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BARRIER METHODS FOR EQUILIBRIUM PROBLEMS

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Abstract. We consider several versions of the barrier method for a general equilibrium problem with nonlinear constraints in a reflexive Banach space setting. We suggest weak coercivity conditions instead of (generalized) monotonicity, including one containing a perturbed barrier function, in order to entail solutions for the equilibrium problem. We obtain convergence properties of the method under mild assumptions.

1. Introduction

Let D be a set in a reflexive Banach space E and let $\phi : D \times D \to \mathbb{R}$ be an equilibrium bifunction, i.e. $\phi(x, x) = 0$ for every $x \in D$. The equilibrium problem (EP for short) is to find a point $x^* \in D$ such that

$$\phi(x^*, y) \geq 0 \quad \forall y \in D. \tag{1.1}$$

EPs give a suitable common format for investigation of various applied problems arising in Economics, Mathematical Physics, Engineering and other fields. Besides, EPs are closely related with other general problems in Nonlinear Analysis, such as fixed point, game equilibrium, variational inequality, and optimization problems; see e.g. [2,5,11,12] and references therein. It is well known that most solution methods for EPs were substantiated under certain (generalized) monotonicity/convexity conditions. In particular, these conditions were also utilized for various penalty and regularization methods; see e.g. [1, 10, 18, 21]. However, there exist less restrictive coercivity conditions, which provide existence of solutions for EPs; see e.g. [3, 4, 16].

Rather recently, it was proposed to utilize the general coercivity conditions for external penalty methods for EPs; see [14, 15]. Moreover, similar conditions were taken for the regularized versions of these methods. The absence of restrictive concordance rules for penalty and regularization parameters is an essential feature of these methods.

It is known that the barrier (or internal penalty) method gives another fundamental approach for finding solutions of various problems with complex nonlinear constraints and even have certain preferences, such as the feasibility of the iterative sequence; see e.g. [8, 21]. At the same time, the substantiation of the barrier

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method meets additional difficulties since the auxiliary function is undefined on the boundary of the feasible set.

In this paper, we intend to develop the approach from [14, 15] for barrier methods applied to EP (1.1) with nonlinear constraints. We establish some results of weak convergence of the penalized and regularized methods without monotonicity assumptions via mild coercivity conditions.

2. Basic preliminaries

We first recall some basic definitions and auxiliary properties. Let X be a nonempty subset of a Banach space E and $\varphi : X \to \mathbb{R}$ be a function. In the sequel, for every $\gamma \in \mathbb{R}$, we will denote by $\ell_\gamma(\varphi)$ the set

$$\ell_\gamma(\varphi) := \{x \in X : \varphi(x) \leq \gamma\}.$$

The function φ is said to be

(a) *convex* on a convex set $K \subseteq X$, if

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y), \quad \forall x, y \in K \text{ and } \forall \alpha \in [0,1];$$

(b) *uniformly convex* on a convex set $K \subseteq X$, if there exists a continuous and increasing function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta(0) = 0$ such that

$$\begin{aligned}\varphi(\alpha x + (1-\alpha)y) \leq \;& \alpha\varphi(x) + (1-\alpha)\varphi(y) - \alpha(1-\alpha)\theta(\|x-y\|),\\ & \forall x, y \in K \text{ and } \forall \alpha \in [0,1];\end{aligned}$$

(c) *coercive* with respect to a set $K \subseteq X$ if

$$\varphi(x) \to +\infty \quad \text{as} \quad \|x\| \to \infty,\ x \in K;$$

(d) *weakly coercive* with respect to a set $K \subseteq X$ if there exists a number γ such that the set $\ell_\gamma(\varphi) \cap K$ is nonempty and bounded;

(e) *upper (lower) semicontinuous* at a point $z \in X$, if, for each sequence $\{x^k\} \to z$, $x^k \in X$, it holds that

$$\limsup_{k\to\infty} \varphi(x^k) \leq \varphi(z) \quad (\liminf_{k\to\infty} \varphi(x^k) \geq \varphi(z));$$

(f) *weakly upper (lower) semicontinuous* at a point $z \in X$, if, for each sequence $\{x^k\} \xrightarrow{w} z$, $x^k \in X$, it holds that

$$\limsup_{k\to\infty} \varphi(x^k) \leq \varphi(z) \quad (\liminf_{k\to\infty} \varphi(x^k) \geq \varphi(z)).$$

Here and below $\{x^k\} \to z$ ($\{x^k\} \xrightarrow{w} z$) denotes the strong (weak) convergence of $\{x^k\}$ to z. We say that any of the properties (e) (f) holds on a set $K \subseteq X$, if it holds at any point of K. From the definitions we have

$$(b) \Longrightarrow (a),\ (b) \Longrightarrow (c) \Longrightarrow (d),\ \text{and } (f) \Longrightarrow (e),$$

but the reverse implications are not true in general.

In this section, we will consider EP (1.1) under the following basic assumptions.

(B1) *D is a nonempty, convex and closed subset of E and $\phi : D \times D \to \mathbb{R}$ is an equilibrium bifunction such that $\phi(\cdot, y)$ is weakly upper semicontinuous for each $y \in D$ and $\phi(x, \cdot)$ is convex for each $x \in D$.*

If we add the boundedness of D, EP (1.1) will have a solution due to the simple specialization of the well-known existence result due to Ky Fan; see [6]. It suffices to take the weak topology in E.

Proposition 2.1. *If* **(B1)** *holds and D is bounded, then EP (1.1) has a solution.*

However, we are interested in investigating the general unbounded case. Then, we should utilize a suitable coercivity condition. Let $\mu : E \to \mathbb{R}$ denote a lower semicontinuous and convex function $\mu : E \to \mathbb{R}$, which is weakly coercive with respect to the set D. For instance, we can take μ to be a uniformly convex function. Note that, under the assumptions above on μ, all the sets $\ell_s(\mu) \cap D$, $s \in \mathbb{R}$, are either empty, or bounded (see [7] and also, for instance, [20, Chapter 3, Theorem 3.14]).

The next existence result was proved in [15], Theorem 3.1 under the following mild coercivity condition:

(C1) *There exist a number r such that, for any point $x \in D \setminus \ell_r(\mu)$, there is a point $z \in D$ with*

$$\min\{\phi(x,z), \mu(z) - \mu(x)\} < 0 \quad \text{and} \quad \max\{\phi(x,z), \mu(z) - \mu(x)\} \leq 0. \tag{2.1}$$

Note that under the assumptions **(B1)** and **(C1)**, the set $\ell_r(\mu) \cap D$ is nonempty (see Lemma 3.2 in [15]).

Theorem 2.2. *Let us assume that the set D, the bifunction ϕ, and the function μ satisfy the conditions in* **(B1)** *and* **(C1)***. Then EP (1.1) has a solution.*

If we replace (2.1) in **(C1)** with the stronger condition

$$\mu(z) \leq \mu(x) \quad \text{and} \quad \phi(x,z) < 0, \tag{2.2}$$

then we can show, in addition, that all the solutions belong to the bounded set $D \cap \ell_r(\mu)$. In general, the further coercivity conditions **(C2)**–**(C4)** considered in this paper will lead to a stronger result, namely, the existence of solutions, which belong to a bounded set, without proving the boundedness of the whole solution set.

3. The classical barrier method

Throughout the next sections we will consider EP (1.1) under the following basic assumptions **(B2)**:

- *D is a nonempty set of the form*

$$D = V \cap W, \tag{3.1}$$

  where W and V are convex and closed sets in a reflexive Banach space E such that the set

$$D^0 = V \cap \mathrm{int} W$$

  is nonempty;
- *$\phi : V \times V \to \mathbb{R}$ is an equilibrium bifunction such that $\phi(\cdot, y)$ is weakly upper semicontinuous for each $y \in V$ and $\phi(x, \cdot)$ is convex for each $x \in V$;*

- $\mu : E \to \mathbb{R}$ *is lower semicontinuous, convex and weakly coercive with respect to the set* D.

A barrier function B with respect to W is a function enjoying the following properties:

(i): $B : \text{int}W \to \mathbb{R}$ is nonnegative, convex and lower semicontinuous;
(ii): $B(x) \to +\infty$ as $x \to \Gamma$, where $\Gamma = \partial W$.

We will denote by D_M the subset of D^0 given by

$$D_M := \ell_M(B) \cap V.$$

Example 3.1.

(i): In the particular case of a set W described via functional constraints, i.e.

$$W = \{x \in E \ : \ h_i(x) \leq 0, \ i = 1, \dots, m\},$$

where $h_i : E \to \mathbb{R}$ is convex, $i = 1, \dots, m$, we can set

$$\begin{aligned} B(x) &= -\sum_{i=1}^{m} 1/h_i(x), \\ B(x) &= \sum_{i=1}^{m} \max\left\{-\ln(-h_i(x)), 0\right\}^p, \qquad p \geq 1. \end{aligned}$$

Another possibility could be to take

$$\begin{aligned} B(x) &= -1/h(x), \\ B(x) &= \max\left\{-\ln(-h(x)), 0\right\}^p, \quad p \geq 1, \end{aligned}$$

where $h(x) = \max_{i=1,\dots,m}\{h_i(x)\}$ (see [8, 19, 21]).

(ii): In case of a finite dimensional space E, the function

$$B(x) = \max\left\{-\ln d_W(x), 0\right\}^p, \quad p \geq 1,$$

where the distance function $d_W : W \to \mathbb{R}_+$ is given by

$$d_W(x) = d(x, \Gamma) = \min_{z \in \Gamma}\{\|x - z\|\},$$

is a convex barrier function on W (see [9], Theorem 6.20).

(iii): The perturbed (regularized) barrier function

$$\widetilde{B}(x) = B(x) + \mu(x),$$

where μ is, in addition, nonnegative, seems also suitable since it is weakly coercive with respect to D and it can be used instead of μ for the penalized problems. We will discuss this approach in Section 4.

The main result of this section relies on the following coercivity condition:

(C2) *There exist a number* r *such that, for every* $x \in D^0 \setminus \ell_r(\mu)$ *there exists* $z \in D^0$ *such that*

$$\min\{\phi(x,z), B(z) - B(x)\} < 0 \text{ and } \max\{\phi(x,z), B(z) - B(x), \mu(z) - \mu(x)\} \leq 0$$

We now introduce, for any positive τ, the *penalized equilibrium problem* EP_τ defined as follows: given a barrier function B with respect to W, let $\phi_\tau : D^0 \times D^0 \to \mathbb{R}$ be given by

$$\phi_\tau(x, y) = \phi(x, y) + \tau(B(y) - B(x)).$$

Then, EP_τ is to find a point $x^*_\tau \in D^0$ such that

$$\phi_\tau(x^*_\tau, y) \geq 0, \quad \forall y \in D^0. \tag{3.2}$$

Our plan is, first, to prove the existence of solutions to the penalized equilibrium problem EP_τ (3.2) for every τ, then to show that for any sequence $\tau_l \downarrow 0$, we are able to reach a solution of the original equilibrium problem (1.1) as a limit point of a sequence of solutions of problems EP_{τ_l}.

Lemma 3.2. *Fix $\tau > 0$. If the conditions in* **(B2)** *and* **(C2)** *hold, then the penalized problem EP_τ admits a solution, which belongs to $\ell_r(\mu) \bigcap D^0$.*

Proof. We divide the proof in three steps:

Step 1. Since $D^0 \neq \emptyset$, there exists a positive M such that the set D_M is non empty. We start by proving the existence of solutions of the *reduced equilibrium problem*

$$\phi_\tau(x, y) \geq 0, \quad \forall y \in D_M \tag{3.3}$$

by applying Theorem 2.2 to the set D_M and the bifunction ϕ_τ.

Note that **(B1)** is fulfilled and, in addition, ϕ_τ satisfies the coercivity condition **(C1)** on D_M with respect to the function μ and the scalar r given in **(C2)**. Indeed, if $x \in D_M \setminus \ell_r(\mu)$, then $x \in D^0 \setminus \ell_r(\mu)$ and, from **(C2)**, there exists $z \in D^0$, such that $B(z) \leq B(x) \leq M$, implying that $z \in D_M$. Moreover, from the condition $\min\{\phi(x, z), B(z) - B(x)\} < 0$, it follows easily that $\phi_\tau(x, z) < 0$ and thus (2.2) holds. Therefore, by the remark below Theorem 2.2, the equilibrium problem (3.3) admits solutions and all these solutions belong to the bounded set $D_M \cap \ell_r(\mu)$.

Step 2. Let now $M_k \uparrow +\infty$, and denote by $x_\tau(M_k)$ a solution of the equilibrium problem (3.3) on D_{M_k}. Since $D \cap \ell_r(\mu)$ is convex, closed, and bounded, and $x_\tau(M_k) \in D \cap \ell_r(\mu)$ for every k, without loss of generality we can assume that

$$x_\tau(M_k) \xrightarrow{w} x^*_\tau \in D \cap \ell_r(\mu)$$

as $k \to +\infty$.

It is easy to prove that $x^*_\tau \in D^0$. Indeed, fix $y \in D_{M_1}$. By observing that $D_{M_k} \subseteq D_{M_{k+1}}$, for every $k \geq 1$, we get for any k

$$B(x_\tau(M_k)) \leq \frac{1}{\tau}\phi(x_\tau(M_k), y) + B(y)$$

Since every weakly upper semicontinuous function admits a maximum on every closed, convex and bounded set by the Weierstrass theorem, the boundedness of $\{\phi(x_\tau(M_k), y)\}$ implies the boundedness of $\{B(x_\tau(M_k))\}$. Since $B(x^*_\tau) \leq \liminf_{k\to\infty} B(x_\tau(M_k)) < +\infty$, we get $x^*_\tau \in \mathrm{int}W$. Moreover, $x^*_\tau \in V$, and thus $x^*_\tau \in D^0$.

Step 3. Finally, we show that x^*_τ is a solution of the equilibrium problem (3.2). Suppose, by contradiction, that there exists $y \in D^0$ such that

$$\phi(x^*_\tau, y) + \tau(B(y) - B(x^*_\tau)) < 0.$$

Then, since $y \in D_{M_{\overline{k}}}$ for a suitable $\overline{k}$,

$$\phi(x_\tau(M_k), y) + \tau(B(y) - B(x_\tau(M_k))) \geq 0$$

for all $k \geq \overline{k}$. By the weak upper semicontinuity of $\phi(\cdot, y)$ and weak lower semicontinuity of B, we get

$$\phi(x^*_\tau, y) + \tau(B(y) - B(x^*_\tau)) \geq 0,$$

a contradiction. □

We are now ready to prove the basic theorem for the barrier method.

Theorem 3.3. *Let the assumptions in* **(B2)** *and* **(C2)** *hold and let B be a barrier function with respect to W. Then:*

(i) *EP (1.1), (3.1) has a solution;*
(ii) *EP_τ (3.2) has a solution for each $\tau > 0$ and all these solutions belong to $\ell_r(\mu) \bigcap D^0$;*
(iii) *Each sequence $\{x^*_{\tau_l}\}$ of solutions of EP_{τ_l} with $\tau_l \downarrow 0$ has weak limit points, all these weak limit points belong to $\ell_r(\mu) \bigcap D$ and are solutions of EP (1.1), (3.1).*

Proof. By Lemma 3.2 we know that for every $\tau > 0$ the penalized equilibrium problem EP_τ (3.2) admits a solution $x^*_\tau \in D^0 \cap \ell_r(\mu)$. Besides, if there exists another solution $\bar{x}_\tau \in D^0 \setminus \ell_r(\mu)$, then, by **(C2)** there exists $z \in D^0 \cap \ell_r(\mu)$ such that $\phi_\tau(\bar{x}_\tau, z) < 0$, which is a contradiction. Therefore, assertion (ii) holds. Fix $\tau_l \downarrow 0$, and denote by $x^*_{\tau_l}$ a solution in $\ell_r(\mu) \cap D^0$ of EP_{τ_l}. Without loss of generality, we can assume

$$x^*_{\tau_l} \xrightarrow{w} x^* \in D \cap \ell_r(\mu).$$

We show that x^* is a solution to EP (1.1).

First of all, by the assumptions on B,

$$\phi(x^*_{\tau_l}, y) + \tau_l B(y) \geq \tau_l B(x^*_{\tau_l}) \geq 0 \quad \forall y \in D^0.$$

By the upper semicontinuity of $\phi(\cdot, y)$, we have

$$\phi(x^*, y) \geq \limsup_{l \to +\infty} (\phi(x^*_{\tau_l}, y) + \tau_l B(y)) \geq 0 \quad \forall y \in D^0.$$

Assume now, by contradiction, that there exists $\overline{y} \in \Gamma$ such that $\phi(x^*, \overline{y}) < 0$. Let $y' \in D^0$; then, by convexity of $\phi(x, \cdot)$, since $(1-\lambda)y' + \lambda\overline{y} \in D^0$ for $\lambda \in [0, 1)$, we get

$$0 \leq \phi(x^*, (1-\lambda)y' + \lambda\overline{y}) \leq (1-\lambda)\phi(x^*, y') + \lambda\phi(x^*, \overline{y}) < 0$$

for λ close enough to 1, a contradiction. Therefore, assertion (iii) holds too. Assertion (i) follows from (iii). □

4. The regularized barrier method

In this section we will consider the *penalized and regularized equilibrium problem* $(\widetilde{EP}_\tau)$: find a point $x^*_\tau \in D^0$ such that

$$\varphi_\tau(x^*_\tau, y) \geq 0, \quad \forall y \in D^0, \tag{4.1}$$

where

$$\varphi_\tau(x, y) = \phi(x, y) + \tau(\widetilde{B}(y) - \widetilde{B}(x)),$$

$\widetilde{B}(x) = B(x) + \mu(x)$ and the function μ is, in addition, nonnegative. That is, the barrier function now combines both penalty and regularized terms. Note that, since $\ell_s(\widetilde{B}) \subseteq \ell_s(\mu)$, and B is a barrier function with respect to W, there exists $\widetilde{r}$ such that $\ell_{\widetilde{r}}(\widetilde{B}) \cap D$ is nonempty and bounded, i.e., $\widetilde{B}$ is weakly coercive with respect to D.

The aim of this section is to provide a new existence result relying on the following coercivity condition:

(C3) *There exists a number r such that, for every $x \in D^0 \setminus \ell_r(\widetilde{B})$, there exists $z \in D^0$ such that*

$$\min\{\phi(x,z), \widetilde{B}(z) - \widetilde{B}(x)\} < 0 \text{ and } \max\{\phi(x,z), \widetilde{B}(z) - \widetilde{B}(x)\} \leq 0.$$

Theorem 4.1. *Given a barrier function B with respect to W, let $\widetilde{B}(x) = B(x) + \mu(x)$ where μ is nonnegative. Let the assumptions in* **(B2)** *and* **(C3)** *hold. Then:*

(i) *EP (1.1), (3.1) has a solution;*

(ii) *$\widetilde{EP}_\tau$ (4.1) has a solution for each $\tau > 0$ and all these solutions belong to $\ell_r(\widetilde{B}) \bigcap D^0$;*

(iii) *Each sequence $\{x^*_{\tau_l}\}$ of solutions of $\widetilde{EP}_{\tau_l}$ with $\tau_l \downarrow 0$ has weak limit points, all these weak limit points belong to $\ell_r(\widetilde{B}) \bigcap D$ and are solutions of EP (1.1), (3.1).*

Proof. Let us consider the set $\widetilde{D}_M := V \cap \ell_M(\widetilde{B}) \subseteq D^0$, where $M \geq \max(\widetilde{r}, r)$. Since $\widetilde{D}_M$ is nonempty, closed, convex, and bounded, and φ_τ is a bifunction satisfying the conditions in **(B1)** on $\widetilde{D}_M$, by the well-known result by Ky Fan there exists a solution to the equilibrium problem on $\widetilde{D}_M$, say $x_\tau(M)$. By the coercivity condition **(C3)**, $x_\tau(M) \in \ell_r(\widetilde{B})$. Indeed, if $x_\tau(M) \in D^0 \setminus \ell_r(\widetilde{B})$, there exists $y \in \widetilde{D}_M$ such that $\varphi_\tau(x_\tau(M), y) < 0$, a contradiction.

Let $M_j \uparrow +\infty$; without loss of generality, $x_\tau(M_j) \overset{w}{\to} x^*_\tau \in \widetilde{D}_r \subset D^0$. Following the lines of Steps 2 and 3 in Lemma 3.2, we can show that x^*_τ is a solution of (4.1). Therefore, part (ii) holds.

Furthermore, arguing as in the proof of Theorem 3.3, we can also prove that for $\tau_l \downarrow 0$, $\{x^*_{\tau_l}\}$ has weak limit points in $\widetilde{D}_r$ which are solutions of EP (1.1). Therefore, part (iii) holds too. Part (i) follows from (iii). □

Observe that Theorem 4.1 gives the existence result for EP (1.1), (3.1) under somewhat different conditions in comparison with those in Theorem 3.3.

We can obtain a similar convergence result for a modified coercivity condition weaker than **(C2)**:

(C4) *There exists a number r such that, for every $x \in D^0 \setminus \ell_r(\mu)$, there exists $z \in D^0$ such that*

$$\min\{\phi(x,z), B(z) - B(x), \mu(z) - \mu(x)\} < 0$$

and

$$\max\{\phi(x,z), B(z) - B(x), \mu(z) - \mu(x)\} \leq 0.$$

Theorem 4.2. *Let the assumptions in* **(B2)** *and* **(C4)** *hold, with μ nonnegative, and let B be a barrier function with respect to W. Then:*

(i) *EP (1.1), (3.1) has a solution;*
(ii) $\widetilde{EP}_\tau$ *(4.1) has a solution for each* $\tau > 0$ *and all these solutions belong to* $\ell_r(\mu) \bigcap D^0$*;*
(iii) *Each sequence* $\{x^*_{\tau_l}\}$ *of solutions of* $\widetilde{EP}_{\tau_l}$ *with* $\tau_l \downarrow 0$ *has weak limit points, all these weak limit points belong to* $\ell_r(\mu) \bigcap D$ *and are solutions of EP (1.1), (3.1).*

Proof. Fix $M > 0$. Let us consider the reduced EP: find a point $x_\tau(M) \in D_M$ such that

$$\varphi_\tau(x_\tau(M), y) \geq 0, \quad \forall y \in D_M. \tag{4.2}$$

Clearly, **(B1)** is fulfilled and, in addition, φ_τ satisfies the coercivity condition **(C1)** on D_M with respect to the function μ and the scalar r given in **(C4)**. Hence, EP (4.2) has a solution due to Theorem 2.2, besides, by the remark below Theorem 2.2, all its solutions belong to the bounded set $D_M \cap \ell_r(\mu)$. Next, following the lines of Steps 2 and 3 in Lemma 3.2 and replacing B with $\widetilde{B}$ where necessary, we can show that

$$x_\tau(M_k) \overset{w}{\to} x^*_\tau \in D^0 \cap \ell_r(\mu)$$

as $k \to +\infty$, for any sequence $M_k \uparrow +\infty$, and that x^*_τ solves EP (4.1). Therefore, part (ii) holds. The same substitution in the proof of Theorem 3.3 justifies part (iii). Part (i) follows directly from (iii). □

5. Comparison with other coercivity conditions

In this section we will show that our coercivity conditions **(C1)**–**(C4)** are weaker than other ones recently found in literature. In [16], in a finite-dimensional setting E, the following weak coercivity condition was suggested for establishing existence results:

(G) There exists a convex function $\mu : E \to \mathbb{R}$ which is weakly coercive with respect to the set D, and a number r such that for any point $x \in D \setminus \ell_r(\mu)$ with
$$\inf_{y \in \ell_r(\mu)} \phi(x, y) \geq 0,$$
there is a point $z \in D$, with $\mu(z) < \mu(x)$, and $\phi(x, z) \leq 0$.

However, **(G)** implies **(C1)**. To prove this, fix the same $r \in \mathbb{R}$ and $x \in D \setminus \ell_r(\mu)$, and let us consider two possible cases:

(i): $\phi(x, y) \geq 0$ for every $y \in \ell_r(\mu)$: taking the same z we see that **(C1)** is fulfilled;
(ii): there exists $\overline{y} \in \ell_r(\mu)$ such that $\phi(x, \overline{y}) < 0$. In this case, choose $z = \overline{y}$.

It should be noted that condition **(G)** implies certain weakened coercivity conditions in the case where the initial space admits a partition into a Cartesian product of spaces so that the usual coercivity holds only with respect to a selected subspace; see [16] for more details. These relaxed coercivity conditions are useful in the study of various iterative solution methods; see e.g. [17].

The following well known coercivity condition was used for establishing convergence of penalty methods (see e.g. [10, 13, 18]):

(C) There exists a point $\tilde{x} \in D^0$ such that

$$\phi(x, \tilde{x}) \to -\infty \quad \text{as} \quad \|x - \tilde{x}\| \to \infty, \ x \in D^0.$$

We conclude by showing the relationship between **(C)** and **(C1)**–**(C4)**.

If we set $\mu(x) = \|x - \tilde{x}\|$, then **(C)** clearly implies **(C1)**. Indeed, the set

$$D' = \left\{x \in D^0 \ : \ \phi(x, \tilde{x}) \geq 0\right\}$$

is contained $\ell_{\rho'}(\mu)$ for some positive ρ'. Thus **(C1)** holds with $r = (\rho')$ and $z = \tilde{x}$.

Next, if the barrier function B is weakly coercive with respect to D^0, the set $\ell_{B(\tilde{x}}(B) \cap D^0$ is bounded, therefore it is included in $\ell_{\rho''}(\mu)$, for a suitable ρ''. Taking $\rho = \max\{\rho', \rho''\}$, $z = \tilde{x}$, and $r = \rho$, we see that **(C2)** holds. Therefore, **(C4)** holds too.

Let us go back to condition **(C3)**. We recall that the function $\widetilde{B}(x) = B(x) + \mu(x)$ is weakly coercive with respect to D and that the set D' is bounded. If we choose the functions B and μ such that D' is contained in some set $\ell_r(\widetilde{B})$, then **(C3)** holds true.

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