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ANALYSIS OF CONVEX ISOPERIMETRIC TYPE PROBLEMS

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Abstract. This is a short overview of the present-day status of the functional analytical approach to isoperimetric-type problems over convex surfaces.

To the loving memory of my friend and coauthor Alex Rubinov

Our first paper [5] with Alex Rubinov appeared in 1969 in a mimeographed periodical collection *Optimal Planning*. This Russian collection was edited by Leonid Kantorovich and published in Akademgorodok near Novosibirsk by the Institute of Mathematics of the Siberian Division of the USSR. Today it is the Sobolev Institute named after the founder and first director Sergeĭ Sobolev. This collection was practically unknown beyond Russia as well as the ideas we were engrossed those years. Recall that even the term *convex analysis* was practically unknown then since the Rockafellar definitive book [10] was published only in 1970. Only in 1972 the results of this paper became available to a wide readership because they were included in the survey [6] in the best mathematical journal of Russia.

The background of the paper was the question that is philosophical by nature: "Why are some geometrical tricks applied to the problems of isoperimetric type while analysis has powerful techniques of optimization?" If we can answer this question then the new vistas become open—we can deal with the geometrical problems with however many constraints which are untractable in geometry. For instance, we still cannot solve the *internal isoperimetric problem* of finding a surface enclosing a maximum volume and having a given surface area among those that lie within a fixed body.

Since convexity was the banner of that epoch in optimization, we consider the isoperimetric-type problems only over the set $\mathcal{V}_N$ of convex figures, compact convex sets in the Euclidean space $\mathbb{R}^N$. Of course, $\mathcal{V}_N$ is a cone under Minkowski addition and positive homotheties. Moreover, $\mathcal{V}_N$ is an upper semilattice and a semigroup with cancellation and so it embeds into a vector lattice. The latter turns out dense in the space of continuous functions $C(S_{N-1})$ on the unit Euclidean sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. We were slightly disappointed when we found out that all these were known to Alexandr Alexandrov [1]. Alexandrov extended the volume $V(\mathfrak{x})$ of a convex body $\mathfrak{x}$ to the positive cone of $C(S_{N-1})$ using the envelopes of support functions below continuous functions. His ingenious trick settled

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all for the Minkowski problem. This was done in 1938 but still is one of the summits of convexity. In fact, it was Alexandrov who suggested a functional-analytical approach to extremal problems for convex surfaces. It should be noted that the extreme rays of $\mathfrak{x}$ are dense in $\mathcal{V}_N$, and so to follow it directly in the general setting is impossible without the description of the polar cone of $\mathcal{V}_N$ which was found in [2].

Later research has extended this approach and recently some new vista were open that combined the ideas of Pareto optimality and convex geometry. This article is a short overview of these vistas.

Convex Bodies, Balls, and Dual Cones

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure $\mathfrak{x}$ in $\mathbb{R}^N$ and its *support function* $\mathfrak{x}(z) := \sup\{(x, z) \mid x \in \mathfrak{x}\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in $\mathcal{V}_N$.

The Minkowski duality, assigning to $\mathfrak{x}$ the support function $\mathfrak{x}(\cdot)$ of $\mathfrak{x}$ makes $\mathcal{V}_N$ into a cone in $C(S_{N-1})$, the boundary of the unit ball $\mathfrak{z}_N$. The *linear span* $[\mathcal{V}_N]$ of $\mathcal{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.

The study of this space stems from the pioneering breakthrough of Alexandrov and the further insights of Radström, Hörmander, and Pinsker (see [5]).

A measure μ *linearly majorizes* or *dominates* a measure ν on S_{N-1} provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved in 1954 (see [8]) that

$$\int\limits_{S_{N-1}} p\,d\mu \geq \int\limits_{S_{N-1}} p\,d\nu$$

for each sublinear functional p on $\mathbb{R}^N$ if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions. The converse of the Reshetnyak result appeared in [2]. Note that majorization is a vast subject [7].

Alexandrov proved the unique existence of a translate of a convex body given its surface area function, thus completing the solution of the Minkowski problem. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons.

Each Alexandrov measure is a translation-invariant additive functional over the cone $\mathcal{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathcal{A}_N$.

Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ means that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. So, $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on $\mathcal{V}_N$ of the possibility of inserting one figure into the other by translation.

The sum of the surface area measures of $\mathfrak{x}$ and $\mathfrak{y}$ generates the unique class $\mathfrak{x}\#\mathfrak{y}$ of translates which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$. There is no need in discriminating between a convex figure, the coset of its translates in $\mathcal{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathcal{A}_N$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Let $[\mathcal{A}_N]$ be the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant measures, in fact, the linear span of the set of Alexandrov measures.

$C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu\rangle = \frac{1}{N}\int\limits_{S_{N-1}} f d\mu$$
$$(f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathcal{A}_N]).$$

For $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y}\rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$.

Consider the set $\operatorname{Sym}\mathcal{V}_N$ of centrally symmetric cosets of convex compact sets. Clearly, a translation-invariant linear functional f is positive over $\operatorname{Sym}\mathcal{V}_N$ if and only if the *symmetrization* $\operatorname{Sym}(f)$ is positive over $\mathcal{V}_N$. Here $\operatorname{Sym}(f)$ is the dual of the descent of the even part operator on the factor-space, since the symmetrization of a measure is the dual of the even part operator over $C(S_{N-1})$. We will denote the even part operator, its descent and dual by the same symbol $\operatorname{Sym}(\cdot)$.

Given a cone K in a vector space X in duality with another vector space Y, the *dual* of K is

$$K^* := \{y \in Y \mid (\forall x \in K)\ \langle x, y\rangle \geq 0\}.$$

To a convex subset U of X and $\bar{x} \in U$ there corresponds

$$U_{\bar{x}} := \operatorname{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\},$$

the *cone of feasible directions* of U at $\bar{x}$.

Let $\bar{\mathfrak{x}} \in \mathcal{A}_N$. Then the dual $\mathcal{A}^*_{N,\bar{\mathfrak{x}}}$ of the cone of feasible directions of $\mathcal{A}_N$ at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathcal{A}^*_{N,\bar{\mathfrak{x}}} = \{f \in \mathcal{A}^*_N \mid \langle \bar{\mathfrak{x}}, f\rangle = 0\}.$$

The description of the dual of the feasible cones are well known:

Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}^*_N \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(2) *If* $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;
(4) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ *for* $\bar{\mathfrak{x}} \in \mathcal{V}_N$.

From this the dual cones are available in the case of Minkowski balls.

Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \operatorname{Sym}\mathcal{V}^*_N \leftrightarrow \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \operatorname{Sym}(\mu(\mathfrak{y}))$;
(2) *If* $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \operatorname{Sym}(\mu(\mathfrak{y}))$;
(3) $\operatorname{Sym}(\mathfrak{x}) \geq_{\mathbb{R}^2} \operatorname{Sym}(\mathfrak{y}) \leftrightarrow \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^2} \operatorname{Sym}(\mu(\mathfrak{y}))$;
(4) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \operatorname{Sym}\mathcal{V}^*_{N,\bar{\mathfrak{x}}}$ *then* $\operatorname{Sym}(\mathfrak{y}) =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ *for* $\bar{\mathfrak{x}} \in \operatorname{Sym}\mathcal{V}_N$.

Alexandrov extended the volume to the positive cone of $C(S_{N-1})$ by the formula $V(f) := \langle f, \mu(\mathrm{co}(f))\rangle$ with $\mathrm{co}(f)$ the envelope of support functions below f. He observed that $V(f) = V(\mathrm{co}(f))$ and showed that the gradient of $V(\cdot)$ at $\mathfrak{x}$ is proportional to $\mu(\mathfrak{x})$ and so minimizing $\langle \cdot, \mu\rangle$ over $\{V = 1\}$ will yield the equality $\mu = \mu(\mathfrak{x})$ by the Lagrange multiplier rule. But this idea fails since the interior of $\mathcal{V}_N$ is empty. The fact that DC-functions are dense in $C(S_{N-1})$ is not helpful at all.

The obvious limitations of the Lagrange multiplier rule are immaterial in the case of convex programs. It should be emphasized that the classical isoperimetric problem is not a Minkowski convex program in dimensions greater than 2. The convex counterpart is the Urysohn problem of maximizing volume given integral breadth [11]. The constraints of inclusion type are convex in the Minkowski structure, which opens way to complete solution of new classes of Urysohn-type problems.

External Urysohn Problem: Among the convex figures, circumscribing $\mathfrak{x}_0$ and having integral breadth fixed, find a convex body of greatest volume.

A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all z in the support of μ, i.e.* $z \in \mathrm{spt}(\mu)$.

If $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then $\bar{\mathfrak{x}}$ is a *spherical lens* and μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}.

Pareto Optimization over Minkowski Balls

Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

By way of example, consider a few multiple criteria problems of isoperimetric type.

Vector Isoperimetric Problem over Minkowski Balls: Given are some convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_M$. Find a symmetric convex body $\mathfrak{x}$ encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \mathfrak{y}_1), \dots, V_1(\mathfrak{x}, \mathfrak{y}_M)$. In symbols,

$$\mathfrak{x} \in \mathrm{Sym}(\mathcal{A}_N);\ \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}});\ (\langle \mathfrak{y}_1, \mathfrak{x}\rangle, \dots, \langle \mathfrak{y}_M, \mathfrak{x}\rangle) \to \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure.

Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form $\bar{\mathfrak{x}} = \alpha_1 \mathrm{Sym}(\mathfrak{y}_1) + \dots + \alpha_m \mathrm{Sym}(\mathfrak{y}_m)$, where $\alpha_1, \dots, \alpha_m$ are positive reals.

Internal Urysohn Problem with Flattening over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \operatorname{Sym}\mathcal{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of $\mathfrak{x}$ and minimize the breadth of $\mathfrak{x}$ in the flattening direction: $\mathfrak{x} \in \operatorname{Sym}\mathcal{V}_N$; $\mathfrak{x} \subset \mathfrak{x}_0$; $\langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle$; $(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf$.

For a feasible symmetric convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction $\bar{z}$ over Minkowski balls it is necessary and sufficient that there be positive reals α and β together with a convex figure $\mathfrak{x}$ satisfying

$$\mu(\bar{\mathfrak{x}}) = \operatorname{Sym}(\mu(\mathfrak{x})) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x})).$$

Rotational Symmetry: Assume that a plane convex figure $\mathfrak{x}_0 \in \mathcal{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator $\bar{z}$. Assume further that $\mathfrak{x}_{00}$ is the result of rotating $\mathfrak{x}_0$ around the symmetry axis $A_{\bar{z}}$ in $\mathbb{R}^3$. Consider the problem:

$$\mathfrak{x} \in \mathcal{V}_3;$$
$$\mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}};$$
$$\mathfrak{x} \supset \mathfrak{x}_{00};\ \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle;$$
$$(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$$

Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.

Little is known about similar problems in arbitrary dimensions. The planar case is rediscovered in recent years (see, for instance, [9]).

External Urysohn Problem with Flattening over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and flattening direction $\bar{z} \in S_{N-1}$. Considering Minkowski balls $\mathfrak{x} \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimize breadth in the flattening direction: $\mathfrak{x} \in \operatorname{Sym}\mathcal{V}_N$; $\mathfrak{x} \supset \mathfrak{x}_0$; $\langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle$; $(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf$.

For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening over Minkowski balls it is necessary and sufficient that there be positive reals α and β together with a convex figure $\mathfrak{x}$ satisfying

$$\mu(\bar{\mathfrak{x}}) + \operatorname{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$V(\bar{\mathfrak{x}}) + V_1(\operatorname{Sym}(\mathfrak{x}), \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x})).$$

For more details see [3] and [4].

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