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BALL COVERING PROPERTY FOR UNCONDITIONAL DIRECT SUM OF BANACH SPACES

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Abstract. Let (X_k) be a sequence of Banach spaces and (M_n) be a sequence of semi-normalized Orlicz functions satisfying the uniform Δ_2-condition at zero. We show that $X = (\sum X_k)_{\ell_{(M_n)}}$ has the ball-covering property if and only if each X_k has the ball-covering property. Let E be a Banach space with an 1-unconditional basis (e_n) which satisfies a block upper-p estimate for some $1 < p < \infty$. Then $(\sum X_k)_E$ has the ball-covering property if and only if every X_k has the ball-covering property. Similar results are proved for Banach spaces with the strong ball-covering property.

1. Introduction

A Banach space X is said to have the ball-covering property (BCP) if its unit sphere can be covered by countably many closed, or equivalently open balls off the origin. The ball-covering property has been investigated intensively recently and shown to be different from, but closely related to the topological properties of Banach spaces.

To see that the ball-covering property is not a topological property, one should refer to a result of L. Cheng, Q. Cheng and X. Liu [1] which shows that ℓ_∞ can be renormed to fail the BCP (while ℓ_∞ itself clearly has the BCP). So the BCP is not preserved under linear isomorphisms. A much simpler proof for this purpose was given by Z. Luo and B. Zheng [5] who showed that $L_\infty[0,1]$ fails the BCP. Moreover, V. P. Fonf and C. Zanco [2] obtained a quantitative result by proving that X^* is w^*-separable, then X can be $(1+\epsilon)$-equivalently renormed to have the BCP. This result reveals the fact that the BCP is a geometric property deeply related to the topological properties of Banach spaces.

It is clear from definition that every separable Banach space has the BCP. In [3], M. Liu, R. Liu, J. Lu and B. Zheng provided several examples of non-separable Banach spaces with the BCP. It was shown that if K is a locally compact Hausdorff space and X is a Banach space, then $C_0(K)$ has the BCP if and only if K has countable π-basis. Moreover, $C_0(K,X)$ has the BCP if and only if K has a countable

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π-basis and X has the BCP. It was also proved that $B(c_0)$, $B(\ell_1)$ and every subspace containing finite rank operators in $B(\ell_p)$ for $1 < p < \infty$ have the BCP.

Stability of a certain property under direct sum is a typical question in Banach space theory. Let (X_k) be a sequence of Banach spaces and E be a Banach space with an 1-unconditional basis (e_n). The direct sum of (X_k) with respect to E is the Banach space consisting of infinite sequences (x_k) with $x_k \in X_k$ under the norm

$$\|(x_k)\| := \Big\| \sum \|x_k\| e_k \Big\|.$$

It was proved in [4] by Z. Luo, J. Liu and B. Wang that for $1 \le p \le \infty$, $X \oplus_p Y$ has the BCP if and only if both X and Y have the BCP. Z. Luo and B. Zheng [5] extended the results to infinite direct sum by showing that $(\sum X_k)_E$ has the BCP if and only if each X_k has the BCP, where E is either a Lorentz sequence space, a separable Orlicz sequence space or ℓ_∞. It was also shown in [5] that if (Ω, Σ, μ) is a separable measure space, then the space of Bochner integrable functions $L^p(\mu, X)$ has the BCP if and only if X has the BCP. Similar results were extended in [6] by the same authors to spaces with the strong BCP and uniform BCP.

In the results above, the direct sums were taking over Lorentz sequence spaces or Orlicz sequence spaces which have symmetric bases. A natural question is

Question: Let E be a Banach space with an 1-unconditional basis and (X_k) be a sequence of Banach spaces with the BCP. Does $(\sum X_k)_E$ have the BCP?

In this paper, we will give affirmative answers to the question for two classes of Banach spaces E. One class contains modular sequence spaces that satisfy the uniform Δ_2-conditions at zero. The other class consists of Banach spaces E whose 1-unconditional bases satisfy block upper-p estimate for some $1 < p < \infty$.

If the radii of the balls that cover the unit sphere is bounded, then we say that the space has the strong ball-covering property (SBCP). In [6], Z. Luo and B. Zheng showed that the BCP and SBCP are not equivalent in general. Similar results will be extended to spaces with the SBCP. Throughout the paper, we will use $B(x, r)$ to denote the open ball centered at x with radius r.

2. Main results

Definition 2.1. Let (M_n) be a sequence of Orlicz functions. The space $\ell_{(M_n)}$ is the Banach space of all sequences $x = (a_n)$ with $\sum M_n(|a_n|/\rho) < \infty$ for some $\rho > 0$, equipped with the norm

$$\|x\| = \inf\{\rho > 0 : \sum M_n(|a_n|/\rho) \le 1\}.$$

The space $\ell_{(M_m)}$ is called a modular sequence space.

Definition 2.2. A sequence of Orlicz functions (M_n) is said to satisfy the uniform Δ_2-condition at zero if there exists a constant $K < \infty$ and an integer n_0 such that $M_n(2t)/M_n(t) \le K$ for all $n \ge n_0$ and $0 < t \le 1/2$.

Remark 2.3. Let (M_n) be a sequence of Orlicz functions that satisfies the uniform Δ_2-condition. Then the unit vector basis of $\ell_{(M_n)}$ form a boundedly complete unconditional basis and $\ell_{(M_n)}$ is separable.

Lemma 2.4. *Let X be a Banach space. Then X has the BCP if and only if there exists a countable collection $\{x_n\} \subset X\backslash\{0\}$ such that*

$$S_X \subset \bigcup_n B(x_n, \|x_n\|)\,.$$

Lemma 2.5. *Let X be a Banach space and $x \in X\backslash\{0\}$. If $0 < s < t$, then*

$$B(sx, \|sx\|) \subset B(tx, \|tx\|)$$

and

$$\|sx\| - \|y - sx\| \leq \|tx\| - \|y - tx\| \text{ for } y \in X\,.$$

Lemma 2.6. *Let X be a Banach space and $\{x_n\}$ be a collection of BCP points of X. Then*

$$X\backslash\{0\} = \bigcup_{m,n} B(mx_n, \|mx_n\|)\,.$$

A sequence of Orlicz functions (M_n) is called semi-normalized if the sequence (τ_n) with $M_n(\tau_n) = 1$ is bounded and bounded away from 0.

Theorem 2.7. *Let (X_k) be a sequence of Banach spaces and (M_n) be a sequence of semi-normalized Orlicz functions satisfying the uniform Δ_2-condition at zero. Then $X = (\sum X_k)_{\ell_{(M_n)}}$ has the BCP if and only if each X_k has the BCP.*

Proof. Necessity follows easily from definition and we only prove sufficiency. For each $k \in \mathbb{N}$, let $x_{k,0} = 0$ and $(x_{k,n})_{n\geq 1}$ be a sequence of points in X_k so that $S_{X_k} \subset \bigcup_n B(x_{k,n}, \|x_{k,n}\|)$. Let $\mathcal{A}$ be a subset of $X = (\sum X_k)_{\ell_{(M_n)}}$ defined by

$$\mathcal{A} = \{(z_1, z_2, \ldots, z_m, 0, 0, \ldots) : m \in \mathbb{N}, z_k \in (x_{k,n})_{n\geq 0}, 1 \leq k \leq m\}.$$

It is clear that $\mathcal{A}$ is countable and we will show that $\mathcal{A}$ is the desired BCP points of X.

Let $x = (x_k) \in S_X$. We need to find a $z \in \mathcal{A}$ so that

$$x \in B(z, \|z\|).$$

For each $k \in \mathbb{N}$, if $x_k = 0$, set $n_k = 0$; otherwise, we can find an x_{k,n_k} so that

$$\frac{x_k}{\|x_k\|} \in B(x_{k,n_k}, \|x_{k,n_k}\|)$$

or equivalently,

$$x_k \in B(\|x_k\|x_{k,n_k}, \|x_k\|\|x_{k,n_k}\|).$$

By Lemma 2.5, $B(\|x_k\|x_{k,n_k}, \|x_k\|\|x_{k,n_k}\|) \subset B(x_{k,n_k}, \|x_{k,n_k}\|)$. Hence we have

$$x_k \in B(x_{k,n_k}, \|x_{k,n_k}\|).$$

If $x = (x_k)$ is finitely supported, then there is an $N \in \mathbb{N}$ so that $x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$. Let

$$z = (x_{1,n_1}, x_{2,n_2}, \ldots, x_{N,n_N}, 0, 0, \ldots).$$

We have $x \in B(z, \|z\|)$ by the 1-unconditionality of the unit vector basis of $\ell_{(M_n)}$.

From now on, we assume x is not finitely supported. Let

$$\epsilon_n = \Big\|\sum_{k>n} x_k\Big\|.$$

Then $\epsilon_n > 0$ and decreases to 0. It follows from the definition of ϵ_n that

$$\sum_{k>n} M_k\Big(\frac{\|x_k\|}{\epsilon_n}\Big) = 1.$$

Let (τ_n) be the sequence of reals such that $M_n(\tau_n) = 1$ for all n. Since (M_n) is seminormalized, there exist positive real numbers L and U so that

$$L = \inf\{\tau_n\}, \quad U = \sup\{\tau_n\}.$$

Let

$$\epsilon = \|x_{1,n_1}\| - \|x_{1,n_1} - x_1\|$$

and choose $m \in \mathbb{N}$ big enough so that

$$\epsilon_m < \min\Big\{1, \frac{\epsilon}{2U}\Big\}.$$

By Lemma 2.5 and the definition of ϵ, without loss of generality, we may assume that

$$\|x_{1,n_1}\| > \max\{1, U\}.$$

Let

$$z = (x_{1,n_1}, x_{2,n_2}, \dots, x_{m,n_m}, 0, 0, \dots).$$

By Lemma 2.5 again, we may also assume that

$$\frac{1}{2} < M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big) \le \sum_{k=1}^{m} M_k\Big(\frac{\|x_{k,n_k}\|}{\|z\|}\Big) = 1 \le M_1(U).$$

It follows from above that

$$\|x_{1,n_1}\| \le U\|z\| \text{ and } \|z\| > 1.$$

So it is enough to show that

$$\|x - z\| < \|z\|,$$

which is equivalent to

$$\sum_{k=1}^{m} M_k\Big(\frac{\|x_{k,n_k} - x_k\|}{\|z\|}\Big) + \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big) < \sum_{k=1}^{m} M_k\Big(\frac{\|x_{k,n_k}\|}{\|z\|}\Big) = 1\,,$$

i.e.

$$\sum_{k=1}^{m} \Big\{M_k\Big(\frac{\|x_{k,n_k}\|}{\|z\|}\Big) - M_k\Big(\frac{\|x_{k,n_k} - x_k\|}{\|z\|}\Big)\Big\} > \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big)\,. \tag{2.1}$$

Since $\epsilon_m < 1$ and $\|z\| > 1$, we have

$$\sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big) = \sum_{k>m} M_k\Big(\frac{\varepsilon_m}{\|z\|} \cdot \frac{\|x_k\|}{\varepsilon_m}\Big) \le \frac{\varepsilon_m}{\|z\|} \cdot \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\varepsilon_m}\Big) = \frac{\varepsilon_m}{\|z\|}\,. \tag{2.2}$$

Since

$$\epsilon = \|x_{1,n_1}\| - \|x_{1,n_1} - x_1\|,$$

$$1 - \frac{\|x_{1,n_1} - x_1\|}{\|x_{1,n_1}\|} = \frac{\epsilon}{\|x_{1,n_1}\|}\,.$$

So we have

$$\begin{aligned}
&\sum_{k=1}^{m}\Big\{M_k\Big(\frac{\|x_{k,n_k}\|}{\|z\|}\Big)-M_k\Big(\frac{\|x_{k,n_k}-x_k\|}{\|z\|}\Big)\Big\}\\
\geq\ & M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)-M_1\Big(\frac{\|x_{1,n_1}-x_1\|}{\|z\|}\Big)\\
=\ & M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)-M_1\Big(\frac{\|x_{1,n_1}-x_1\|}{\|x_{1,n_1}\|}\cdot\frac{\|x_{1,n_1}\|}{\|z\|}\Big)\\
\geq\ & M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)-\frac{\|x_{1,n_1}-x_1\|}{\|x_{1,n_1}\|}\cdot M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)\\
=\ & \Big(1-\frac{\|x_{1,n_1}-x_1\|}{\|x_{1,n_1}\|}\Big)\cdot M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)\\
=\ & \frac{\epsilon}{\|x_{1,n_1}\|}\cdot M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)\\
\geq\ & \frac{\epsilon}{U\|z\|}\cdot M_1\Big(\frac{\|x_{1,n_1}\|}{\|z\|}\Big)\\
\geq\ & \frac{\epsilon}{2U\|z\|}\\
\geq\ & \frac{\epsilon_m}{\|z\|}
\end{aligned}$$

Combining the results above with Equation 2.1 and Equation 2.2, we have $\|x-z\|<\|z\|$. This finishes the proof.

□

Corollary 2.8. *Let (X_n) be a sequence of Banach spaces with the BCP and (N_n) be a sequence of Orlicz functions so that $\ell_{(N_n)}$ is separable. Then $(\sum X_n)_{\ell_{(N_n)}}$ can be renormed to satisfy the BCP.*

Let (e_n) be an unconditional basic sequence and $1<p<\infty$. A sequence (x_n) is said to be a block sequence of (e_n) if $x_n=\sum_{k=t_n}^{t_{n+1}-1}b_ke_k$, where (b_k) is a sequence of real numbers and (t_n) is an increasing sequence of positive integers. We say that (e_n) satisfies a block upper-p estimate if there exists a constant $C\geq 1$ so that for all $(a_k)\subset\mathbb{R}$ and block sequences (x_n) of (e_n),

$$\Big\|\sum a_kx_k\Big\|\leq C\Big(\sum|a_k|^p\Big)^{\frac{1}{p}}.$$

In this case, (e_n) is said to have a block upper-p estimate with constant C.

Theorem 2.9. *Let (X_n) be a sequence of Banach spaces with the BCP. Let E be a Banach space with an 1-unconditional basis (e_n) which has a block upper-p estimate with constant 1 for some $1<p<\infty$, then $(\sum X_n)_E$ has the BCP.*

Proof. For each $k\in\mathbb{N}$, let $x_{k,0}=0$ and $(x_{k,n})_{n\geq1}$ be a sequence of points in X_k so that $S_{X_k}\subset\bigcup_n B(x_{k,n},\|x_{k,n}\|)$. Let $\mathcal{A}$ be a subset of $X=(\sum X_k)_E$ defined by

$$\mathcal{A}=\{(z_1,z_2,\dots,z_m,0,0,\dots):m\in\mathbb{N},z_k\in\{jx_{k,n}\}_{n\geq0},1\leq k\leq m,j\in\mathbb{N}\}.$$

It is clear that $\mathcal{A}$ is countable and we will show that $\mathcal{A}$ is the desired BCP points of X. Since elements in X with finite support can be easily covered by balls with centers in $\mathcal{A}$. We will only consider $x = (x_k) \in X$ which are not finitely supported. Let $\epsilon_n = \|\sum_{k>n} x_k\|$. Then (ϵ_n) is a sequence of positive numbers decreasing to 0.

For each $k \in \mathbb{N}$, if $x_k = 0$, set $n_k = 0$; otherwise, we can find an x_{k,n_k} so that

$$x_k \in B(x_{k,n_k}, \|x_{k,n_k}\|).$$

Let $\delta = \|x_{1,n_1}\| - \|x_1 - x_{1,n_1}\| > 0$. Since ϵ_n decreases to 0, we can pick an $m \in \mathbb{N}$ so that

$$\epsilon_m < \frac{\delta}{2}.$$

Fix this particular δ. By Lemma 2.5, we can choose a big enough natural number M so that $x_1 \in B(Mx_{1,n_1}, M\|x_{1,n_1}\|)$, $\|Mx_{1,n_1}\| - \|Mx_{1,n_1} - x_1\| \geq \delta$ and

$$\|Mx_{1,n_1} - x_1\| > \left(\frac{2}{\delta}\right)^{\frac{1}{p-1}} \cdot \left\|\sum_{k=2}^{m} \|x_{k,n_k}\| e_k\right\|^{\frac{p}{p-1}} - \frac{\delta}{2}.$$

Let $z = (Mx_{1,n_1}, x_{2,n_2}, \ldots, x_{m,n_m}, 0, 0, \ldots)$. It is clear that $z \in \mathcal{A}$. Next we will show that

$$\|z - x\| < \|z\|.$$

By the 1-unconditionality of (e_n) and type p of E, we have

$$\begin{aligned}
\|z - x\| &= \left\| \|Mx_{1,n_1} - x_1\| e_1 + \sum_{k=2}^{m} \|x_{k,n_k} - x_k\| e_k + \sum_{k>m} \|x_k\| e_k \right\| \\
&\leq \left\| \|Mx_{1,n_1} - x_1\| e_1 + \sum_{k=2}^{m} \|x_{k,n_k} - x_k\| e_k \right\| + \epsilon_m \\
&\leq \left\| \|Mx_{1,n_1} - x_1\| e_1 + \sum_{k=2}^{m} \|x_{k,n_k} - x_k\| e_k \right\| + \frac{\delta}{2} \\
&\leq \left(\left\| Mx_1 - x_{1,n_1} \right\|^p + \left\| \sum_{k=2}^{m} \|x_{k,n_k} - x_k\| e_k \right\|^p \right)^{\frac{1}{p}} + \frac{\delta}{2} \\
&< \left(\|Mx_1 - x_{1,n_1}\|^p + \frac{\delta}{2}\left(\|Mx_1 - x_{1,n_1}\| + \frac{\delta}{2} \right)^{p-1} \right)^{\frac{1}{p}} + \frac{\delta}{2} \\
&< \left(\|Mx_1 - x_{1,n_1}\| \cdot \left(\|Mx_1 - x_{1,n_1}\| + \frac{\delta}{2} \right)^{p-1} + \frac{\delta}{2}\left(\|Mx_1 - x_{1,n_1}\| + \frac{\delta}{2} \right)^{p-1} \right)^{\frac{1}{p}} + \frac{\delta}{2} \\
&< \left(\|Mx_1 - x_{1,n_1}\| + \frac{\delta}{2} \right) + \frac{\delta}{2} \\
&= \|Mx_1 - x_{1,n_1}\| + \delta \\
&\leq \|Mx_{1,n_1}\| \\
&\leq \|z\|.
\end{aligned}$$

□

Theorem 2.10. *Let (X_k) be a sequence Banach spaces so that each X_k has the r-SBCP for some $r \geq 1$ independent of k. Let (M_n) be a sequence of Orlicz functions. Then $(\sum X_k)_{\ell_{(M_n)}}$ has the r-SBCP.*

Proof. For each $k \in \mathbb{N}$, let $x_{k,0} = 0$ and $(x_{k,n})_{n\geq 1}$ be a sequence of points in X_k so that $S_{X_k} \subset \bigcup_n B(x_{k,n}, \|x_{k,n}\|)$ and $1 \leq \|x_{k,n}\| \leq r$ for all $k, n \in \mathbb{N}$. Let $\mathcal{A}$ be a subset of $X = (\sum X_k)_{\ell_{(M_n)}}$ defined by

$$\mathcal{A} = \{(s_1 z_1, s_2 z_2, \dots, s_m z_m, 0, 0, \dots) : \\ m \in \mathbb{N}, \{s_k\}_{k=1}^m \in \mathcal{S}, z_k \in \{x_{k,n}\}_{n\geq 0}, 1 \leq k \leq m\}.$$

where $\mathcal{S}$ is the set of finite sequences of positive rational numbers $(s_k)_{k=1}^m$ with $\sum_{k=1}^m M_k(s_k) \leq 1$. It is clear that $\mathcal{A}$ is countable and for any $z = (s_1 z_1, s_2 z_2, \dots, s_m z_m, 0, 0, \dots)$,

$$\|z\| \leq \sup_{1\leq k\leq m} \{\|z_k\|\} \sum_{k=1}^m M_k(s_k) \leq r.$$

We will show that $\mathcal{A}$ is the desired set of BCP points of X.

Let $x = (x_k) \in S_X$. We need to find a $z \in \mathcal{A}$ so that

$$x \in B(z, \|z\|).$$

For each $k \in \mathbb{N}$, if $x_k = 0$, set $n_k = 0$; otherwise, we can find an x_{k,n_k} so that

$$x_k \in B(\|x_k\| x_{k,n_k}, \|x_k\| \|x_{k,n_k}\|).$$

Without loss of generality, we assume x is not finitely supported. Let

$$\epsilon_n = \Big\| \sum_{k>n} x_k \Big\|.$$

Then $\epsilon_n > 0$ and decreases to 0. It follows from the definition of ϵ_n that

$$\sum_{k>n} M_k\Big(\frac{\|x_k\|}{\epsilon_n}\Big) = 1.$$

Pick $\epsilon > 0$ so that

$$\epsilon < \|x_1\| \|x_{1,n_1}\| - \Big\| \|x_1\| x_{1,n_1} - x_1 \Big\|.$$

Then we have

$$1 - \frac{\Big\| \|x_1\| x_{1,n_1} - x_1 \Big\|}{\|x_1\| \|x_{1,n_1}\|} > \frac{\epsilon}{\|x_1\| \|x_{1,n_1}\|}\,. \tag{2.3}$$

Select a positive rational number s_1 so that

$$\frac{1}{2} \|x_1\| < s_1 \leq \|x_1\|$$

and

$$1 - \frac{\big\| s_1 x_{1,n_1} - x_1 \big\|}{s_1 \|x_{1,n_1}\|} > \frac{\epsilon}{s_1 \|x_{1,n_1}\|}\,. \tag{2.4}$$

Choose an $m \in \mathbb{N}$ so big that

$$\epsilon_m < \epsilon M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{r}\Big).$$

By the definition of $\mathcal{S}$ and the choices of (x_{k,n_k}), we can pick $s_2, s_3, \dots, s_m$ such that $(s_k)_{k=1}^m \in \mathcal{S}$. Let

$$z = (s_1 x_{1,n_1}, s_2 x_{2,n_2}, \dots, s_m x_{m,n_m}, 0, 0, \dots) \in \mathcal{A},$$

and

$$\|s_k x_{k,n_k}\| > \|s_k x_{k,n_k} - x_k\|, \quad \forall 1 \le k \le m.$$

So it is enough to show that

$$\|x - z\| < \|z\|,$$

which is equivalent to

$$\sum_{k=1}^{m} M_k\Big(\frac{\|s_k x_{k,n_k} - x_k\|}{\|z\|}\Big) + \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big) < \sum_{k=1}^{m} M_k\Big(\frac{\|s_k x_{k,n_k}\|}{\|z\|}\Big) = 1\,,$$

i.e.

$$\sum_{k=1}^{m} \Big\{ M_k\Big(\frac{\|s_k x_{k,n_k}\|}{\|z\|}\Big) - M_k\Big(\frac{\|s_k x_{k,n_k} - x_k\|}{\|z\|}\Big)\Big\} > \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big)\,. \tag{2.5}$$

Without loss of generality, we may assume that $\epsilon_m < 1$ and $\|z\| > 1$. Then we have

$$\sum_{k>m} M_k\Big(\frac{\|x_k\|}{\|z\|}\Big) = \sum_{k>m} M_k\Big(\frac{\varepsilon_m}{\|z\|} \cdot \frac{\|x_k\|}{\varepsilon_m}\Big) \le \frac{\varepsilon_m}{\|z\|} \cdot \sum_{k>m} M_k\Big(\frac{\|x_k\|}{\varepsilon_m}\Big) = \frac{\varepsilon_m}{\|z\|}\,. \tag{2.6}$$

Now we start our estimate.

$$\begin{aligned}
&\sum_{k=1}^{m} \Big\{ M_k\Big(\frac{\|s_k x_{k,n_k}\|}{\|z\|}\Big) - M_k\Big(\frac{\|s_k x_{k,n_k} - x_k\|}{\|z\|}\Big)\Big\} \\
&\ge M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) - M_1\Big(\frac{\|s_1 x_{1,n_1} - x_1\|}{\|z\|}\Big) \\
&= M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) - M_1\Big(\frac{\|s_1 x_{1,n_1} - x_1\|}{\|s_1 x_{1,n_1}\|} \cdot \frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) \\
&\ge M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) - \frac{\|s_1 x_{1,n_1} - x_1\|}{\|s_1 x_{1,n_1}\|} \cdot M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) \\
&= \Big(1 - \frac{\|s_1 x_{1,n_1} - x_1\|}{s_1\|x_{1,n_1}\|}\Big) \cdot M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) \\
&> \frac{\epsilon}{s_1\|x_{1,n_1}\|} \cdot M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{\|z\|}\Big) \\
&\ge \frac{\epsilon}{s_1\|x_{1,n_1}\|} \cdot M_1\Big(\frac{\|s_1 x_{1,n_1}\|}{r}\Big) \\
&> \frac{\epsilon_m}{s_1\|x_{1,n_1}\|} \\
&\ge \frac{\epsilon_m}{\|z\|}
\end{aligned}$$

Combining the results above with Equation 2.5 and Equation 2.6, we have $\|x - z\| < \|z\|$. This finishes the proof. □

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