0 + l_1 + 3, x, u) - \delta_2/8\\
&+ I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) - \delta_2/8\\
&- 2\mu(f)(L_0 + l_0 + l_1 + 3) - \pi^f(x_*(0)) - \pi_-^f(\bar{x}_*(0)).
\end{aligned} \tag{10.39}
$$

In view of (9.1) and (10.39),

$$\begin{aligned}\delta + 3\delta_0/2 + 3\delta_2/4 + \delta_1/8 \ge\ & \phi_h(x(0), x(T)) \\ & - \phi_h(x_*(0), \bar{x}_*(0)) - \pi^f(x(0)) - \pi^f_-(x(T)) \\ & + I^f(0, L_0 + l_0 + l_1 + 3, x, u) \\ & + I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) \\ & - 2\mu(f)(L_0 + l_0 + l_1 + 3).\end{aligned} \tag{10.40}$$

Set

$$y(t) = x(T-t),\ v(t) = u(T-t),\ t \in [0, T]. \tag{10.41}$$

In view of (3.6) and (10.41),

$$I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) = I^f(0, L_0 + l_0 + l_1 + 3, y, v). \tag{10.42}$$

By (10.4), (10.29) and (10.40)-(10.42),

$$\begin{aligned}&\delta + 3\delta_0/2 + 3\delta_2/4 + \delta_1/8 \\ &\ge \phi_h(x(0), x(T)) - \inf(\phi_h) \\ &\quad + I^f(0, L_0 + l_0 + l_1 + 3, x, u) - \mu(f)(L_0 + l_0 + l_1 + 3) \\ &\quad - \pi^f(x(0)) + \pi^f(x(L_0 + l_0 + l_1 + 3)) - \pi^f(x(L_0 + l_0 + l_1 + 3)) \\ &\quad + I^f(0, L_0 + l_0 + l_1 + 3, y, v) \\ &\quad - \pi^f_-(y(0)) + \pi^f_-(y(L_0 + l_0 + l_1 + 3)) \\ &\quad - \mu(f)(L_0 + l_0 + l_1 + 3) - \pi^f_-(x(T - L_0 - l_0 - l_1 - 3)) \\ &\ge \phi_h(x(0), x(T)) - \inf(\phi_h) \\ &\quad + I^f(0, L_0 + l_0 + l_1 + 3, x, u) \\ &\quad - \mu(f)(L_0 + l_0 + l_1 + 3) - \pi^f(x(0)) + \pi^f(x(L_0 + l_0 + l_1 + 3)) \\ &\quad + I^f(0, L_0 + l_0 + l_1 + 3, y, v) \\ &\quad - \pi^f_-(y(0)) + \pi^f_-(y(L_0 + l_0 + l_1 + 3)) \\ &\quad - \mu(f)(L_0 + l_0 + l_1 + 3) - \delta_0/4.\end{aligned} \tag{10.43}$$

Proposition 3.1, (10.41) and (10.43) imply that

$$4\delta_0 \ge \phi_h(x(0), x(T)) - \inf(\phi_h), \tag{10.44}$$

$$\begin{aligned}4\delta_0 &\ge I^f(0, L_0 + l_0 + l_1 + 3, x, u) \\ &\quad - \mu(f)(L_0 + l_0 + l_1 + 3) - \pi^f(x(0)) + \pi^f(x(L_0 + l_0 + l_1 + 3)) \\ &\ge I^f(0, L, x, u) - \mu(f)L_0 - \pi^f(x(0)) + \pi^f(x(L_0)),\end{aligned} \tag{10.45}$$

$$\begin{aligned}4\delta_0 &\ge I^f(0, L_0 + l_0 + l_1 + 3, y, v) \\ &\quad - \pi^f_-(y(0)) + \pi^f_-(y(L_0 + l_0 + l_1 + 3)) - \mu(f)(L_0 + l_0 + l_1 + 3) \\ &\ge I^f(0, L_0, y, v) - \pi^f_-(y(0)) + \pi^f_-(y(L_0)) - \mu(f)L_0.\end{aligned} \tag{10.46}$$

By (10.44)-(10.46) and property (P5), there exist an (f, A, B)-overtaking optimal pair $(x_1^*, u_1^*) \in X(A, B, 0, \infty)$ and an $(f, -A, -B)$-overtaking optimal pair $(x_2^*, u_2^*) \in X(-A, -B, 0, \infty)$ such that

$$\phi_h(x_1^*(0), x_2^*(0) = \inf(\phi_h)$$

and that for all $t \in [0, L_0]$,

$$\epsilon \geq |x(t) - x_1^*(t)|, \ \epsilon \geq |y(t) - x_2^*(t)| = |x(T-t) - x_2^*(t)|.$$

Theorem 5.3 is proved.

References

[1] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[2] S. M. Aseev and V. M. Veliov, *Maximum principle for infinite-horizon optimal control problems with dominating discount*, Dynamics of Continuous, Discrete and Impulsive Systems, Ser. B **19** (2012), 43–63.

[3] S. Aubry and P. Y. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions I*, Physica D **8** (1983), 381–422.

[4] M. Bachir and J. Blot, *Infinite dimensional infinite-horizon Pontryagin principles for discrete-time problems*, Set-Valued and Variational Analysis **23** (2015), 43–54.

[5] J. Baumeister, A. Leitao and G. N. Silva, *On the value function for nonautonomous optimal control problem with infinite horizon*, Systems Control Lett. **56** (2007), 188–196.

[6] J. Blot, *Infinite-horizon Pontryagin principles without invertibility*, J. Nonlinear Convex Anal. **10** (2009), 177–189.

[7] J. Blot and P. Cartigny, *Optimality in infinite-horizon variational problems under sign conditions*, J. Optim. Theory Appl. **106** (2000), 411–419.

[8] J. Blot and N. Hayek, *Infinite-Horizon Optimal Control in the Discrete-Time Framework*, SpringerBriefs in Optimization, New York, 2014.

[9] I. Bright, *A reduction of topological infinite-horizon optimization to periodic optimization in a class of compact 2-manifolds*, Journal of Mathematical Analysis and Applications **394** (2012), 84–101.

[10] D. A. Carlson, *The existence of catching-up optimal solutions for a class of infinite horizon optimal control problems with time delay,* SIAM Journal on Control and Optimization **28** (1990), 402–422.

[11] D. A. Carlson, A. Haurie and A. Jabrane, *Existence of overtaking solutions to infinite dimensional control problems on unbounded time intervals*, SIAM J. Control Optim. **25** (1987), 1517–541.

[12] D. A. Carlson, A. Haurie and A. Leizarowitz, *Infinite Horizon Optimal Control*, Berlin, Springer-Verlag, 1991.

[13] P. Cartigny and P. Michel, *On a sufficient transversality condition for infinite horizon optimal control problems*, Automatica J. IFAC **39** (2003), 1007–1010.

[14] T. Damm, L. Grune, M. Stieler and K. Worthmann, *An exponential turnpike theorem for dissipative discrete time optimal control problems*, SIAM Journal on Control and Optimization **52** (2014), 1935–1957.

[15] V. A. De Oliveira and G. N. Silva, *Optimality conditions for infinite horizon control problems with state constraints*, Nonlinear Analysis **71** (2009), 1788–1795.

[16] Z. Dzalilov, A. M Rubinov and P. E. Kloeden, *Lyapunov sequences and a turnpike theorem without convexity*, Set-Valued Analysis **6** (1998), 277–302.

[17] V. Gaitsgory, L. Grune and N. Thatcher, *Stabilization with discounted optimal control*, Systems and Control Letters **82** (2015), 91–98.

[18] V. Gaitsgory, S. Rossomakhine and N. Thatcher, *Approximate solution of the HJB inequality related to the infinite horizon optimal control problem with discounting*, Dynamics Contin. Discrete Impuls. Syst., Series B **19** (2012), 65–92.

[19] N. Hayek, *Infinite horizon multiobjective optimal control problems in the discrete time case*, Optimization **60** (2011), 509–529.

[20] H. Jasso-Fuentes and O. Hernandez-Lerma, *Characterizations of overtaking optimality for controlled diffusion processes*, Appl. Math. Optim. **57** (2008), 349–369.

[21] M. Ali Khan and A. J. Zaslavski, *On two classical turnpike results for the Robinson-Solow-Srinivisan (RSS) model*, Adv. in Math. Econom. **13** (2010), 47–97.

[22] D. V. Khlopin, *Necessary conditions of overtaking equilibrium for infinite horizon differential games*, Mat. Teor. Igr Pril. **5** (2013), 105–136.

[23] D. V. Khlopin, *Necessity of vanishing shadow price in infinite horizon control problems*, J Dyn Control Syst **19** (2013), 519–552.

[24] V. Kolokoltsov and W. Yang, *The turnpike theorems for Markov games*, Dynamic Games and Applications **2** (2012), 294–312.

[25] A. Leizarowitz, *Infinite horizon autonomous systems with unbounded cost*, Appl. Math. and Opt. **13** (1985), 19–43.

[26] A. Leizarowitz, *Tracking nonperiodic trajectories with the overtaking criterion*, Appl. Math. and Opt. **14** (1986), 155–171.

[27] A. Leizarowitz and V. J. Mizel, *One dimensional infinite horizon variational problems arising in continuum mechanics*, Arch. Rational Mech. Anal. **106** (1989).

[28] V. Lykina, S. Pickenhain and M. Wagner, *Different interpretations of the improper integral objective in an infinite horizon control problem*, J. Math. Anal. Appl. **340** (2008), 498–510.

[29] V. L. Makarov and A. M. Rubinov, *Mathematical Theory of Economic Dynamics and Equilibria*, New York, Springer-Verlag, 1977.

[30] A. B. Malinowska, N. Martins and D. F. M. Torres, *Transversality conditions for infinite horizon variational problems on time scales*, Optimization Lett. **5** (2011), 41–53.

[31] M. Mammadov, *Turnpike theorem for an infinite horizon optimal control problem with time delay*, SIAM Journal on Control and Optimization **52** (2014), 420–438.

[32] M. Marcus and A. J. Zaslavski, *The structure of extremals of a class of second order variational problems*, Ann. Inst. H. Poincaré, Anal. non Lineare **16** (1999), 593–629.

[33] L. W. McKenzie, *Turnpike theory*, Econometrica **44** (1976), 841–866.

[34] B. S. Mordukhovich, *Optimal control and feedback design of state-constrained parabolic systems in uncertainly conditions*, Appl. Analysis **90** (2011), 1075–1109.

[35] B. S. Mordukhovich and I. Shvartsman, *Optimization and feedback control of constrained parabolic systems under uncertain perturbations*, in: Optimal control, stabilization and nonsmooth analysis. Lecture Notes Control Inform. Sci., Springer, 2004, pp. 121–132.

[36] E. Ocana Anaya, P. Cartigny and P. Loisel, *Singular infinite horizon calculus of variations. Applications to fisheries management*, J. Nonlinear Convex Anal. **10** (2009), 157–176.

[37] S. Pickenhain, V. Lykina and M. Wagner, *On the lower semicontinuity of functionals involving Lebesgue or improper Riemann integrals in infinite horizon optimal control problems*, Control Cybernet. **37** (2008), 451–468.

[38] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim. **51** (2013), 4242–4273.

[39] A. M. Rubinov, *Superlinear Multivalued Mappings and Their Applications to Problems of Mathematical Economics*, Nauka, Leningrad, 1980.

[40] A. M. Rubinov, *Economic dynamics*, J. Soviet Math. **26** (1984), 1975–2012.

[41] P. A. Samuelson, *A catenary turnpike theorem involving consumption and the golden rule*, Amer. Econom. Rev. **55** (1965), 486–496.

[42] E. Trelat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations **218** (2015), 81–114.

[43] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, New York, 2006.

[44] A. J. Zaslavski, *The existence and structure of approximate solutions of dynamic discrete time zero-sum games*, Journal of Nonlinear and Convex Analysis **12** (2011), 49–68.

[45] A. J. Zaslavski, *Nonconvex Optimal Control and Variational Problems*, Springer Optimization and Its Applications, New York, 2013.

[46] A. J. Zaslavski, *Structure of Solutions of Variational Problems,* SpringerBriefs in Optimization, New York, 2013.
[47] A. J. Zaslavski, *Structure of Approximate Solutions of Optimal Control Problems,* SpringerBriefs in Optimization, New York, 2013.
[48] A. J. Zaslavski, *Turnpike Phenomenon and Infinite Horizon Optimal Control*, Springer Optimization and Its Applications, New York, 2014.
[49] A. J. Zaslavski, *Turnpike Theory of Continuous-Time Linear Optimal Control Problems*, Springer Optimization and Its Applications, Springer, Cham-Heidelberg-New York-Dordrecht-London, 2015.

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