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NON-CENTRAL SECTIONS OF THE REGULAR n-SIMPLEX

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Dedicated to the memory of Nicole Tomczak-Jaegermann, friend and coauthor

Abstract. We show that the maximal non-central hyperplane sections of the regular n-simplex of side-length $\sqrt{2}$ at a fixed distance t to the centroid are those parallel to a face of the simplex, if $\sqrt{\frac{n-2}{3(n+1)}} < t \leq \sqrt{\frac{n-1}{2(n+1)}}$ and $n \geq 5$. For $n = 4$, the same is true in a slightly smaller range for t. This adds to a previous result for $\sqrt{\frac{n-1}{2(n+1)}} < t \leq \sqrt{\frac{n}{n+1}}$. For $n = 2, 3$, we determine the maximal and the minimal sections for all distances t to the centroid.

1. Introduction

The Busemann-Petty problem and the hyperplane conjecture of Bourgain initiated an investigation of extremal hyperplane sections of convex bodies. This became a very active area in convex geometry and geometric tomography.

K. Ball [1] found the maximal hyperplane section of the n-cube in a celebrated paper. The minimal ones had been determined by Hadwiger [5] and Hensley [6]. Meyer and Pajor [11] settled the maximal section problem for the l_p^n-balls, if $0 < p < 2$, and the minimal one for the l_1^n-ball, which was extended by Koldobsky [7] to l_p^n-balls for $0 < p < 2$. Important progress in the case $2 < p < \infty$ was made recently by Eskenazis, Nayar and Tkocz [3]. Webb [16] found the maximal central sections of the regular n-simplex. The paper by Nayar and Tkocz [13] gives an excellent survey on extremal sections of classical convex bodies.

For non-central sections at distance t to the centroid, not too many results are known. Moody, Stone, Zach and Zvavitch [12] proved that the maximal sections of the n-cube at very large distances from the origin are those perpendicular to the main diagonal. Pournin [14] showed the same result for slightly smaller distances. Liu and Tkocz [10] proved a corresponding result for the l_1^n-ball, if the distance t to the origin satisfies $\frac{1}{\sqrt{2}} < t \leq 1$. König [9] extended this to $\frac{1}{\sqrt{3}} < t \leq \frac{1}{\sqrt{2}}$. Maximal non-central sections of the regular n-simplex of side-length $\sqrt{2}$ were studied by König [8] for relatively large distances t to the centroid. In this paper, we extend this result to smaller distances t for $n \geq 4$, resulting in the range $\sqrt{\frac{n-2}{3(n+1)}} < t \leq \sqrt{\frac{n}{n+1}}$. Note that for $t > \sqrt{\frac{n}{n+1}}$, the intersection is empty. The maximal sections are those parallel to faces, i.e. those perpendicular to the diagonals from the centroid

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to the vertices of the simplex, in dimension $n = 4$ in a slightly smaller range for t. The proof uses a volume formula of Dirksen [2] which extended a result of Webb [16]. This formula involves alternating terms with removable singularities, if $\sqrt{\frac{n-2}{3(n+1)}} < t \le \sqrt{\frac{n-1}{2(n+1)}}$, resulting in difficulties to find extrema of section volumes.

For small dimensions $n = 2, 3$, we determine the maximal and the minimal hyperplane sections for all possible distances $|t| \le \sqrt{\frac{n}{n+1}}$.

To formulate our results precisely, we introduce some definitions and notations. Let $\mathbb{N} := \{ n \in \mathbb{Z} \mid n \ge 1 \}$ denote the positive integers. For the regular n-simplex $K = \Delta^n$ we use its representation in $\mathbb{R}^{n+1}$

$$\Delta^n := \{x \in \mathbb{R}^{n+1} \mid x = (x_j)_{j=1}^{n+1},\ x_j \ge 0,\ \sum_{j=1}^{n+1} x_j = 1\}.$$

Let $a \in S^n := \{ x = (x_j)_{j=1}^{n+1} \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} x_j^2 = 1 \}$ be a direction vector and $t \in \mathbb{R}$. We will always assume that $\sum_{j=1}^{n+1} a_j = 0$ so that the centroid $c = \frac{1}{n+1}(1, \cdots, 1) \in \Delta^n$ is in the hyperplane $a^\perp$. By $H(a,t) := \{x \in \Delta^n \mid \langle a, x\rangle = t\} = \{ta\} + a^\perp$ we denote the hyperplane orthogonal to a at distance t to the centroid. Thus $c \in H_0(a)$. Then

$$A(a,t) := \mathrm{vol}_{n-1}(H(a,t) \cap \Delta^n) = \mathrm{vol}_{n-1}(\{x \in \Delta^n \mid \langle a, x\rangle = t\})$$

is the *parallel section function* of Δ^n. We aim to find the extrema of $A(\cdot, t)$ on S^n for fixed t. For $t \ne 0$ we have non-central sections.

Note that $\sqrt{\frac{n}{n+1}}$ is the distance of the vertices $e_j = (0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0)$ of Δ^n to the centroid c; it is the maximum coordinates of $a \in S^n$ with $\sum_{j=1}^{n+1} a_j = 0$ can attain, and in this case all other coordinates are equal to $-\frac{1}{\sqrt{n(n+1)}}$. The side-length of the simplex Δ^n is $\sqrt{2}$, its height $\sqrt{\frac{n+1}{n}}$ and its volume $\frac{\sqrt{n+1}}{n!}$. The distance of the centroid to the midpoint of edges of Δ^n is $\sqrt{\frac{n-1}{2(n+1)}}$. By symmetry, we may assume that $a = (a_j)_{j=1}^{n+1} \in S^n$ satisfies $a_1 \ge a_2 \cdots \ge a_{n+1}$. Throughout this paper, we make the following

General Assumption. A direction vector $a = (a_j)_{j=1}^{n+1} \in \mathbb{R}^n$ is always assumed to satisfy $\sum_{j=1}^{n+1} a_j^2 = 1$, $\sum_{j=1}^{n+1} a_j = 0$ and $a_1 \ge a_2 \cdots \ge a_{n+1}$.

Note that under this assumption always $a_{n+1} < 0$. We introduce the sets $H_+(a,t) = \{ x \in \Delta^n \mid \langle a, x\rangle > t \}$ and $H_-(a,t) = \{ x \in \Delta^n \mid \langle a, x\rangle < t \}$. For $\sqrt{\frac{n-1}{2(n+1)}} < t < \sqrt{\frac{n}{n+1}}$ the set $H_+(a,t)$ contains just one vertex, namely e_1. Further, $\sqrt{\frac{n-k+1}{k(n+1)}}$ is the distance from the centroid of Δ^n to the centroid of a $(k-1)$-boundary simplex. For $\sqrt{\frac{n-k}{(k+1)(n+1)}} < t < \sqrt{\frac{n-k+1}{k(n+1)}}$ the set $H_+(a,t)$ contains at most k vertices of Δ^n. This follows from Lemma 2.1 below. If $a_1 < t$ and $x \in \Delta^n$, $t > \sum_{j=1}^{n+1} a_j x_j = \langle a, x\rangle$, hence $H(a,t) = \emptyset$. If $a_1 = t$ and $x \in \Delta^n$,

$t > \sum_{j=1}^{n+1} a_j x_j = \langle a, x\rangle$ as well, unless $x = e_1$. Then $H(a,t) = \{e_1\}$ and $A(a,t) = 0$. Therefore, if $A(a,t) > 0$, the General Assumption implies that $a_1 > t$.

The unit vectors in the direction from the centroid of Δ^n to the centroid $\frac{e_1+\cdots+e_k}{k}$ of a (particular) boundary $(k-1)$-simplex are

$$a^{(k)} := \sqrt{\frac{k(n-k+1)}{n+1}}(\underbrace{\frac{1}{k},\dots,\frac{1}{k}}_{k},\underbrace{-\frac{1}{n-k+1},\dots,-\frac{1}{n-k+1}}_{n-k+1}) \in S^n \ ,$$

$k = 1, \cdots, n$. Thus $a^{(1)}$ points from the centroid of Δ^n to a vertex, $a^{(2)}$ to the midpoint of an edge and $a^{(3)}$ to the center of a boundary triangle. A hyperplane orthogonal to the "main diagonal" $a^{(1)}$ is parallel to a boundary $(n-1)$-face of Δ^n. Note that $a^{(n-k+1)}$ is essentially equal to $-a^{(k)}$, up to a permutation of coordinates. Since $H_t(a) = H_{-t}(-a)$, we have

$$(1.1) \qquad A(a,t) = A(-a,-t) \quad , \quad A(a^{(k)},t) = A(a^{(n-k+1)},-t) \ , \ k = 1,\dots,n \ .$$

We now formulate our main results.

Theorem 1.1. *Let $n \ge 5$, Δ^n be the regular n-simplex and assume that $t \in \mathbb{R}$ satisfies $\sqrt{\frac{n-2}{3(n+1)}} < t \le \sqrt{\frac{n}{n+1}}$. Let $a \in S^n$ with $\sum_{j=1}^{n+1} a_j = 0$. Then $H_t(a^{(1)})$ is a maximal hyperplane section of Δ^n at distance t to the centroid, i.e.*

$$A(a,t) \le A(a^{(1)},t) = \frac{\sqrt{n+1}}{(n-1)!}\left(\frac{n}{n+1}\right)^{n/2}\left(\sqrt{\frac{n}{n+1}} - t\right)^{n-1} .$$

All maximal sections at distance t are parallel to a boundary $(n-1)$-face, and the orthogonal vector is a permutation of $a^{(1)}$. For $n = 4$, the same is true if $0.3877 \simeq \tilde{t} < t < \sqrt{\frac{4}{5}}$. For $n = 4$ and $\sqrt{\frac{n-2}{3(n+1)}} = \sqrt{\frac{2}{15}} \le t < \tilde{t}$, $a^{(2)}$ yields a maximal hyperplane at distance t to the centroid. We have $A(a^{(2)},\tilde{t}) = A(a^{(1)},\tilde{t})$.

The range $\sqrt{\frac{n-1}{2(n+1)}} < t \le \sqrt{\frac{n}{n+1}}$ was already covered by a result in König [8]. As in the results of Moody, Stone, Zach and Zvavitch [12] for the n-cube and of Liu and Tkocz [10] and König [9] for the l_1^n-ball, the maximal hyperplanes far from the barycenter are those perpendicular to the main diagonal. By König [8], $a^{(1)}$ is a local maximum of $A(\cdot,t)$ if $\frac{2n+1}{n(n+2)}\sqrt{\frac{n}{n+1}} < t$. It can be routinely verified that $\frac{2n+1}{n(n+2)}\sqrt{\frac{n}{n+1}} \le \sqrt{\frac{n-2}{3(n+1)}}$ for all $n \ge 4$. The value $\tilde{t}$ is the solution of a cubic equation.

For $t = 0$, by Webb [16] the maximal central sections of Δ^n are those containing $(n-1)$ vertices and the midpoint of the remaining two vertices, orthogonal to vectors like $\frac{1}{\sqrt{2}}(1,0,\dots,0,-1)$. Filliman [4] stated without proof that the minimal central sections are those parallel to a face of Δ^n. Recently, Tang [15] proved an asymptotically sharp lower bound for the volume of central sections of Δ^n.

For dimensions $n = 2, 3$, we give the extrema of $A(\cdot,t)$ for all t:

Proposition 1.2. *For $n = 2$ and $0 \le t \le \sqrt{\frac{2}{3}}$, let*

$$a^{[t]} := \left(\frac{1}{2}(\sqrt{2-3t^2} - t), t, -\frac{1}{2}(t + \sqrt{2-3t^2})\right) \in S^2 \quad \textit{and}$$

$$a^{\{t\}} := (\tilde{a_1}, b_+, b_-) \in S^2, \tilde{a_1} := t + \sqrt{t^2 - \frac{1}{6}}, b_\pm := -\frac{\tilde{a_1}}{2} \pm \frac{1}{2}\sqrt{2 - 3\tilde{a_1}^2} \ .$$

Then the extrema of $A(\cdot, t)$ on S^2 are given by

$$Maximum : \left\{ \begin{array}{ll} a^{[t]} \quad , 0 \le t \le \frac{1}{\sqrt{6}} : & A(a^{[t]}, t) = \sqrt{3}\frac{1}{\sqrt{2-3t^2}} \\ a^{\{t\}} \ , \frac{1}{\sqrt{6}} < t \le \frac{5}{4}\frac{1}{\sqrt{6}} : & A(a^{\{t\}}, t) = \frac{1}{2\sqrt{3}}\frac{1}{t+\sqrt{t^2-1/6}} \\ a^{(1)} \ , \frac{5}{4}\frac{1}{\sqrt{6}} \le t < \sqrt{\frac{2}{3}} : & A(a^{(1)}, t) = \frac{2\sqrt{2}}{3} - \frac{2}{\sqrt{3}}t \end{array} \right\} ,$$

$$Minimum : \left\{ \ a^{(1)} \ , \ 0 \le t \le \frac{1}{\sqrt{6}} : \quad A(a^{(1)}, t) = \frac{2\sqrt{2}}{3} - \frac{2}{\sqrt{3}}t \ \right\} .$$

Partially, this can be found already in Dirksen [2] and König [8]. For $0 \le t \le \frac{1}{\sqrt{6}}$, $a^{[t]}$ turns continuously from $a^{[0]} = \frac{1}{\sqrt{2}}(1, 0, -1)$ as given by Webb's result [16] to $a^{(2)} = \frac{1}{\sqrt{6}}(1, 1, -2)$, and for $\frac{1}{\sqrt{6}} < t \le \frac{5}{4}\frac{1}{\sqrt{6}}$, $a^{\{t\}}$ turns continuously from $a^{(2)} = \frac{1}{\sqrt{6}}(1, 1, -2)$ to $a^{(1)} = \frac{1}{\sqrt{6}}(2, -1, -1)$. Up to a permutation of coordinates, $a^{(2)}$ is just $-a^{(1)}$. The distance of the center to a side of the triangle Δ^2 is $t = \frac{1}{\sqrt{6}}$. Enlarging t a bit, turning the side slightly and moving it partially outside of Δ^2, intersects Δ^2 in about $\frac{1}{2}$ of its length. Thus there is a discontinuity of $A(a^{\max}, \cdot)$ at $t = \frac{1}{\sqrt{6}}$, where $a^{\max}$ denotes the direction of largest value of the parallel section function,

$$A(a^{(2)}, \frac{1}{\sqrt{6}}) = \sqrt{2} > \lim_{t \searrow 1/\sqrt{6}} A(a^{\{t\}}, t) = \frac{1}{\sqrt{2}} \ .$$

Proposition 1.3. *For $n = 3$ and $0 \le t \le \frac{\sqrt{3}}{2}$, let $t_0 := \frac{9+4\sqrt{16-6\sqrt{3}}}{2(3\sqrt{3}+16)} \simeq 0.43575$, $t_1 := \frac{\sqrt{6}+1}{10} \simeq 0.34495$ and*

$$a^{[t]} := \left(\frac{1}{2}\sqrt{2 - 8t^2} - t, t, t, -(\frac{1}{2}\sqrt{2 - 8t^2} + t)\right) \in S^3 \quad and$$

$$a^{\{t\}} := (\tilde{a_1}, \tilde{a_1}, b_+, b_-) \in S^3, b_\pm := -\tilde{a_1} \pm \frac{1}{2}\sqrt{2 - 8\tilde{a_1}^2} \ ,$$

where $t \in [\frac{1}{2\sqrt{3}}, t_1]$ and $\tilde{a_1} \in [\frac{1}{2\sqrt{3}}, \frac{1}{2}]$ is the unique solution of $\phi(a_1) = t$,

$$\phi(a_1) := \frac{a_1(5 - 12a_1^2) + (6a_1^2 - \frac{1}{2})\sqrt{28a_1^2 - 1}}{1 + 36a_1^2} \ .$$

Then the extrema of $A(\cdot, t)$ on S^3 are given by

$$Maximum : \left\{ \begin{array}{ll} a^{[t]} \ , \ 0 \le t \le \frac{1}{2\sqrt{3}} : & A(a^{[t]}, t) = \frac{1}{\sqrt{2-8t^2}} \\ a^{\{t\}} \ , \frac{1}{2\sqrt{3}} < t \le t_1 & \\ a^{(2)} \ , \ \ t_1 \le t \le t_0 : & A(a^{(2)}, t) = \frac{1}{2} - 2t^2 \\ a^{(1)} \ , t_0 \le t < \frac{\sqrt{3}}{2} : & A(a^{(1)}, t) = \frac{3\sqrt{3}}{8}(\frac{\sqrt{3}}{2} - t)^2 \end{array} \right\} ,$$

$$Minimum : \left\{ \ a^{(1)} \ , \ 0 \le t \le \frac{1}{2\sqrt{3}} : \quad A(a^{(1)}, t) = \frac{3\sqrt{3}}{8}(\frac{\sqrt{3}}{2} - t)^2 \ \right\} .$$

For $\frac{1}{2\sqrt{3}} < t \le t_1$, $A(a^{\{t\}}, t) = \psi(\tilde{a_1})$, $\psi(a_1) := \frac{4\left[a_1(5+52a_1^2)-(2a_1^2+\frac{1}{2})\sqrt{28a_1^2-1}\right]}{(1+36a_1^2)^2}$. Again, $a^{[t]}$ turns continuously from $a^{[0]} = \frac{1}{\sqrt{2}}(1,0,0,-1)$ as given by Webb [16] to $a^{(3)} = \frac{1}{2\sqrt{3}}(1,1,1,-3)$ which is $-a^{(1)}$ up to a permutation of coordinates. For $\frac{1}{2\sqrt{3}} < t \le t_1$, $a^{\{t\}}$ turns continuously from $a^{(3)}$ to $a^{(2)} = \frac{1}{2}(1,1,-1,-1)$. Again there is a discontinuity at the distance of the center to a boundary triangle $t = \frac{1}{2\sqrt{3}}$,

$$A(a^{(3)}, \frac{1}{2\sqrt{3}}) = \frac{\sqrt{3}}{2} > \lim_{t \searrow 1/2\sqrt{3}} A(a^{\{t\}}, t) = \frac{5}{9}\frac{\sqrt{3}}{2} = \frac{5}{18}\sqrt{3} \ .$$

The last value stems from the fact that for $t \searrow \frac{1}{2\sqrt{3}}$, also $\tilde{a_1} \searrow \frac{1}{2\sqrt{3}}$. The factor $\frac{5}{9}$ is easily explained: Dividing a boundary triangle of Δ^3 by a line through its center parallel to a side into a triangle and a trapezoid , the area of the trapezoid is $\frac{5}{9}$-th of the area of the triangle itself.

In the next chapter we state a general formula for $A(a,t)$ which involves alternating terms with removable singularities and give some consequences, e.g. a formula for $A(a^{(k)}, t)$. After some preparations in chapter 3, we prove Theorem 1.1 in chapter 4. The proof uses some ideas also found in König [9], though there are essential differences due to the non-symmetry of Δ^n and the additional constraint $\sum_{j=1}^{n+1} a_j = 0$. Basically, we show that there are at most three different coordinates in critical points $a \in S^{n-1}$ of $A(\cdot, t)$ in addition to the largest two coordinates, which is then reduced to two with predetermined multiplicity. This leads to specific functions of the largest two coordinates (a_1, a_2) of a to be investigated. We then apply some monotonicity result proved in chapter 3. In chapter 5 we verify Propositions 1.2 and 1.3.

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2. A volume formula for sections of the n-simplex

The general assumption on a implies

Lemma 2.1. *Let $a \in S^n$, $\sum_{j=1}^{n+1} a_j = 0$, $a_1 \ge a_2 \ge \cdots \ge a_{n+1}$ and $1 \le k \le n$.*

i) *If $a_1 \ge \cdots \ge a_k \ge 0$, $a_k \le \sqrt{\frac{n-k+1}{k(n+1)}}$. In particular, $a_1 \le \sqrt{\frac{n}{n+1}}$. If $a_1 > a_2$, $a_3 < \sqrt{\frac{n-2}{3(n+1)}}$.*

ii) *If $a_1 \ge a_2 \ge 0$, $a_1 \le \psi(a_2) := \sqrt{\frac{n-1}{n}}\sqrt{1 - \frac{n+1}{n}a_2^2} - \frac{a_2}{n}$. ψ is a decreasing function with $\psi^2 = Id$, $a_1 = \psi(a_2)$ if and only if $a_2 = \psi(a_1)$.*

Therefore, if $\sqrt{\frac{n-k}{(k+1)(n+1)}} < t \le \sqrt{\frac{n-k+1}{k(n+1)}}$, as assumed in Theorem 1.1 for $k = 2$ or $k = 1$, $a_{k+1} < t$ and there are $\le k$ vertices e_j of Δ^n in $H_+(a,t)$.

We will use the function ψ throughout the paper.

Proof. i) Using the Cauchy Schwarz inequality, we find that
$\sum_{j=1}^{k} a_j = -\sum_{j=k+1}^{n+1} a_j \le \sqrt{n-k+1}\sqrt{\sum_{j=k+1}^{n+1} a_j^2} = \sqrt{n-k+1}\sqrt{1-\sum_{j=1}^{k} a_j^2}$, which together with $ka_k \le \sum_{j=1}^{k} a_j$ and $ka_k^2 \le \sum_{j=1}^{k} a_j^2$ implies $a_k \le \sqrt{\frac{n-k+1}{k(n+1)}}$.

ii) For $k = 2$, the previous inequalities yield $(a_1 + a_2)^2 \le (n-1)(1-(a_1^2 + a_2^2))$. This quadratic inequality implies $a_1 \le \sqrt{\frac{n-1}{n}}\sqrt{1-\frac{n+1}{n}a_2^2} - \frac{a_2}{n} =: \psi(a_2)$. Since $\psi'(a_2) = -\sqrt{\frac{n-1}{n}}\frac{(n+1)/na}{\sqrt{1-(n+1)/na^2}} - \frac{1}{n} < 0$, ψ is strictly decreasing. We have $a_1 = \psi(a_2)$ if and only if $(a_1 + a_2)^2 = (n-1)(1-(a_1^2 + a_2^2))$, which is symmetric in a_1 and a_2. $\square$

We need an explicit formula for $A(a,t)$. Put $x_+ := \max(x, 0)$ for any $x \in \mathbb{R}$. Then:

Proposition 2.2. *Let $n \in \mathbb{N}$, $t \in \mathbb{R}$ with $|t| \le \sqrt{\frac{n}{n+1}}$ and $a \in S^n$ with $\sum_{j=1}^{n+1} a_j = 0$. Assume that $a_1 > a_2 > \cdots > a_{n+1}$. Then*

$$A(a,t) = \frac{\sqrt{n+1}}{(n-1)!}\sum_{j=1}^{n+1}\frac{(a_j - t)_+^{n-1}}{\prod_{k=1,k\ne j}^{n+1}(a_j - a_k)} . \tag{2.1}$$

This is Corollary 2.4 of Dirksen [2]. It is stated there for sequences $a \in S^n$ without $\sum_{j=1}^{n+1} a_j = 0$, but explained how to transform it into the above formula. Formula (2.1) generalizes a result of Webb [16] for central sections. The proof uses the Fourier transform technique explained in Koldobsky's book [7] leading to $A(a,t) = \frac{1}{2\pi}\int_{\mathbb{R}}\prod_{k=1}^{n+1}\frac{1}{1+i(a_k-t)s}\, ds$, which is evaluated using the residue theorem.

By (1.1), $A(a,t) = A(-a,-t)$. Possibly exchanging (a,t) with $(-a,-t)$, we may apply (2.1) with at most $[\frac{n+1}{2}]$ non-zero terms, thus with less than n non-zero terms. For the standard vectors $a^{(k)}$ we have the following result.

Proposition 2.3. *Let $1 \le k \le n$ and $-\sqrt{\frac{k}{(n-k+1)(n+1)}} < t < \sqrt{\frac{n-k+1}{k(n+1)}}$. Then* $A(a^{(k)},t) = \frac{\sqrt{n+1}}{(n-1)!}\binom{n-1}{k-1}\sqrt{\frac{k(n-k+1)}{n+1}}^{\,n}\left(t+\sqrt{\frac{k}{(n-k+1)(n+1)}}\right)^{k-1}\left(\sqrt{\frac{n-k+1}{k(n+1)}}-t\right)^{n-k}$.

Since $|\langle a^{(k)}, e_j\rangle| \ge \sqrt{\frac{k}{(n-k+1)(n+1)}}$ for all $j = 1, \ldots, n+1$, $A(a^{(k)},t) = 0$ for $t < -\sqrt{\frac{k}{(n-k+1)(n+1)}}$.

Proof. By continuity, formula (2.1) also holds for all sequences $a \in S^n$ with $a_1 > a_2 > \cdots > a_k > t \ge a_{k+1}, \cdots, a_{n+1}$ where some or all of the a_l with $l > k$ may be equal. For $k = 1$, equation (2.1) immediately implies the claim. For $a_1 > a_2 > t \ge a_3, \ldots, a_{n+1}$,

$$A(a,t) = \frac{\sqrt{n+1}}{(n-1)!}\frac{1}{a_1 - a_2}\left(\frac{(a_1 - t)^{n-1}}{\prod_{j=3}^{n+1}(a_1 - a_j)} - \frac{(a_2 - t)^{n-1}}{\prod_{j=3}^{n+1}(a_2 - a_j)}\right) .$$

For $k = 2$, with $a^{(2)} = (b, b, c, \cdots, c)$, $b := \sqrt{\frac{n-1}{2(n+1)}}$, $c := -\sqrt{\frac{2}{(n-1)(n+1)}}$, let $a(\varepsilon) := \frac{1}{\sqrt{1-\varepsilon^2}}(b+\varepsilon, b-\varepsilon, c, \cdots, c)$, $\sum_{j=1}^{n+1} a(\varepsilon)_j = 0$, $\sum_{j=1}^{n+1} a(\varepsilon)_j^2 = 1$, $a(\varepsilon) \to a^{(2)}$ and

$A(a(\varepsilon),t) \to A(a^{(2)},t)$ as $\varepsilon \to 0$. The continuous extension of (2.1) to $a(\varepsilon)$ then contains two alternating non-zero summands with $b+\varepsilon = a_1$ and $b-\varepsilon = a_2$, provided that $b-\varepsilon > t$. We find for $t < \sqrt{\frac{n-1}{2(n+1)}}$

$$\frac{(n-1)!}{\sqrt{n+1}} A(a^{(2)},t) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left(\frac{(b-t+\varepsilon)^{n-1}}{(b-c+\varepsilon)^{n-1}} - \frac{(b-t-\varepsilon)^{n-1}}{(b-c-\varepsilon)^{n-1}} \right) = g'(b) \ ,$$

where $g(x) := \frac{(x-t)^{n-1}}{(x-c)^{n-1}}$, $g'(x) = (n-1)(t-c)\frac{(x-t)^{n-2}}{(x-c)^n}$,
$g'(b) = (n-1)\left(t + \sqrt{\frac{2}{(n-1)(n+1)}}\right)\left(\sqrt{\frac{2(n-1)}{n+1}}\right)^n \left(\sqrt{\frac{n-1}{2(n+1)}} - t\right)^{n-2}$, since $\frac{1}{b-c} = \sqrt{\frac{2(n-1)}{n+1}}$.

For $k > 2$ and $a_k = (\underbrace{b,\dots,b}_{k},\underbrace{c,\dots,c}_{n+1-k})$, $b := \sqrt{\frac{n-k+1}{k(n+1)}}$, $c := -\sqrt{\frac{k}{(n-k+1)(n+1)}}$ and $t < b$ we consider k equidistant points of step-size ε around b, $b + j\varepsilon$, $j = -[\frac{k}{2}],\dots,[\frac{k}{2}]$, to define $a(\varepsilon)$ similarly, and then (2.1) yields a $(k-1)$-st order ε-step size difference $\Delta_\varepsilon^{(k-1)} g$ of the function $g(x) := \frac{(x-t)^{n-1}}{(x-c)^{n-k+1}}$, namely

$$\frac{(n-1)!}{\sqrt{n+1}} A(a^{(k)},t) = \lim_{\varepsilon\to 0} \frac{1}{(k-1)!} \frac{1}{\varepsilon^{k-1}} \Delta_\varepsilon^{(k-1)} g(b) = \frac{1}{(k-1)!} g^{(k-1)}(b) \ .$$

The power in the denominator of g is reduced by the fact that there are $(k-1)$ products of differences of values near b resulting in the $(k-1)$-st order difference. One verifies by induction that

$$\frac{1}{(k-1)!} g^{(k-1)}(x) = \binom{n-1}{k-1} (t-c)^{k-1} \frac{(x-t)^{n-k}}{(x-c)^n} \ ,$$

and with $\frac{1}{b-c} = \sqrt{\frac{k(n-k+1)}{n+1}}$ we have that
$A(a^{(k)},t) = \frac{\sqrt{n+1}}{(n-1)!}\binom{n-1}{k-1}\sqrt{\frac{k(n-k+1)}{n+1}}^n \left(t + \sqrt{\frac{k}{(n-k+1)(n+1)}}\right)^{k-1} \left(\sqrt{\frac{n-k+1}{k(n+1)}} - t\right)^{n-k}$. □

In particular, $A(a^{(1)},t) = \frac{\sqrt{n+1}}{(n-1)!}\left(\sqrt{\frac{n}{n+1}}\right)^n \left(\sqrt{\frac{n}{n+1}} - t\right)^{n-1}$, $t < \sqrt{\frac{n}{n+1}}$,
$A(a^{(2)},t) = \frac{\sqrt{n+1}}{(n-2)!}\left(\sqrt{\frac{2(n-1)}{n+1}}\right)^n \left(\sqrt{\frac{2}{(n-1)(n+1)}} + t\right)\left(\sqrt{\frac{n-1}{2(n+1)}} - t\right)^{n-2}$, $t < \sqrt{\frac{n-1}{2(n+1)}}$.

For all $\frac{c}{\sqrt{n(n+1)}} < t < \sqrt{\frac{n}{n+1}}$, $A(a^{(1)},t) > A(a^{(2)},t)$ is true asymptotically for $c \simeq 2.6363$: $A\left(a^{(1)}, \frac{c}{\sqrt{n(n+1)}}\right) - A\left(a^{(2)}, \frac{c}{\sqrt{n(n+1)}}\right) = O(\frac{1}{n})$ leads to the requirement $2(\sqrt{2}+c) = \exp(1+(\sqrt{2}-1)c)$, which has only one positive solution $c \simeq 2.6363$. Let t_n be the positive solution in dimension n of $A(a^{(1)},t_n) = A(a^{(2)},t_n)$ less than $\sqrt{\frac{n-1}{2(n+1)}}$. Then $t_3 \simeq 0.4357$, $t_4 \simeq 0.3877$ and $t_5 \simeq 0.3426$.

3. Prerequisites

In this chapter we prove two results which are needed in the proof of Theorem 1.1 in the next chapter. The first one is used to reduce the number of different coordinates of extremal critical points of $A(\cdot, t)$, the second is a monotonicity result for $A(a, t)$.

Proposition 3.1. *Let* $n \geq 4$, $\sqrt{\frac{n-2}{3(n+1)}} < t < a_2 < a_1$, $p, r, s \in \mathbb{N}$ *and* $c, d, e \in \mathbb{R}$ *be such that* $a_1 + a_2 + c + d + e = 0$, $a_1^2 + a_2^2 + \frac{c^2}{p} + \frac{d^2}{r} + \frac{e^2}{s} = 1$, $p + r + s = n - 1$. *Then*

$$\left(\frac{a_1 - t}{a_2 - t}\right)^{n-1} > \left(\frac{a_1 - \frac{c}{p}}{a_2 - \frac{c}{p}}\right)^{p+1} \left(\frac{a_1 - \frac{d}{r}}{a_2 - \frac{d}{r}}\right)^{r+1} \left(\frac{a_1 - \frac{e}{s}}{a_2 - \frac{e}{s}}\right)^{s+1} .$$

For the proof we need the

Lemma 3.2. *Let* $n \geq 4$, $\sqrt{\frac{n-2}{3(n+1)}} < t < a_2 < a_1$ *and* $c \in \mathbb{R}$. *Then:*

(i) *If* $c \leq 0$, $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-c}{a_2-c}\right)^3$.

(ii) *If* $c \leq g(n) := 2\sqrt{\frac{n-2}{3(n+1)}} - \sqrt{\frac{n-1}{2(n+1)}}$, $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-c}{a_2-c}\right)^2$.

Proof. (i) Since $\left(\frac{a_1-c}{a_2-c}\right)$ is decreasing in c, it suffices to prove (i) for $c = 0$. This means $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1}{a_2}\right)^3$ or equivalently $t(a_1^3 - a_2^3) - a_1 a_2 (a_1^2 - a_2^2) > 0$, which is equivalent to $t > \frac{a_1 a_2 (a_1+a_2)}{(a_1-a_2)^2 + 3a_1 a_2}$. By Lemma 2.1 $a_1 \leq \psi(a_2)$, hence $a_1 + a_2 \leq \psi(a_2) + a_2 = \sqrt{\frac{n-1}{n}}\sqrt{1 - \frac{n+1}{n} a_2^2} + \frac{n-1}{n} a_2 =: \gamma(a_2)$, $\gamma'(a_2) = -\sqrt{\frac{n-1}{n}}\frac{n+1}{n}\frac{a_2}{\sqrt{1-\frac{n+1}{n}a_2^2}} + \frac{n-1}{n} \geq 0$, since $\frac{n-1}{n} \geq \frac{(\frac{n+1}{n})^2 a_2^2}{1-\frac{n+1}{n}a_2^2}$ is satisfied in view of $a_2 \leq \sqrt{\frac{n-1}{2(n+1)}}$ which holds by Lemma 2.1. Thus γ is increasing in a_2, so that $a_1 + a_2 \leq \gamma(\sqrt{\frac{n-1}{2(n+1)}}) = \sqrt{\frac{2(n-1)}{n+1}}$. Therefore for $n \geq 4$

$$\frac{a_1 a_2 (a_1 + a_2)}{(a_1 - a_2)^2 + 3a_1 a_2} < \frac{a_1 + a_2}{3} \leq \frac{1}{3}\sqrt{\frac{2(n-1)}{n+1}} \leq \sqrt{\frac{n-2}{3(n+1)}} < t$$

is satisfied and (i) holds for $c \leq 0$.

(ii) In view of (i), we may assume that $c > 0$. The claim $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-c}{a_2-c}\right)^2$ is equivalent to $(a_1 - a_2)[(a_1 + a_2 - 2c)t + c^2 - a_1 a_2] > 0$. Therefore we need $t > \frac{a_1 a_2 - c^2}{a_1 + a_2 - 2c} =: h(c)$. Since $h'(c) = 2\frac{(a_1-c)(a_2-c)}{(a_1+a_2-2c)^2} > 0$, if $c < a_2 < a_1$, h is increasing in $[0, a_2]$. Note here that $c \leq g(n) < t < a_2$. Now $c_0 = t - \sqrt{(a_1 - t)(a_2 - t)}$ is the solution of $t = h(c) = \frac{a_1 a_2 - c^2}{(a_1 + a_2 - 2c)}$ with $c_0 < t < a_2$. Thus for $c < c_0$, $h(c) < t$. By the arithmetic-geometric mean inequality $\sqrt{(a_1 - t)(a_2 - t)} \leq \frac{a_1+a_2}{2} - t \leq \sqrt{\frac{n-1}{2(n+1)}} - t$ and hence $c_0 = t - \sqrt{(a_1 - t)(a_2 - t)} \geq 2t - \sqrt{\frac{n-1}{2(n+1)}} > 2\sqrt{\frac{n-2}{3(n+1)}} - \sqrt{\frac{n-1}{2(n+1)}} = g(n)$. Thus, if $c \leq g(n)$, $c < c_0$ and $h(c) < t$ is satisfied. $\square$

Proof of Proposition 3.1. Assume that $\frac{c}{p} \leq \frac{d}{r} \leq \frac{e}{s}$. Then $c < 0$. By assumption $\sqrt{\frac{n-2}{3(n+1)}} < a_2 < a_1$ and by Lemma 2.1 i), $\frac{e}{s} \leq \sqrt{\frac{n-2}{3(n+1)}}$. We distinguish some cases for $n \geq 4$:

a) If $\frac{d}{r}, \frac{e}{s} > g(n)$, $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-\frac{c}{p}}{a_2-\frac{c}{p}}\right)^3$, $\left(\frac{a_1-t}{a_2-t}\right) > \max\left(\left(\frac{a_1-\frac{d}{r}}{a_2-\frac{d}{r}}\right), \left(\frac{a_1-\frac{e}{s}}{a_2-\frac{e}{s}}\right)\right)$ and the claim will follow from $\frac{p+1}{3} + r + 1 + s + 1 \le n - 1 = p + r + s$, i.e. $p \ge 4$. This is satisfied: Since $c < 0$, $|c| = a_1 + a_2 + d + e \ge 2\sqrt{\frac{n-2}{3(n+1)}} + (r+s)g(n) = 2(1+r+s)\sqrt{\frac{n-2}{3(n+1)}} - (r+s)\sqrt{\frac{n-1}{2(n+1)}}$ and $\frac{c^2}{p} = 1 - a_1^2 - a_2^2 - r\left(\frac{d}{r}\right)^2 - s\left(\frac{e}{s}\right)^2 \le 1 - \frac{2}{3}\frac{n-2}{n+1} - (r+s)g(n)^2$, implying

$$p \ge \frac{\left(2(1+r+s)\sqrt{\frac{n-2}{3(n+1)}} - (r+s)\sqrt{\frac{n-1}{2(n+1)}}\right)^2}{1 - \frac{2}{3}\frac{n-2}{n+1} - (r+s)g(n)^2} . \tag{3.1}$$

Since $r, s \in \mathbb{N}$, $r + s \ge 2$. For $n = 5$ this yields $p \ge 3.035$, i.e. $p \ge 4$. The right side of (3.1) is increasing in $r + s$ and in n. Thus $p \ge 4$ holds for all $n \ge 5$. For $n = 4$, $p + r + s = 3$ means $p = r = s = 1$. But (3.1) yields $p > 1$. Thus this case is impossible for $n = 4$.

b) If $\frac{d}{r} \le g(n) < \frac{e}{s}$, we have $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-\frac{d}{r}}{a_2-\frac{d}{r}}\right)^2$ and the claim will follow from $\frac{p+1}{3} + \frac{r+1}{2} + s + 1 \le n - 1 = p + r + s$, i.e. $11 \le 4p + 3r$. This requires $p \ge 2$ or $p = 1$ and $r \ge 3$. We check this similarly as above, if $d \ge 0$: Then $|c| \ge a_1 + a_2 + d + e \ge 2\sqrt{\frac{n-2}{3(n+1)}} + s\ g(n)$, $\frac{c^2}{p} \le 1 - \frac{2}{3}\frac{n-2}{n+1} - s\ g(n)^2$. This yields (3.1) with $r + s$ replaced by s. For $s = 1$, $n = 4$, (3.1) yields $p \ge 1.19$, i.e. $p \ge 2$ is satisfied, as needed. For $s > 1$ or $n > 4$, p will be even larger. For $n = 4$, this case is impossible, since $p + r + s = n - 1 = 3$ requires $p = r = s = 1$.

If $d < 0$, we only need for the claim that $\frac{p+1}{3} + \frac{r+1}{3} + s + 1 \le n - 1 = p + r + s$, which means $p + r \ge 3$. We show that $p = r = 1$ is impossible so that $p + r \ge 3$ will follow. If $p = r = 1$, $0 = a_1 + a_2 + c + d + e \ge 2\sqrt{\frac{n-2}{3(n+1)}} + s\ g(n) + c + d \ge 2(1+s)\sqrt{\frac{n-2}{3(n+1)}} - s\sqrt{\frac{n-1}{2(n+1)}} - \sqrt{\frac{2}{3}\frac{n+7}{n+1}} =: \gamma(s, n)$, using $c^2 + d^2 \le 1 - a_1^2 - a_2^2 \le \frac{n+7}{3(n+1)}$, $|c + d| \le \sqrt{\frac{2}{3}\frac{n+7}{n+1}}$. For $s = 1$, $\gamma(s, n) > 0$, if $n \ge 6$. For $n = 5$, we need $4 = p + r + s = 2 + s$, i.e. $s = 2$. However, also $\gamma(2, 5) > 0$. Thus $p + r \ge 3$ is satisfied. The case $n = 4$ needs a separate argument.

c) If $\frac{d}{r}, \frac{e}{s} \le g(n)$, we have that $\left(\frac{a_1-t}{a_2-t}\right) > \max\left(\left(\frac{a_1-\frac{d}{r}}{a_2-\frac{d}{r}}\right)^2, \left(\frac{a_1-\frac{e}{s}}{a_2-\frac{e}{s}}\right)^2\right)$, and we need $\frac{p+1}{3} + \frac{r+1}{2} + \frac{s+1}{2} \le n - 1 = p + r + s$, i.e. $8 \le 4p + 3(r + s)$, which is always satisfied, also for $n = 4$.

d) For $n = 4$, case b) with $d < 0$ was left open. Then $d, e = -\frac{a_1+a_2+c}{2} \pm \frac{1}{2}\sqrt{2 - 3(a_1^2 + a_2^2 + c^2) - 2(a_1 a_2 + a_1 c + a_2 c)}$. Calculation shows

$$\left(\frac{a_1 - d}{a_2 - d}\right)\left(\frac{a_1 - e}{a_2 - e}\right) = 1 + (a_1 - a_2)\frac{2(a_1 + a_2) + c}{(a_1 + a_2 + c)^2 + 2a_2^2 - a_1 c - \frac{1}{2}} ,$$

$$\left(\frac{a_1 - t}{a_2 - t}\right) = 1 + (a_1 - a_2)\frac{1}{a_2 - t} .$$

We will prove that $\left(\frac{a_1-t}{a_2-t}\right) > \left(\frac{a_1-d}{a_2-d}\right)\left(\frac{a_1-e}{a_2-e}\right)$. The claim of Proposition 3.1 will then follow from the inequality $\frac{2}{3}+2<3$ for the exponents. We have to show that $\phi(a_1,a_2) := \frac{(a_1+a_2+c)^2+2a_2^2-a_1c-\frac{1}{2}}{2(a_1+a_2)+c} > a_2-t$. Using $a_1^2+a_2^2 \geq \frac{4}{15}$, $|c| \leq \sqrt{\frac{11}{15}}$, it can be checked that $\frac{\partial\phi}{\partial a_1} = \frac{1+2(a_1^2-a_2^2)+4a_1a_2+2a_1c-c^2}{(2(a_1+a_2)+c)^2} \geq 0$. Therefore $\phi(a_1,a_2) \geq \phi(a_2,a_2) = \frac{(2a_2+c)^2+2a_2^2-a_2c-\frac{1}{2}}{4a_2+c} =: k(c)$. We want to minimize $k(c)$ on the interval $c\in[-1,0]$. We have $k'(c) = \frac{6a_2^2+8a_2c+c^2+\frac{1}{2}}{(4a_2+c)^2} = 0$ for $c_\pm = -4a_2 \pm \frac{1}{2}\sqrt{40a_2^2-2}$. Only c_+ satisfies $c_+ \geq -1$ as needed for $a\in S^4$, and c_+ is the minimum of k. We have $k(c_+) = \sqrt{40a_2^2-2}-5a_2$, hence $\phi(a_1,a_2) \geq \sqrt{40a_2^2-2}-5a_2$. It is easy to see that this is $> a_2 - \sqrt{\frac{2}{15}} > a_2-t$. □

The second auxiliary result concerns the monotonicity of a very special function appearing in the proof of Theorem 1.1 in the situation that $(n-2)$ of the coordinates of a critical point $a\in S^n$ of $A(\cdot,t)$ with fixed $a_1>a_2>t$ coincide.

Proposition 3.3. *Let $n\geq 4$, $\sqrt{\frac{n-2}{3(n+1)}} \leq t < a_2 < a_1$. Let*

$$W = W(a_1,a_2) := \sqrt{(n-1)(1-a_1^2-a_2^2)-(a_1+a_2)^2}\ ,$$

$$F = F(a_1,a_2) := \frac{(a_1-t)^{n-1}}{\left(a_1+\frac{a_1+a_2}{n-1}+\frac{1}{\sqrt{n-2}(n-1)}W\right)^{n-2}\left(a_1+\frac{a_1+a_2}{n-1}-\frac{\sqrt{n-2}}{n-1}W\right)}\ ,$$

$$G = G(a_1,a_2) := \frac{(a_2-t)^{n-1}}{\left(a_2+\frac{a_1+a_2}{n-1}+\frac{1}{\sqrt{n-2}(n-1)}W\right)^{n-2}\left(a_2+\frac{a_1+a_2}{n-1}-\frac{\sqrt{n-2}}{n-1}W\right)}\ ,$$

and $f = f(a_1,a_2) := \frac{F(a_1,a_2)-G(a_1,a_2)}{a_1-a_2}$. Then for fixed a_1, f is strictly increasing in a_2.

By Lemma 2.1 $a_2 < \psi(a_1)$, and this guarantees that W is well-defined. For $a_2=\psi(a_1)$, $W = W(a_1,\psi(a_1)) = 0$. W decreases if a_2 increases. For the proof we need

Lemma 3.4. *Let $n\geq 4$, $\sqrt{\frac{n-2}{3(n+1)}} \leq t < a_2 < a_1$ and $W(a_1,a_2) := \sqrt{(n-1)(1-a_1^2-a_2^2)-(a_1+a_2)^2}$, $c := -\frac{a_1+a_2}{n-1} - \frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)$ and $d := -\frac{a_1+a_2}{n-1}+\frac{\sqrt{n-2}}{n-1}W(a_1,a_2)$. Then :*

(a) $a_2-d \geq 3(a_2-t)+(a_1-a_2)$,
(b) $a_2-c \geq 5(a_2-t)+\frac{a_1-a_2}{n-1}$,
(c) $\frac{a_1-c}{a_2-c} < \frac{3}{2}$.

Proof. (a) Let $\delta := a_2-t$, $\varepsilon := a_1-t$. We claim that $t-d\geq\delta+\varepsilon$. Inserting δ,ε into $t-d$, this is equivalent to $t-d = \frac{n+1}{n-1}t+\frac{\delta+\varepsilon}{n-1}-\frac{\sqrt{n-2}}{n-1}W(a_1,a_2) \geq \delta+\varepsilon$ which means $\sqrt{n-2}\,W(a_1,a_2) \le (n+1)t-(n-2)(\delta+\varepsilon)$. This is equivalent to

$$\begin{aligned}(n-2)&\left[n-1-2(n+1)(t^2+(\delta+\varepsilon)t)-n(\delta^2+\varepsilon^2)-2\delta\varepsilon\right]\\ &\le (n+1)^2t^2-2(n-2)(n+1)(\delta+\varepsilon)t+(n-2)^2(\delta+\varepsilon)^2\ .\end{aligned}$$

The terms with $(\delta+\varepsilon)t$ on both sides cancel, and the inequality is clearly satisfied if $(n-2)(n-1) \le t^2((n+1)^2+2(n-2)(n+1)) = t^2\,3(n+1)(n-1)$, i.e. $t^2 \ge \frac{n-2}{3(n+1)}$, which holds by assumption on t. Hence $t-d \ge a_1+a_2-2t = 2(a_2-t)+(a_1-a_2)$, $a_2-d = a_2-t+t-d \ge 3(a_2-t)+(a_1-a_2)$.

(b) We have $t-c = t+\frac{a_1+a_2}{n-1}+\frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2) \ge t+\frac{a_1+a_2}{n-1} = t+\frac{2}{n-1}a_2+\frac{a_1-a_2}{n-1}$. We claim that $t+\frac{2}{n-1}a_2 \ge 4(a_2-t)$ holds. This is equivalent to $t \ge (\frac{4}{5}-\frac{2}{5}\frac{1}{n-1})a_2$. Since $t \ge \sqrt{\frac{n-2}{3(n+1)}} = \sqrt{\frac{2}{3}\frac{n-2}{n-1}}\sqrt{\frac{n-1}{2(n+1)}} \ge \sqrt{\frac{2}{3}\frac{n-2}{n-1}}\ a_2$, $\sqrt{\frac{2}{3}\frac{n-2}{n-1}} \ge \frac{4}{5}-\frac{2}{5}\frac{1}{n-1}$ will imply the claim, and this is true for all $n \ge 4$. Hence $a_2-c = a_2-t+t-c \ge 5(a_2-t)+\frac{a_1-a_2}{n-1}$.

(c) By Lemma 2.1 $\frac{a_1}{a_2} \le \frac{\psi(a_2)}{a_2} = \sqrt{\frac{n-1}{n}}\sqrt{\frac{1}{a_2^2}-\frac{n+1}{n}}-\frac{1}{n}$. Using $a_2 \ge \sqrt{\frac{n-2}{3(n+1)}}$, we find $y := \frac{a_1}{a_2} \le \frac{n+1}{n}\sqrt{\frac{n-1}{n-2}}\sqrt{2}-\frac{1}{n}$. Since $\frac{a_1+x}{a_2+x}$ is decreasing for $x \ge 0$ and $-c \ge 0$, $-c \ge \frac{a_1+a_2}{n-1}$, we conclude $\frac{a_1-c}{a_2-c} \le \frac{a_1+\frac{a_1+a_2}{n-1}}{a_2+\frac{a_1+a_2}{n-1}} = \frac{ny+1}{n+y}$. We claim that $\frac{ny+1}{n+y} < \frac{3}{2}$. This is equivalent to $y < \frac{3n-2}{2n-3}$. Using the above estimate for y, it suffices to show $\sqrt{\frac{2(n-1)}{n-2}} < \frac{n}{n+1}\left(\frac{3n-2}{2n-3}+\frac{1}{n}\right) = 3\frac{n-1}{2n-3}$, or equivalently, $2(2n-3)^2 < 9(n-1)(n-2)$, which holds for all $n \ge 4$. Hence $\frac{a_1-c}{a_2-c} < \frac{3}{2}$ for all $n \ge 4$. □

Remark. By Lemma 3.4 (a), $a_2+\frac{a_1+a_2}{n-1}-\frac{\sqrt{n-2}}{n-1}W = a_2-d > 0$, so that the denominators of $F(a_1,a_2)$ and $G(a_1,a_2)$ in Proposition 3.3 are positive and $F(a_1,a_2) > 0$, $G(a_1,a_2) > 0$.

Proof of Proposition 3.3. We denote by $'$ the derivative with respect to a_2. Then

$$\begin{aligned}&f'(a_1,a_2)\\ &= \frac{F(a_1,a_2)(1+(a_1-a_2)(\ln F)'(a_1,a_2))-G(a_1,a_2)(1+(a_1-a_2)(\ln G)'(a_1,a_2))}{(a_1-a_2)^2}\ .\end{aligned}$$

We will show that $f'(a_1,a_2) \ge 0$. This is equivalent to

$$\frac{F(a_1,a_2)}{G(a_1,a_2)}\left[1+(a_1-a_2)(\ln F)'(a_1,a_2)\right] \ge 1+(a_1-a_2)(\ln G)'(a_1,a_2)\ .$$

We have $W'(a_1,a_2) = -\frac{na_2+a_1}{W(a_1,a_2)}$. Let $u := a_1+\frac{a_1+a_2}{n-1}$, $v := a_2+\frac{a_1+a_2}{n-1}$. Then

$$(\ln F)'(a_1,a_2) = -\left(\frac{\frac{n-2}{n-1}-\frac{\sqrt{n-2}}{n-1}\frac{na_2+a_1}{W(a_1,a_2)}}{u+\frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)}+\frac{\frac{1}{n-1}+\frac{\sqrt{n-2}}{n-1}\frac{na_2+a_1}{W(a_1,a_2)}}{u-\frac{\sqrt{n-2}}{n-1}W(a_1,a_2)}\right)\ .$$

A tedious calculation with the common denominator

$$h(u) := \left(u+\frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)\right)\left(u-\frac{\sqrt{n-2}}{n-1}W(a_1,a_2)\right)$$

shows that this is equal to $(\ln F)'(a_1,a_2) = -\frac{h(u)-h(v)}{(a_1-a_2)h(u)}$. Therefore

$$1+(a_1-a_2)(\ln F)'(a_1,a_2) = 1-\frac{h(u)-h(v)}{h(u)} = \frac{h(v)}{h(u)} = \left(\frac{a_2-c}{a_1-c}\right)\left(\frac{a_2-d}{a_1-d}\right) ,$$

where $c := -\frac{a_1+a_2}{n-1} - \frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)$, $d := -\frac{a_1+a_2}{n-1} + \frac{\sqrt{n-2}}{n-1}W(a_1,a_2)$. We have $a_1+a_2+(n-2)c+d=0$, $a_1^2+a_2^2+(n-2)c^2+d^2=1$. We get

$$\frac{F(a_1,a_2)}{G(a_1,a_2)}\left[1+(a_1-a_2)(\ln F)'(a_1,a_2)\right] = \left(\frac{a_1-t}{a_2-t}\right)^{n-1}\left(\frac{a_2-c}{a_1-c}\right)^{n-1}\left(\frac{a_2-d}{a_1-d}\right)^2$$

and we have to show that this is $\geq 1+(a_1-a_2)(\ln G)'(a_1,a_2)$. We have

$$(\ln G)'(a_1,a_2) = \frac{n-1}{a_2-t} - \frac{(n-2)\frac{n}{n-1} - \frac{\sqrt{n-2}}{n-1}\frac{na_2+a_1}{W(a_1,a_2)}}{y+\frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)} - \frac{\frac{n}{n-1}+\frac{\sqrt{n-2}}{n-1}\frac{na_2+a_1}{W(a_1,a_2)}}{y-\frac{\sqrt{n-2}}{n-1}W(a_1,a_2)} .$$

Using that $c-d = -\frac{1}{\sqrt{n-2}}W(a_1,a_2)$, $a_2-c = y+\frac{1}{\sqrt{n-2}(n-1)}W(a_1,a_2)$ and $a_2-d = y-\frac{\sqrt{n-2}}{n-1}W(a_1,a_2)$, we find after some calculation

$$(\ln G)'(a_1,a_2) = (n-1)\left(\frac{1}{a_2-t}-\frac{1}{a_2-c}\right) - \frac{2}{a_2-d} .$$

We have to show

$$\text{(3.2)}\qquad \left(\frac{a_1-t}{a_2-t}\right)^{n-1} > \left(\frac{a_1-c}{a_2-c}\right)^{n-1}\left(\frac{a_1-d}{a_2-d}\right)^2 \times$$
$$\times\left(1+(a_1-a_2)\left[(n-1)(\frac{1}{a_2-t}-\frac{1}{a_2-c})-\frac{2}{a_2-d}\right]\right)$$

Let $x := \frac{a_1-a_2}{a_2-t} > 0$, $y := \frac{a_1-a_2}{a_2-c} > 0$ and $z := \frac{a_1-a_2}{a_2-d} > 0$. Then we need

$$\text{(3.3)}\qquad (1+x)^{n-1} > (1+y)^{n-1}(1+z)^2\left(1+(n-1)(x-y)-2z\right) .$$

By Lemma 3.4, $y \leq \frac{x}{5}$ and $z \leq \frac{a_1-a_2}{3(a_2-t)+(a_1-a_2)} = \frac{x}{3+x}$. Now the right side of (3.3) is increasing in z and y, since their derivatives satisfy

$$2(1+z)\left[1+(n-1)(x-y)-2z-(1+z)\right] \geq 2(1+z)\left[\frac{4}{5}(n-1)x-\frac{3x}{3+x}\right] > 0 ,$$

$$(n-1)(1+y)^{n-2}\left[1+(n-1)(x-y)-2z-(1+y)\right]$$
$$\geq (n-1)(1+y)^{n-2}\left[(n-1)\frac{4}{5}x-\frac{2x}{3+x}-\frac{x}{5}\right] > 0 .$$

Therefore (3.3) will be satisfied if for all $x>0$

(3.4)
$$\phi_n(x) := (1+x)^{n-1} - \left(1+\frac{x}{5}\right)^{n-1}\left(1+\frac{x}{3+x}\right)^2\left(1+(n-1)\frac{4}{5}x-\frac{2x}{3+x}\right) > 0$$

holds. We use induction on n to show that $\phi_n(x)>0$ for all $n\geq 5$ and all $0<x\leq\frac{5}{2}$. For $n=5$, expansion shows that $\phi_5(x) = \frac{x^2}{(1+\frac{x}{3})^3}\sum_{j=0}^{6}\alpha_j x^j$, $\alpha_0 = \frac{7}{5}$, $\alpha_j \geq 0$ for

$j = 1, \cdots, 5$, $\alpha_6 = -\frac{64}{84375} < 0$. Since $\alpha_6(\frac{5}{2})^6 = \frac{5}{27} < \frac{1}{5}$, $\phi_5(x) > \frac{6}{5}\frac{x^2}{(1+\frac{x}{3})^3} > 0$. For $n \geq 5$, assume that $\phi_n(x) > 0$ for all $0 < x \leq \frac{5}{2}$. Then

$$\begin{aligned}
&\phi_{n+1}(x) - \phi_n(x) \\
&= x(1+x)^{n-1} - x\left(1+\frac{x}{5}\right)^{n-1}\left(1+\frac{x}{3+x}\right)^2\left(1+\frac{4}{25}nx - \frac{2}{5}\frac{x}{3+x}\right) \\
&> x\phi_n(x) + x\left(1+\frac{x}{5}\right)^{n-1}\left(1+\frac{x}{3+x}\right)^2\left(\frac{16}{25}nx - \frac{4}{3}x\right) > 0 \ .
\end{aligned}$$

For $x \geq \frac{5}{2}$, $\frac{x}{5} \geq \frac{1}{2}$, we use that by Lemma 3.4 (c), $\frac{a_1-c}{a_2-c} = 1+y < \frac{3}{2}$, $y < \frac{1}{2}$. Then (3.3) will be satisfied if

$$\Phi_n(x) := \left(\frac{2}{3}(1+x)\right)^{n-1} - \left(1+\frac{x}{3+x}\right)^2\left(1+(n-1)(x-\frac{1}{2}) - \frac{2x}{3+x}\right) > 0$$

holds. For $n = 5$, $\Phi_5(x) = \frac{\sum_{j=0}^7 \beta_j (x-\frac{5}{2})^j}{(1+\frac{x}{3})^3}$ with $\beta_j > 0$ for all $j = 0, \cdots, 7$, $\beta_0 > 75$. Thus $\Phi_5(x) > 0$ for all $x \geq \frac{5}{2}$. For $n \geq 5$ and $x \geq \frac{5}{2}$,

$$\begin{aligned}
\Phi_{n+1}(x) \geq \frac{7}{3}\left(\frac{2}{3}(1+x)\right)^{n-1} &- \left(1+\frac{x}{3+x}\right)^2\left(1+(n-1)(x-\frac{1}{2}) - \frac{2x}{3+x}\right) \\
&- \left(1+\frac{x}{3+x}\right)^2\left(x-\frac{1}{2}\right) \ .
\end{aligned}$$

Thus $\Phi_n(x) > 0$ implies $\Phi_{n+1}(x) > 0$, since for $x \geq \frac{5}{2}$ and $n \geq 5$ we have $\frac{4}{3}\left(\frac{2}{3}(1+x)\right)^{n-1} > \frac{1}{3}(1+x)^3 > \frac{1}{3}(1+x)^2(x-\frac{1}{2}) > (1+\frac{x}{3})^2(x-\frac{1}{2}) > \left(1+\frac{x}{3+x}\right)^2(x-\frac{1}{2})$. Therefore we have verified (3.3) and (3.2) for all $n \geq 5$.

For $n = 4$, we use that by Lemma 3.4 we actually have a better estimate for y, namely $y \leq \frac{a_1-a_2}{5(a_2-t)+\frac{a_1-a_2}{3}} = \frac{x}{5+\frac{x}{3}}$. Then we need only that for $x > 0$

$$\tilde{\phi_4}(x) = (1+x)^3 - \left(1+\frac{x}{5+\frac{x}{3}}\right)^3\left(1+\frac{x}{3+x}\right)^2\left(1+3x-3\frac{x}{5+\frac{x}{3}} - \frac{2x}{3+x}\right) > 0 \ .$$

Calculation shows that $\tilde{\phi_4}(x) = \frac{x^2}{(1+\frac{x}{3})^3(1+\frac{x}{15})^4}\sum_{j=0}^8 \gamma_j x^j$, $\gamma_j > 0$ for all $j = 0, \cdots, 8$ with $\gamma_0 = \frac{13}{75} > \frac{1}{6}$. Thus $\tilde{\phi_4}(x) > \frac{1}{6}\frac{x^2}{(1+\frac{x}{3})^3(1+\frac{x}{15})^4} > 0$ for all $x > 0$. Hence inequality (3.3) holds for $n = 4$, too. □

4. Proof of Theorem 1.1

Proof. The case $\sqrt{\frac{n-1}{2(n+1)}} < t < \sqrt{\frac{n}{n+1}}$ is covered by Theorem 1.1 of [8]. Hence assume that $\sqrt{\frac{n-2}{3(n+1)}} < t \leq \sqrt{\frac{n-1}{2(n+1)}}$. We may assume $A(a,t) > 0$. As mentioned in the Introduction, the general assumption and $A(a,t) > 0$ implies that $a_1 > t$ holds. By Lemma 2.1 $a_3 \leq \sqrt{\frac{n-2}{3(n+1)}} < t$, $a_2 \leq \sqrt{\frac{n-1}{2(n+1)}}$ and $a_1 \leq \sqrt{\frac{n}{n+1}}$. We have to consider two cases: $a_1 > t \geq a_2, \ldots, a_{n+1}$ and $a_1 > a_2 > t \geq a_3, \ldots, a_{n+1}$, with limiting case $a_1 = a_2$. In the first case, formula (2.1) contains only one non-zero summand, in the second case two non-zero summands. In the second case $a_2 > \sqrt{\frac{n-2}{3(n+1)}}$, and by Lemma 2.1 $a_1 \le \psi(a_2) < \sqrt{\frac{2}{3}}$.

i) Assume first that $a_1 > t \ge a_2, \dots, a_{n+1}$. By continuity, (2.1) is also valid if some or all of the values $a_2, \dots, a_{n+1}$ coincide. To find the maximum of $A(a,t) = \frac{\sqrt{n+1}}{(n-1)!} \frac{(a_1-t)^{n-1}}{\prod_{j=2}^{n+1}(a_1-a_j)}$ in $\Omega := \{(a_1, a_2, \dots, a_{n+1}) \mid a_1 > t \ge a_2, \dots, a_{n+1}\}$ relative to the constraints $\sum_{j=1}^{n+1} a_j = 0$, $\sum_{j=1}^{n+1} a_j^2 = 1$, we first look for the critical points of $A(\cdot, t)$ in the interior of Ω, i.e. when $a_2, \dots, a_{n+1} < t$. We show later in ii) that the maximum of $A(\cdot, t)$ is not attained on the boundary of Ω. The Lagrange function

$$L(a,t) = (n-1)\ln(a_1 - t) - \sum_{j=2}^{n+1} \ln(a_1 - a_j) + \frac{\lambda}{2}\left(\sum_{j=1}^{n+1} a_j^2 - 1\right) + \mu \left(\sum_{j=1}^{n+1} a_j\right)$$

gives the critical points of $A(\cdot, t)$ relative to the above constraints,

$$\frac{\partial L}{\partial a_1} = \frac{n-1}{a_1 - t} - \sum_{j=2}^{n+1} \frac{1}{a_1 - a_j} + \lambda a_1 + \mu = 0 \ ,$$
$$\frac{\partial L}{\partial a_l} = \frac{1}{a_1 - a_l} + \lambda a_l + \mu = 0 \ , \ l = 2, \dots, n+1 \ .$$

As shown in [8], this implies $\mu = -\frac{n-1}{n+1}\frac{1}{a_1-t}$, $\lambda = n - (n-1)\frac{a_1}{a_1-t}$ and that the a_l, $l \ge 2$ are solutions of the quadratic equation $x^2 - px - q = 0$, $p = a_1 - \frac{n-1}{n+1}\frac{1}{nt-a_1}$, $q = \frac{(n+1)t-2a_1}{(n+1)(nt-a_1)}$. Thus there are at most two different values among the a_l's, $l \ge 2$. In fact, as shown in [8], for $t > \frac{2}{\sqrt{n(n+1)}}$, there is only one possible solution x_- since the other one x_+ does not satisfy $x_+ \le t$. In this case, the only critical point of $A(\cdot, t)$ is $a^{(1)} = \sqrt{\frac{n}{n+1}}(1, -\frac{1}{n}, \dots, -\frac{1}{n})$. It is a (local) maximum by Proposition 1.3 of [8], and hence the absolute maximum of $A(\cdot, t)$. The condition $t > \frac{2}{\sqrt{n(n+1)}}$ is satisfied for all $n \ge 5$, since then $\sqrt{\frac{n-2}{3(n+1)}} > \frac{2}{\sqrt{n(n+1)}}$. For $n = 4$, the condition might be violated, and then there might be two different values among (a_2, a_3, a_4, a_5) in the case of a critical point a. Suppose $c := a_2$ appears r times and a_5 s times, $r + s = 4$. Then $a_1 + rc + sd = 0$, $a_1^2 + rc^2 + sd^2 = 1$. If $r = s = 2$, $c = -\frac{1}{4}\left(a_1 + \sqrt{4 - 5a_1^2}\right) < d = -\frac{1}{4}\left(a_1 - \sqrt{4 - 5a_1^2}\right)$. Inserting these values into the equation $\frac{1}{a_1-c} + \lambda c + \mu = 0$ with the above values for $\lambda = \lambda(t)$ and $\mu = \mu(t)$ gives a formula for t, namely $t = \frac{5a_1^2+2}{20a_1}$, $a_1 = 2t + \frac{1}{5}\sqrt{100t^2 - 10}$, which is $> \sqrt{\frac{4}{5}}$ for $t \ge 0.34$. However, $0.365 < \sqrt{\frac{2}{15}} < t$ by our assumption on t for $n = 4$. Thus the case $r = s = 2$ is impossible. Similarly, also $r = 1, s = 3$ is impossible. This leaves $r = 3, s = 1$, $c = -\frac{1}{4}\left(a_1 + \frac{1}{3}\sqrt{12 - 15a_1^2}\right) < d = -\frac{1}{4}\left(a_1 - \sqrt{12 - 15a_1^2}\right)$. In this case $t = \frac{a_1(80a_1^2+17)+3\sqrt{12-15a_1^2}}{40(8a_1^2-1)} =: \phi(a_1)$. The function ϕ is strictly decreasing in $[\sqrt{\frac{2}{15}}, \sqrt{\frac{4}{5}}]$. One has $\phi(a_1) > \sqrt{\frac{3}{10}}$ for $a_1 < \sqrt{\frac{3}{10}}$, hence $\phi(a_1) = t$ has no solution $a_1 \in [\sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}})$. For $\sqrt{\frac{3}{10}} \le a_1 \le \sqrt{\frac{4}{5}}$, one has $\phi(\sqrt{\frac{3}{10}}) = \sqrt{\frac{3}{10}}$, and ϕ is bijective $\phi : [\sqrt{\frac{3}{10}}, \sqrt{\frac{4}{5}}] \to [\frac{3}{4}\frac{1}{\sqrt{5}}, \sqrt{\frac{3}{10}}]$. The equation $t = \phi(a_1) = d$ has the solutions $\bar{a} = \sqrt{\frac{3}{10}} \simeq 0.548$ and $\tilde{a} = \frac{13}{2}\frac{1}{\sqrt{95}} \simeq 0.667$. For $\bar{a} < a_1 < \tilde{a}$ we have $d > t$ which is impossible. This leaves the possibility that $a_1 \in \left[\frac{13}{2}\frac{1}{\sqrt{95}}, \sqrt{\frac{4}{5}}\right]$ with $c < d \le t$. In this case, $0.335 \simeq \frac{3}{4}\frac{1}{\sqrt{5}} < t < \phi(\tilde{a}) = \frac{4}{\sqrt{95}} \simeq 0.411$. The value $\sqrt{\frac{2}{15}} \simeq 0.365$ is in this range. For $a = (a_1, c, c, c, d) \in S^4$ with $t \in [\sqrt{\frac{2}{15}}, \frac{4}{\sqrt{95}}]$, we have $A(a,t) = A(a, \phi(a_1)) = \frac{\sqrt{5}}{6}\frac{27}{1000}\frac{9a_1(260a_1^4 - 128a_1^2 + 17) - 3(20a_1^4 + 8a_1^2 - 3)\sqrt{12 - 15a_1^2}}{(8a_1^2 - 1)^3}$. Comparing this with

$$A(a^{(1)}, t) = A(a^{(1)}, \phi(a_1)) = \frac{\sqrt{5}}{6}\frac{16}{25}\left(\sqrt{\frac{4}{5}} - \frac{a_1(80a_1^2 + 17) + 3\sqrt{12 - 15a_1^2}}{40(8a_1^2 - 1)}\right)^3,$$

(tedious) estimates show that $A(a, \phi(a_1)) < A(a^{(1)}, \phi(a_1))$ for all $a_1 \in \left[\frac{13}{2}\frac{1}{\sqrt{95}}, \sqrt{\frac{4}{5}}\right)$, with equality for $a_1 = \sqrt{\frac{4}{5}}$. Thus $A(a,t) < A(a^{(1)}, t)$ for all $\sqrt{\frac{2}{15}} \le t < \frac{4}{\sqrt{95}}$. In fact, for $t < \tilde{t} \simeq 0.3877$, $A(a^{(1)}, t) < A(a^{(2)}, t)$, with $\tilde{t} \in [\sqrt{\frac{2}{15}}, \frac{4}{\sqrt{95}}]$.

ii) The boundary of Ω in i) is $\Lambda = \{(a_1, a_3, \dots, a_{n+1}) \mid a_1 > t = a_2 > a_3, \dots, a_{n+1}\}$. Note that by Lemma 2.1 i) we have $a_3, \dots, a_{n+1} < \sqrt{\frac{n-2}{3(n+1)}} < t$. For $t = a_2$ we have $A(a,t) = \frac{\sqrt{n+1}}{(n-1)!}\frac{(a_1 - t)^{n-2}}{\prod_{j=3}^{n+1}(a_1 - a_j)}$ with constraints $\sum_{j=1, j\neq 2}^{n+1} a_j = -t$, $\sum_{j=1, j\neq 2}^{n+1} a_j^2 = 1 - t^2$. The corresponding Lagrange function yields the necessary conditions for critical points of $A(\cdot, t)$ in Λ relative to the constraints

$$\frac{\partial L}{\partial a_1} = \frac{n-2}{a_1 - t} - \sum_{j=3}^{n+1}\frac{1}{a_1 - a_j} + \lambda a_1 + \mu = 0 ,$$

$$\frac{\partial L}{\partial a_l} = \frac{1}{a_1 - a_l} + \lambda a_l + \mu = 0 , \ l = 3, \dots, n+1 , \tag{4.1}$$

implying for $l, k \in \{3, \dots, n+1\}$

$$\frac{a_l - a_k}{(a_1 - a_l)(a_1 - a_k)} = \frac{1}{a_1 - a_l} - \frac{1}{a_1 - a_k} = \lambda(a_k - a_l) .$$

If there are $l \neq k$ with $a_l \neq a_k$, $\lambda = -\frac{1}{(a_1 - a_l)(a_1 - a_k)}$ and $\mu = -\frac{a_1 - a_l - a_k}{(a_1 - a_l)(a_1 - a_k)}$. Therefore there are at most two different values among the a_l, $l \in \{3, \dots, n+1\}$. We claim that there is only one value. Suppose to the contrary that $c := a_l \neq a_k =: d$ satisfy (4.1) with multiplicities p and q, i.e. $p + q = n - 1$, $a_1 + pc + qd = -t$, $a_1^2 + pc^2 + qd^2 = 1 - t^2$. Inserting the values of λ and μ into the first equation of (4.1) yields

$$\frac{n-2}{a_1 - t} - \frac{p+1}{a_1 - c} - \frac{q+1}{a_1 - d} = 0 ,$$

or equivalently

(4.2) $F := (n-2)(a_1-c)(a_1-d) - (a-t)[(p+1)(a_1-d)+(q+1)(a_1-d)] = 0$.

However, we will show that $F > 0$ holds. The quadratic equations for c and d imply with $W := (n-1)(1-a_1^2-t^2)-(a_1+t)^2$, assuming that $c > d$,

$$c = -\frac{a_1+t}{n-1} + \frac{1}{n-1}\sqrt{\frac{q}{p}}\sqrt{W} \ , \ d = -\frac{a_1+t}{n-1} - \frac{1}{n-1}\sqrt{\frac{p}{q}}\sqrt{W} \ ,$$

$$a_1 - c = a_1 + \frac{a_1+t}{n-1} - \frac{1}{n-1}\sqrt{\frac{q}{p}}\sqrt{W} \ , a_1 - d = a_1 + \frac{a_1+t}{n-1} + \frac{1}{n-1}\sqrt{\frac{p}{q}}\sqrt{W} \ .$$

Inserting these values into (4.2) and using $\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}} = \frac{p-q}{\sqrt{pq}}$, $(p+1)\sqrt{\frac{p}{q}} - (q+1)\sqrt{\frac{q}{p}} = n\frac{p-q}{\sqrt{pq}}$, calculation yields

$$\begin{aligned}(n-1)^2 F = &\big((n+1)[(2n-3)t^2 + a_1((n^2-3)t - na_1)] - (n-1)(n-2)\big) \\ &+ \frac{p-q}{\sqrt{pq}}((n^2-2)t - na_1)\sqrt{W} \ .\end{aligned}$$

Since $a_1 \leq \psi(t) = \sqrt{\frac{n-1}{n}}\sqrt{1-\frac{n+1}{n}t^2} - \frac{t}{n} < 1.916t < 2t$ for all $n \geq 4$, $(n^2 - 3)t - na_1 > 0$. Further, $a_1((n^2-3)t - na_1)$ is strictly increasing as a function of $a_1 \in [t, \psi(t)]$ for all $n \geq 5$, and for $n = 4$ attains its strict minimum at $a_1 = t$. For $a_1 > t \geq \sqrt{\frac{n-2}{3(n+1)}}$, $W < \frac{n+1}{3}$. To show $F > 0$, the worst case is $p = 1$, $q = n-2$, since then $\frac{p-q}{\sqrt{pq}} = -\frac{n-3}{\sqrt{n-2}}$ attains its (negative) minimum. Replacing in this case first W by $\frac{n+1}{3}$ and then a_1 by t, we conclude

$$(n-1)^2 F > (n-2)\left([(n+3)(n+1)t^2 - (n-1)] - (n-3)(n+1)\sqrt{\frac{n+1}{3(n-2)}}t\right) \ .$$

The right side is increasing in t, its derivative being positive for $t \geq \sqrt{\frac{n-2}{3(n+1)}} =: t_0$. Note that t_0 depends on n, but for simplicity of notation we will omit the index n. Replacing t by $\sqrt{\frac{n-2}{3(n+1)}}$, the right side becomes zero and we conclude that $F > 0$.

Therefore all a_l, $l \geq 3$ satisfying (4.1) have to coincide, $a_3 = \cdots a_{n+1} = -\frac{a_1+t}{n-1}$, $W = 0$, $a_1 = \psi(t)$ and for $a = (a_1, t, a_3, \ldots, a_3)$

$$\frac{(n-1)!}{\sqrt{n+1}} A(a,t) = \frac{(a_1-t)^{n-2}}{(a_1+\frac{a_1+t}{n-1})^{n-1}} = \left(\frac{n-1}{n}\right)^{n-1} \frac{(1-\frac{t}{\psi(t)})^{n-2}}{\psi(t)(1+\frac{1}{n}\frac{t}{\psi(t)})^{n-1}} \ .$$

For $n \geq 4$, $\psi(t) \geq 0.6992$ and $\psi(t)(1+\frac{1}{n}\frac{t}{\psi(t)})^{n-1} \geq (1+\frac{1}{1.916n})^{n-1} > 1.01 > 1$, $n \geq 4$. Since $t \geq t_0$, $\psi(t) \leq \psi(t_0)$, implying

$$\frac{(n-1)!}{\sqrt{n+1}} A(a,t) < \left(\frac{n-1}{n}\right)^{n-1} \left(1 - \frac{t}{\psi(t_0)}\right)^{n-2} \ .$$

We compare this with

$$\frac{(n-1)!}{\sqrt{n+1}} A(a^{(1)},t) = \left(\frac{n}{n+1}\right)^{n-\frac{1}{2}} \left(1-\sqrt{\frac{n+1}{n}}t\right)^{n-1} ,$$

and want to show that $A(a,t) < A(a^{(1)},t)$ for all $t \in [\sqrt{\frac{n-2}{3(n+1)}}, \sqrt{\frac{n-1}{2(n+1)}}]$ and $n \geq 5$ and for $t \in [0.37, \sqrt{\frac{3}{10}}]$ for $n = 4$. The quotient $h_n(t) = \frac{(1-\frac{t}{\psi(t_0)})^{n-2}}{(1-\sqrt{\frac{n+1}{n}}t)^{n-1}}$ is decreasing in $[\sqrt{\frac{n-2}{3(n+1)}}, \sqrt{\frac{n-1}{2(n+1)}}]$, since

$$(\ln h_n)'(t) = \frac{-(n-2)(\sqrt{\frac{n+1}{n}} - t) + (n-1)(\psi(t_0)-t)}{(\sqrt{\frac{n+1}{n}} - t)(\psi(t_0)-t)} < 0 .$$

The last inequality holds since for $\phi(t) = -(n-2)(\sqrt{\frac{n+1}{n}} - t) + (n-1)(\psi(t_0)-t)$ we have $\phi'(t) = -1 < 0$, $\phi(t_0) \leq -\frac{1}{20}$, $n \geq 4$. Therefore with $\sqrt{\frac{n+1}{n}}t_0 = \sqrt{\frac{n-2}{3n}}$

$$\frac{A(a,t)}{A(a^{(1)},t)} \leq \frac{A(a,t_0)}{A(a^{(1)},t_0)} < \left(\frac{n^2-1}{n^2}\right)^{n-1} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} \frac{(1-\frac{t_0}{\psi(t_0)})^{n-2}}{(1-\sqrt{\frac{n-2}{3n}})^{n-1}} .$$

We have $\frac{\psi(t_0)}{t_0} = \frac{n+1}{n}\sqrt{2\frac{n-1}{n-2}} - \frac{1}{n} < \frac{3n-2}{2n-3}$, the latter inequality being equivalent to $\sqrt{2\frac{n-1}{n-2}} < 3\frac{n-1}{3n-2}$, $2(2n-3)^2 < 9(n-1)(n-2)$, which is true for all $n \geq 4$. Hence

$$\frac{A(a,t)}{A(a^{(1)},t)} < \left(\frac{n^2-1}{n^2}\right)^{n-1} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} \frac{(1-\frac{2n-3}{3n-2})^{n-2}}{(1-\sqrt{\frac{n-2}{3n}})^{n-1}} ,$$

which is < 1 for all $n \geq 5$, the right side being decreasing in n, as can be seen by taking the logarithmic derivative with respect to n: The base in the numerator tends to $\frac{1}{3}$, the one in the denominator to $1 - \frac{1}{\sqrt{3}} > \frac{1}{3}$ for $n \to \infty$. For $n = 4$, $\frac{A(a,t)}{A(a^{(1)},t)} < 1$ for all $t \geq 0.37$. For $t \in [\sqrt{\frac{2}{15}}, 0.37]$ with $\sqrt{\frac{2}{15}} \simeq 0.3651$ we have $A(a,t) < A(a^{(2)},t)$, since the formulas for $A(a,t)$ and $A(a^{(2)},t)$ show for these t and $n = 4$

$$A(a,t) \leq A(a, \sqrt{\frac{2}{15}}) = \frac{3}{16}\sqrt{2} - \frac{3}{32}\sqrt{6} < 0.0355 < 0.0374 < A(a^{(2)}, 0.37) \leq A(a^{(2)},t) .$$

Therefore the maximum of $A(\cdot,t)$ is not attained on the boundary of Ω in i).

Actually, for $\sqrt{\frac{2}{15}} \leq t < 0.3668$ we have $A(a^{(1)},t) < A(a,t) < A^{(2)},t)$, when $n = 4$.

iii) Assume next that $a_1 > a_2 > t > \sqrt{\frac{n-2}{3(n+1)}} \geq a_3, \dots, a_{n+1}$. Then by Proposition 2.2

$$\frac{(n-1)!}{\sqrt{n+1}} A(a,t) = \frac{1}{a_1-a_2} \left(\frac{(a_1-t)^{n-1}}{\prod_{j=3}^{n+1}(a_1-a_j)} - \frac{(a_2-t)^{n-1}}{\prod_{j=3}^{n+1}(a_2-a_j)} \right) =: \frac{B(a,t)-C(a,t)}{a_1-a_2} .$$

By continuity, this formula also holds if some or all of the a_j, $j \geq 3$ coincide. Fix $a_1 > a_2 > t$ and consider $(a_1 - a_2)A(a,t)$ as a function of $(a_3, \dots, a_{n+1})$ with constraints $\sum_{j=3}^{n+1} a_j = -(a_1 + a_2)$, $\sum_{j=3}^{n+1} a_j^2 = 1 - (a_1^2 + a_2^2)$. The Lagrange function $L(a, \lambda, \mu) := B(a,t) - C(a,t) + \frac{\lambda}{2}(\sum_{j=3}^{n+1} a_j^2 - 1 + (a_1^2 + a_2^2)) + \mu(\sum_{j=3}^{n+1} a_j + (a_1 + a_2))$ yields the critical points of A relative to the constraints via

$$\frac{\partial L}{\partial a_l} = \frac{B(a,t)}{a_1 - a_l} - \frac{C(a,t)}{a_2 - a_l} + \lambda a_l + \mu = 0 \ , \ l = 3, \dots, n+1 \ .$$

This implies that all coordinates a_l satisfy the cubic equation in $x = a_l$

$$B(a,t)(a_2 - x) - C(a,t)(a_1 - x) + \lambda x(a_1 - x)(a_2 - x) + \mu(a_1 - x)(a_2 - x) = 0 \ .$$

Therefore there are at most three different values among the coordinates a_l, $l \geq 3$ of a critical point $a \in S^n$. Suppose these occur with multiplicities p, r, s, $p + r + s = n - 1$, and call them $(x = a_l =)$ $\frac{c}{p}$, $\frac{d}{r}$, $\frac{e}{s}$. Then $a_1 + a_2 + c + d + e = 0$, $a_1^2 + a_2^2 + \frac{c^2}{p} + \frac{d^2}{r} + \frac{e^2}{s} = 1$ and

$$\begin{aligned} &\frac{(n-1)!}{\sqrt{n+1}} A(a,t) \\ &= \frac{1}{a_1 - a_2} \left(\frac{(a_1 - t)^{n-1}}{(a_1 - \frac{c}{p})^p (a_1 - \frac{d}{r})^r (a_1 - \frac{e}{s})^s} - \frac{(a_2 - t)^{n-1}}{(a_2 - \frac{c}{p})^p (a_2 - \frac{d}{r})^r (a_2 - \frac{e}{s})^s} \right) \\ &=: \frac{B - C}{a_1 - a_2} \ . \end{aligned}$$

Suppose that $\frac{c}{p} < \frac{d}{r} < \frac{e}{s}$. Then $c < 0$ and by Lemma 2.1 $a_1 \leq \sqrt{\frac{n}{n+1}}$, $a_2 \leq \sqrt{\frac{n-1}{2(n+1)}}$. We perturb the values c, d, e by $\delta > \varepsilon := \theta\delta > 0$, $\theta := \frac{\frac{d}{r} - \frac{c}{p}}{\frac{e}{s} - \frac{c}{p}} \in (0,1)$ in the following way: $c' = c + (\delta - \varepsilon)$, $d' = d - \delta$, $e' = e + \varepsilon$. Then $a_1 + a_2 + c' + d' + e' = 0$, $a_1^2 + a_2^2 + \frac{c'^2}{p} + \frac{d'^2}{r} + \frac{e'^2}{s} = 1 + O(\delta^2)$, since $\frac{c}{p}(\delta - \varepsilon) - \frac{d}{r}\delta + \frac{e}{s}\varepsilon = 0$. These values are independent of a_1. Calculation shows that

$$g_{a_1}(\delta) := \frac{1}{(a_1 - \frac{c}{p} - (1-\theta)\frac{\delta}{p})^p} \frac{1}{(a_1 - \frac{d}{r} + \frac{\delta}{r})^r} \frac{1}{(a_1 - \frac{e}{s} - \theta\frac{\delta}{s})^s}$$

satisfies

$$g_{a_1}(\delta) = g_{a_1}(0) + \frac{(\frac{e}{s} - \frac{d}{r})(\frac{d}{r} - \frac{c}{p})}{(a_1 - \frac{c}{p})^{p+1}(a_1 - \frac{d}{r})^{r+1}(a_1 - \frac{e}{s})^{s+1}} \delta + O(\delta^2) \ ,$$

where the terms in the numerator are positive. (The value of ε could be modified $\varepsilon = \theta\delta + O(\delta^2)$ to guarantee that $a_1^2 + a_2^2 + \frac{c'^2}{p} + \frac{d'^2}{r} + \frac{e'^2}{s} = 1$ holds precisely. The additional second order perturbation yields the same result for $g_{a_1}(\delta)$.) Let $B(\delta) := (a_1 - t)^{n-1} g_{a_1}(\delta)$ and $C(\delta) := (a_2 - t)^{n-1} g_{a_2}(\delta)$. We claim that the first order perturbation of B, $B(\delta) - B(0)$, is bigger than the one of C, $C(\delta) - C(0)$, so that $B(\delta) - C(\delta) > B(0) - C(0)$, as $\delta \to 0$ (up to $O(\delta^2)$). This is equivalent to the inequality $(a_1 - t)^{n-1} g_{a_1}(\delta) > (a_2 - t)^{n-1} g_{a_2}(\delta)$ for small δ, i.e.

$$\left(\frac{a_1 - t}{a_2 - t}\right)^{n-1} > \left(\frac{a_1 - \frac{c}{p}}{a_2 - \frac{c}{p}}\right)^{p+1} \left(\frac{a_1 - \frac{d}{r}}{a_2 - \frac{d}{r}}\right)^{r+1} \left(\frac{a_1 - \frac{e}{s}}{a_2 - \frac{e}{s}}\right)^{s+1} ,$$

$p + r + s = n - 1$. By Proposition 3.1 this inequality is true. Therefore, fixing $a_1 > a_2 > t$, $A(a, t)$ increases if c is replaced by $c + \delta_1$, d by $d - \delta_2$ and e by $e + \delta_3$ for small $\delta_1, \delta_2, \delta_3 > 0$ depending on each other. This holds independently of t. Therefore the maximum of $A(a, t)$ relative to fixed $a_1 > a_2 > t$ occurs for $\frac{c}{p} = \frac{d}{r} < \frac{e}{s}$. Further, we claim that in the maximum case, the multiplicity of $\frac{e}{s}$ should be $s = 1$. If $s \geq 2$, the vector a would have p coordinates $\frac{c}{p}$ and s coordinates $\frac{e}{s}$ with $\frac{c}{p} < \frac{e}{s}$, $p+s = n-1$, $c+e = -(a_1+a_2)$, $\frac{c^2}{p} + \frac{e^2}{s} = 1 - a_1^2 - a_2^2$. Define a perturbed vector a' having p coordinates $\frac{c'}{p} = \frac{c+\varepsilon(\delta)}{p}$, $s-1$ coordinates $\frac{d'}{s-1} = \frac{e}{s} - \frac{\delta}{s-1}$ and 1 coordinate $e' = \frac{e}{s} + \delta - \varepsilon(\delta)$, where $\varepsilon(\delta) = \frac{1}{2} \frac{s}{s-1} \frac{\delta^2}{\frac{e}{s} - \frac{c}{p}}$ for small $\delta > 0$, $0 < \varepsilon(\delta) < \delta$. Then $c'+d'+e' = c+e = -(a_1+a_2)$ and $\frac{c'^2}{p} + \frac{d'^2}{s-1} + e'^2 = \frac{c^2}{p} + \frac{e^2}{s} + O(\delta^3) = 1 - a_1^2 - a_2^2 + O(\delta^3)$. Let

$$g_{a_1}(\delta) := \frac{1}{(a_1 - \frac{c}{p} - \frac{\varepsilon(\delta)}{p})^p} \frac{1}{(a_1 - \frac{e}{s} + \frac{\delta}{s-1})^{s-1}} \frac{1}{(a_1 - \frac{e}{s} - \delta + \varepsilon(\delta))^s} .$$

Calculation shows that

$$g_{a_1}(\delta) - g_{a_1}(0) = \frac{1}{2} \frac{\frac{e}{s} - \frac{c}{p}}{(a_1 - \frac{c}{p})^{p+1} (a_1 - \frac{e}{s})^{s+2}} + O(\delta^3) ,$$

with $\frac{e}{s} - \frac{c}{p} > 0$ being independent of a_1. We remark that if $\varepsilon(\delta)$ is chosen as the solution of the quadratic equation implied by requiring that a' satisfies the constraints exactly, and not only up to $O(\delta^3)$, the same is true. The given value $\varepsilon(\delta)$ is an $O(\delta^2)$-approximation of this solution. Again we have $B(\delta) - C(\delta) > B(0) - C(0)$, i.e. $A(a', t) > A(a, t)$, since

$$\left(\frac{a_1 - t}{a_2 - t}\right)^{n-1} > \left(\frac{a_1 - \frac{c}{p}}{a_2 - \frac{c}{p}}\right)^{p+1} \left(\frac{a_1 - \frac{e}{s}}{a_2 - \frac{e}{s}}\right)^{s+2} .$$

The last inequality follows from Proposition 3.1 with the choice $r = s - 1$, $s = 1$, $c + (s-1)\frac{e}{s} + \frac{e}{s} = -(a_1 + a_2)$ and $p(\frac{c}{p})^2 + (s-1)(\frac{e}{s})^2 + (\frac{e}{s})^2 = \frac{c^2}{p} + \frac{e^2}{s} = 1 - a_1^2 - a_2^2$. The vector a' again has three different coordinates, the largest one of multiplicity 1. Applying the first case then shows that in the maximum case the two smaller coordinates should coincide.

For e of multiplicity 1 and $\frac{c}{p}$ of multiplicity $n - 2$ we have

$$\frac{(n-1)!}{\sqrt{n+1}} A(a, t) = \frac{1}{a_1 - a_2} \left(\frac{(a_1 - t)^{n-1}}{(a_1 - \frac{c}{n-2})^{n-2} (a_1 - e)} - \frac{(a_2 - t)^{n-1}}{(a_2 - \frac{c}{n-2})^{n-2} (a_2 - e)} \right) ,$$

with $a_1 + a_2 + c + e = 0$, $a_1^2 + a_2^2 + \frac{c^2}{n-2} + e^2 = 1$. This yields $c = -\frac{n-2}{n-1}(a_1 + a_2) - \frac{\sqrt{n-2}}{n-1} W$, $e = -\frac{1}{n-1}(a_1 + a_2) + \frac{\sqrt{n-2}}{n-1} W$, where $W = \sqrt{(n-1)(1 - a_1^2 - a_2^2) - (a_1 + a_2)^2}$, with $\frac{c}{n-2} < e$, $c < 0$. The square root W is well-defined, since by Lemma 2.1

$a_2 \le \psi(a_1) := \sqrt{\frac{n-1}{n}}\sqrt{1-\frac{n+1}{n}a_1^2} - \frac{a_1}{n}$ or equivalently $a_1 \le \psi(a_2)$. In fact, ψ is strictly decreasing and $\psi : [\sqrt{\frac{n-2}{3(n+1)}}, \psi(\sqrt{\frac{n-2}{3(n+1)}})] \to [\sqrt{\frac{n-2}{3(n+1)}}, \psi(\sqrt{\frac{n-2}{3(n+1)}})]$ is bijective with $\psi^{-1} = \psi$ and $\psi(\sqrt{\frac{n-2}{3(n+1)}}) = \sqrt{\frac{2}{3}}\frac{\sqrt{n^2-1}}{n} - \frac{1}{n}\sqrt{\frac{n-2}{3(n+1)}} < \sqrt{\frac{2}{3}}$. Further, $\psi(\sqrt{\frac{n-1}{2(n+1)}}) = \sqrt{\frac{n-1}{2(n+1)}}$. Inserting the values of c and e, we have that

(4.3)
$$\begin{aligned}\frac{(n-1)!}{\sqrt{n+1}}A(a,t) &= \frac{1}{a_1-a_2}\Big(\frac{(a_1-t)^{n-1}}{(a_1+\frac{a_1+a_2}{n-1}+\frac{1}{\sqrt{n-2}(n-1)}W)^{n-2}(a_1+\frac{a_1+a_2}{n-1}-\frac{\sqrt{n-2}}{n-1}W)} \\ &\qquad - \frac{(a_2-t)^{n-1}}{(a_2+\frac{a_1+a_2}{n-1}+\frac{1}{\sqrt{n-2}(n-1)}W)^{n-2}(a_2+\frac{a_1+a_2}{n-1}-\frac{\sqrt{n-2}}{n-1}W)}\Big) \\ &=: \frac{1}{a_1-a_2}\left(B(a_1,a_2)-C(a_1,a_2)\right) .\end{aligned}$$

By Proposition 3.3 $\frac{B(a_1,a_2)-C(a_1,a_2)}{a_1-a_2}$ is increasing in a_2. Thus the maximum of $A(a,t)$ occurs either for $a_2 \to a_1$, if $a_1 \le \sqrt{\frac{n-1}{2(n+1)}}$ or for $a_2 = \psi(a_1)$, if $a_1 > \sqrt{\frac{n-1}{2(n+1)}}$. Recall here that $\sqrt{\frac{n-2}{3(n+1)}} < a_2 \le \sqrt{\frac{n-1}{2(n+1)}}$, so that in the latter situation $a_2 = a_1$ is impossible.

iv) We first consider the subcase of iii) that $t < a_1 \le \sqrt{\frac{n-1}{2(n+1)}}$ and $\sqrt{\frac{n-2}{3(n+1)}} < a_2$. Then the limit $a_2 \to a_1$ can be taken in (4.3) using l'Hospital's rule, yielding with $V(n,a_1) := \sqrt{n-1-2(n+1)a_1^2}$ and $\tilde{a} = (a_1, a_1, c(a_1), \cdots, c(a_1), d(a_1))$, $c(a_1) = -\frac{2a_1}{n-1} - \frac{V(n,a_1)}{\sqrt{n-2}(n+1)}$ and $d(a_1) = -\frac{2a_1}{n-1} + \frac{\sqrt{n-2}V(n,a_1)}{n+1}$ that

(4.4)
$$\begin{aligned}&(\frac{n+1}{n-1})^n\frac{(n-1)!}{\sqrt{n+1}}A(\tilde{a},t) = \\ &= \frac{(a_1-t)^{n-2}}{\left(a_1+\frac{1}{\sqrt{n-2}(n+1)}V(n,a_1)\right)^{n-1}\left(a_1-\frac{\sqrt{n-2}}{n+1}V(n,a_1)\right)^2}M(n,t,a_1) =: f(n,t,a_1) ,\end{aligned}$$

where $M(n,a_1) := 4a_1^2 + (n-1)a_1t - \frac{n-1}{n+1} - \frac{n-3}{\sqrt{n-2}(n+1)}[2a_1 + (n-1)t]V(n,a_1)$. We claim that $f(n,t,a_1)$ is strictly increasing as a function of a_1. Let

$$\begin{aligned}f_1(n,t,a_1) &:= \frac{(a_1-t)^{n-3}}{\left(a_1+\frac{1}{\sqrt{n-2}(n+1)}V(n,a_1)\right)^{n-2}\left(a_1-\frac{\sqrt{n-2}}{n+1}V(n,a_1)\right)} \\ f_2(n,t,a_1) &:= \frac{(a_1-t)}{\left(a_1+\frac{1}{\sqrt{n-2}(n+1)}V(n,a_1)\right)\left(a_1-\frac{\sqrt{n-2}}{n+1}V(n,a_1)\right)}M(n,t,a_1) .\end{aligned}$$

Then $f(n,t,a_1) = f_1(n,t,a_1)f_2(n,t,a_2)$. We will show that for $n \ge 5$ the functions f_1 and f_2 are both increasing in a_1. For $n = 4$ we have to multiply f_1 by $\sqrt{a_1-t}$ and divide f_2 by $\sqrt{a_1-t}$ to get the same result. Since $f_1, f_2 > 0$, it suffices to prove that $(\ln f_i)'(n,t,a_1) > 0$ for $a_1 > t$ and $i = 1,2$, where $'$ denotes the derivative with respect to a_1. As for f_1 we have, using $V'(n,a_1) = -\frac{2(n+1)a_1}{V(n,a_1)}$,

$$\begin{aligned}(\ln f_1)'(n,t,a_1) &= \frac{n-3}{a_1-t} - \frac{n-2-\frac{2a_1\sqrt{n-2}}{V(n,a_1)}}{a_1+\frac{V(n,a_1)}{\sqrt{n-2}(n+1)}} - \frac{1+\frac{2a_1\sqrt{n-2}}{V(n,a_1)}}{a_1-\frac{\sqrt{n-2}V(n,a_1)}{n+1}}\\ &= \frac{n-3}{a_1-t} - \frac{(n-1)A}{a_1A-\frac{n-1}{n+1}}\ ,\ A := (n+3)a_1 - \frac{n-3}{\sqrt{n-2}}V(n,a_1)\ .\end{aligned}$$

This is positive if and only if

$$1 < \frac{n+1}{n-3}\left(t-\frac{2a_1}{n-1}\right)A = \frac{n+1}{n-3}\left(t-\frac{2a_1}{n-1}\right)\left((n+3)a_1-\frac{n-3}{\sqrt{n-2}}V(n,a_1)\right)\ .$$

Assume first that $n \geq 5$ holds. For the last inequality it suffices to verify

$$1 < \frac{n+1}{n-3}\left(t-\frac{2a_1}{n-1}\right)\left((n+3)a_1-\frac{n-3}{\sqrt{n-2}}V(n,a_1)\right) =: g_n(a_1)$$

for all $a_1 \in (\sqrt{\frac{n-2}{3(n+1)}}, \sqrt{\frac{n-1}{2(n+1)}}] =: I_n$, since in this subcase $a_1 \leq \sqrt{\frac{n-1}{2(n+1)}}$ is assumed. We have $g_n(\sqrt{\frac{n-2}{3(n+1)}}) = 1$ and we will show for all $n \geq 5$ that $g_n'(a_1) > 0$ for all $a_1 \in I_n$. We find

$$\begin{aligned}\frac{g_n'(a_1)}{n+3} &= \left(\sqrt{\frac{n-2}{3(n+1)}} - \frac{4a_1}{n-1}\right)\\ &\quad + \frac{2(n-3)}{(n+3)\sqrt{n-2}}\left[\frac{(n+1)a_1}{V(n,a_1)}\left(\sqrt{\frac{n-2}{3(n+1)}}-\frac{2a_1}{n-1}\right)+\frac{V(n,a_1)}{n-1}\right]\ . \qquad (4.5)\end{aligned}$$

Since $\sqrt{\frac{n-2}{3(n+1)}} - \frac{4a_1}{n-1} \geq \sqrt{\frac{n-2}{3(n+1)}} - 2\sqrt{\frac{2}{n^2-1}} =: k(n) > 0$ if and only if $(n-1)(n-2) > 24$, i.e. $n \geq 7$, $g_n'(a_1) > 0$ holds for all $n \geq 7$, since the bracket $[\cdot]$ is always positive. For $n = 5,6$, $k(5) = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \simeq -0.169$, $k(6) = 2(\frac{1}{\sqrt{21}} - \sqrt{\frac{2}{35}}) \simeq -0.0417$. For $n = 5,6$ we have to estimate $[\cdot]$ from below. Easy estimates show that $a_1(\sqrt{\frac{n-2}{3(n+1)}} - \frac{2a_1}{n-1}) \geq \alpha_n$, where $\alpha_5 = \frac{\sqrt{2}-1}{6}$, $\alpha_6 = \frac{4}{35}$ for all $a_1 \in I_n$. Therefore the bracket $[\cdot]$ in (4.5) is $\geq \phi(V(n,a_1))$, where $\phi(s) := \frac{A}{s} + Bs$, $s > 0$, with $A = (n+1)\alpha_n$ and $B = \frac{1}{n-1}$. The minimum of ϕ in $(0,\infty)$ is $2\sqrt{AB} = 2\sqrt{\frac{n+1}{n-1}\alpha_n}$, attained at $s = \sqrt{\frac{A}{B}}$. Hence for $n = 5,6$, $\frac{g_n'(a_1)}{n+3} \geq \frac{2(n-3)}{(n+3)\sqrt{n-2}} 2\sqrt{\frac{n+1}{n-1}\alpha_n}$. Inserting the values for n, α_n yields that $g_5'(a_1) > \frac{2}{15} > 0$ and $g_6'(a_1) > 2 > 0$. Thus $g_n(a_1) > 1$ for $a_1 \in I_n$ and hence f_1 is strictly increasing for $n \geq 5$. Similarly for $n = 4$, $\sqrt{a_1 - t}\ f_1(n,t,a_1)$

is increasing in a_1. Concerning f_2 we have

$$\begin{aligned} N(n,t,a_1) &:= M'(n,t,a_1) = 8a_1 + (n-1)t \\ &\quad + 2\frac{n-3}{\sqrt{n-2}\, V(n,a_1)} \left(4a_1^2 + (n-1)a_1 t - \frac{n-1}{n+1}\right) , \\ M_2(n,t,a_1) &:= (n+1)\left(a_1 + \frac{V(n,a_1)}{\sqrt{n-2}\,(n+1)}\right)\left(a_1 - \frac{\sqrt{n-2}\, V(n,a_1)}{n+1}\right) \\ &= (n+3)a_1^2 - \frac{n-3}{\sqrt{n-2}} a_1 V(n,a_1) - \frac{n-1}{n+1} . \end{aligned}$$

We get

$$\begin{aligned} (\ln f_2)'(n,t,a_1) &= \frac{1}{a_1 - t} - \frac{2(n+3)a_1 - \frac{n-3}{\sqrt{n-2}} V(n,a_1) + 2\frac{(n-3)(n+1)}{\sqrt{n-2}} \frac{a_1^2}{V(n,a_1)}}{M_2(n,t,a_1)} \\ &\quad + \frac{N(n,t,a_1)}{M(n,t,a_1)} \\ &= \left(\frac{1}{a_1 - t} - \frac{2(n+3)a_1 - \frac{n-3}{\sqrt{n-2}} V(n,a_1)}{M_2(n,t,a_1)} \right) \\ &\quad + \left(\frac{N(n,t,a_1)}{M(n,t,a_1)} - \frac{N_2(n,t,a_1)}{M_2(n,t,a_1)} \right) , \end{aligned}$$

with $N_2(n,t,a_1) := 2\frac{(n-3)(n+1)}{\sqrt{n-2}} \frac{a_1^2}{V(n,a_1)}$. We claim that both terms in the brackets are positive. Calculation shows that the first term in brackets is positive if and only if $t > \frac{(n+3)a_1^2 + \frac{n-1}{n+1}}{2(n+3)a_1 - \frac{n-3}{\sqrt{n-2}} V(n,a_1)} =: H_n(a_1)$. We have $H_n(\sqrt{\frac{n-2}{3(n+1)}}) = \sqrt{\frac{n-2}{3(n+1)}}$ and claim that for all $n \geq 5$, the function H_n is strictly decreasing in I_n. Then for $a_1 > t$, $H_n(a_1) < \sqrt{\frac{n-2}{3(n+1)}} \leq t$ will be satisfied. As for H_n', we find with $\frac{n-1}{n+1} \geq 2a_1^2$

$$\begin{aligned} (\ln H_n)'(a_1) &= \frac{2(n+3)a_1}{(n+3)a_1^2 + \frac{n-1}{n+1}} - \frac{1 + \frac{n-3}{(n+3)\sqrt{n-2}} \frac{(n+1)a_1}{V(n,a_1)}}{2(n+3)a_1 - \frac{n-3}{\sqrt{n-2}} V(n,a_1)} \\ &\leq \frac{1}{(n+5)a_1} - \frac{1 + \frac{n-3}{(n+3)\sqrt{n-2}} \frac{(n+1)a_1}{V(n,a_1)}}{2(n+3)a_1 - \frac{n-3}{\sqrt{n-2}} V(n,a_1)} . \end{aligned}$$

Calculation shows that the latter is negative if and only if $1 \leq \frac{n-3}{\sqrt{n-2}}(\frac{V(n,a_1)}{(n+1)a_1} + \frac{n+5}{n+3}\frac{a_1}{V(n,a_1)}) = \frac{n-3}{\sqrt{n-2}}\phi(\frac{a_1}{V(n,a_1)}) =: l_n(a_1)$ with $A := \frac{1}{n+1}$ and $B := \frac{n+5}{n+3}$ in terms of the above function ϕ. Therefore $l_n(a_1) \geq \frac{n-3}{\sqrt{n-2}} 2\sqrt{AB} = \frac{n-3}{\sqrt{n-2}} 2\sqrt{\frac{n+5}{(n+1)(n+3)}} =: \Phi(n)$. We have that Φ is increasing with $\Phi(5) = \frac{\sqrt{10}}{3} > 1$. Therefore H_n is decreasing for all $n \geq 5$. The second term in brackets is positive if and only if $g(n,t,a_1) := N(n,t,a_1)M_2(n,t,a_1) - N_2(n,t,a_1)M(n,t,a_1) >$

0. Calculation shows

$$
\begin{aligned}
g(n,t,a_1) = &\left((n+1)(n+3)a_1^2 - \frac{(n-1)^2}{n+1}\right) t + \\
&\frac{2a_1}{n-2}\left(2(n^2+8n-21)a_1^2 + \frac{n-1}{n+1}(n^2-10n+17)\right) \\
&- \frac{n-3}{\sqrt{n-2}(n+1)} V(n,a_1)\left[8(n+2)a_1^2 + (n-1)(n+3)a_1 t - 2\frac{n-1}{n+1}\right] .
\end{aligned}
$$

Since $V(n,a_1)$ is decreasing in a_1, g is obviously strictly increasing in a_1 and in t. We have for $t = a_1 = \sqrt{\frac{n-2}{3(n+1)}}$ that $g(n,t,a_1) = 0$. Thus for $t < a_1$, $g(n,t,a_1) > 0$, and hence f_2 is increasing in a_1, too. For $n = 4$, $\frac{f_2(n,t,a_1)}{\sqrt{a_1 - t}}$ is increasing in a_1 and hence $f(4,t,a_1)$ is increasing in a_1, too.

We conclude that for $\sqrt{\frac{n-2}{3(n+1)}} \le t < a_1 < \sqrt{\frac{n-1}{2(n+1)}}$, $A(\tilde{a},t)$ is increasing in a_1. Since for $a_1 \to \sqrt{\frac{n-1}{2(n+1)}}$, $V(n,a_1) \to 0$ and hence $\tilde{a} \to a^{(2)}$, we find from (4.4)
$A(\tilde{a},t) \le A(a^{(2)},t) = \frac{\sqrt{n+1}}{(n-2)!}\left(\sqrt{\frac{2(n-1)}{n+1}}\right)^n \left(\sqrt{\frac{2}{(n-1)(n+1)}} + t\right)\left(\sqrt{\frac{n-1}{2(n+1)}} - t\right)^{n-2}$,
if $t < \sqrt{\frac{n-1}{2(n+1)}}$, coinciding, of course, with the value for $A(a^{(2)},t)$ given in Proposition 2.3. Therefore in this case $a^{(2)}$ yields the maximum of $A(\cdot,t)$.

v) Secondly, consider the subcase of iii) that $a_1 > \sqrt{\frac{n-1}{2(n+1)}} =: t_1$. By Lemma 2.1, $a_2 \le \psi(a_1) \le \psi(t_1) = t_1$. Thus $a_1 > t_1 \ge a_2 > t > t_0 := \sqrt{\frac{n-2}{3(n+1)}}$. Since $A(\cdot,t)$ is increasing in a_2, the maximum occurs for $a_2 = \psi(a_1)$. Then $W = 0$ and

$$
\text{(4.6)} \qquad \frac{(n-1)!}{\sqrt{n+1}} A(a,t) = \frac{\left(\frac{a_1 - t}{a_1 + \frac{a_1+a_2}{n-1}}\right)^{n-1} - \left(\frac{a_2 - t}{a_2 + \frac{a_1+a_2}{n-1}}\right)^{n-1}}{a_1 - a_2} =: B(a_1,a_2,t) .
$$

For $a_1 > t_1$, $\psi'(a_1) = -\frac{n+1}{n}\frac{\sqrt{n-1}a_1}{\sqrt{n-(n+1)a_1^2}} - \frac{1}{n} \le -1$. Since $\psi(t_1) = t_1$, we have for $a_1 > t_1$ and a suitable $\theta \in (t_1, a_1)$

$$
a_1 - a_2 = a_1 - \psi(a_1) = a_1 - t_1 - (\psi(a_1) - \psi(t_1)) = (a_1 - t_1)(1 - \psi'(\theta)) \ge 2(a_1 - t_1) .
$$

We will show for $n \ge 5$ that $A(a,t) < A(a^{(1)},t)$, considering two cases: first, $a_1 \ge \sqrt{\frac{n-\frac{1}{2}}{2(n+1)}} =: t_2$ and secondly $t_1 < a_1 < t_2$. In the first case, a_1 and $a_2 = \psi(a_1)$ are sufficiently far apart that we can omit the second term in $B(a_1,a_2,t)$. In the second case, we use the mean value theorem to estimate $B(a_1,a_2,t)$. The second case is investigated in part vi) below, the case $n = 4$ in part vii).

Thus let $n \geq 5$ and $a_1 \geq t_2 > t_1 > a_2 = \psi(a_1) > t > t_0$. By (4.6)

(4.7)
$$\frac{(n-1)!}{\sqrt{n+1}} A(a,t) \leq \frac{1}{2(a_1-t_1)} \left(\frac{a_1 - t}{a_1 + \frac{a_1+\psi(a_1)}{n-1}} \right)^{n-1} = \frac{(\frac{n-1}{n})^{n-1}}{2(a_1-t_1)} \left(\frac{1-\frac{t}{a_1}}{1+\frac{1}{n}\frac{\psi(a_1)}{a_1}} \right)^{n-1}$$
$$\leq \frac{(\frac{n-1}{n})^{n-1}}{2(a_1-t_1)} \left(\frac{1-\frac{t}{a_1}}{1+\frac{1}{n}\frac{t}{a_1}} \right)^{n-1} =: L \ ,$$

Since $\psi^2 = \mathrm{Id}$ and ψ is decreasing, this implies $a_1 = \psi(a_2) \leq \psi(t_0) = \sqrt{\frac{2}{3}}\frac{\sqrt{n^2-1}}{n} - \frac{1}{n}t_0 < \sqrt{\frac{2}{3}}\sqrt{\frac{n}{n+1}} =: t_3$. The last inequality follows from $\frac{n^2-1}{n^2} < (\sqrt{1-\frac{1}{n+1}} + \frac{1}{n}t_0)^2 = 1 - \frac{1}{n+1} + \frac{t_0^2}{n^2} + \frac{2}{\sqrt{3}}\frac{1}{n+1}\sqrt{\frac{n-2}{n}}$, which is implied by $1 - \frac{1}{n} < \frac{2}{\sqrt{3}}\sqrt{\frac{n-2}{n}}$ for $n \geq 5$. We want to show that L is strictly less than

$$\frac{(n-1)!}{\sqrt{n+1}} A(a^{(1)},t) = \left(\frac{n}{n+1}\right)^{n-\frac{1}{2}} \left(1 - \sqrt{\frac{n+1}{n}}t\right)^{n-1} =: R \ ,$$

i.e. that (4.6) does not yield the maximum of $A(\cdot,t)$. We have

$$\frac{L}{R} = \frac{(\frac{n^2-1}{n^2})^{n-\frac{1}{2}}\sqrt{\frac{n}{n-1}}}{2(a_1-t_1)} \left(\frac{1-\frac{t}{a_1}}{(1-\sqrt{\frac{n+1}{n}}t)(1+\frac{1}{n}\frac{t}{a_1})} \right)^{n-1} .$$

Now $h(t) := \ln(\frac{1-\frac{t}{a_1}}{(1-\sqrt{\frac{n+1}{n}}t)(1+\frac{1}{n}\frac{t}{a_1})})$ is decreasing in t, since $h'(t) = -\frac{1}{a_1-t} + \frac{1}{\sqrt{\frac{n}{n+1}}-t} - \frac{\frac{1}{n}}{a_1+\frac{1}{n}t} = -\frac{\sqrt{\frac{n}{n+1}}-a_1}{(a_1-t)(\sqrt{\frac{n}{n+1}}-t)} - \frac{\frac{1}{n}}{a_1+\frac{1}{n}t} < 0$, so that h is maximal for $t = t_0 := \sqrt{\frac{n-2}{3(n+1)}}$. Therefore

$$\frac{L}{R} \leq \frac{(\frac{n^2-1}{n^2})^{n-\frac{1}{2}}\sqrt{\frac{n}{n-1}}}{2(a_1-t_1)} \left(\frac{1-\frac{t_0}{a_1}}{(1-\sqrt{\frac{n+1}{n}}t_0)(1+\frac{1}{n}\frac{t_0}{a_1})} \right)^{n-1} =: Q_n(a_1).$$

We claim that $\phi(a_1) := \frac{1}{a_1-t_1}\left(\frac{1-\frac{t_0}{a_1}}{(1-\sqrt{\frac{n+1}{n}}t_0)(1+\frac{1}{n}\frac{t_0}{a_1})} \right)^{n-1}$ is first decreasing and then increasing in $(t_1, t_3]$. This will imply $Q_n(a_1) \leq \max(Q_n(t_2), Q_n(t_3))$ for all $a_1 \in [t_2, t_3]$. The logarithmic derivative of ϕ is

$$(\ln\phi)'(a_1) = -\frac{1}{a_1-t_1} + \frac{(n-1)t_0}{a_1(a_1-t_0)} + \frac{(n-1)t_0}{a_1(na_1+t_0)}$$
$$= \frac{nt_0a_1 - a_1^2 - (n-1)t_0t_1}{(a_1(a_1-t_0)(a_1-t_1)} + \frac{(n-1)t_0}{a_1(na_1+t_0)} .$$

The first term is negative for $a_1 > t_1$ close to t_1, being singular at t_1. The quadratic term

$$q(a_1) = nt_0a_1 - a_1^2 - (n-1)t_0t_1 = \sqrt{\frac{n-2}{n+1}}\left(\frac{na_1}{\sqrt{3}} - \frac{(n-1)^{\frac{3}{2}}}{\sqrt{6}(n+1)^{\frac{3}{2}}}\right) - a_1^2$$

is positive at $a_1 = t_3 = \sqrt{\frac{2}{3}\frac{n}{n+1}}$. The second zero of q is near $nt_0 - t_1$, clearly > 1 for $n \geq 5$. Thus the first term in $(\ln\phi)'(a_1)$ has exactly one zero $\tilde{a} \in (t_1, t_3)$. It is decreasing in modulus for $a_1 \in (t_1, \tilde{a})$, since q is increasing there, with negative values, and the denominator $a_1(a_1 - t_0)(a_1 - t_1)$ is increasing, too. Therefore also $(\ln\phi)'(a_1)$ has exactly one zero $\bar{a} \in (t_1, t_3)$ with $\bar{a} < \tilde{a}$, since we add a second positive term. This means that ϕ is first decreasing and then increasing in (t_1, t_3). Therefore

$$\frac{L}{R} \leq Q_n(a_1) \leq \max(Q_n(t_2), Q_n(t_3)) \ , \ a_1 \in [t_2, t_3] \ .$$

To estimate $Q_n(t_3)$, we use $\frac{1}{2(t_3-t_1)} = \frac{\sqrt{n+1}}{n+3}(\sqrt{6}\sqrt{n} + \frac{3}{\sqrt{2}}\sqrt{n-1}) \leq (\sqrt{6} + \frac{3}{\sqrt{2}})\frac{n+\frac{1}{2}}{n+3} < 4.6 - \frac{11}{n+3}$. Logarithmic differentiation shows that $f_1(n) := (1 + \frac{1}{\sqrt{2}n}\sqrt{\frac{n-2}{n}})^{n-1}$ is increasing in n, so that $f_1(n) \geq f_1(5) > \frac{3}{2}$ for $n \geq 5$. We also use $f_2(n) := (\frac{n^2-1}{n^2})^{n-\frac{1}{2}}\sqrt{\frac{n}{n-1}} < 1$. Note that $\frac{t_0}{t_3} = \frac{1}{\sqrt{2}}\sqrt{\frac{n-2}{n}}$ and $\sqrt{\frac{n+1}{n}}t_0 = \frac{1}{\sqrt{3}}\sqrt{\frac{n-2}{n}}$. The function $f_3(\alpha) := \frac{1-\frac{\alpha}{\sqrt{2}}}{1-\frac{\alpha}{\sqrt{3}}}$ is decreasing in $\alpha \in (0,1)$, since $(\ln f_2)'(\alpha) = -\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}-\alpha)(\sqrt{2}-\alpha)} < 0$. Applying this for $\alpha = \sqrt{\frac{n-2}{n}}$, $f_4(n) := \frac{1-\sqrt{\frac{n-2}{2n}}}{1-\sqrt{\frac{n-2}{3n}}}$ satisfies $f_4(n) \leq f_4(5) < 0.82$ for all $n \geq 5$. We conclude that

$$Q_n(t_3) < (4.6 - \frac{11}{n+3})\frac{f_4(n)^{n-1}}{f_1(n)} < \frac{2}{3}(4.6 - \frac{11}{n+3})0.82^{n-1} =: f_5(n) \ ,$$

which is decreasing in n, with $f_5(n) \leq f_5(5) < 0.98 < 1$ for all $n \geq 5$.

As for $Q_n(t_2)$, we have $\frac{t_0}{t_2} = \sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}}$, $\sqrt{\frac{n+1}{n}}t_0 = \sqrt{\frac{n-2}{3n}}$. By the arithmetic-geometric mean inequality $\frac{1}{2(t_2-t_1)} = \sqrt{2}\sqrt{n+1}(\sqrt{n-\frac{1}{2}} + \sqrt{n-1}) \leq 2\sqrt{2}n + \frac{1}{2\sqrt{2}}$. Therefore

$$Q_n(t_2) \leq (2\sqrt{2}n + \frac{1}{2\sqrt{2}})\left(\frac{n^2-1}{n^2}\right)^{n-\frac{1}{2}}\sqrt{\frac{n}{n-1}}\left(\frac{1-\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}}}{(1-\sqrt{\frac{1}{3}\frac{n-2}{n}})(1+\frac{1}{n}\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}})}\right)^{n-1} .$$

Let $f_6(n) := \frac{1-\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}}}{1-\sqrt{\frac{1}{3}\frac{n-2}{n}}}$. Then $f_6(5) < 0.604$ and we claim that $f_6(n) < \frac{3}{5}$ for all $n \geq 6$. This is equivalent to $2 < 5\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}} - 3\sqrt{\frac{1}{3}\frac{n-2}{n}} =: f_7(n)$. Now $\sqrt{\frac{n-2}{n-\frac{1}{2}}} \geq 1 - \frac{1}{n}$, $\sqrt{\frac{n-2}{n}} \leq 1 - \frac{1}{n}$, so that $f_7(n) \geq (5\sqrt{\frac{2}{3}} - \sqrt{3})(1 - \frac{1}{n}) > 2$ for all $n \geq 7$. Also $f_7(6) > 2$. Further, $f_8(n) := (1 + \frac{1}{n}\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}})^{n-1}$ is increasing in n, being essentially of the form $(1+\sqrt{\frac{2}{3}}\frac{1}{n}(1+O(\frac{1}{n})))^n$, with $f_8(n) \geq f_8(5) > 1.64$ for all $n \geq 5$. Hence for $n \geq 6$, we have $Q_n(t_2) < \frac{2\sqrt{2}n+\frac{1}{2\sqrt{2}}}{1.64}\left(\frac{3}{5}\right)^{n-1} =: f_9(n)$, which is decreasing in n, with $f_9(n) \leq f_9(6) < 0.83 < 1$ for $n \geq 6$. For $n = 5$, the estimate has to be refined: first, replace $t_2 \simeq 0.6124$ by $\tilde{t_2} = \sqrt{\frac{n-\frac{2}{5}}{2(n+1)}} \simeq 0.6191$, with $\frac{1}{2(\tilde{t_2}-t_1)} < 12$ and the corresponding $\tilde{f_6}(5) < 0.617$, $\tilde{f_8}(5) > 1.64$. We also use $f_2(5) < 0.931$. Then for $n = 5$, $Q_n(\tilde{t_2}) < \frac{12}{1.64}\, 0.931 \cdot 0.617^4 < 0.99 < 1$. This leaves the estimate for $Q_n(a_1)$ for $a_1 \in [t_2, \tilde{t_2}]$, $n = 5$. In this situation, we may improve the estimate $1+\frac{1}{n}\frac{\psi(a_1)}{a_1} \geq 1+\frac{1}{n}\frac{t}{a_1}$ leading to L, by using $\frac{\psi(a_1)}{a_1} \geq \frac{\psi(\tilde{t_2})}{\tilde{t_2}} > 0.86$ and $(1+\frac{1}{n}\frac{\psi(a_1)}{a_1})^{n-1} > (1+\frac{0.86}{n})^{n-1} > 1.88$, $n = 5$. Then for $a_1 \in [t_2, \tilde{t_2}]$ and $n = 5$, $\frac{L}{R} < \frac{10\sqrt{2}+\frac{1}{2\sqrt{2}}}{1.88}\, 0.931 \cdot 0.604^4 < 0.96 < 1$. This shows $A(a,t) < A(a^{(1)},t)$ also for $n = 5$.

vi) We now study the case $n \geq 5$ and $t_0 < t < a_2 = \psi(a_1) < t_1 < a_1 < t_2$. Again, we estimate the right side $B(a_1,a_2,t)$ in (4.6). Let $\gamma := \frac{a_1+a_2}{n-1}$ and consider $g(x) := (\frac{x-t}{x+\gamma})^{n-1}$. Then $g'(x) = (n-1)\frac{(x-t)^{n-2}}{(x+\gamma)^n}(t+\gamma)$, $g''(x) = (n-1)\frac{(x-t)^{n-3}}{(x+\gamma)^{n+1}}(t+\gamma)(n(t+\gamma)-2(x+\gamma))$. Since $4t > 2a_1$, $2t > a_2$, $g''(x) > 0$ for all $x \in (a_2,a_1)$. Therefore g' is increasing in (a_2,a_1) and we find from (4.6) with suitable $\theta \in (a_2,a_1)$

$$B(a_1,a_2,t) = g'(\theta) \leq g'(a_1) = (n-1)\frac{(a_1-t)^{n-2}}{(a_1+\frac{a_1+a_2}{n-1})^n}\left(t+\frac{a_1+a_2}{n-1}\right) .$$

To estimate $B(a_1,a_2,t)$, we will use $a_1+a_2 = a_1+\psi(a_1) \leq 2t_1$. This follows from $\psi'(x) \leq -1$, $x \geq t_1$. Further, we claim that $\frac{a_2}{a_1} \geq 1-\frac{3}{5}\frac{1}{n}$. Since ψ is decreasing, $\frac{a_2}{a_1} \geq \frac{\psi(t_2)}{t_2} = \frac{\sqrt{n^2-1}}{n}\sqrt{\frac{2n+1}{2n-1}}-\frac{1}{n}$. Our claim will follow from $\frac{\sqrt{n^2-1}}{n}\sqrt{\frac{2n+1}{2n-1}} \geq 1+\frac{2}{5}\frac{1}{n}$. This is true, since for $n \geq 5$ we have $\frac{n^2-1}{n}\frac{2n+1}{2n-1}-1 \geq \frac{13}{15}\frac{1}{n} \geq \frac{4}{5}\frac{1}{n}+\frac{4}{25}\frac{1}{n^2}$. Thus $(1+\frac{1}{n}\frac{a_2}{a_1})^n \geq (1+\frac{1}{n}(1-\frac{3}{5}\frac{1}{n}))^n \geq (1+\frac{22}{25}\frac{1}{n})^n > \frac{20}{9}$ for $n \geq 5$. We find

$$\begin{aligned} B(a_1,a_2,t) &\leq (n-1)\left(\frac{n-1}{n}\right)^n\left(t+\frac{2t_1}{n-1}\right)\frac{(1-\frac{t}{a_1})^{n-2}}{a_1^2(1+\frac{1}{n}\frac{a_2}{a_1})^n} \\ &\leq \frac{9}{20}(n-1)\left(\frac{n-1}{n}\right)^n\left(t+\frac{2t_1}{n-1}\right)\frac{(1-\frac{t}{a_1})^{n-2}}{a_1^2} . \end{aligned}$$

The function $k(a_1) := \frac{(1-\frac{t}{a_1})^{n-2}}{a_1^2}$ is increasing in a_1, since $(\ln k)'(a_1) = \frac{nt-2a_1}{a_1(a_1-t)} > 0$. Therefore

$$B(a_1,a_2,t) \leq \frac{9}{20}(n-1)\left(\frac{n-1}{n}\right)^n\left(t+\frac{2t_1}{n-1}\right)\frac{(1-\frac{t}{t_2})^{n-2}}{t_2^2} =: L .$$

Again we claim that $A(a,t) < A(a^{(1)},t)$. This will follow from $\frac{L}{R} < 1$ with R as in v). We have

$$\frac{L}{R} = \frac{9}{20}(n-1)\left(\frac{n^2-1}{n^2}\right)^n\sqrt{\frac{n}{n-1}}\frac{t+\frac{2t_1}{n-1}}{t_2^2}\frac{(1-\frac{t}{t_2})^{n-2}}{(1-\sqrt{\frac{n+1}{n}}t)^{n-1}} .$$

The function $h(t) := (t+\delta)\frac{(1-\frac{t}{a_1})^{n-2}}{(1-\sqrt{\frac{n+1}{n}}t)^{n-1}}$ is decreasing in t, for any fixed $\delta \in (0,1)$, since $(\ln h)'(t) = \frac{1}{t+\delta} - \frac{n-2}{a_1-t} + \frac{n-1}{\sqrt{\frac{n}{n+1}}-t} = \frac{1}{t+\delta} + \frac{1}{\sqrt{\frac{n}{n+1}}-t} - (n-2)\frac{\sqrt{\frac{n}{n+1}}-a_1}{(a_1-t)(\sqrt{\frac{n}{n+1}}-t)} < 0$. Thus $\frac{L}{R}$ will be maximal for $t = t_0$, when $\frac{t_0}{t_2} = \sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}}$, $\sqrt{\frac{n+1}{n}}t_0 = \sqrt{\frac{1}{3}\frac{n-2}{n}}$ and then, also using $(\frac{n^2-1}{n^2})^n < \exp(-\frac{1}{n})$,

$$\frac{L}{R} \leq \frac{9}{20}(n-1)\exp(-\frac{1}{n})\left(\frac{2}{\sqrt{3}}\frac{\sqrt{n(n-2)}}{n-\frac{1}{2}} + 2\sqrt{2}\frac{\sqrt{\frac{n}{n-1}}}{n-\frac{1}{2}}\right)\frac{(1-\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}})^{n-2}}{(1-\sqrt{\frac{1}{3}\frac{n-2}{n}})^{n-1}} =: \alpha(n)\,.$$

This gives $\frac{L}{R} \leq \alpha(5) < 0.99 < 1$ for $n = 5$. For $n > 5$, we use $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $|x| < 1$ to estimate $\frac{2}{\sqrt{3}}\frac{\sqrt{n(n-2)}}{n-\frac{1}{2}} + 2\sqrt{2}\frac{\sqrt{\frac{n}{n-1}}}{n-\frac{1}{2}} \leq \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}n} + \frac{2\sqrt{2}}{n-1} < \frac{8}{5}$ for all $n \geq 7$. For $n = 6$, the left side is also $< \frac{8}{5}$. Therefore

$$\frac{L}{R} \leq \frac{18}{25}(n-1)\exp(-\frac{1}{n})\frac{(1-\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}})^{n-2}}{(1-\sqrt{\frac{1}{3}\frac{n-2}{n}})^{n-1}}\,.$$

We know from part v) that $\frac{1-\sqrt{\frac{2}{3}\frac{n-2}{n-\frac{1}{2}}}}{1-\sqrt{\frac{1}{3}\frac{n-2}{n}}} < \frac{3}{5}$ for $n \geq 6$. Using also $1 - \sqrt{\frac{1}{3}\frac{n-2}{n}} \geq 1 - \frac{1}{\sqrt{3}} > \frac{21}{50}$, we get for all $n \geq 6$ that $\frac{L}{R} \leq \frac{12}{7}\exp(-\frac{1}{n})(n-1)\left(\frac{3}{5}\right)^{n-2} =: \beta(n)$. Since $(\ln\beta)'(n) = \frac{1}{n^2} + \frac{1}{n-1} - \ln(\frac{5}{3}) < 0$ for all $n \geq 6$, β is decreasing with $\frac{L}{R} \leq \beta(n) \leq \beta(6) < 0.95 < 1$ for all $n \geq 6$. Therefore $A(a,t) < A(a^{(1)},t)$ for all $n \geq 5$.

vii) Last we consider the dimension $n = 4$ and $a_1 > t_1 = \sqrt{\frac{3}{10}}$, $a_2 = \psi(a_1) > t > t_0 = \sqrt{\frac{2}{15}}$. We have

$$\frac{6}{\sqrt{5}}A(a,t) = \frac{1}{a_1-a_2}\left(\frac{(a_1-t)^3}{(a_1+\frac{a_1+a_2}{3})^3} - \frac{(a_2-t)^3}{(a_2+\frac{a_1+a_2}{3})^3}\right)\,. \tag{4.8}$$

This is equal to $\frac{1}{a_1-a_2}\int_{a_2}^{a_1}\frac{(x-t)^2}{(x+\alpha)^4}\,dx\,(t+\alpha)$, $\alpha = \frac{a_1+a_2}{3} \geq \beta := 0.3548$, since $h(x) = \frac{(x-t)^3}{(x+\alpha)^3}$ has the derivative $h'(x) = \frac{(x-t)^2}{(x+\alpha)^4}(t+\alpha)$. Let $t_0 := 0.404$. We show that $\frac{A(a,t)}{A(a^{(1)},t)}$ is decreasing for $t \in [t_0, \sqrt{\frac{3}{10}}]$, where $\frac{6}{\sqrt{5}}A(a^{(1)},t) = \frac{16}{25}\left(\sqrt{\frac{4}{5}}-t\right)^3$. By the integral formula, it suffices to prove that $k(t) := \frac{(x-t)^2}{(\sqrt{\frac{4}{5}}-t)^3}(t+\beta)$ is decreasing in t for any $x \leq a_1 \leq \psi(t)$. Note here that $t \leq a_2 = \psi(a_1)$ implies $a_1 \leq \psi(t)$ since

$\psi^2 = \mathrm{Id}$ and ψ is decreasing. We have for $x \le \psi(t_0) \le 0.6717$

$$(\ln k)'(t) = \frac{-2}{x-t} + \frac{3}{\sqrt{\frac{4}{5}} - t} + \frac{1}{t+\beta}$$

$$= -\frac{2\sqrt{\frac{4}{5}} + t - 3x}{(x-t)\left(\sqrt{\frac{4}{5}} - t\right)} + \frac{1}{t+\beta} \le -\frac{0.1777}{0.268 \cdot 0.491} + \frac{1}{0.758} \le -0.03 < 0 \ .$$

Therefore k is decreasing for $t \in [t_0, \sqrt{\frac{3}{10}}]$. Thus if we prove that $A(a, t_0) \le A(a^{(1)}, t_0)$ holds, $A(a, t) \le A(a^{(1)}, t)$ will follow for all $t \in [t_0, \sqrt{\frac{3}{10}}]$.

Calculation shows that the formal singularity in (4.8) can be removed; we find with $a_2 = \psi(a_1)$

$$\begin{aligned} \frac{6}{\sqrt{5}} A(a,t) &= \left(\frac{t + \frac{a_1+a_2}{3}}{(a_1 + \frac{a_1+a_2}{3})(a_2 + \frac{a_1+a_2}{3})} \right)^3 (a_1 - a_2)^2 \\ &\quad + \frac{3(t + \frac{a_1+a_2}{3})}{(a_1 + \frac{a_1+a_2}{3})^2 (a_2 + \frac{a_1+a_2}{3})^2} (a_1 - t)(a_2 - t) \\ &=: f_1(t, a_1, a_2) + f_2(t, a_1, a_2) =: f(t, a_1, a_2) \ . \end{aligned} \tag{4.9}$$

We claim that $f(t, a_1, \psi(a_1))$ is decreasing in $a_1 \in [\sqrt{\frac{3}{10}}, \psi(\sqrt{\frac{3}{10}})] \simeq [0.5477, 0.6993]$ for all $t \in [\sqrt{\frac{2}{15}}, t_0]$. Then the maximum for these t in (4.9) will be attained for $a_1 = a_2 = \sqrt{\frac{3}{10}}$, when $f_1 = 0$ and $f_2 = \frac{6}{\sqrt{5}} A(a^{(2)}, t)$. Therefore for $t \in [\sqrt{\frac{2}{15}}, t_0]$, $A(a, t) \le A(a^{(2)}, t)$, with equality for $a_1 = \sqrt{\frac{3}{10}}$. For $t > \tilde{t} \simeq 0.3877$, $A(a^{(2)}, t) < A(a^{(1)}, t)$, so that, in particular, $A(a, t_0) < A(a^{(1)}, t_0)$ holds as required above.

Actually, for all $t \in [\sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}]$, $f_2(t, a_1, \psi(a_1))$ is decreasing in $a_1 \in [\sqrt{\frac{3}{10}}, \psi(t)]$ and $f_1(t, a_1, \psi(a_1))$ is increasing. For $t \in [\sqrt{\frac{2}{15}}, t_0]$, f_2 is decreasing faster than f_1 is increasing (up from 0). One has for a_1 near $\sqrt{\frac{3}{10}}$

$$\begin{aligned} &f(t, a_1, \psi(a_1)) = \frac{6}{\sqrt{5}} A(a^{(2)}, t) \\ &- \frac{54}{125} \left(4t - \sqrt{\frac{2}{15}}\right) \left(\frac{7}{3} + 10\sqrt{\frac{2}{15}} t - 20t^2\right) \left(a_1 - \sqrt{\frac{3}{10}}\right)^2 + O\left(\left(a_1 - \sqrt{\frac{3}{10}}\right)^3\right) \ . \end{aligned}$$

This means that close to $\sqrt{\frac{3}{10}}$, f is decreasing for all t with $t \le \frac{1}{4}(\sqrt{2} + \sqrt{\frac{2}{15}}) \simeq 0.4448$. For $t \le t_0$, f is decreasing in the full interval $[\sqrt{\frac{3}{10}}, \psi(\sqrt{\frac{3}{10}})]$, as a tedious calculation of the derivative $\frac{d}{da_1} f(t, a_1, \psi(a_1))$ shows, using that $\psi(a_1) \le -1$ for $a_1 \in [\sqrt{\frac{3}{10}}, \psi(\sqrt{\frac{3}{10}})]$. We do not give the details of the calculation. The conclusion is $A(a,t) \le \max(A(a^{(1)},t), A(a^{(2)},t))$, with equality for $t \le \tilde{t}$ and $a_1 = a_2$ and strict inequality for $t > \tilde{t}$. $\square$

Remark. For $0.411 < t < 0.444$, f in part vii) is first decreasing and then increasing in $[\sqrt{\frac{3}{10}}, \psi(t)]$. For $t > 0.445$ f is increasing in the full interval. The situation for $n = 4$ is slightly different than for $n \ge 5$, in the sense that in the second case $a_1 > a_2 > t$, $t \le \tilde{t}$ leads to a maximal situation $A(a,t) = A(a^{(2)},t)$ in the limit $a_1 = a_2 = \sqrt{\frac{n-1}{2(n+1)}}$ for $n = 4$, whereas for $n \ge 5$ it does not. The above quotient $\frac{A(a,t)}{A(a^{(1)},t)}$ is actually also decreasing for $t \in [\sqrt{\frac{2}{15}}, t_0]$, but the quotient may be > 1 for $t < \tilde{t}$.

5. Dimensions $n = 2$ and $n = 3$

We now prove Propositions 1.2 and 1.3 and start with the easy case of dimension 2.

Proof of Proposition 1.2. Let $a \in S^2 \subset \mathbb{R}^3$ with $\sum_{j=1}^3 a_j = 0$, $t < \sqrt{\frac{2}{3}}$ and $a_1 > t$. Then $a_2 + a_3 = -a_1$, $a_2^2 + a_3^2 = 1 - a_1^2$ imply that $a_{2,3} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{2 - 3a_1^2}$, $(a_1 - a_2)(a_1 - a_3) = 3a_1^2 - \frac{1}{2}$. By Proposition 2.2 $A(a,t) = 2\sqrt{3}\frac{a_1 - t}{6a_1^2 - 1}$.

If $e_2, e_3 \in H_-(a,t) = \{x \in \Delta^2 | \langle a, x\rangle < t\}$, we have by Lemma 2.1 that $2t > a_2 + a_3 = -a_1 \ge -\sqrt{\frac{2}{3}}$, $t > -\frac{1}{\sqrt{6}}$. Therefore, if $t \le -\frac{1}{\sqrt{6}}$, only one vertex e_j can be in $H_-(a,t)$. Using $A(a,t) = A(-a,-t)$, it suffices to consider only one non-zero term in formula (2.1) for $A(a,t)$ when i) $\frac{1}{\sqrt{6}} \le t < \sqrt{\frac{2}{3}}$ or ii) $t \in [0, \frac{1}{\sqrt{6}}]$ or $-t \in [0, \frac{1}{\sqrt{6}}]$.

We have $\frac{d}{da_1}\frac{a_1 - t}{6a_1^2 - 1} = -\frac{6a_1^2 + 1 - 12a_1 t}{(6a_1^2 - 1)^2}$. This is zero for $a_1 = t + \sqrt{t^2 - 1/6}$, recalling $a_1 > t$. For $t > \frac{5}{4}\frac{1}{\sqrt{6}}$, $a_1 > \sqrt{\frac{2}{3}}$ which is impossible. For $t > \frac{5}{4}\frac{1}{\sqrt{6}}$, $\frac{d}{da_1}A(\cdot,t) < 0$, and $A(a,t)$ is maximal for $a_1 = \sqrt{\frac{2}{3}}$, $a_2 = a_3 = -\frac{1}{2\sqrt{3}}$, i.e. $a = a^{(1)}$. For $\frac{1}{\sqrt{6}} < t < \frac{5}{4}\frac{1}{\sqrt{6}}$, $\frac{1}{\sqrt{6}} < a_1 := t + \sqrt{t^2 - 1/6} < \sqrt{\frac{2}{3}}$, and $\frac{x - t}{6x^2 - 1}$ is increasing for $x < a_1$ and decreasing for $x > a_1$. Thus $a^{\{t\}} \in S^2$ yields the maximum of $A(\cdot,t)$ in this range of t. For $0 \le t \le \frac{1}{\sqrt{6}}$, again $\frac{d}{da_1}A(\cdot,t) < 0$ and $A(\cdot,t)$ attains its minimum for the maximal value of a_1, i.e. for $a^{(1)}$.

As for the maximum of $A(\cdot,t)$ for $0 \le t \le \frac{1}{\sqrt{6}}$, we have to consider $-t$, i.e. $A(a,t) = 2\sqrt{3}\frac{a_1 + t}{6a_1^2 - 1}$. Then $\frac{d}{da_1}\frac{a_1 + t}{6a_1^2 - 1} = -\frac{6a_1^2 + 1 + 12a_1 t}{(6a_1^2 - 1)^2} < 0$ and $A(\cdot,t)$ is decreasing in a_1. We need $a_1 > -t \ge a_2, a_3$. Thus $a_2 + t \le 0$ with $a_2 = \frac{1}{2}(\sqrt{2 - 3a_1^2} - a_1)$. This implies $a_1 \ge \frac{1}{2}(\sqrt{2 - 3t^2} + t)$, and in the maximum case $a_1 = \frac{1}{2}(\sqrt{2 - 3t^2} + t)$, $a_2 = -t$ and $a_3 = -\frac{1}{2}(\sqrt{2 - 3t^2} - t)$: Replacing $-t = |t|$ again by t yields the maximum $a^{[t]}$ stated in Proposition 1.2. $\square$

In dimension 3 there are more critical points of $A(a,t)$.

Proof of Proposition 1.3. Let $a \in S^3 \subset \mathbb{R}^4$ with $\sum_{j=1}^4 a_j = 0$ and $t \le \frac{\sqrt{3}}{2}$. If $e_2, e_3, e_4 \in H_-(a,t)$, we get using Lemma 2.1 i) that $3t > a_2+a_3+a_4 = -a_1 \ge -\frac{\sqrt{3}}{2}$, $t > -\frac{1}{2\sqrt{3}}$. Therefore, if $t \le -\frac{1}{2\sqrt{3}}$, only one or two vertices of Δ^3 can be in $H_-(a,t)$. Using $A(a,t) = A(-a,-t)$, it suffices to consider one or two non-zero terms in formula (2.1) for $A(a,t)$ when i) $\frac{1}{2\sqrt{3}} \le t < \frac{\sqrt{3}}{2}$ or ii) $t \in [0, \frac{1}{2\sqrt{3}}]$ or $-t \in [0, \frac{1}{2\sqrt{3}}]$. Thus we have to consider two possible cases in the volume formula (2.1): if $a_1 > t \ge a_2, a_3, a_4$, there is only one non-zero term in (2.1); if $a_1 > a_2 > t \ge a_3, a_4$, there are two terms, exchanging possibly t and $-t$ when $|t| \le \frac{1}{2\sqrt{3}}$.

a) i) Consider first the case $a_1 > t \ge a_2, a_3, a_4$. As in the proof of Theorem 1.1, there are at most two different coordinates among (a_2, a_3, a_4). Letting $a_2 = a_3 =: c$, $a_4 =: d$, we have $a_1 + 2c + d = 0$, $a_1^2 + 2c^2 + d^2 = 1$. We have two possibilities

$$\left\{ \begin{array}{ll} c_+ = & -\frac{1}{3}a_1 + \frac{1}{6}\sqrt{6-8a_1^2} \\ d_+ = & -\frac{1}{3}a_1 - \frac{1}{3}\sqrt{6-8a_1^2} \end{array} \right\} \quad , \quad \left\{ \begin{array}{ll} c_- = & -\frac{1}{3}a_1 - \frac{1}{6}\sqrt{6-8a_1^2} \\ d_- = & -\frac{1}{3}a_1 + \frac{1}{3}\sqrt{6-8a_1^2} \end{array} \right\} .$$

For (c_+, d_+), $c_+ < a_1$ requires $a_1 > \frac{1}{2\sqrt{3}}$ and if $a_1 > t \ge \frac{1}{2\sqrt{3}}$, $c_+ < t$ is satisfied. If $0 \le t < \frac{1}{2\sqrt{3}}$, for $c_+ < t$ we need $a_1 > \frac{1}{2}\sqrt{2-8t^2} - t$.

For (c_-, d_-), $d_- < a_1$ requires $a_1 > \frac{1}{2}$ and if $a_1 > t \ge \frac{1}{2}$, $d_- < t$ is satisfied. If $0 \le t \le \frac{1}{2}$, for $d_- < t$ we need $a_1 > \frac{1}{3}\sqrt{6-8t^2} - \frac{t}{3}$. Proposition 2.2 yields $A(a,t) = f_\pm(a_1,t) = \frac{(a_1-t)^2}{(a_1-c_\pm)^2(a_1-d_\pm)}$. We find with $' = \frac{d}{da_1}$
$(\ln f_\pm)'(a_1,t) = \frac{4}{a_1-t}\frac{1}{4a_1 \pm \sqrt{6-8a_1^2}}\frac{1}{8a_1 \mp \sqrt{6-8a_1^2}}[\pm(3t-a_1)\sqrt{6-8a_1^2} + 30a_1t - 10a_1^2 - 3]$. The denominators are positive, if $a_1 > \frac{1}{2\sqrt{3}}$ or $a_1 > \frac{1}{2}$, respectively.

In the case of f_+, let $t_0 := \frac{7}{10}\frac{1}{\sqrt{3}} \simeq 0.4041$. Since the factor $F(a_1,t) = (3t-a_1)\sqrt{6-8a_1^2} + 30a_1t - 10a_1^2 - 3$ is increasing in t, its minimum for $t \ge t_0$ is in t_0, with $F(a_1,t_0) = (\frac{7}{10}\sqrt{3} - a_1)\sqrt{6-8a_1^2} + 7\sqrt{3}a_1 - 10a_1^2 - 3$, which is decreasing in a_1, with value 0 in $a_1 = \frac{\sqrt{3}}{2}$. Therefore $(\ln f_+)' \ge 0$ for all $\frac{1}{2\sqrt{3}} \le a_1 \le \frac{\sqrt{3}}{2}$ and $t \ge t_0$. Hence for $t \ge t_0$, the maximum of f_+ is in $a_1 = \frac{\sqrt{3}}{2}$, i.e. $a^{(1)}$ attains the maximum. The minimum here is irrelevant, since $t \ge t_0 > \frac{1}{2\sqrt{3}}$ and the the minimum of $A(\cdot,t)$ is zero.

For $\frac{1}{2\sqrt{3}} < t < \frac{7}{10}\frac{1}{\sqrt{3}} = t_0$, $F(a_1,t) = 0$ if and only if $t = \frac{a_1}{3} + \frac{1}{10a_1+\sqrt{6-8a_1^2}} =: \phi(a_1)$. The function ϕ is strictly increasing and bijective $\phi : [\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}] \to [\frac{1}{2\sqrt{3}}, \frac{7}{10}\frac{1}{\sqrt{3}}]$. For $\frac{1}{2\sqrt{3}} < t < \frac{7}{10}\frac{1}{\sqrt{3}}$, let a_0 be the unique solution of $t = \phi(a_1)$. Then f_+ is increasing for $a_1 < a_0$ and decreasing for $a_0 < a_1$, hence attains its maximum at a_0: In this case, $A(\cdot,t)$ attains its maximum at $\tilde{a} := (a_0, c(a_0), c(a_0), d(a_0))$, $A(\tilde{a},t) = \frac{(a_0-t)^2}{(a_0-c(a_0))^2(a_0-d(a_0))} = \frac{(a_0-\phi(a_0))^2}{(a_0-c(a_0))^2(a_0-d(a_0))} =: \Phi(a_0)$. It can be checked that for $a_0 \ge \frac{1}{3}$, this is strictly smaller than $A(a^{(2)},t) = A(a^{(2)},\phi(a_0)) = \frac{1}{2} - 2\phi(a_0)^2$, so that $\tilde{a}$ does not yield the absolute maximum of $A(\cdot,t)$ for $t \ge \phi(\frac{1}{3}) = \frac{22+\sqrt{19}}{90} \simeq 0.2928$. However, for t very close to $\frac{1}{2\sqrt{3}} \simeq 0.2887$, $A(\tilde{a},t) > A(a^{(2)},t)$. But in this case $A(\tilde{a},t) < A(a^{\{t\}},t)$ for the vector $a^{\{t\}}$ given in Proposition 1.3 and considered in part b) i) below. Actually, Φ is decreasing in a_0 and $A(a^{\{t\}},t)$ is decreasing in t, so that for $\frac{1}{2\sqrt{3}} < t < \phi(\frac{1}{3}) \simeq 0.2928$, $A(\tilde{a},t) = \Phi(a_0) < \Phi(\frac{1}{2\sqrt{3}}) = \frac{16}{75}\sqrt{3} \simeq 0.3695 < 0.4069 \simeq A(a^{\{(\phi(\frac{1}{3})\}}, \phi(\frac{1}{3})) \le A(a^{\{t\}},t)$, and again $\tilde{a}$ does not yield the maximum of $A(\cdot,t)$. We have $\lim_{s\searrow\frac{1}{2\sqrt{3}}} A(a^{\{s\}},s) = \frac{5}{18}\sqrt{3}$, see part b) i) below.

For $0 \le t < \frac{1}{2\sqrt{3}}$, $F(a_1,t) < 0$: since it is increasing in t, the maximum of F is attained at $t = \frac{1}{2\sqrt{3}}$ and $F(a_1, \frac{1}{2\sqrt{3}}) = (\frac{\sqrt{3}}{2} - a_1)\sqrt{6-8a_1^2} + 5\sqrt{3}a_1 - 10a_1^2 - 3$ is decreasing in a_1 and 0 in $a_1 = \frac{1}{2\sqrt{3}}$. Thus $(\ln f_+)' \le 0$ for all $\frac{1}{2\sqrt{3}} \le a_1 \le \frac{\sqrt{3}}{2}$ and $0 \le t \le \frac{1}{2\sqrt{3}}$. Hence the minimum of $A = f_+$ for all $0 \le t \le \frac{1}{2\sqrt{3}}$ is in $a_1 = \frac{\sqrt{3}}{2}$, i.e. $a^{(1)}$ attains the minimum of $A(\cdot,t)$. The maximum of $A(\cdot,t) = f_+$ is attained at the minimal possible value $a_1 = \frac{1}{2}\sqrt{2-8a_1^2} - t$, and the extremal vector is

$$a^{[t]} = \left(\frac{1}{2}\sqrt{2-8a_1^2} - t, t, t, -\frac{1}{2}\sqrt{2-8a_1^2} - t\right)$$

with $A(a^{[t]},t) = \frac{1}{\sqrt{2-8t^2}}$.

ii) In the case of f_-, let $F(a,t) = (a_1 - 3t)\sqrt{6-8a_1^2} + 30a_1t - 10a_1^2 - 3$. Recall $a_1 > \frac{1}{2}$. Then $\frac{d}{dt}F(a_1,t) = 30a_1 - 3\sqrt{6-8a_1^2} > 0$ for $a_1 > \frac{1}{2}$. Hence F is increasing in t. For $0 \le t \le t_0 = \frac{7}{10}\frac{1}{\sqrt{3}}$, f is maximal in t_0, with $F(a_1,t_0) = (a_1 - \frac{7}{10}\sqrt{3})\sqrt{6-8a_1^2} + 7\sqrt{3}a_1 - 10a_1^2 - 3$, which is increasing in a_1, $F(\frac{\sqrt{3}}{2}, t_0) = 0$. Therefore $(\ln f_-)' \le 0$ for all $0 \le t \le t_0$ and $\frac{1}{2} \le a_1 \le \frac{\sqrt{3}}{2}$, and f_- attains its maximum at $a_1 = \frac{1}{2}$, i.e. $a^{(2)} = \frac{1}{2}(1,-1,-1,1)$ attains the maximum of $A(\cdot,t)$ in the case of f_- for $0 \le t \le t_0$. The minimum is in $a_1 = \frac{\sqrt{3}}{2}$, with vector $a^{(1)}$.

For $t_0 < t < \frac{1}{2}$, we have $F < 0$ in $\frac{1}{2} < a_1 < a_0$ and $F > 0$ in $a_0 < a_1 < \frac{\sqrt{3}}{2}$, where a_0 is the solution of $t = \frac{3+10a_1^2 - a - \sqrt{6-8a_1^2}}{3(10a_1 - \sqrt{6-8a_1^2})} =: \phi(a_1)$. Here $\phi : [\frac{1}{2}, \frac{\sqrt{3}}{2}] \to [t_0, \frac{1}{2}]$ is bijective and strictly decreasing, so that there is a unique solution $a_1 \in [\frac{1}{2}, \frac{\sqrt{3}}{2}]$. The minimal possible value of a_1 is $\tilde{a} = \frac{1}{3}\sqrt{6-8t^2} - \frac{t}{3}$. Moreover, $\phi(\tilde{a}) > t = \phi(a_0)$. Since ϕ is decreasing, $\tilde{a} < a_0$. Hence the maximum of f_- is attained either in $a_1 = \tilde{a}$ or in $a_1 = \frac{\sqrt{3}}{2}$. For $t \le \bar{t} \simeq 0.4259$ this occurs in $\tilde{a}$, for $t \ge \bar{t}$ in $\frac{\sqrt{3}}{2}$. In the latter case, the maximum of $A(\cdot,t)$ is in $a^{(1)}$. However, for $t_0 < t < \bar{t}$, $f_-(\tilde{a},t) < \frac{1}{2} - 2t^2 = A(a^{(2)},t)$, so that the maximum of f_- for $t_0 < t < \frac{1}{2}$ is attained either in $a^{(1)}$ or in $a^{(2)}$.

b) i) Next we consider the second case $a_1 > a_2 > t \ge a_3, a_4$. Then $a_3 + a_4 = -(a_1 + a_2)$, $a_3^2 + a_4^2 = 1 - a_1^2 - a_2^2$ implies $a_{3,4} = -\frac{a_1+a_2}{2} \pm \frac{1}{2}\sqrt{2(1-a_1^2-a_2^2) - (a_1+a_2)^2}$, where $2(1-a_1^2-a_2^2) - (a_1+a_2)^2 \ge 0$ requires $a_1 \le \frac{1}{3}\sqrt{6-8a_2^2} - \frac{a_2}{3} =: \psi(a_2) \le \sqrt{\frac{2}{3}}$. We also have $t < a_2 \le \frac{1}{2}$ $(= \sqrt{\frac{n-1}{2(n+1)}})$, as shown in Lemma 2.1. Therefore $t < a_2 \le \frac{1}{2}$, $a_1 \le \sqrt{\frac{2}{3}}$. Proposition 2.2 yields with $(a_1 - a_3)(a_1 - a_4) = 3a_1^2 + 2a_1a_2 + a_2^2 - \frac{1}{2}$,

$(a_2-a_3)(a_2-a_4) = a_1^2+2a_1a_2+3a_2^2-\frac{1}{2}$ that

(5.1)
$$A(a,t) = f(a_1,a_2,t) := \frac{1}{a_1-a_2}\Big(\frac{(a_1-t)^2}{3a_1^2+2a_1a_2+a_2^2-\frac{1}{2}} - \frac{(a_2-t)^2}{a_1^2+2a_1a_2+3a_2^2-\frac{1}{2}}\Big) .$$

The critical points of $A(\cdot,t)$ satisfy $\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = 0$. In particular,
$0 = (\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2})(a_1,a_2,t) = \frac{(a_1-a_2)[4t(a_1+a_2)-(1-2a_1^2-2a_2^2)][t(1-2a_1^2-2a_2^2)+2(a_1+a_2)^3-(a_1+a_2)]}{(3a_1^2+2a_1a_2+a_2^2-\frac{1}{2})(a_1^2+2a_1a_2+3a_2^2-\frac{1}{2})}$.
Therefore either $a_1 = a_2$ (in the limit $a_2 \to a_1$) or

$$(5.2) \qquad t = \frac{1-2(a_1+a_2)^2+4a_1a_2}{4(a_1+a_2)} \text{ or } t = \frac{(a_1+a_2)(1-2(a_1+a_2)^2)}{1-2(a_1+a_2)^2+4a_1a_2} .$$

First, suppose that $t \ge \frac{1}{2\sqrt{3}}$. Then there is no solution (a_1,a_2) of (5.2), since for $\frac{1}{2\sqrt{3}} \le t < a_2 < a_1 \le \sqrt{\frac{2}{3}}$ we have in the first case $\frac{1-2(a_1+a_2)^2+4a_1a_2}{4(a_1+a_2)} < \frac{1}{2\sqrt{3}}$ because $a_1^2+a_2^2+\frac{a_1+a_2}{\sqrt{3}}-\frac{1}{2} = (a_2-\frac{1}{2\sqrt{3}})(2a_2+\sqrt{3})+(2a_2+\frac{1}{\sqrt{3}})(a_1-a_2)+(a_1-a_2)^2 > 0$ and in the second case also $\frac{(a_1+a_2)(1-2(a_1+a_2)^2)}{1-2(a_1+a_2)^2+4a_1a_2} < \frac{1}{2\sqrt{3}}$, if $2a_1^2+2a_2^2 < 1$, with equality for $a_1 = a_2 = \frac{1}{2\sqrt{3}}$, which is not attained since $a_1 > a_2$. If $2a_1^2+2a_2^2 > 1$, the left side is > 3 and thus cannot be equal to $t \le \sqrt{\frac{2}{3}}$, either.

Hence for $t \ge \frac{1}{2\sqrt{3}}$ the only critical points of $A(\cdot,t)$ occur for $a_1 = a_2$ (limiting case). Then $a_1 = a_2 \le \frac{1}{2}$ and by l'Hospital's rule

(5.3)
$$\lim_{a_2 \nearrow a_1} f(a_1,a_2,t) = \frac{4}{(12a_1^2-1)^2}\big(a_1(8a_1^2-1)+t(1-4a_1^2)-4t^2a_1\big) =: F(a_1,t) .$$

For $a_1 = \frac{1}{2}$ we get $F(\frac{1}{2},t) = \frac{1}{2}-2t^2$, with $a = a^{(2)} = \frac{1}{2}(1,1,-1,-1)$. The critical points of F satisfy

$$\frac{\partial F}{\partial a_1} = \frac{4}{(12a_1^2-1)^3}\big[4(1+36a_1^2)t^2 - 8a_1(5-12a_1^2)t + (1+12a_1^2-96a_1^4)\big] = 0$$

with solutions

$$(5.4) \qquad t_\pm = t_\pm(a_1) = \frac{1}{1+36a_1^2}\left[a_1(5-12a_1^2) \pm \frac{1}{2}(12a_1^2-1)\sqrt{28a_1^2-1}\right] .$$

Both functions t_+ and t_- are bijective in the following ranges, $t_+ : [\frac{1}{2\sqrt{3}},\frac{1}{2}] \to [\frac{1}{2\sqrt{3}},\frac{\sqrt{6}+1}{10}]$, $t_- : [\frac{1}{2\sqrt{3}},\frac{1}{2}] \to [-\frac{\sqrt{6}-1}{10},\frac{1}{2\sqrt{3}}]$, where t_+ is increasing and t_- decreasing. Here $\frac{\sqrt{6}+1}{10} \simeq 0.3449$, $-\frac{\sqrt{6}-1}{10} \simeq -0.1449$. Then $t_\pm'(a_1)$ has the same sign as $\pm a_1(3024a_1^4+504a_1^2-31)-(216a_1^4+108a_1^2-5/2)\sqrt{28a_1^2-1}$. For t_+ this is non-negative and zero in $a_1 = \frac{1}{2\sqrt{3}}$.

For $t > \frac{1}{2\sqrt{3}}$, (5.4) admits a solution a_1 only for $t_+ = t_+(a_1)$. Moreover $t \le t_1 := \frac{\sqrt{6}+1}{10}$ is required to have a solution. Hence the solution interval for $t = t_+(a_1)$ is $t \in [\frac{1}{2\sqrt{3}},\frac{\sqrt{6}+1}{10}]$, with $a_1 \in [\frac{1}{2\sqrt{3}},\frac{1}{2}]$.

For $t > t_1$, $\frac{\partial F}{\partial a_1} > 0$ and F is increasing, with maximum in $a_1 = \frac{1}{2}$, i.e. for $a^{(2)}$: The absolute maximum of $A(\cdot, t)$ is then in $a^{(1)}$ or in $a^{(2)}$. Define $t_0 := \frac{9+4\sqrt{16-6\sqrt{3}}}{2(3\sqrt{3}+16)} \simeq 0.4357$. For $t > t_0$, the maximum of $A(\cdot, t)$ is in $a^{(1)}$, for $t < t_0$, in $a^{(2)}$. For $\frac{1}{2\sqrt{3}} < t < t_1$, solve $t = t_+(\bar{a_1})$ with $\bar{a_1} \in (\frac{1}{2\sqrt{3}}, \frac{1}{2})$. Then $\frac{\partial F}{\partial a_1}$ is positive for $\frac{1}{2\sqrt{3}} < a_1 < \bar{a_1}$ and negative for $\bar{a_1} < a_1 < \frac{1}{2}$. Hence $F(\cdot, t)$ attains its maximum in $\bar{a_1}$, with a larger value than in $a_1 = \frac{1}{2}$, i.e. larger than for $a^{(2)}$. This yields the maximum $a^{\{t\}}$ of $A(\cdot, t)$ given in Proposition 1.3.

For $t \searrow \frac{1}{2\sqrt{3}}$ we have $a_1 \searrow \frac{1}{2\sqrt{3}}$ and $F(a_1, t) \to \frac{5}{9}\frac{\sqrt{3}}{2} = \frac{5}{9}A(a^{(1)}, t)$.

ii) Secondly, we suppose that $0 \le t < \frac{1}{2\sqrt{3}}$ in the case $a_1 > a_2 > t \ge a_3, a_4$. Generally $a \in S^3$, $a_1 > 0$, $\sum_{j=1}^4 a_j = 0$ requires $a_1 \ge \frac{1}{2\sqrt{3}}$. Then $t_+(a_1) \ge \frac{1}{2\sqrt{3}}$, so that $t_+(a_1) = t$ is impossible. However, $t_-(a_1) = t$ has a solution as long as $t_-(a_1) \ge 0$. This means $a_1 \le \frac{1}{4}\sqrt{1 + \sqrt{\frac{11}{3}}} =: \tilde{a_1} \simeq 0.4268$.

But first, we consider the case $a_1 = a_2 \le \frac{1}{2}$. Then $t \ge a_3 = -a_1 + \frac{1}{2}\sqrt{2 - 8a_1^2}$ is required. For $a_1 \in [\frac{1}{2\sqrt{3}}, \frac{1}{2}]$, $t_-(a_1) \ge -a_1 + \frac{1}{2}\sqrt{2 - 8a_1^2}$ is satisfied: The difference is zero in $a_1 = \frac{1}{2\sqrt{3}}$ and increasing in a_1. We have for the function F given in (5.3) that

$$F(a_1, t_-(a_1)) = \frac{4}{(1 + 36a_1^2)^2}\left[a_1(5 + 52a_1^2) + (2a_1^2 + \frac{1}{2})\sqrt{28a_1^2 - 1}\right] . \tag{5.5}$$

In part a) we had for $a^{[t]}$ that $A(a^{[t]}, t) = \frac{1}{\sqrt{2-8t^2}} =: \Phi(t)$ and then

$$\Phi(t_-(a_1)) = \frac{1 + 36a_1^2}{2\sqrt{(1 + 4a_1^2)(1 + 36a_1^2) - 2a_1(12a_1^2 - 1)(12a_1^2 - 5)(8a_1 + \sqrt{28a_1^2 - 1})}} .$$

One can check that $\Phi(t_-(a_1) > F(a_1, t_-(a_1))$ for all $a_1 \in (\frac{1}{2\sqrt{3}}, \tilde{a_1}]$, with equality for $a_1 = \frac{1}{2\sqrt{3}}$. Therefore $a^{[t]}$ is also the maximum of $A(\cdot, t)$ in this case.

Next, we study the cases $t = \frac{1-2(a_1+a_2)^2+4a_1a_2}{4(a_1+a_2)}$ and $t = \frac{(a_1+a_2)(1-2(a_1+a_2)^2)}{1-2(a_1+a_2)^2+4a_1a_2}$ of (5.2) with $0 \le t = t_-(a_1) < \frac{1}{2\sqrt{3}}$.

We claim that the second case is impossible, since $a_3 \le t$ would not be satisfied: Let $x := a_1 + a_2$, $y := 4a_1a_2$. Then $a_3 = -\frac{x}{2} + \frac{1}{2}\sqrt{2 - 3x^2 + y}$ and $t = \frac{x(1-2x^2)}{1-2x^2+y}$, with $\frac{1}{2\sqrt{3}} \le x \le \frac{1}{\sqrt{2}}$ and $0 \le y \le x^2 \le \frac{1}{2}$. Assume $a_3 \le t$. We show the contradiction $a_3 > t$ which is equivalent to

$$(1 - 2x^2 + y)(\sqrt{2 - 3x^2 + y} - x) > 2x(1 - 2x^2) . \tag{5.6}$$

We claim that $y > 2x^2 + 2x\sqrt{1 - 2x^2} - 1$. This will imply (5.6): Since the left side of (5.6) is strictly increasing in y, (5.6) follows from
$2x\sqrt{1 - 2x^2}(\sqrt{1 - x^2 + 2x\sqrt{1 - 2x^2}} - x) \ge 2x(1 - 2x^2)$ which is true with equality sign since $(x + \sqrt{1 - 2x^2})^2 = 1 - x^2 + 2x\sqrt{1 - 2x^2}$.

Now $y > 2x^2 + 2x\sqrt{1-2x^2} - 1$ is equivalent to

$$\left(1 - 2(a_1^2 + a_2^2)\right)^2 - 4(a_1 + a_2)^2 \left(1 - 2(a_1 + a_2)^2\right)$$
$$= \left(6a_1^2 - 1 + 4a_1a_2 + 2a_2^2\right)\left(2a_1^2 - 1 + 4a_1a_2 + 6a_2^2\right) > 0 \ ,$$

which is true since $a_1 > a_2 > -\frac{a_1}{3} + \frac{1}{6}\sqrt{6 - 8a_1^2}$ is satisfied: the assumption $a_2 > t \geq a_3 = -\frac{a_1+a_2}{2} + \frac{1}{2}\sqrt{2 - 3(a_1^2 + a_2^2) - (a_1 + a_2)^2}$ yields $a_2 > -\frac{a_1}{3} + \frac{1}{6}\sqrt{6 - 8a_1^2}$. Therefore the claim for y is true and the contradiction $a_3 > t$ would follow.

This leaves the case $t = \frac{1-2(a_1+a_2)^2+4a_1a_2}{4(a_1+a_2)}$. Calculation using (5.1) shows that then $A(a,t) = \frac{1}{2(a_1+a_2)}$. This is smaller than $A(a^{[t]}, t) = \frac{1}{\sqrt{2-8t^2}}$, since $2 - 8t^2 < 4(a_1+a_2)^2$ is satisfied in view of $\left(1 - 2(a_1^2 + a_2^2)\right)^2 - 4(a_1 + a_2)^2\left(1 - 2(a_1 + a_2)^2\right) > 0$, as we just showed.

iii) This leaves only the minimum case for $0 \leq t < \frac{1}{2\sqrt{3}}$. For $a_1 = a_2$ with $\frac{1}{2\sqrt{3}} \leq a_1 = a_2 \leq \frac{1}{2}$, we have by (5.5) for $t = t_-(a_1)$

$$F(a_1, t_-(a_1)) = \frac{4}{(1 + 36a_1^2)^2}\left[a_1(5 + 52a_1^2) + (2a_1^2 + \frac{1}{2})\sqrt{28a_1^2 - 1}\right] \ .$$

This is decreasing in a_1. On the other hand, $A(a^{(1)}, t) = \left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{\sqrt{3}}{2} - t_-(a_1)\right)^2$ is increasing in a_1. For the maximal possible $a_1 \leq \tilde{a_1} \simeq 0.4268$ (zero of t_-) we have $t_-(\tilde{a_1}) = 0$ and $A(a^{(1)}, 0) = \frac{9}{32}\sqrt{3} \simeq 0.4871 < F(a_1, 0) \simeq 0.5551$. Hence $A(a^{(1)}, t) < F(a_1, t)$ for all $0 \leq t < \frac{1}{2\sqrt{3}}$, and $a^{(1)}$ is the minimum of $A(\cdot, t)$.

In the second case $t = t(a_1, a_2) = \frac{1-2(a_1+a_2)^2+4a_1a_2}{4(a_1+a_2)}$, as above $A(a,t) = \frac{1}{2(a_1+a_2)}$. This is larger than $A(a^{(1)}, t) = \left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{\sqrt{3}}{2} - t(a_1, a_2)\right)^2$: Using $a_2 \geq \frac{1}{6}\sqrt{6 - 8a_1^2} - \frac{a_1}{3}$, $a_2 > 0$, the difference $A(a,t) - A(a^{(1)}, t)$ is minimal for $a_1 = a_2 = \tilde{a_1}$ which then is $0.0986 > 0$. This case does not lead to the minimum of $A(\cdot, t)$. □

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