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2-ROTUND NORMS FOR GENERALIZED BAERNSTEIN SPACES AND THEIR DUALS

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Dedicated to the memory of Professor Nicole Tomczak-Jaegermann

Abstract. We consider a generalized Baernstein space associated to a compact family of finite subsets of an uncountable set. We show that for certain transfinitely defined families such spaces admit an equivalent 2-rotund norm. We also show that for an arbitrary family the dual space admits an equivalent 2-rotund norm.

1. Introduction

The notions of 2-rotund and weakly 2-rotund norms were introduced by Milman [13] and are defined as follows.

Definition 1.1. Let X be a Banach space. We say that a norm $\|\cdot\|$ on X is 2-rotund ($2R$) (resp. weakly 2-rotund ($W2R$)) if for every $(x_n) \subset X$ such that $\|x_n\| \leq 1$ $(n \geq 1)$ and

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2,$$

there exists $x \in X$ such that $x = \lim_{n\to\infty} x_n$ strongly (resp. weakly).

It follows from a characterization of reflexivity due to James [10] that if X admits an equivalent $W2R$ norm then X is reflexive. Hájek and Johanis proved the converse: every reflexive Banach space admits an equivalent $W2R$ norm [9]. Odell and Schlumprecht [14] proved that every separable reflexive Banach space X admits an equivalent $2R$ norm (cf. [8]). However, it is an open question whether every reflexive Banach space admits an equivalent $2R$ norm.

Let Γ be an infinite set. Throughout, $\mathcal{F}$ denotes a collection of finite subsets of Γ satisfying the following:

- $\mathcal{F}$ contains all singletons;
- $\mathcal{F}$ is hereditary, i.e., if $F \in \mathcal{F}$ and $G \subseteq F$ then $G \in \mathcal{F}$;
- $\mathcal{F}$ is compact, i.e., $\{1_F \colon F \in \mathcal{F}\}$ is a compact subset of $\{0,1\}^\Gamma$ in the topology of pointwise convergence.

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Let $(e_\gamma)_{\gamma\in\Gamma}$ denote the unit vector basis of $c_{00}(\Gamma)$ and let (e^*_γ) denote the dual basis. We define a norm $\|\cdot\|$ on $c_{00}(\Gamma)$ as follows:

$$\|\sum a_\gamma e_\gamma\| = \sup(\sum_{i=1}^{n}(\sum_{\gamma\in F_i}|a_\gamma|)^2)^{1/2}, \tag{1.1}$$

where the supremum is taken over all $n \geq 1$ and all disjoint $F_i \in \mathcal{F}$ $(1 \leq i \leq n)$.

The generalized Baernstein space $(B(\mathcal{F}), \|\cdot\|)$ is the completion of $c_{00}(\Gamma)$ with respect to $\|\cdot\|$. Note that $(e_\gamma)_{\gamma\in\Gamma}$ is a 1-unconditional basis of $B(\mathcal{F})$ and that $\|\cdot\|$ satisfies a lower 2-estimate for disjointly supported vectors x, y:

$$\|x + y\|^2 \geq \|x\|^2 + \|y\|^2. \tag{1.2}$$

The first space of this type was introduced by Baernstein [3] with $\Gamma = \mathbb{N}$ and $\mathcal{F} = \mathcal{S}_1 = \{E \subset \mathbb{N} \colon |E| \leq \min E\}$ (the Schreier family), with the extra assumption that $\max F_i < \min F_{i+1}$ for $1 \leq i \leq n-1$ in (1.1). It was the first example of a reflexive Banach space with a normalized basis (weakly null by reflexivity) whose arithmetic means do not converge strongly to zero.

The space $B(\mathcal{F})$ is reflexive (for arbitrary Γ and $\mathcal{F}$). For completeness we present a proof at the end of the paper.

The norm of $(B(\mathcal{F}, \|\cdot\|)$ and its dual norm $\|\cdot\|_*$ are not $2R$ in general. For example, for the original Baernstein space, we have

$$\|e_n + e_m\| = 2, \quad \|e^*_3 + e^*_n\|_* = 1, \|(e^*_3 + e^*_n) + (e^*_3 + e^*_m)\|_* = 2 \quad (m, n > 3),$$

and hence $\|\cdot\|$ and $\|\cdot\|_*$ are not $2R$ norms.

The following question is open to the best of our knowledge.

Question 1. *Suppose Γ is uncountable. Does $B(\mathcal{F})$ have an equivalent $2R$ norm?*

In Section 2, motivated by the Schreier hierarchy introduced in [1], we present a general method for defining, for each countable ordinal α, a family $\mathcal{F}_\alpha$ for certain uncountable Γ. The construction is similar to that of the transfinitely defined families introduced in [2]. In Section 3 we prove that, for each countable ordinal α, $B(\mathcal{F}_\alpha)$ has an equivalent $2R$ norm.

In Section 4 we prove, for arbitrary Γ and $\mathcal{F}$, that $B(\mathcal{F})^*$ admits an equivalent $2R$ norm. The renorming is essentially the same as the $W2R$ renorming given in [9].

As an application of these results we prove that the space constructed by Kutzarova and Troyanski [12] (based on a family of sets introduced in [4]) which does not admit an equivalent norm that is either uniformly rotund in every direction or uniformly differentiable in every direction does admit an equivalent $2R$ norm.

In forthcoming articles we prove positive results for other classes of spaces. In particular, in [7] we consider the existence of equivalent symmetric $2R$ norms for spaces with a symmetric basis.

2. Transfinitely defined families

- Let S be any set of cardinality at least 2 and let $\overline{S} := S^{\mathbb{N}}$.

- For distinct $p = (p(i))_{i=1}^{\infty} \in \overline{S}$ and $q = (q(i))_{i=1}^{\infty} \in \overline{S}$, let $d(p,q) = 1$ if $p(1) \neq q(1)$ and, for $k \geq 2$, let $d(p,q) = k$ if $p(k) \neq q(k)$ and $p(j) = q(j)$ for $1 \leq j \leq k-1$.
- For $A \subset \overline{S}$, with $|A| \geq 2$, let

$$A^{\sharp} = \min\{d(p,q) \colon p, q \in A, p \neq q\}.$$

We define, for each countable ordinal α, a hereditary family $\mathcal{F}_\alpha$ of finite subsets of $\overline{S}$.

- Let

$$\mathcal{F}_0 = \{\emptyset\} \cup \{\{p\} \colon p \in \overline{S}\}.$$

- If $k \geq 1$ and $\mathcal{F}$ is any collection of finite subsets of $\overline{S}$ satisfying the conditions set out in the Introduction, let

$$\mathcal{F}^{(k)} = \mathcal{F}_0 \cup \{A \in \mathcal{F} \colon A^{\sharp} \geq k\}.$$

Note that since $\mathcal{F}$ is hereditary, $\mathcal{F}^{(k)}$ is also hereditary.

- If $\alpha = \beta^+$ is a successor ordinal, let $\mathcal{F}_\alpha$ be any hereditary family satisfying the following:
  - $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$.
  - If $A \in \mathcal{F}_\alpha$ and $|A| \geq 2$, then there exist $A_i \in \mathcal{F}_\beta$ $(1 \leq i \leq A^{\sharp})$ such that

$$A = \cup_{i=1}^{A^{\sharp}} A_i.$$

- If α is a limit ordinal, choose $\alpha_r \uparrow \alpha$ $(r \geq 1)$ and define

$$\mathcal{F}_\alpha = \cup_{r=1}^{\infty} \mathcal{F}_{\alpha_r}^{(r)}.$$

Note that, for each $k \geq 1$,

$$\mathcal{F}_\alpha^{(k)} = \cup_{r=1}^{\infty} \mathcal{F}_{\alpha_r}^{(r \vee k)},$$

where $r \vee k := \max(r,k)$.

3. $B(\mathcal{F}_\alpha^{(k)})$ admits an equivalent $2R$ norm

Theorem 3.1. *For each countable ordinal α and $k \geq 1$, $B(\mathcal{F}_\alpha^{(k)})$ admits a $2R$ renorming.*

We shall use the following characterization of 2-rotundity (see e.g., [6, II.6.4] or [9]): $\|\cdot\|$ is a $2R$ norm on X if for all $(x_n) \subset X$ such that

$$\lim_{m,n\to\infty} [\|x_m + x_n\|^2 - 2(\|x_m\|^2 + \|x_n\|^2)] = 0, \tag{3.1}$$

there exists $x \in X$ such that $x = \lim_{n\to\infty} x_n$ strongly.

For $x \in B(\mathcal{F})$, the support of x, denoted $\operatorname{supp} x$, is defined by

$$\operatorname{supp} x = \{\gamma \in \overline{S} : e_\gamma^*(x) \neq 0\}.$$

Let $\|\cdot\|_{\alpha,k}$ denote the norm in $B(\mathcal{F}_\alpha^{(k)})$.

Lemma 3.2. *Let α be a limit ordinal (with $\alpha_r \uparrow \alpha$ as above) and let $k \geq 1$. Suppose that $\|x_n\|_{\alpha,k} \leq 1$ $(n \geq 1)$ and that*

$$\lim_{m,n\to\infty} \|x_m + x_n\|_{\alpha,k} = 2. \tag{3.2}$$

Then, for some $r \geq 1$, $\limsup_{n\to\infty} \|x_n\|_{\alpha_r, r\vee k} > 0$.

Proof. Suppose, to derive a contradiction, that $\lim_{n\to\infty} \|x_n\|_{\alpha_r,r\vee k} = 0$ for all $r \geq 1$; in particular, $x_n \to 0$ in $\ell_2(\overline{S})$. Hence, by a gliding hump argument, approximating by finitely disjointly supported vectors, and after passing to a subsequence and relabelling, we may assume that $\operatorname{supp} x_n$ is finite and that $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset$ if $m \neq n$.

Fix $n \geq 1$ and $F \in \mathcal{F}_\alpha^{(k)}$ satisfying

$$|F \cap \operatorname{supp} x_n| \geq 2.$$

Let

$$N = \max\{d(p,q) \colon p, q \in \operatorname{supp} x_n, p \neq q\}.$$

It follows that $F^\sharp \leq N$, and hence

$$F \in \cup_{r=1}^N \mathcal{F}_{\alpha_r}^{(r\vee k)}.$$

Let

$$x_n = \sum a_\gamma e_\gamma$$

and, for $m > n$,

$$x_m = \sum b_\gamma^m e_\gamma.$$

Since $x_n \to 0$ in $\ell_2(\overline{S})$,

$$\lim_{n\to\infty} \sum a_\gamma^2 = 0. \tag{3.3}$$

Since the supports of the x_m's are disjoint, we may assume that $a_\gamma \geq 0$ and $b_\gamma^m \geq 0$.

By assumption, $\|x_m\|_{\alpha_r,r\vee k} \to 0$ as $m \to \infty$ for all $r \geq 1$. Hence

$$\lim_{m\to\infty} \sum_{\gamma\in F} b_\gamma^m = 0 \tag{3.4}$$

uniformly over all $F \in \mathcal{F}_\alpha^{(k)}$ satisfying $|F \cap \operatorname{supp} x_n| \geq 2$.

Note that if $F_1, F_2, \dots, F_s$ are *disjoint* sets in $\mathcal{F}_\alpha^{(k)}$ satisfying $|F_i \cap \operatorname{supp} x_n| \geq 2$ $(1 \leq i \leq s)$ then $s \leq |\operatorname{supp} x_n|$. Hence (3.4) implies that

$$\sum_{\gamma\in\cup_{i=1}^s F_i} b_\gamma^m \to 0 \tag{3.5}$$

as $m \to \infty$ *uniformly* over all such collections $(F_i)_{i=1}^s$. Let $A_i = \sum_{\gamma \in F_i} a_\gamma$ and let $B_i^m = \sum_{\gamma \in F_i} b_\gamma^m$. Then

$$\begin{aligned}\sum_{i=1}^s (A_i + B_i^m)^2 &= \sum_{i=1}^s (A_i^2 + (B_i^m)^2 + 2A_i B_i^m) \\ &\leq \sum_{i=1}^s A_i^2 + (\sum_{i=1}^s B_i^m)^2 + 2(\sum_{i=1}^s B_i^m)(\sum A_i^2)^{1/2} \\ &\leq \sum_{i=1}^s A_i^2 + (\sum_{i=1}^s B_i^m)^2 + 2\|x_n\|_{\alpha,k} \sum_{i=1}^s B_i^m \\ &\leq \sum_{i=1}^s A_i^2 + (\sum_{i=1}^s B_i^m)^2 + 2\sum_{i=1}^s B_i^m.\end{aligned}$$

Note that (3.5) implies that $\sum_{i=1}^s B_i^m \to 0$ as $m \to \infty$ uniformly over all such $(F_i)_{i=1}^s$. Let $\varepsilon > 0$. It follows that for all $m \geq M(n, \varepsilon)$,

$$\sum_{i=1}^s (A_i + B_i^m)^2 < \sum_{i=1}^s A_i^2 + \varepsilon \leq \|x_n\|_{\alpha_k} + \varepsilon \leq 1 + \varepsilon \tag{3.6}$$

uniformly over all $(F_i)_{i=1}^s$. Moreover, it follows from (3.3) that for all $n \geq N(\varepsilon)$

$$\sum a_\gamma^2 < \varepsilon^2.$$

Let $J \subset \operatorname{supp} x_n$. Consider a disjoint collection consisting of $G_i \in \mathcal{F}_\alpha^{(k)}$ $(1 \leq i \leq t)$ and $G_\lambda \in \mathcal{F}_\alpha^{(k)}$ $(\lambda \in J)$ satisfying $G_i \cap \operatorname{supp}(x_n) = \emptyset$ and $G_\lambda \cap \operatorname{supp} x_n = \{\lambda\}$ $(\lambda \in J)$. Let $C_i^m = \sum_{\gamma \in G_i} b_\gamma^m$ and $C_\lambda^m = \sum_{\gamma \in G_\lambda} b_\gamma^m$. Then for all $m > n > N(\varepsilon)$,

$$\begin{aligned}\sum_{\lambda \in J} (a_\lambda + C_\lambda^m)^2 &\leq \sum_{\lambda \in J} a_\gamma^2 + \sum_{\lambda \in J} (C_\lambda^m)^2 + 2(\sum_{\lambda \in J} a_\lambda^2)^{1/2} (\sum_{\lambda \in J} (C_\lambda^m)^2)^{1/2} \\ &\leq \varepsilon + \sum_{\lambda \in J} (C_\lambda^m)^2 + 2\varepsilon \|x_m\|_{\alpha,k} \\ &\leq \varepsilon + \sum_{\lambda \in J} (C_\lambda^m)^2 + 2\varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^t (C_i^m)^2 + \sum_{\lambda \in J} (a_\lambda + C_\lambda^m)^2 &\leq 3\varepsilon + \sum_{i=1}^t (C_i^m)^2 + \sum_{\lambda \in J} (C_\lambda^m)^2 \\ &\leq 3\varepsilon + \|x_m\|_{\alpha,k}^2.\end{aligned} \tag{3.7}$$

Hence, combining (3.6) and (3.7), for all $n \geq N(\varepsilon)$ and $m > M(n, \varepsilon)$,

$$\|x_n + x_m\|_{\alpha,k}^2 \leq 2 + 4\varepsilon. \tag{3.8}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n + x_m\|_{\alpha,k} \leq \sqrt{2}, \tag{3.9}$$

which contradicts (3.2). $\square$

The following analogue for successor ordinals has a similar (but simpler) proof.

Lemma 3.3. *Let $\alpha = \beta^+$ be a successor ordinal. Suppose that $\|x_n\|_{\alpha,k} \leq 1$ $(n \geq 1)$ and that*

$$\lim_{m,n\to\infty} \|x_m + x_n\|_{\alpha,k} = 2. \tag{3.10}$$

Then

$$\limsup_{n\to\infty} \|x_n\|_{\beta,k} > 0.$$

Remark 3.4. (3.9) shows that Lemma 3.2 and Lemma 3.3 can be strengthened by replacing (3.2) and (3.10) by

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n + x_m\|_{\alpha,k} > \sqrt{2}.$$

The proof of the following lemma uses the fact that Hilbert space $(\ell_2, |\cdot|)$ is uniformly convex; specifically, for $0 < \varepsilon < 2$,

$$|x| \leq 1, |y| \leq 1, |x - y| = \varepsilon \Rightarrow |\frac{x+y}{2}| \leq 1 - \frac{\varepsilon^2}{8}. \tag{3.11}$$

We will also use the following notation: for $x = \sum_{\gamma\in\overline{S}} x_\gamma e_\gamma$ and disjoint sets $F_i \subset \overline{S}$ $(1 \leq i \leq n)$,

$$|(x; F_1, \dots, F_n)|_2 := (\sum_{i=1}^{n} (\sum_{\gamma\in F_i} x_\gamma)^2)^{1/2}.$$

Note that if $x \geq 0$, then

$$\|x\|_{\alpha,k} = \sup |(x; F_1, \dots, F_n)|_2,$$

where the supremum is taken over all $n \geq 1$ and disjoint $F_i \in \mathcal{F}_\alpha^{(k)}$.

Lemma 3.5. *Let α be a limit ordinal (with $\alpha_r \uparrow \alpha$ as above) and let $k \geq 1$. Suppose that $\|x_n\|_{\alpha,k} \leq 1$ $(n \geq 1)$, that*

$$\lim_{m,n\to\infty} \|x_m + x_n\|_{\alpha,k} = 2, \tag{3.12}$$

and that there exists $x \in \ell_2(\overline{S})$ such that, for each $r \geq 1$,

$$\lim_{n\to\infty} \|x_n - x\|_{\alpha_r, r\vee k} = 0. \tag{3.13}$$

Then $\lim_{n\to\infty} \|x_n - x\|_{\alpha,k} = 0$.

Proof. Note that

$$\|x\|_{\alpha,k} \leq \limsup_{n\to\infty} \|x_n\|_{\alpha,k} \leq 1,$$

since $x_n \to x$ pointwise. Suppose, to derive a contradiction, that the conclusion is false. Then, after passing to a subsequence and relabelling, we may assume that

$$\lim_{n\to\infty} \|x_n - x\|_{\alpha,k} = \delta > 0.$$

Let $x'_n = x_n - x$. By assumption, for all $r \geq 1$,

$$\lim_{n\to\infty} \|x'_n\|_{\alpha_r, r\vee k} = 0.$$

Let $\varepsilon > 0$. Choose a finitely supported vector y such that

$$\|x - y\|_{\alpha,k} < \frac{\varepsilon^2}{10}.$$

By a gliding hump argument, passing to a further subsequence and relabelling, we may choose disjointly supported vectors y_n ($n \geq 1$), each with finite support disjoint from the support of y, such that $\|x'_n - y_n\|_{\alpha,k} \to 0$ as $n \to \infty$ and, for all $m, n \geq 1$,

$$\|y + y_n\|_{\alpha,k} \leq 1,$$

and also

$$\|2y + y_n + y_m\|_{\alpha,k} > 2 - \frac{\varepsilon^2}{4}.$$

Hence

$$\lim_{n\to\infty} \|y_n\|_{\alpha,k} = \delta,$$

and, for all $r \geq 1$,

$$\lim_{n\to\infty} \|y_n\|_{\alpha_r, r\vee k} = \lim_{n\to\infty} \|x'_n\|_{\alpha_r, r\vee k} = 0. \tag{3.14}$$

Without loss of generality, we may assume that $y \geq 0$ and $y_n \geq 0$ for all $n \geq 1$. Fix $n \geq 1$ and let $m > n$. Suppose that $2y+y_n+y_m$ is normed by disjoint sets $F_1, \dots, F_u$ in $\mathcal{F}_{\alpha,k}$ (we suppress the dependence of F_i on n and m to simplify notation), i.e.,

$$|(2y + y_n + y_m; F_1, \dots, F_u)|_2 = \|2y + y_n + y_m\|_{\alpha,k} > 2 - \frac{\varepsilon^2}{4}.$$

Since

$$|(y + y_n; F_1, \dots, F_u)|_2 \leq \|y + y_n\|_{\alpha,k} \leq 1$$

and

$$|(y + y_m; F_1, \dots, F_u)|_2 \leq \|y + y_m\|_{\alpha,k} \leq 1,$$

the uniform convexity of ℓ_2 yields

$$|(y_n - y_m; F_1, \dots, F_u)|_2 < \varepsilon.$$

We may assume that $F_1, \dots, F_s$ have nonempty intersection with *both* $\operatorname{supp} y$ and $\operatorname{supp} y_n$, that $F_{s+1}, \dots, F_t$ intersect $\operatorname{supp} y$ but *not* $\operatorname{supp} y_n$, and that $F_{t+1}, \dots, F_u$ do *not* intersect $\operatorname{supp} y$. Note that $s \leq |\operatorname{supp} y|$ and $|F_i \cap \operatorname{supp}(y + y_n)| \geq 2$ for $1 \leq i \leq s$. Hence, repeating the argument used to prove (3.5), we deduce that

$$\lim_{m\to\infty} \sum_{\gamma \in \cup_{i=1}^s F_i} b_\gamma^m = 0 \tag{3.15}$$

for $y_m = \sum b_\gamma^m e_\gamma$. Hence

$$|(y_m; F_1, \dots, F_s)|_2 < \frac{\varepsilon}{2}$$

for all $m > M_1(n, \varepsilon)$.

Note that y_n vanishes on F_i for $s+1 \le i \le t$. Hence, for all $m > M_1(n,\varepsilon)$,

$$\begin{aligned}|(y_n + y_m; F_1, \dots, F_t)|_2 &= (\sum_{i=1}^{s}(\sum_{\gamma\in F_i}(b_\gamma^n + b_\gamma^m))^2 + \sum_{i=s+1}^{t}(\sum_{\gamma\in F_i} b_\gamma^m)^2)^{1/2} \\ &\le (\sum_{i=1}^{s}(\sum_{\gamma\in F_i}(b_\gamma^n - b_\gamma^m))^2 + \sum_{i=s+1}^{t}(\sum_{\gamma\in F_i} b_\gamma^m)^2)^{1/2} \\ &+ 2(\sum_{i=1}^{s}(\sum_{\gamma\in F_i} b_\gamma^m)^2)^{1/2}\end{aligned}$$

(by the triangle inequality in ℓ_2)

$$\begin{aligned}&= |(y_n - y_m; F_1, \dots, F_t)|_2 + 2|(y_m; F_1, \dots F_s)|_2 \\ &\le \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

So

$$\begin{aligned}|(2y + y_n + y_m; F_1, \dots, F_t)|_2 &\le 2|(y; F_1, \dots, F_t)|_2 + |(y_n + y_m; F_1, \dots, F_t)|_2 \\ &\le 2\|y\|_{\alpha,k} + 2\varepsilon.\end{aligned}$$

Thus,

(3.16)

$$\begin{aligned}(2 - \frac{\varepsilon^2}{4})^2 &< |(2y + y_n + y_m; F_1, \dots, F_u)|_2^2 \\ &= |(2y + y_n + y_m; F_1, \dots, F_t)|_2^2 + |(y_n + y_m; F_{t+1}, \dots, F_u)|_2^2\end{aligned}$$

(since y vanishes on F_i for $t+1 \le i \le u$)

$$\le (2\|y\|_{\alpha,k} + 2\varepsilon)^2 + \|y_n + y_m\|_{\alpha,k}^2.$$

Since y, y_n, and y_m are disjointly supported, we have

$$\|y\|_{\alpha,k}^2 + \|y_n\|_{\alpha,k}^2 \le \|y + y_n\|_{\alpha,k}^2 \le 1 \tag{3.17}$$

and

$$\|y\|_{\alpha,k}^2 + \|y_m\|_{\alpha,k}^2 \le \|y + y_m\|_{\alpha,k}^2 \le 1. \tag{3.18}$$

Combining (3.16), (3.17), and (3.18),

$$\begin{aligned}4\|y\|_{\alpha,k}^2 + 2(\|y_n\|_{\alpha,k}^2 + \|y_m\|_{\alpha,k}^2) &\le 4 \\ &= (2 - \frac{\varepsilon^2}{4})^2 + \varepsilon^2 - \frac{\varepsilon^4}{16} \\ &\le (2\|y\|_{\alpha,k} + 2\varepsilon)^2 + \|y_n + y_m\|_{\alpha,k}^2 + \varepsilon^2 \\ &\le 4\|y\|_{\alpha,k}^2 + \|y_n + y_m\|_{\alpha,k}^2 + (8\varepsilon + 5\varepsilon^2).\end{aligned}$$

Hence for all $m > M_1(n,\varepsilon)$,

$$\|y_n + y_m\|_{\alpha,k}^2 + (8\varepsilon + 5\varepsilon^2) \ge 2(\|y_n\|_{\alpha,k}^2 + \|y_m\|_{\alpha,k}^2). \tag{3.19}$$

Now suppose ε is chosen so that $8\varepsilon + 5\varepsilon^2 < 2\delta^2$. Since $\lim_{n\to\infty} \|y_n\|_{\alpha,k} = \delta$, it follows from (3.19) that

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \|y_n + y_m\|_{\alpha,k} > \sqrt{2}\delta.$$

which contradicts Remark 3.4 since, for all $r \geq 1$,

$$\lim_{n\to\infty} \|y_n\|_{\alpha_r, r\vee k} = 0.$$

□

The following analogue for successor ordinals has a similar (but simpler) proof.

Lemma 3.6. *Let $\alpha = \beta^+$ be a successor ordinal. Suppose that $\|x_n\|_{\alpha,k} \leq 1$ ($n \geq 1$), that*

$$\lim_{m,n\to\infty} \|x_m + x_n\|_{\alpha,k} = 2, \tag{3.20}$$

and that there exists $x \in \ell_2(\overline{S})$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{\beta,k} = 0. \tag{3.21}$$

Then $\lim_{n\to\infty} \|x_n - x\|_{\alpha,k} = 0$.

Proof of Theorem 3.1. We will prove the result for a fixed α and for all $k \geq 1$ by transfinite induction on α. The result clearly holds for $\alpha = 0$ since $B(\mathcal{F}_0^{(k)}) = B(\mathcal{F}_0) = \ell_2(\overline{S})$ for all $k \geq 1$. So suppose the result holds for all $\beta < \alpha$ and for all $k \geq 1$.

Case I: α is a limit ordinal. So $\mathcal{F}_\alpha^{(k)} = \cup_{r=1}^\infty \mathcal{F}_{\alpha_r}^{(r\vee k)}$, where $\alpha_r \uparrow \alpha$. By inductive hypothesis, each $B(\mathcal{F}_{\alpha_r}^{(r\vee k)})$ admits an equivalent $2R$ norm $|||\cdot|||_{\alpha_r, r\vee k}$. Note that

$$|||\cdot|||_{\alpha_r, r\vee k} \leq C_r \|\cdot\|_{\alpha,k}$$

for some $C_r < \infty$. Thus,

$$|||\cdot|||_{\alpha,k}^2 := \|\cdot\|_{\alpha,k}^2 + \sum_{r=1}^\infty \frac{1}{2^r C_r^2} |||\cdot|||_{\alpha_r, r\vee k}^2$$

defines an equivalent norm $|||\cdot|||_{\alpha,k}$ on $B(\mathcal{F}_\alpha^{(k)})$. Let us show that $|||\cdot|||_{\alpha,k}$ is a $2R$ norm. Suppose that $(x_n) \subset B(\mathcal{F}_\alpha^{(k)})$ satisfies

$$\lim_{m,n\to\infty} |||x_n + x_m|||_{\alpha,k}^2 - 2(|||x_n|||_{\alpha,k}^2 + |||x_m|||_{\alpha,k}^2) = 0.$$

Note that

$$\begin{aligned} &|||x_n + x_m|||_{\alpha,k}^2 - 2(|||x_n|||_{\alpha,k}^2 + |||x_m|||_{\alpha,k}^2) \\ &\leq -(\|x_n\|_{\alpha,k} - \|x_m\|_{\alpha,k})^2 - \sum_{r=1}^\infty \frac{1}{2^r C_r^2} (|||x_n|||_{\alpha_r,k} - |||x_m|||_{\alpha+r,k})^2. \end{aligned}$$

It follows that $\lim_{n\to\infty} \|x_n\|_{\alpha,k} = L$ for some $L \geq 0$, that

$$\lim_{m,n\to\infty} \|x_n + x_m\|_{\alpha,k}^2 - 2(\|x_n\|_{\alpha,k}^2 + \|x_m\|_{\alpha,k}^2) = 0, \tag{3.22}$$

and that, for all $r \geq 1$,

$$\lim_{m,n\to\infty} \|x_n + x_m\|^2_{\alpha_r,k\vee r} - 2(\|x_n\|^2_{\alpha_r,k\vee r} + \|x_m\|^2_{\alpha_r,k\vee r}) = 0.$$

Since each $\|\cdot\|_{\alpha_r,r\vee k}$ is a $2R$ norm, it follows from (3.1) that there exists $x \in \ell_2(\overline{S})$ such that, for all $r \geq 1$,

$$\lim_{n\to\infty} \|x_n - x\|_{\alpha_r,r\vee k} = 0.$$

Moreover, (3.22) implies that

$$\lim_{m,n\to\infty} \|x_n + x_m\|_{\alpha,k} = 2L.$$

So, by Lemma 3.5,

$$\lim_{n\to\infty} \|x_n - x\|_{\alpha,k} = 0,$$

and hence

$$\lim_{n\to\infty} \|x_n - x\|_{\alpha,k} = 0$$

as desired.

Case II: $\alpha = \beta^+$ is a successor ordinal. The proof is very similar to the limit ordinal case. By the inductive hypothesis, $B(\mathcal{F}^{(k)}_\beta)$ admits an equivalent $2R$ norm $\|\cdot\|_{\beta,k}$. Let

$$\|\cdot\|^2_{\alpha,k} = \|\cdot\|^2_{\alpha,k} + \|\cdot\|^2_{\beta,k}.$$

Using Lemma 3.6 instead of Lemma 3.5 and repeating the argument of Case I shows that $\|\cdot\|_{\alpha,k}$ is a $2R$ norm. □

4. $B(\mathcal{F})^*$ admits an equivalent $2R$ norm

Let $\mathcal{F}$ be a compact, hereditary family of finite subsets of an infinite set Γ containing all singleton sets. We prove in Section 5 that $(B(\mathcal{F}), \|\cdot\|)$ is reflexive. Day [5] introduced the norm $\|\cdot\|_{\text{Day}}$ on $c_0(\Gamma)$ defined by

$$\|\sum a_\gamma e_\gamma\|_{\text{Day}} = \sup(\sum_{i=1}^n 4^{-i}|a_{\gamma_i}|^2)^{1/2},$$

where the supremum is taken over all $n \geq 1$ and all choices of distinct $\gamma_i \in \Gamma$ $(1 \leq i \leq n)$. We define an equivalent norm on $B(\mathcal{F})^*$ thus:

$$\|x\|^2 = \|x\|^2_* + \|x\|^2_{\text{Day}} \qquad (x \in B(\mathcal{F})^*).$$

The following result is essentially due to Hájek and Johannis. It is a consequence of Theorem 3 and Corollary 4 of [9] and the reflexivity of $B(\mathcal{F})^*$.

Lemma 4.1. *Suppose $(y_n) \subset B(\mathcal{F})^*$ satisfies*

$$\lim_{m,n\to\infty} \|y_n + y_m\|^2 - 2(\|y_n\|^2 + \|y_m\|^2) = 0. \tag{4.1}$$

Then there exists $y \in B(\mathcal{F})^*$ such that

$$y_n \to y \quad \textit{weakly as } n \to \infty$$

and

$$\lim_{n\to\infty} \|y_n - y\|_\infty = 0.$$

Dualizing (1.2), the dual space $(B(\mathcal{F})^*, \|\cdot\|_*)$ satisfies an upper 2-estimate for disjointly supported vectors $x, y \in B(\mathcal{F})^*$:

$$\|x + y\|_*^2 \le \|x\|_*^2 + \|y\|_*^2.$$

Moreover, for all $x \in B(\mathcal{F})^*$ and $F \in \mathcal{F}$,

$$\|x \cdot 1_F\|_* = \|x \cdot 1_F\|_\infty \le \|x\|_\infty.$$

Lemma 4.2. *Suppose that y and y_n have disjoint finite supports ($n \ge 1$), that*

$$\|y\|_* = \|y_n\|_* = 1 \qquad (n \ge 1),$$

and that

$$\lim_{n\to\infty} \|y_n\|_\infty = 0.$$

Then, for all $\delta > 0$,

$$\lim_{n\to\infty} \|y + \delta y_n\|_* = (1 + \delta^2)^{1/2}.$$

Proof. We may assume that $y \ge 0$ and $y_n \ge 0$. Choose positive norming vectors $x, x_n \in B(\mathcal{F})$ with

$$(x, y) = \|x\| = \|y\|_* = 1, \quad (x_n, y_n) = \|x_n\| = \|y_n\|_* = 1,$$

where $(\cdot, \cdot)$ denotes the duality pairing for $B(\mathcal{F}) \times B(\mathcal{F})^*$. Note that x and x_n have disjoint finite supports ($n \ge 1$). Fixing $n \ge 1$, choose disjoint $F_i \in \mathcal{F}$ ($1 \le i \le N$) such that

$$\|x + \delta x_n\| = |(x + \delta x_n; F_1, \ldots, F_N)|_2.$$

We may assume that only $F_1, \ldots, F_k$ have non-empty intersection with both $\operatorname{supp} x$ and $\operatorname{supp} x_n$. Note that

$$k \le M := |\operatorname{supp} x|.$$

For each $1 \le i \le k$,

$$\|y_n \cdot 1_{F_i}\|_* \le \|y_n\|_\infty.$$

Hence

$$\sum_{i=1}^{k} \|y_n \cdot 1_{F_i}\|_* \le M\|y_n\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Let $F = \cup_{i=1}^k F_i$. (To simplify notation we suppress the dependence of F on n.) Then

$$\begin{aligned}\|x_n - x_n \cdot 1_F\| &\ge (x_n - x_n \cdot 1_F, y_n) \\ &= (x_n, y_n - y_n \cdot 1_F) \\ &= 1 - (x_n, y_n \cdot 1_F) \\ &\ge 1 - \sum_{i=1}^{k} \|y_n \cdot 1_{F_i}\|_* \\ &\to 1 \quad \text{as } n \to \infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty} (\|x_n\|^2 - \|x_n - x_n \cdot 1_F\|^2) = 1 - \lim_{n\to\infty} \|x_n - x_n \cdot 1_F\|^2 = 0.$$

Since $(B(\mathcal{F}, \|\cdot\|)$ satisfies a lower 2-estimate, it follows that $\|x_n \cdot 1_F\| \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}1 + \delta^2 &\le \liminf_{n\to\infty} \|x + \delta x_n\|^2 \\ &\le \limsup_{n\to\infty} \|x + \delta x_n\|^2 \\ &= \limsup_{n\to\infty} |(x + \delta x_n; F_1, \dots F_N)|_2^2 \\ &= \limsup_{n\to\infty} |(x + \delta x_n - \delta x_n \cdot 1_F; F_1, \dots, F_N)|_2^2 \\ &\le \limsup_{n\to\infty} [\|x\|^2 + \delta^2 \|x_n - x_n \cdot 1_F\|^2]\end{aligned}$$

(since no F_i intersects both $\operatorname{supp} x$ and $\operatorname{supp}(x_n - x_n \cdot 1_F)$)

$$= 1 + \delta^2.$$

Thus,

$$\lim_{n\to\infty} \|x + \delta x_n\|^2 = 1 + \delta^2,$$

and hence

$$(1 + \delta^2)^{1/2} \ge \limsup_{n\to\infty} \|y + \delta y_n\|_*$$

(since $\|\cdot\|_*$ satisfies an upper 2-estimate)

$$\begin{aligned}&\ge \liminf_{n\to\infty} \|y + \delta y_n\|_* \\ &\ge \liminf_{n\to\infty} \frac{(x + \delta x_n, y + \delta y_n)}{\|x + \delta x_n\|} \\ &= \frac{1 + \delta^2}{(1 + \delta^2)^{1/2}} \\ &= (1 + \delta^2)^{1/2}.\end{aligned}$$

□

Theorem 4.3. *$|\!|\!|\cdot|\!|\!|$ is an equivalent $2R$ norm for $B(\mathcal{F})^*$.*

Proof. Suppose $(y_n) \subset B(\mathcal{F})^*$ satisfies (4.1). By Lemma 4.1 there exists $y \in B(\mathcal{F})^*$ such that $y = w - \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} \|y_n - y\|_\infty = 0$. Suppose, to derive a contradiction, that (y_n) does not converge strongly to y. Passing to a subsequence and relabelling, we may assume that $y_n = y + z_n$ where

$$\lim_{n\to\infty} \|z_n\|_* = \delta > 0, \lim_{n\to\infty} \|z_n\|_\infty = 0,$$

and that the following limits exist:

$$\lim_{n\to\infty} \|y + z_n\|_*, \lim_{n,m\to\infty} \|2y + z_n + z_m\|_*.$$

Let $\varepsilon > 0$. By passing to a further subsequence, a gliding hump argument and the fact that $\lim_{n\to\infty} \|z_n\|_\infty = 0$ show that there exist vectors y' and z'_n $(n \ge 1)$ with disjoint finite supports such that

$$\|y - y'\|_* < \varepsilon, \lim_{n\to\infty} \|z_n - z'_n\|_* = 0. \tag{4.2}$$

Note that (4.1) implies that

$$\lim_{m,n\to\infty} \|2y + z_n + z_m\|_*^2 - 2(\|y + z_n\|_*^2 + \|y + z_m\|_*^2) = 0. \tag{4.3}$$

Since $\lim_{n\to\infty} \|z'_n\|_\infty = 0$, Lemma 4.2 yields

$$\lim_{n\to\infty} \|y' + z'_n\|_*^2 = \|y'\|_*^2 + \lim_{n\to\infty} \|z'_n\|_*^2 = \|y'\|_*^2 + \delta^2.$$

Since $(B(\mathcal{F}, \|\cdot\|_*)$ satisfies an upper 2-estimate,

$$\lim_{n,m\to\infty} \|2y' + z'_n + z'_m\|_*^2 \le 4\|y'\|_*^2 + 2\delta^2.$$

Hence

$$\begin{aligned}&\limsup_{n,m\to\infty} [\|2y' + z'_n + z'_m\|_*^2 - 2(\|y' + z'_n\|_*^2 + \|y' + z'_m\|_*^2)]\\ &\le 4\|y'\|_*^2 + 2\delta^2 - 2(2\|y'\|_*^2 + 2\delta^2)\\ &= -2\delta^2,\end{aligned}$$

which contradicts (4.2) and (4.3) provided ε is sufficiently small. □

Based on a family of sets introduced in [4], Kutzarova and Troyanski [12] constructed a Banach space Y which does not admit an equivalent norm that is uniformly rotund or uniformly differentiable in every direction. As an application of our results, we show that Y does admit an equivalent $2R$ norm.

Corollary 4.4. *The Banach space Y defined in* [12] *admits an equivalent $2R$ norm.*

Proof. The space Y is defined as $X \oplus X^*$, where X is defined below.

Let $S = \mathbb{N}$. Let $\mathcal{F}_1$ be the collection of all finite subsets F of $\overline{S}$ such that, if $|F| \ge 2$, then for all $p \in F$, $p(1) = 1$ and $p(i) \in \{1, 2, \dots, i-1\}$ for all $i \ge 2$ and such that there exists $m \ge 3$ such that for all distinct $p, q \in F$, $p(i) = q(i)$ for all $1 \le i \le m-1$ and $p(m) \ne q(m)$, which implies that $F^\sharp = m$ and $|F| = m-1$. Let $\|\cdot\|_1$ denote the norm of $B(\mathcal{F}_1)$. The space X is defined to be the closed linear span of

$$\{e_p \colon p(1) = 1, p(i) \in \{1, 2, \dots, i-1\}, i \ge 2\}$$

in $B(\mathcal{F}_1)$. The successor case of the proof of Theorem 3.1 shows that

$$|\!|\!|\cdot|\!|\!|^2 = \|\cdot\|_1^2 + \|\cdot\|_{\ell_2(\overline{S})}^2$$

is an equivalent $2R$ norm on $B(\mathcal{F}_1)$. Hence $|\!|\!|\cdot|\!|\!|$ restricts to an equivalent $2R$ norm on X. By Theorem 4.3, $B(\mathcal{F}_1)^*$ admits an equivalent $2R$ norm. Note that X^* is isomorphic to a quotient space of $B(\mathcal{F}_1)^*$. It is easily seen that a quotient norm of a $2R$ norm is $2R$. Hence X^* admits an equivalent $2R$ norm, $|\!|\!|\cdot|\!|\!|'$ say. Finally,

$$\|(x, x^*)\| = \sqrt{|\!|\!|x|\!|\!|^2 + |\!|\!|x^*|\!|\!|'^2} \qquad ((x, x^*) \in X \oplus X^*)$$

is an equivalent $2R$ norm on $X \oplus X^* = Y$. □

5. $B(\mathcal{F})$ IS REFLEXIVE

Theorem 5.1. *For arbitrary* Γ *and* $\mathcal{F}$*,* $B(\mathcal{F})$ *is reflexive.*

Proof. First, we consider the case $\Gamma = \mathbb{N}$. Let (e_n) denote the unit vector basis of $B(\mathcal{F})$.

Let $(F_i)_{i=1}^{\infty} \subset \mathcal{F}$ be a collection of disjoint elements of $\mathcal{F}$ and suppose that $\sum_{i=1}^{\infty} |a_i|^2 \le 1$. For $x = \sum_{i=1}^{\infty} x_i e_i \in B(\mathcal{F})$,

$$|\sum_{i=1}^{\infty} a_i(\sum_{j\in F_i} x_j)| \le (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}\|x\| \le \|x\|. \tag{5.1}$$

Hence we may identify $\sum_{i=1}^{\infty} a_i 1_{F_i} \in \ell_\infty$ with the element k in the unit ball of $B(\mathcal{F})^*$ defined by (5.1).

Suppose $x \in B(\mathcal{F})$ has finite support and that

$$\|x\| = (\sum_{i=1}^{n}(\sum_{j\in G_i} |x_j|)^2)^{1/2}$$

for disjoint $G_i \in \mathcal{F}$. There exist nonnegative $a_1, \dots, a_n$ with $\sum_{i=1}^{n} a_i^2 = 1$ such that

$$\sum_{i=1}^{n} a_i(\sum_{j\in G_i} |x_j|) = \|x\|,$$

and there exist $H_i \subseteq G_i$ $(1 \le i \le n)$ such that

$$\sum_{i=1}^{n} a_i|\sum_{j\in H_i} x_j| \ge \frac{1}{2}\sum_{i=1}^{n} a_i(\sum_{j\in G_i} |x_j|) = \frac{1}{2}\|x\|.$$

Note that $H_i \in \mathcal{F}$ since $\mathcal{F}$ is hereditary. Hence the collection of linear functionals with a representation of the form

$$k = \sum_{i=1}^{\infty} a_i 1_{F_i} \in \ell_\infty \qquad (\sum_{i=1}^{\infty} |a_i|^2 = 1, (F_i)_{i=1}^{\infty} \text{ disjoint sets in } \mathcal{F})$$

is a 2-norming set for $B(\mathcal{F})$. It follows that the discretized collection

$$K = \{\sum_{r=1}^{\infty} \pm 2^{-s(r)} 1_{F_r} \colon \sum_{r=1}^{\infty} 2^{-2s(r)} \le 1\}.$$

is a 4-norming set.

Let us show that $K \subset \ell_\infty$ is compact in the topology of pointwise convergence on ℓ_∞. For $n \ge 1$, let

$$k_n = \sum_{r=1}^{\infty} 2^{-r}(1_{U_r^n} - 1_{V_r^n}),$$

where $U_r^n = \cup_{i=1}^{p(n,r)} F_i^n$ and $V_r^n = \cup_{i=1}^{q(n,r)} G_i^n$, and for each $n \ge 1$,

$$\{F_{r,i}^n, G_{r,j}^n \colon r \ge 1, 1 \le i \le p(n,r), 1 \le j \le q(n,r)\}$$

is a collection of nonempty disjoint elements of $\mathcal{F}$, and

$$\sum_{r=1}^{\infty} 2^{-2r}(p(n,r)+q(n,r)) \leq 1.$$

In particular, $p(n,r)+q(n,r) \leq 2^{2r}$ for all $n, r \geq 1$. By a diagonal argument, passing to a subsequence and relabelling, we may assume that

$$p(n,r) = p_r, q(n,r) = q_r \quad \text{for all } n \geq r.$$

By compactness of $\mathcal{F}$, we may also assume that

$$\lim_{n\to\infty} F^n_{r,i} = F_{r,i} \quad (1 \leq i \leq p_r), \lim_{n\to\infty} G^n_{r,j} = G_{r,j} \quad (1 \leq j \leq q_r),$$

where

$$\{F_{r,i}, G_{r,j} \colon r \geq 1, 1 \leq i \leq p_r, 1 \leq j \leq q_r\}$$

is a collection of disjoint (possibly empty) elements of $\mathcal{F}$ and

$$\sum_{r=1}^{\infty} 2^{-2r}(p_r + q_r) \leq 1.$$

Set $F_r = \cup_{i=1}^{p_r} F_{r,i}$ and $G_r = \cup_{i=1}^{q_r} G_{r,i}$. It follows that

$$k = \sum_{r=1}^{\infty} 2^{-r}(1_{F_r} - 1_{G_r}) \in K$$

and $k = \lim_{n\to\infty} k_n$ pointwise in ℓ_∞. So K is compact (and metrizable) in the topology of pointwise convergence.

For $x \in B(\mathcal{F})$, define $\hat{x} \colon K \to \mathbb{R}$ by $\hat{x}(k) = k(x)$. Suppose that $(k_n) \subset K$ and $k_n \to k$ pointwise in ℓ_∞. Clearly, $\hat{x}(k_n) \to \hat{x}(k)$ when x has finite support. Since the finitely supported vectors are norm-dense in $B(\mathcal{F})$, it follows that $\hat{x}(k_n) \to \hat{x}(k)$ for all $x \in B(\mathcal{F})$, i.e., that $\hat{x}$ is continuous on K. Since K is 4-norming for $B(\mathcal{F})$, the mapping $x \mapsto \hat{x}$ defines a linear isomorphism from $B(\mathcal{F})$ onto a closed subspace of $C(K)$.

Suppose that $(x_n) \subset B(\mathcal{F})$ is bounded and coordinatewise null with respect to (e_n). It follows from (5.1) that

$$\lim_{n\to\infty} \hat{x}_n(k) = 0 \qquad (k \in K).$$

Hence, by the Riesz representation and bounded convergence theorems, $\hat{x}_n \to 0$ weakly in $C(K)$. In particular, if $x_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i$, where $p_{n-1} < p_n$, is a bounded block basis of (e_n), then $(\hat{x}_n)$ is weakly null in $C(K)$. Hence (x_n) is weakly null in $B(\mathcal{F})$, which implies that (e_n) is a *shrinking* basis. On the other hand, since (e_n) satisfies a lower 2-estimate, it is *boundedly complete.* It follows from a theorem of James [11] that $B(\mathcal{F})$ is reflexive.

Next suppose that Γ is uncountable. Let Γ_0 be a countably infinite subset of Γ. Then

$$X_0 = \{x \in B(\mathcal{F}) \colon \operatorname{supp} x \subseteq \Gamma_0\}$$

is the Baernstein space on Γ_0 corresponding to the family $\mathcal{F}_0 = \{F \cap \Gamma_0 \colon F \in \mathcal{F}\}$. By the first part of the proof, X_0 is reflexive. But every separable subspace of $B(\mathcal{F})$

is contained in X_0 for some Γ_0. Hence every separable subspace of $B(\mathcal{F})$ is reflexive, which implies that $B(\mathcal{F})$ is also reflexive since reflexivity is separably determined. $\square$

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