Pure and Applied Functional Analysis
Volume 2, Number 1, 2017, 117–128

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ASYMPTOTIC BEHAVIOR OF THE FOURIER TRANSFORM OF A FUNCTION WITH DERIVATIVE IN A HARDY SPACE

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Abstract. Results on the asymptotic behavior of the cosine and sine Fourier transforms of locally absolutely continuous functions of bounded variation are extended to the case of several variables. For this, the initial one-dimensional results are reconsidered and refined by means of a special balance operator.

1. Introduction

Many of the integrability conditions for the Fourier transform of a function of bounded variation originated from the noteworthy result of Trigub [19] on the asymptotic behavior of the Fourier transform of a convex function. In a recent survey paper [17] integrability of the Fourier transform is considered in the context of belonging to Wiener's algebra and its relations to the theory of Fourier multipliers and comparison of operators. However, the most advanced results on the integrability of the Fourier transform of a function of bounded variation are related to the real Hardy space (see [5, 8, 10, 15]). In fact, the cosine and the sine Fourier transforms,

$$\widehat{f}_c(x) = \int_0^\infty f(t)\cos xt\,dt$$

and

$$\widehat{f}_s(x) = \int_0^\infty f(t)\sin xt\,dt,$$

respectively, are studied separately in dimension one for a function of bounded variation on $\mathbb{R}_+ = [0,\infty)$, vanishing at infinity, $\lim\limits_{t\to\infty} f(t) = 0$ (written $f \in BV_0[0,\infty)$), and locally absolutely continuous on $(0,\infty)$ (written $LAC(0,\infty)$), to wit all these

$$f \in BV_0[0,\infty) \cap LAC(0,\infty).$$

One of such results (see [10]) reads as follows.

Theorem 1.1. *Let $f \in BV_0[0,\infty) \cap LAC(0,\infty)$ and $f' \in H^1_o(\mathbb{R}_+)$. Then the cosine Fourier transform of f is integrable, with*

2010 *Mathematics Subject Classification.* Primary 42B10; Secondary 42A38, 42A50, 42B35, 26A45.

Key words and phrases. Bounded variation, Fourier transform, integrability, Hilbert transform, Hardy space, Hardy variation.

$$\|\widehat{f}_c\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H^1_o(\mathbb{R}_+)};$$

while for the sine Fourier transform an asymptotic formula holds: for $x > 0$,

$$\widehat{f}_s(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + F(x),$$

where

$$\|F\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H^1_o(\mathbb{R}_+)}.$$

Here and in what follows we use the notations "$\lesssim$" and "$\gtrsim$" as abbreviations for "$\le C$" and "$\ge C$", with C being an absolute positive constant. Also, $\mathcal{H}_o$ and $\mathcal{H}_e$ denote the Hilbert transform applied to an odd and even function, respectively. Note that, correspondingly, $H^1_o(\mathbb{R}_+)$ and $H^1_e(\mathbb{R}_+)$ stand for the Hardy type spaces for odd and even functions, respectively, to be specified later.

There are various arguments to justify our interest in this piece, see, e.g., recent works [12] and [13], where much is said on these. Let us mention that the spaces of functions that guarantee the integrability of the Fourier transform considered till recently are of interest by themselves and have applications in other areas of analysis (see, e.g., [3], [20]).

Mention also a recent result in [15], a counterpart of the above Theorem 1.1.

Theorem 1.2. *Let* $f \in BV_0[0,\infty) \cap LAC(0,\infty)$ *and* $f' \in H^1_e(\mathbb{R}_+)$. *Then the sine Fourier transform of* f *is integrable, with*

$$\|\widehat{f}_s\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H^1_e(\mathbb{R}_+)};$$

while for the cosine Fourier transform an asymptotic formula holds: for $x > 0$,

$$\widehat{f}_c(x) = \frac{A}{x} f\left(\frac{\pi}{2x}\right) + \frac{2}{\pi x}\int_0^{\frac{\pi}{2x}} f'(t)\ln\frac{2tx}{\pi}\,dt + F(x),$$

where

$$A = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos t}{t}\,dt - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos t}{t}\,dt\right)$$

and

$$\|F\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H^1_e(\mathbb{R}_+)}.$$

Of course, F here does not mean exactly the same function as above but just stand for something integrable. The reasons why the latter theorem appeared so much later than the former is not the subject of this paper. We mention only that it was not possible until a new proof of Theorem 1.1 appeared (see [14]).

Most of the earlier obtained results have been generalized to the multivariate case, various results of that kind can be found in the survey paper [17]. It is also summarized in [8] or in [13]. All these generalizations fail in sharpness, since too many terms come as the remainder terms. To obtain better extensions is the main goal of this work. We shall deal with

$$\widehat{f}_\eta(x) = \int_{\mathbb{R}^n_+} f(u) \prod_{i:\eta_i=1} \cos x_i u_i \prod_{i:\eta_i=0} \sin x_i u_i \, du, \tag{1.1}$$

where f is a locally absolutely continuous function with bounded Hardy's variation. Discussion on why among a variety of the notions of multidimensional variation the one due to Hardy (and Krause) is the most natural in the considered problems can be found in [13]. Here and in the sequel $\eta = (\eta_1, ..., \eta_n)$ is an n-dimensional vector with the entries either 0 or 1 only. Correspondingly, $|\eta| = \eta_1 + ... + \eta_n$. Such is the vector χ as well. If the only 1 entry is on the j-th place, while the rest are zeros, such a vector will be denoted by e_j. The inequality of vectors is meant coordinate wise. If $\eta = \mathbf{1}$ in (1.1) we have the purely cosine transform, while if $\eta = \mathbf{0}$ we have the purely sine transform, otherwise we have a mixed transform with both cosines and sines. But even in the case where the Fourier transform is the sine in each variable, the only case where a sort of a multidimensional asymptotic formula has been known till recently, the remainder terms are rough in a sense. It is plain to realize that in order to get more advanced multivariate generalizations, the asymptotic relations in Theorems 1.1 and 1.2 should be rewritten in a more precise form. For this, an operator balancing all the terms in these asymptotic relations is introduced in [16]. It is defined by means of a generating function φ and takes on an appropriate function g the value

$$B_\varphi g(x) = \frac{1}{x^2} \int_0^\infty g(\frac{t}{x}) \varphi(t) \, dt.$$

It seems to be very convenient in many situations. For example, for $\varphi(t) = \sin t$, we have $\widehat{g_s}(x) = xB_s g(x)$ and, similarly, $xB_c g(x)$ is the cosine Fourier transform of g. The only (obvious) property of it we will need is that for $g \in L^1(\mathbb{R}_+)$, we have $B_\varphi \in L^1(\mathbb{R}_+)$ provided

$$\int_0^\infty \frac{|\varphi(t)|}{t} \, dt < \infty. \tag{1.2}$$

One more peculiarity that makes new one-dimensional and multi-dimensional results different from the preceding ones is that, in fact, we do not assume the belonging of f' to some Hardy space. This can be done in applications, say, in order to simplify calculations but general relations do not claim for this. What is assumed instead, much less restrictive, is that the inverse formula for the Hilbert transform of f' holds true almost everywhere. In particular, this is the case if the derivative belongs to the corresponding Hardy space, as in Theorems 1.1 and 1.2.

To present our main results, Theorems 3.3 and 4.3, too many prerequisites are needed to do this immediately. For all that, it is of the form

$$\widehat{f}_\eta(x) = \textbf{many leading terms} + \textit{integrable remainder terms.}$$

Some of the leading terms as well as the remainder terms will be combinations of balancing operators with various special functions φ.

The paper is organized as follows. In the next section we present the needed preliminaries. In Sections 3 and 4 we study in detail asymptotic behavior of the sine and cosine Fourier transforms of a function of bounded variation and extend these to the multivariate case. In Section 5 we present certain concluding remarks, in particular, an asymptotic formula is discussed for the most general case.

2. Prerequisites

In this section we give needed notions, definitions and auxiliary results.

2.1. **Hilbert transforms and Hardy spaces.**

The Hilbert transform of an integrable function g is defined by

$$\mathcal{H}g(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{g(t)}{x-t}\,dt, \tag{2.1}$$

where the integral is also understood in the improper (principal value) sense, now as $\lim\limits_{\delta\to 0+}\int_{|t-x|>\delta}$. It is not necessarily integrable, and when it is, we say that g is in the (real) Hardy space $H^1 := H^1(\mathbb{R})$. If $g \in H^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} g(t)\,dt = 0. \tag{2.2}$$

It was apparently first mentioned in [9].

An odd function always satisfies (2.2). However, not every odd integrable function belongs to $H^1(\mathbb{R})$, for counterexamples, see [18] and [11]. When in the definition of the Hilbert transform (2.1) the function g is odd, we will denote this transform by $\mathcal{H}_o$, and it is equal to

$$\mathcal{H}_o g(x) = \frac{2}{\pi}\int_0^\infty \frac{tg(t)}{x^2-t^2}\,dt.$$

If it is integrable, we shall denote the corresponding Hardy space by $H_o^1(\mathbb{R}_+)$, or sometimes simply H_o^1. Correspondingly, when in the definition of the Hilbert transform (2.1) the function g is even, we will denote this transform by $\mathcal{H}_e$, and it is equal to

$$\mathcal{H}_e g(x) = \frac{2}{\pi}\int_0^\infty \frac{xg(t)}{x^2-t^2}\,dt.$$

Symmetrically, we shall denote the corresponding Hardy space by $H_e^1(\mathbb{R}_+)$, or sometimes simply H_e^1. Clearly, to have (2.2) for this class, $\int_0^\infty g(t)\,dt = 0$ should be valid.

There exist clear relations between the even and odd Hilbert transforms (see, e.g., [12]).

Proposition 2.1. *Let $g \in L^1(\mathbb{R}_+)$. Then*

$$\mathcal{H}_e g(x) = \mathcal{H}_o g(x) + \frac{2}{\pi x}\int_0^x g(t)\,dt + G(x), \tag{2.3}$$

where

$$\int_0^\infty |G(x)|\,dx \lesssim \int_0^\infty |g(t)\,dt.$$

.

2.2. **Hardy variation.** One of the simplest and direct generalization of the one-dimensional variation, the Vitali variation, is defined as follows (cf., e.g., [1,4]). Let f be a complex-valued function and

$$\Delta_u f(x) = \left(\prod_{j=1}^n \Delta_{u_j}\right) f(x),$$

with

$$\Delta_{u_j} f(x) = f(x + u_j e_j) - f(x),$$

be a "mixed" difference with respect to the parallelepiped $[x, x+u] = [x_1, x_1+u_1] \times ... \times [x_n, x_n + u_n]$. We will need the following notations. Denote by $\Delta_{u_\eta} f(x)$ the partial difference

$$\Delta_{u_\eta} f(x) = \left(\prod_{j:\eta_j=1} \Delta_{u_j}\right) f(x).$$

Let us take an arbitrary number of non-overlapping parallelepipeds, and form a mixed difference with respect to each of them. Then the Vitali variation is

$$VV(f) = \sup \sum |\Delta_u f(x)|,$$

where the sum and then the least upper bound are taken over all the sets of such nonoverlapping parallelepipeds. For smooth enough functions f, the Vitali variation is expressed as

$$VV(f) = \int\limits_{\mathbb{R}^n} \left|\frac{\partial^n f(x)}{\partial x_1 ... \partial x_n}\right| dx = \int_{\mathbb{R}^n} |D^{\mathbf{1}} f(x)|\,dx.$$

Here and in what follows $D^\eta f$ for $\eta = \mathbf{0} = (0, 0, ..., 0)$ or $\eta = \mathbf{1} = (1, 1, ..., 1)$ mean the function itself and the partial derivative repeatedly in each variable, respectively, where

$$D^\eta f(x) = \left(\prod_{j:\,\eta_j=1} \frac{\partial}{\partial x_j}\right) f(x).$$

However, in many problems Vitali's variation is helpless, because "bad" marginal functions of a smaller number of variables may be added to a function of bounded Vitali's variation. The next notion is free of this disadvantage.

A function f is said to be of bounded Hardy variation, written $f \in VH(f)$, if it is of bounded Vitali variation and is of bounded Vitali variation with respect to any smaller number of variables (see, e.g., [7], [4]; sometimes this notion is also attributed to Krause). The latter will be denoted by $VV_\eta(f) < \infty$, with $\eta \neq \mathbf{1}, \mathbf{0}$. Correspondingly, $VV(f) := VV_{\mathbf{1}}(f)$. In other words, $VH(f) < \infty$ if and only if $VV_\eta(f) < \infty$ for all η, except $\eta = \mathbf{0}$ which is meaningless. However, just for convenience, we can understand $VV_{\mathbf{0}}(f) := f$.

If f is of bounded Vitali variation on $\mathbb{R}^n$ and $\lim\limits_{|x|\to\infty} f(x) = 0$, then functions depending on a smaller number of variables than n are excluded. Such a function is of bounded Hardy variation.

2.3. **Absolute continuity.** In order to present a multidimensional version of Theorems 1.1 and 1.2, we should discuss a multidimensional notion of absolute continuity; see, e.g., [2]. There are several equivalent definitions. It suffices to define such functions to be those representable as

$$f(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} h(u)\,du + \sum_{\eta\neq\mathbf{1}} h_\eta(x_\eta), \tag{2.4}$$

where marginal functions h_η depending on a smaller number of variables than n, in fact, $|\eta| < n$, since $|\eta| = n$ only if $\eta = \mathbf{1}$, are absolutely continuous on $\mathbb{R}^{|\eta|}$. This inductive definition is correct since reduces to the usual absolute continuity on $\mathbb{R}$ for marginal functions of one variable. Locally absolute continuity means absolute continuity on every finite rectangle $[a,b] = [a_1,b_1] \times \dots \times [a_n,b_n]$. In this case, a_1,...,a_n, respectively, should replace $-\infty$ in (2.4).

2.4. **Additional notation.** Certain additional notation is in order. By x_η we denote the $|\eta|$-tuple consisting only of x_j such that $\eta_j = 1$. We denote by $\frac{1}{x}$ the vector $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$. Similarly to the above notation, by $(\frac{1}{x})_\eta$ we denote the $|\eta|$-tuple consisting only of $\frac{1}{x_j}$ for j such that $\eta_j = 1$. If in the multivariate setting one of the operators $\mathcal{H}_o$, B_s, B_C and the like is applied to the j-th variable, it will be denoted by $\mathcal{H}_o^j$, B_s^j, B_C^j, etc. Like the partial derivative above, the other operators applied to the j-th variables for j such that $\eta_j = 1$ will be denoted by means of the superscript η, like

$$\mathcal{H}_o^\eta = \prod_{j:\eta_j=1} \mathcal{H}_o^j, \qquad B_s^\eta = \prod_{j:\eta_j=1} B_s^j, \qquad B_C^\eta = \prod_{j:\eta_j=1} B_C^j.$$

3. Generalizations of Theorem 1.1

It turns out that the theorem in question can be expressed in a much more precise form.

Theorem 3.1. *Let $f \in BV_0[0,\infty) \cap LAC(0,\infty)$. Let almost everywhere*

$$-\mathcal{H}_o(\mathcal{H}_e f')(t) = f'(t).$$

Then for the cosine Fourier transform of f, we have

$$\widehat{f}_c(x) = \int_0^\infty f(t)\cos xt\,dt = -B_s f'(x), \tag{3.1}$$

while for the sine Fourier transform an asymptotic formula holds: for $x > 0$,

$$\widehat{f}_s(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + B_s(\mathcal{H}_o f')(x) + B_S f'(x), \tag{3.2}$$

where B_S is generated by the function

$$S(t) = \frac{2}{\pi}\begin{cases} -t\int_0^\infty \frac{\sin s}{s(s+t)}\,ds, & 0 < t < \frac{\pi}{2}, \\ \\ \int_0^\infty \frac{\sin s}{s+t}\,ds, & t \ge \frac{\pi}{2}. \end{cases}$$

Remark 3.2. This theorem becomes more meaningful if one observes that

$$\|B_s f'\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H_o^1(\mathbb{R}_+)},$$

$$\|B_s(\mathcal{H}_o f')\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H_o^1(\mathbb{R}_+)},$$

both by the well-known extension of Hardy's inequality (see, e.g., [6, (7.24)])

$$\int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\,dx \lesssim \|g\|_{H^1(\mathbb{R})} \tag{3.3}$$

provided $f' \in H_o^1(\mathbb{R}_+)$, and

$$\|B_S f'\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)}.$$

Proof. One should accurately keep all the bounds that give $F(x)$ in [14] unestimated. For this, let us recall the main steps of the proof for the sine transform, since for the cosine transform the result follows immediately after integrating by parts. A heavy duty is over (3.3). To make use of it, or, more precisely, to separate $B_s(\mathcal{H}_o f')(x)$ as an isolated summand, the inverse formula for the Hilbert transform is applied to the derivative of f after integrating by parts in the sine Fourier transform. This leads to

$$\frac{1}{x}\int_0^\infty (\mathcal{H}_e f')(t)\sin xt\,dt.$$

To replace $\mathcal{H}_e$ by $\mathcal{H}_o$ before applying (3.3), we use (2.3), wherein the above mentioned estimates appear. More precisely, these are (2.8) in [14] and calculations after it. Changing variables $xt \to t$ in each of the terms makes this group to become $B_S f'(x)$. That they indeed form an integrable remainder term follows from (1.2). The proof is complete. $\square$

We are now in a position to formulate and prove our first main result.

Theorem 3.3. *Let $f:\mathbb{R}^n_+\to\mathbb{C}$ be of bounded Hardy's variation on $\mathbb{R}^n_+$ and f vanishes at infinity along with all $D^\eta f$ except $\eta=\mathbf{1}$. Let also f and the same $D^\eta f$ be locally absolutely continuous in the above sense. In addition, let for all the derivatives the inverse formula for the Hilbert transform holds almost everywhere in each variable. Then*

$$
\begin{aligned}
(-1)^{|\eta|}\widehat{f}_\eta(x) &= \left(\prod_{j:\eta_j=0}\frac{1}{x_j}\right) B_s^\eta D^\eta f\left(x_\eta,\left(\frac{\pi}{2x}\right)_{\mathbf{1}-\eta}\right)\\
&+\sum_{\substack{\chi:\chi_i=0\ \ if\ \ \eta_i=1,\\ \chi\neq\mathbf{0},\mathbf{1}}}\left(\prod_{j:\chi_j=1}\frac{1}{x_j}\right)\prod_{j:(\mathbf{1}-\eta-\chi)_{\mathbf{j}}=\mathbf{1}}\left(B_s^j\mathcal{H}_o^j\frac{\partial}{\partial x_j}+B_S^j\frac{\partial}{\partial x_j}\right)\\
&\times B_s^\eta D^\eta f\left(x_{\mathbf{1}-\chi},\left(\frac{\pi}{2x}\right)_\chi\right)\\
&+\prod_{j:(\mathbf{1}-\eta)_{\mathbf{j}}=\mathbf{1}}\left(B_s^j\mathcal{H}_o^j\frac{\partial}{\partial x_j}+B_S^j\frac{\partial}{\partial x_j}\right)B_s^\eta D^\eta f(x).
\end{aligned}
\tag{3.4}
$$

Proof. In fact, the proof is the application of either (3.1) or (3.2) in each variable. It is convenient to apply $|\eta|$ times (3.1) to (1.1) first. By this, we get

$$\widehat{f}_\eta(x)=(-1)^{|\eta|}\int_{\mathbb{R}_+^{n-|\eta|}}B_S^\eta D^\eta f(x_\eta,u_{\mathbf{1}-\eta})\prod_{i:\eta_i=0}\sin x_iu_i\,du_{\mathbf{1}-\eta}.$$

We then apply (3.2) in each of the variables in the remained purely sine Fourier transform. Observe that the cases where $\chi=\mathbf{0},\mathbf{1}$ are written separately in (3.4). The proof is complete. □

One can see that the point is no the proof of Theorem 3.3, it is just superposition of one-dimensional results. The latter becomes possible because of utilizing the operator B_φ.

The last term in (3.4) can be made the (integrable) remainder term by assuming $D^{\mathbf{1}}f$ to belong to the product Hardy space (for details, see, e.g., [13]). And in general it is the only such remainder term. The rest of the terms are various types of the leading terms. One cannot get rid of them if wishes to stay in the most general setting of all functions of bounded Hardy variation. They or some of them disappear (except of the first one, of course), or, more precisely, become of remainder type, if one restricts oneself to certain subspaces of the space of functions of bounded Hardy variation, see, e.g., [13].

4. Generalizations of Theorem 1.2

Like in the previous section, the theorem under consideration can also be expressed in a much more precise form. The details are very similar to those above, hence we omit them and just formulate the results.

Theorem 4.1. *Let $f\in BV_0[0,\infty)\cap LAC(0,\infty)$. Let almost everywhere*

$$-\mathcal{H}_e(\mathcal{H}_o f')(t) = f'(t).$$

Then for the sine Fourier transform of f, we have

$$\widehat{f}_s(x) = \int_0^\infty f(t)\sin xt\,dt = B_c f'(x),$$

while for the cosine Fourier transform an asymptotic formula holds: for $x > 0$,

$$\widehat{f}_c(x) = B_L f'(x) + B_c(\mathcal{H}_e f')(x) + B_C f'(x),$$

where B_L is generated by the function

$$L(t) = \begin{cases} A + \frac{2}{\pi}\ln\frac{2t}{\pi}, & 0 < t < \frac{\pi}{2}, \\ \\ 0, & t \ge \frac{\pi}{2}, \end{cases}$$

and B_C is generated by the function

$$C(t) = -\frac{2}{\pi}\begin{cases} (\cos t - 1)\int_t^\infty \frac{\cos s}{s}\,ds & \\ +\sin t\int_t^\infty \frac{\sin s}{s}\,ds + \int_0^t \frac{\cos s - 1}{s}\,ds, & 0 < t < \frac{\pi}{2}, \\ \\ \int_0^\infty \frac{\cos s}{s+t}\,ds, & t \ge \frac{\pi}{2}. \end{cases}$$

Remark 4.2. This theorem becomes more meaningful if one observes that

$$\|B_c f'\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H_e^1(\mathbb{R}_+)},$$

$$\|B_c(\mathcal{H}_e f')\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H_e^1(\mathbb{R}_+)}$$

both by (3.3) provided $f' \in H_e^1(\mathbb{R}_+)$, and

$$\|B_C f'\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)}.$$

And one technical detail is worth to be mentioned. In Theorem 1.2 we have two leading terms. Here, it is convenient to rewrite exactly the same two terms as one by introducing a special function L. It is possible, since, by absolute continuity and cancelation property (2.2),

$$\frac{A}{x} f\left(\frac{\pi}{2x}\right) = \frac{A}{x}\int_0^{\frac{\pi}{2x}} f'(t)\,dt.$$

What is unusual here is that even the leading term is expressed by means of the balance operator.

We are now in a position to formulate and prove our second main result, a counterpart of the first one.

Theorem 4.3. *Let $f : \mathbb{R}^n_+ \to \mathbb{C}$ be of bounded Hardy's variation on $\mathbb{R}^n_+$ and f vanishes at infinity along with all $D^\eta f$ except $\eta = \mathbf{1}$. Let also f and the same $D^\eta f$ be locally absolutely continuous in the above sense. In addition, let for all the derivatives the inverse formula for the Hilbert transform holds almost everywhere in each variable. Then*

$$
\begin{aligned}
\widehat{f_\eta}(x) &= B_L^\eta B_c^{\mathbf{1}-\eta} D^{\mathbf{1}} f(x) \\
&+ \sum_{\substack{\chi:\chi_i=0 \ if \ \eta_i=0,\\ \chi\neq \mathbf{0},\mathbf{1}}} B_L^\chi \prod_{j:(\eta-\chi)_j=1} \left(B_c^j \mathcal{H}_e^j \frac{\partial}{\partial x_j} + B_C^j \frac{\partial}{\partial x_j} \right) B_c^{\mathbf{1}-\eta} D^{\mathbf{1}-\eta+\chi} f(x) \\
&+ \prod_{j:\eta_j=1} \left(B_c^j \mathcal{H}_e^j \frac{\partial}{\partial x_j} + B_C^j \frac{\partial}{\partial x_j} \right) B_c^{\mathbf{1}-\eta} D^{\mathbf{1}-\eta} f(x).
\end{aligned}
$$

5. The most general situation

If dimension is high enough, $n \geq 4$, one can imagine a combination of all the four opportunities. The general formulation is too superfluous, therefore to feel the flavor of such a situation, we give a four-dimensional version. Of course, the assumptions are the same, that is we deal with $f : \mathbb{R}^4_+ \to \mathbb{C}$ of bounded Hardy's variation on $\mathbb{R}^4_+$ and vanishing at infinity along with all $D^\eta f$ except $\eta = \mathbf{1}$. Let also f and the same $D^\eta f$ be locally absolutely continuous. In addition, let for all the derivatives the inverse formula for the Hilbert transform holds almost everywhere in each variable. We present each of the four steps for the asymptotic representation of the corresponding Fourier transform separately as follows:

$$
\begin{aligned}
\widehat{f}_{(1,1,0,0)}(x) &= \int_{\mathbb{R}^4_+} f(u) \cos x_1u_1 \cos x_2u_2 \sin x_3u_3 \sin x_4u_4 \, du \\
&= -\int_{\mathbb{R}^3_+} B_s^1 \frac{\partial}{\partial x_1} f(x_1, u_2, u_3, u_4) \cos x_2u_2 \sin x_3u_3 \sin x_4u_4 \, du_2du_3du_4 \\
&= -\int_{\mathbb{R}^2_+} B_s^1 B_c^3 \frac{\partial^2}{\partial x_1 \partial x_3} f(x_1, u_2, x_3, u_4) \cos x_2u_2 \sin x_4u_4 \, du_2du_4 \\
= -\frac{1}{x_4} &\int_{\mathbb{R}_+} B_s^1 B_c^3 \frac{\partial^2}{\partial x_1 \partial x_3} f\left(x_1, u_2, x_3, \frac{\pi}{2x_4}\right) \cos x_2u_2 \, du_2 \\
&- \int_{\mathbb{R}_+} B_s^1 B_c^3 B_s^4 \mathcal{H}_0^4 \frac{\partial^3}{\partial x_1 \partial x_3 \partial x_4} f(x_1, u_2, x_3, x_4) \cos x_2u_2 \, du_2 \\
&- \int_{\mathbb{R}_+} B_s^1 B_c^3 B_S^4 \frac{\partial^3}{\partial x_1 \partial x_3 \partial x_4} f(x_1, u_2, x_3, x_4) \cos x_2u_2 \, du_2,
\end{aligned}
$$

and finally we get

$$
\widehat{f}_{(1,1,0,0)}(x) = \int_{\mathbb{R}^4_+} f(u) \cos x_1u_1 \cos x_2u_2 \sin x_3u_3 \sin x_4u_4 \, du
$$

$$
\begin{aligned}
&= -\frac{1}{x_4} B_L^2 B_s^1 B_c^3 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} f\left(x_1, x_2, x_3, \frac{\pi}{2x_4}\right) \\
&\quad - \frac{1}{x_4} B_c^2 \mathcal{H}_e^2 B_s^1 B_c^3 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} f\left(x_1, x_2, x_3, \frac{\pi}{2x_4}\right) \\
&\quad - \frac{1}{x_4} B_C^2 B_s^1 B_c^3 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} f\left(x_1, x_2, x_3, \frac{\pi}{2x_4}\right) \\
&\qquad - B_L^2 B_s^1 B_c^3 B_s^4 \mathcal{H}_o^4 D^{\mathbf{1}} f(x) - B_c^2 \mathcal{H}_e^2 B_s^1 B_c^3 B_s^4 \mathcal{H}_o^4 D^{\mathbf{1}} f(x) \\
&\qquad - B_C^2 B_s^1 B_c^3 B_s^4 \mathcal{H}_o^4 D^{\mathbf{1}} f(x) - B_L^2 B_s^1 B_c^3 B_S^4 D^{\mathbf{1}} f(x) \\
&\qquad - B_c^2 \mathcal{H}_e^2 B_s^1 B_c^3 B_S^4 D^{\mathbf{1}} f(x) - B_C^2 B_s^1 B_c^3 B_S^4 D^{\mathbf{1}} f(x).
\end{aligned}
$$

The last formula can be made shorter in two ways. First, we know from the one-dimensional theory that the terms where the Hilbert transform appears are integrable by assuming that these Hilbert transforms are integrable, or, in other words, the corresponding derivative belongs to a Hardy space. Under such assumption these terms may be denoted as a function integrable over $\mathbb{R}_+^4$. Second, some other terms may be claimed to be integrable, which is a very special assumption that makes the asymptotic formula rough to certain extent.

The same concerns the formulas from Theorems 1.1 and 1.2, which, as mentioned, leads to a notion of product Hardy space, which we are not going to discuss here; see, e.g., [13].

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Manuscript received July 25 2016
revised August 22 2016

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