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VERTEX GENERATED POLYTOPES

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In warm memory of Nicole, whose spirit
and kindness continue to inspire us all.

Abstract. In this paper we define and investigate a class of polytopes which we call "vertex generated" consisting of polytopes which are the average of their 0 and n dimensional faces. We show many results regarding this class, among them: that the class contains all zonotopes, that it is dense in dimension $n = 2$, that any polytope can be summed with a zonotope so that the sum is in this class, and that a strong form of the celebrated "Maurey Lemma" holds for polytopes in this class. We introduce for every polytope a parameter which measures how far it is from being vertex-generated, and show that when this parameter is small, strong covering properties hold.

1. Introduction

Given a polytope $P \subset \mathbb{R}^n$ we let $V(P)$ denote the set of its vertices. Our starting point is the following simple observation which follows directly from Caratheodory's theorem: for every polytope $P \subseteq \mathbb{R}^n$,

$$P = \frac{n}{n+1}P + \frac{1}{n+1}V(P). \tag{1.1}$$

Here $A + B = \{a + b : a \in A, b \in B\}$ denotes the Minkowski sum of the sets A and B. Indeed, every $x \in P$ can be written as $x = \sum_{i=1}^{n+1} \lambda_i v_i$ for some $v_i \in V(P)$ and $\lambda_i \geq 0$ with $\sum_{i=1}^{n+1} \lambda_i = 1$. Assuming without loss of generality that λ_i are decreasing, so that $\lambda_1 \geq 1/(n+1)$, we get $x = v_1/(n+1) + ny/(n+1)$ where $y = ((n+1)/n)(\sum_{i=2}^{n+1} \lambda_i v_i + (\lambda_1 - 1/(n+1))v_1) \in P$.

In [10] Schneider defined the following parameter, measuring in an affine way the deviation of a compact set A from its convex hull:

$$\begin{aligned} c(A) &= \inf\{t \geq 0 : A + t\,\mathrm{conv}(A) \text{ is convex}\} \\ &= \inf\{t \geq 0 : A + t\,\mathrm{conv}(A) = (1+t)\mathrm{conv}(A)\}. \end{aligned}$$

In this note the set A will be the set of vertices of a given polytope, and it will be more convenient to use a slightly different normalization. We define

Definition 1.1. For a polytope $P \subseteq \mathbb{R}^n$ define

$$\lambda(P) = \sup\{\lambda : P = (1-\lambda)P + \lambda V(P)\}.$$

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Thus, $\lambda(P) = 1/(1 + c(V(P)))$. The above observation (1.1) implies that for any polytope $P \subseteq \mathbb{R}^n$ one has $\lambda(P) \geq 1/(n+1)$ (equiv. $c(A) \leq n$). The only polytopes for which $\lambda(P) = 1/(n+1)$ are simplices. This was proved by Schneider in [10] see [11, Theorem 3.1.9], see Proposition 7.5 for an alternative proof.

We call a polytope $P \subseteq \mathbb{R}^n$ "λ-vertex generated" if $\lambda(P) \geq \lambda$, and we denote the class of λ-vertex generated polytopes by $\mathrm{VG}(\mathbb{R}^n, \lambda)$. Clearly $\mathrm{VG}(\mathbb{R}^n, \lambda)$ is a decreasing family in λ. It is easy to check that $\lambda(P) \leq 1/2$ for any polytope P by considering a 1-dimensional face of P. We let $\mathrm{VG}(\mathbb{R}^n) = \mathrm{VG}(\mathbb{R}^n, 1/2)$ be the smallest of these classes, and call the polytopes in this class "vertex generated". Our main objective in this note is to study the class $\mathrm{VG}(\mathbb{R}^n)$, namely polytopes for which

$$P + V(P) = 2P.$$

The name "λ-vertex generated" stems from the fact that the equality in the definition of $\lambda(P)$ can be iterated. As we show in Proposition 6.1, $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ if and only if

$$P = \mathrm{cl}\left(\sum_{i=0}^{\infty} (1-\lambda)^i \lambda V(P) \right),$$

where $\mathrm{cl}(A)$ denotes the closure of the set A.

This paper is devoted to studying some interesting features of these classes. We present several of these in the introduction, and the text contains several other results and observations. Our first main result is that for any λ, the sum of a polytope in $\mathrm{VG}(\mathbb{R}^n, \lambda)$ with a segment is also in $\mathrm{VG}(\mathbb{R}^n, \lambda)$.

Theorem 1.2. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Given $x, y \in \mathbb{R}^n$ denote $\ell = [x, y] = \{(1-\mu)x + \mu y : 0 \leq \mu \leq 1\} \subset \mathbb{R}^n$. For any $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ it holds that $P + \ell \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ as well.*

As a consequence, we get that zonotopes are vertex generated. Recall that a zonotpe in $\mathbb{R}^n$ is the Minkowski sum of a finite number of segments. The class of zonotopes is well studied, see [11] and references therein. Indeed, Theorem 1.2 implies that all zonotopes are vertex generated, since a segment is clearly vertex generated.

Corollary 1.3. *Let $n \geq 1$. Let $Z \subset \mathbb{R}^n$ be a zonotope. Then $Z \in \mathrm{VG}(\mathbb{R}^n)$.*

In the spirit of "generalized zonoids", see [11], we show that all polytopes are so-called "generalized $\mathrm{VG}(\mathbb{R}^n)$", namely that any polytope can be summed with a zonotope and become vertex generated.

Theorem 1.4. *Let $n \geq 1$, and let $P \subseteq \mathbb{R}^n$ be a polytope. Then there exists a zonotope $Z \subseteq \mathbb{R}^n$ such that $P + Z$ is vertex generated.*

A natural question is whether any polytope can be approximated by vertex generated polytopes. We verify this in the plane, and prove that in dimension $n = 2$ vertex generated polytopes are dense with respect to the Hausdorff metric, in the class of all convex bodies.

Theorem 1.5. *Let $P \subseteq \mathbb{R}^2$ be a compact convex set. Then there exists a sequence $(P_m)_{m=1}^{\infty} \subset \mathrm{VG}(\mathbb{R}^2)$ with $P_m \to P$ in the Hausdorff metric.*

Theorem 1.2 implies that the Minkowski sum of a vertex generated polytope and a zonotope remains vertex generated. It is not clear whether $\mathrm{VG}(\mathbb{R}^n, \lambda)$ is generally closed under Minkowski addition (nor whether it is closed under projections). However, we are able to show that this holds for any pair of centrally symmetric polytopes which are in a "generic" position. To formulate the result, let $N_P(v)$ denote the cone of normal vectors of a polytope P at a vertex v, namely

$$N_P(v) = \{y \in \mathbb{R}^n : \langle y, x - v\rangle \leq 0, \ \ \forall x \in P\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$. A "generic pair" is defined as follows.

Definition 1.6. Given two polytopes $P, Q \subseteq \mathbb{R}^n$ with non-empty interior, we say that they are a "generic pair" if, given $v \in V(P)$ and $w \in V(Q)$, it holds that

$$N_P(v) \cap N_Q(w) \neq \emptyset \implies \mathrm{int}(N_P(v)) \cap \mathrm{int}(N_Q(w)) \neq \emptyset.$$

Proposition 1.7. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Suppose that $P, Q \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ are centrally symmetric and form a generic pair. Then $P+Q \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ as well.*

The paper is organized as follows. In Section 2 we show that a λ-vertex generated polytope must have exponentially many vertices and that faces of λ-vertex generated are also λ-vertex generated. In Section 3, we prove Theorem 1.2 in a slightly more general form. In Section 4 we discuss various denseness notions, we prove Theorem 1.5, we show that, with respect to a weaker metric, λ-vertex generated polytopes are closed and we prove Theorem 1.4. Section 5 is devoted to centrally symmetric polytopes. Along with several other observations, we show that a centrally symmetric polytope is λ-vertex generated if and only if all of its faces are, and we prove Proposition 1.7. We include a curious fact regarding other linear variants of vertex generated polytopes, demonstrating that $P - V(P) = P - P$ can occur only when P is centrally symmetric and vertex generated. Section 6 is devoted to the series expansion and covering properties of λ-vertex generated polytopes. Section 7 includes some additional remarks and connected results, as well as a proof of the fact that the simplex is the unique minimizer of $\lambda(P)$.

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2. Some simple obstacles

We will demonstrate in this paper that the class $\mathrm{VG}(\mathbb{R}^n, \lambda)$, for any λ, is quite rich. In the next section we will see that it includes all zonotopes, which follows from the stronger claim which is that if a polytope is λ-vertex generated, then it remains λ-vertex generated also when adding a segment to it. In other words, the class of λ-vertex generated polytopes is closed under the operation of Minkowski summation with a segment (and, by induction, under adding any zonotope).

It is clear, however, that not every polytope is vertex generated (for example, a triangle). In fact, in any dimension n, the simplex is not a member of $\mathrm{VG}(\mathbb{R}^n, \lambda)$

for any $\lambda > 1/(n+1)$, as we show in Proposition 7.5. We also show there that any polytope which is not a simplex, does belong to some $\mathrm{VG}(\mathbb{R}^n, \lambda)$ with $\lambda > 1/(n+1)$.

We begin with some simple observations regarding polytopes in the class $\mathrm{VG}(\mathbb{R}^n, \lambda)$. In what follows we denote the boundary of a polytope $P \subseteq \mathbb{R}^n$ by ∂P.

Lemma 2.1. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. If $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ has non-empty interior then it has at least $1/(1-\lambda)^n$ vertices. Moreover, if $P \in \mathrm{VG}(\mathbb{R}^n)$ has exactly 2^n vertices, then it must be a linear image of the cube.*

Proof. Indeed, by assumption $P \subseteq \bigcup_{v \in V(P)} (\lambda v + (1-\lambda)P)$ and so comparing volumes

$$\mathrm{vol}(P) \leq |V(P)| \, \mathrm{vol}((1-\lambda)P) = |V(P)|(1-\lambda)^n \, \mathrm{vol}(P).$$

This completes the proof of the inequality. Next, assume that $P \in \mathrm{VG}(\mathbb{R}^n)$ has exactly 2^n vertices. This means that each intersection $(\frac{v+P}{2}) \cap (\frac{w+P}{2})$ is non-empty (includes $(v+w)/2$) and has measure zero. Therefore, considering a separating hyperplane, there exists some unit vector $u \in \mathbb{R}^n$ such that u is an outer normal to P at v and $-u$ is an outer normal of P at w. A subset of ∂P with this property (that for any two of its elements one can find such a u) is called an "antipodal set". In other words, the vertices of P form an antipodal set of cardinality 2^n. By a result of Danzer and Grünbaum [5], this happens only if P is a linear image of the cube. This completes the proof. □

We remark that for $\lambda > 1/2$ is it easy to see that any $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ with non-empty interior has strictly more than $1/(1-\lambda)^n$ vertices since the copies $\lambda v + (1-\lambda)P$ cannot have disjoint interiors.

Next, we claim that faces of λ-vertex generated polytopes are also λ-vertex generated. In particular, any polytope with a triangular 2-dimensional face is not vertex generated. We prove a formally stronger claim, namely that if a face of P is included in $(1-\lambda)P + \lambda V(P)$, then the face is λ-vertex generated (even without requiring the whole polytope to be in $\mathrm{VG}(\mathbb{R}^n, \lambda)$).

Lemma 2.2. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Let $P \subseteq \mathbb{R}^n$ be a polytope and let F be a face of P. Assume $F \subseteq (1-\lambda)P + \lambda V(P)$. Then $F = (1-\lambda)F + \lambda V(F)$. In particular, faces of λ-vertex generated polytopes are λ-vertex generated.*

Proof. Let $x \in F$. By our assumption there exist $v \in V(P)$, $y \in P$ such that $x = (1-\lambda)y + \lambda v$. Clearly both y and v belong to F. A vertex of P which belongs to F is also a vertex of F, and the proof is complete. □

Remark 2.3. We note that $\mathrm{VG}(\mathbb{R}^n, \lambda)$ is not closed under intersection with either other λ-vertex generated polytopes or a subspace, because we can translate and rotate two cubes so that their intersection is a simplex, and as all centrally symmetric polytopes are sections of higher-dimensional hyper-cubes.

3. Zonotopes, and Minkowski addition of a segment

In this section we will prove that the sum of a λ-vertex generated polytope and a segment is λ-vertex generated. In fact, we show something slightly more general, namely that for a polytope P all of whose faces are λ-vertex generated, the difference

between the polytope and the average $(1-\lambda)P+\lambda V(P)$ only decreases when passing to $P+\ell$ for any line segment ℓ. More precisely we prove the following.

Theorem 3.1. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Let $P \subseteq \mathbb{R}^n$ be a polytope such that all of its $(n-1)$-dimensional faces are λ-vertex generated. Let $\ell \subseteq \mathbb{R}^n$ be an origin symmetric line segment and let $P' = P + \ell$. Then*

$$P' \setminus \big((1-\lambda)P' + \lambda V(P')\big) \subseteq P \setminus ((1-\lambda)P + \lambda V(P)). \tag{3.1}$$

In particular, if P is λ-vertex generated (so that the right hand side is empty) then so is $P+\ell$ for any segment ℓ.

To prove Theorem 3.1 we need the following simple lemma.

Lemma 3.2. *Let $n \geq 1$, $P \subseteq \mathbb{R}^n$ a polytope, F a face of P and $\theta \in S^{n-1}$ satisfy that $P \cap (\varepsilon\theta + F) = \emptyset$ for any $\varepsilon > 0$. Then for any $c > 0$, $F + c\theta$ is a face of $P + [-c\theta, c\theta]$.*

Proof. We will use the following notation: for a polytope P and a unit vector u, we let F_u denote the face of P in direction u, namely

$$F_u(P) = \{x \in P : \langle u, x\rangle = \sup_{y \in P} \langle u, y\rangle = h_P(u)\}.$$

It is well known (see [11, Theorem 1.7.5]) that Minkowski addition respects this definition, namely that for $u \in S^{n-1}$ and polytopes $P, Q \subseteq \mathbb{R}^n$ one has

$$F_u(P+Q) = F_u(P) + F_u(Q). \tag{3.2}$$

The condition that $P \cap (\varepsilon\theta + F) = \emptyset$ for any $\varepsilon > 0$ is equivalent to the existence of some $u \in S^{n-1}$ which belongs to the normal cone of F (that is, $F = F_u(P)$) and such that $\langle u, \theta\rangle > 0$. Indeed, this follows from the duality between the "support cone" of a convex body at a point and the normal cone, see [11, Section 2.2, eq. (2.2)]. Thus

$$F_u(P + [-c\theta, c\theta]) = F_u(P) + F_u([-c\theta, c\theta]) = F + c\theta.$$

This completes the proof of the lemma. □

We proceed with the proof of Theorem 3.1. In what follows, we denote the relative interior of a set $A \subseteq \mathbb{R}^n$ by $\mathrm{relint}(A)$.

Proof of Theorem 3.1. Given ℓ denote $\ell = [-c\theta, c\theta]$ where $\theta \in S^{n-1}$ and $c > 0$. We start by showing that $(1-\lambda)P + \lambda V(P) \subseteq (1-\lambda)P' + \lambda V(P')$. Let $x \in (1-\lambda)P + \lambda V(P)$, and choose $v \in V(P)$, $y \in P$ such that $(1-\lambda)y + \lambda v = x$. Since $\{v\}$ is a 0-dimensional face of P, either $(v + \mathbb{R}^+\theta) \cap P = \emptyset$ or $(v - \mathbb{R}^+\theta) \cap P = \emptyset$ (or both). By interchanging θ and $-\theta$ we may assume without loss of generality the former, and by Lemma 3.2 we get that $v + c\theta \in V(P+\ell) = V(P')$. Using that $y - \frac{\lambda}{1-\lambda} c\theta \in P + \ell = P'$, we get

$$x = (1-\lambda)y + \lambda v = (1-\lambda)(y - \frac{\lambda}{1-\lambda} c\theta) + \lambda(v + c\theta) \in (1-\lambda)P' + \lambda V(P').$$

Next, let $x \in P' \setminus ((1-\lambda)P' + \lambda V(P'))$. We have just demonstrated that $x \notin (1-\lambda)P + \lambda V(P)$, and are left with showing that $x \in P$. Assume towards a contradiction that $x \notin P$. Since $x \in P + \ell$ it follows that $P \cap (x + [-c\theta, c\theta]) \neq \emptyset$,

and (again, switching to $-\theta$ if needed) we may assume $P\cap(x+[-c\theta,0))\neq\emptyset$. We let $c':=\inf\{r>0: x-r\theta\in P\}$ and $x'=x-c'\theta$. In other words, x' is one of the two intersection points of $x+\mathbb{R}\theta$ and ∂P, the one closer to x. Denote by F the minimal face of P containing x' so that $x'\in\text{relint}(F)$. By our assumption (taking into account Lemma 2.2), $F=(1-\lambda)F+\lambda V(F)$ so that there exist $v\in V(F)$, $y\in F$ such that $(1-\lambda)y+\lambda v=x'$. Since $x'\in\text{relint}(F)$ and c' was minimal, the direction θ must satisfy that there is some $u\in S^{n-1}$ in the normal cone of F with $\langle u,\theta\rangle>0$ and thus $(F+\varepsilon\theta)\cap P=\emptyset$ for any $\varepsilon>0$. We use Lemma 3.2 to conclude that $v+c\theta\in V(P+\ell)$, and since $0<c'\leq c$ we have that $\frac{c'-\lambda c}{1-\lambda}\in[-c,c]$ and so $y+\frac{c'-\lambda c}{1-\lambda}\theta\in P+\ell$. Therefore

$$x=x'+c'\theta=(1-\lambda)(y+\frac{c'-\lambda c}{1-\lambda}\theta)+\lambda(v+c\theta)\in(1-\lambda)P'+\lambda V(P').$$

This contradicts our assumption on x, and the proof is complete. □

Using that a segment is vertex generated, and Theorem 3.1 which implies that when we add a segment to a vertex generated polytope it remains vertex generated, we get that any zonotope in any dimension is vertex generated, proving Corollary 1.3. We will see another simple proof that zonotopes are vertex generated in Section 5.

4. On various denseness notions for $\text{VG}(\mathbb{R}^n)$ and $\text{VG}(\mathbb{R}^n,\lambda)$

It is natural to ask whether vertex generated polytopes are dense within the class of polytopes, or, similarly, within the class of all convex bodies (when considering the Hausdorff metric, this is the same question). We are able to prove this in dimension $n=2$, which is the subject of Section 4.1. In Section 4.2 we define a different metric on polytopes, d_F, which is weaker than the Hausdorff metric $d_F(P,Q)\geq d_H(P,Q)$, and such that with respect to this metric the vertex generated polytopes form a closed set. The metric is quite natural and is given by the Hausdorff distance between the vertex-sets of the polytopes. We consider other similar questions, such as whether one can always add a zonotope to a polytope so that the resulting polytope is vertex generated (we show this is correct, see Theorem 1.4). In the same vein, one can try to add a very *small* polytope such that the resulting polytope is vertex generated, which, if true, would imply density with respect to Hausdorff distance. However, this we show cannot hold, and if a polytope $P\subseteq\mathbb{R}^n$ is not vertex generated then there is some $\varepsilon>0$ such that for all polytopes $Q\subseteq\varepsilon B_2^n$ the polytope $P+Q$ is not vertex generated. These two results are given in Section 4.3.

4.1. Vertex generated polytopes in the plane.

In this section, we focus on planar vertex generated polytopes, showing that they are dense in the class of all planar convex bodies:

Theorem 4.1. *The class* $\text{VG}(\mathbb{R}^2)$ *of vertex generated polytopes in the plane is dense, with respect to the Hausdorff distance, in the class of all planar convex bodies.*

We shall make use of the following lemma (which actually holds in any dimension).

Lemma 4.2. *Let $P \subseteq \mathbb{R}^2$ be a polytope with non-empty interior. For any $u \in \partial P$, there exists $r > 0$ such that*

$$(u + rB_2^2) \cap P = (u + rB_2^2) \cap \frac{1}{2}(P + u).$$

Proof. We denote the 1-dimensional faces of P by $E(P)$, that is, edges of P. Fix some $u \in \partial P$ and denote

$$r = \frac{1}{2} \min\{d(u, E) : E \in E(P), u \notin E\} > 0.$$

Given $x \in (u + rB_2^2) \cap P$, denote by $u' \in \partial P$ the point farthest from u for which $x \in [u, u']$. Clearly $u' \in \partial P$ so that it lies on some edge. It cannot, however, lie only on an edge which includes u, since in such a case we would have that $x \in \partial P$ and u' is a vertex with $x \in [u, u']$, and then u' lies on two edges, one of which does not include u. By our choice of r, we have $d(u, u') \geq 2r$ (again, one can distinguish the two cases, $x \in \partial P$ and $x \in \text{int}(P)$). Since $d(x, u) \leq r$ we see that in fact $x \in [u, (u + u')/2]$. However, $[u, (u + u')/2] \subseteq (P + u)/2$ so that $x \in (P + u)/2$, as needed. □

Proof of Theorem 4.1. Let P be a polytope and $\varepsilon > 0$. Clearly, we may also assume that P is not vertex generated, so that, in particular, P is not centrally-symmetric up to any translation. Indeed, centrally symmetric polytopes in the plane are zonotopes (see Remark 5.3) which are vertex generated, by Corollary 1.3. Our goal is to construct a polytope $Q \in \text{VG}(\mathbb{R}^2)$ such that $d_H(P, Q) \leq \varepsilon$.

By Lemma 4.2, there exists some $\delta > 0$ such that for every vertex v of P it holds that $P \cap \text{int}(v + \delta B_2^2) \subseteq (P + V(P))/2$.

Denote the set of vertices of P by $\{v_i\}_{i=1}^m$, and set $v_{m+1} = v_1$. Assume that the vertices are labeled so that $E_i = (v_i, v_{i+1})$ is an edge of P (i.e., a 1-dimenional face) for each $i \in \{1, \ldots, m\}$. Let η_i be the outer normal of E_i and θ_i be an outer normal of v_i which is different from all the outer normals $\{\eta_j\}_{j=1}^m$. For each $i \in \{1, \ldots, m\}$, consider a circle of sufficiently large radius such that E_i is a chord with corresponding (minor) arc A_i and the following conditions hold as well (see Figure 1):

(C1) $d_H(A_i, E_i) < \varepsilon$,
(C2) $2e_i - A_i \subseteq P$, where $e_i = \frac{v_i + v_{i+1}}{2}$ is the center of the edge E_i,
(C3) $A_i \cap \{v_i + \theta_i^\perp\} = v_i$ and $A_i \cap \{v_{i+1} + \theta_{i+1}^\perp\} = v_{i+1}$.

By (C3), we see that $Q' = \text{conv}(\bigcup_{i=1}^m A_i) \supseteq P$ is a compact convex set, and that every boundary point of Q' is an extremal point of Q'. Moreover, since P is not centrally-symmetric up to any translation, we know that Q' is not centrally-symmetric up to any translation, as well.

Consider the family $\{S_u\}_{u \in \partial Q'} = \{\text{int}(u + Q')/2\}_{u \in \partial Q'}$. We claim that it forms a cover of $\text{int}(Q')$ and therefore a cover of the compact set

$$P' := P \setminus \Big(\bigcup_{v \in V(P)} \text{int}(v + \delta B_2^2) \Big) \subseteq \text{int}(Q'),$$

where $\delta > 0$ is the positive constant chosen at the beginning of the proof.

To show that this is indeed a cover, let $x \in \mathrm{int}(Q')$ and suppose that $x \in (a,b)$ for some $a, b \in \partial Q'$. Note that if $|x-a| < |x-b|$ then $x \in (a,(a+b)/2)$ and hence $x \in S_a$. This means that x is either in some S_u or is the center of every line section of P passing through x. The latter would imply that $P - x$ is centrally-symmetric, and as we assumed that this is not the case, we conclude that $\{S_u\}_{u\in\partial Q'}$ is indeed a cover of Q'.

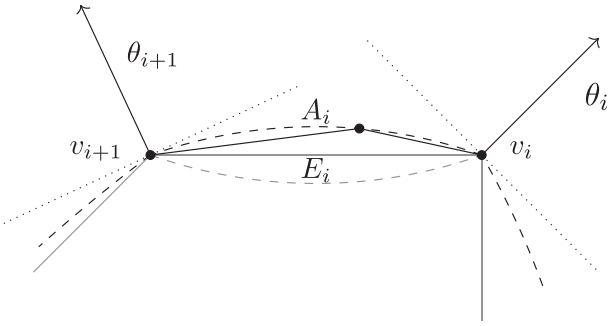


FIGURE 1. The construction of Q'.

By the compactness of P', we may select a finite sub-cover $\{S_{u'_j}\}_{j=1}^n$ of P' such that $V(P) \subseteq \{u'_j\}_{j=1}^n$. Let Q_k be the convex hull of $k \geq n$ points $\{u'_j\}_{j=1}^n \cup \{u_j\}_{j=1}^{k-n}$ on the boundary of Q' such that $d_H(Q_k, Q') \to 0$. As one can verify, using a standard compactness argument, for some k_0 big enough, the family $\{\mathrm{int}(u'_j + Q_{k_0})/2\}_{j=1}^n$ forms a cover of P' as well. We define our desired polytope by $Q = Q_{k_0}$. Clearly, by (C1), we have $d_H(P,Q) < \varepsilon$. Also note that, by our construction, $V(P) \subseteq \{u'_j\}_{j=1}^n \subseteq V(Q)$.

It is left to show that Q is vertex generated. Let $x \in Q$. Suppose first that $x \in P$. If $x \in \mathrm{int}(v + \delta B_2^2)$ for some $v \in V(P)$ then, by our choice of $\delta > 0$, $x \in (P + V(P))/2$, and since $V(P) \subseteq V(Q)$, it follows that $x \in (Q + V(Q))/2$, as well. Otherwise, $x \in P'$ and then, by our choice of $\{u'_j\}$ in the construction, $x \in (u'_j + Q)/2$ for some vertex $u'_j \in V(Q)$.

Suppose next that $x \in Q \setminus P$. In this case, $x \in \mathrm{conv}(A_i \cup E_i)$ for some i. Let $y \in \partial Q$ be the intersection of the ray emanating from e_i and passing through x with ∂Q. Clearly, we also have $y \in \mathrm{conv}\{A_i \cup E_i\}$. If $y \in V(Q)$ then by (C2), $2x - y \in Q$, and hence $x \in \frac{1}{2}(V(Q) + Q)$. Otherwise, let $a, b \in V(Q)$ be the vertices of the edge containing y. By definition, $a, b \in A_i$ and so, by (C2), we have $2e_i - a, 2e_i - b \in Q$. By interchanging the roles of a and b, we may assume that $x \in \mathrm{conv}(e_i, \frac{1}{2}(a+b), a)$ and so $2x - a \in \mathrm{conv}\{2e_i - a, b, a\} \subseteq Q$. Consequently, we get that $x \in \frac{1}{2}(Q + V(Q))$, which completes our proof. □

4.2. **Non-denseness with respect to a non-standard metric.** While the question of denseness of vertex generated polytopes with respect to the Hausdorff metric remains open in dimensions $n \geq 3$, we may define a weaker metric with respect to which vertex generated polytopes form a closed set within the class of all polytopes. This metric, which we denote d_F, is defined for a pair of polytopes as the Hasudorff

distance between the sets of their vertices. Since one may approximate a segment, say, in the Hausdorff metric, by a sequence of triangles which are the convex hulls of the segment with a vertex close to the middle of the segment, we see that d_F is weaker that d_H (as this is an example where $d_H \to 0$ and $d_F \not\to 0$). Let us formally introduce this new metric on polytopes.

Definition 4.3. Let $n \geq 1$. Given polytopes $P, Q \subseteq \mathbb{R}^n$ with vertex sets $V(P)$ and $V(Q)$ respectively, let $d_F(P,Q) = d_H(V(P), V(Q))$ or, equivalently,

$$d_F(P,Q) = \min\{\varepsilon > 0 : V(Q) \subseteq V(P) + \varepsilon B_2^n \text{ and } V(P) \subseteq V(Q) + \varepsilon B_2^n\}.$$

Clearly $d_F(P,Q) \geq d_H(P,Q)$ (using that $V(P) + \varepsilon B_2^n \subseteq P + \varepsilon B_2^n$ and that for a convex set A, a polytope is included in A if and only if all its vertices are in A). It turns out that with respect to the metric d_F and for any λ, the class $\mathrm{VG}(\mathbb{R}^n, \lambda)$ is closed.

Proposition 4.4. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. For any polytope $P \subseteq \mathbb{R}^n$ which is not λ-vertex generated, there exists some $\varepsilon > 0$ such that if $d_F(P,Q) < \varepsilon$ then Q is not λ-vertex generated as well. Equivalently, for polytopes $P, (P_m)_{m=1}^{\infty} \subseteq \mathbb{R}^n$, if $P_m \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ and $d_F(P_m, P) \to 0$ then $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$.*

Proof. Consider a sequence of polytopes $(P_m)_{m=1}^{\infty} \subseteq \mathrm{VG}(\mathbb{R}^n, \lambda)$ and a polytope $P \subseteq \mathbb{R}^n$ such that $d_F(P_m, P) \longrightarrow 0$. In particular $d_H(P_m, P) \to 0$ and P_m are uniformly bounded. Given $x \in P$, one may find a sequence $x_m \in P_m$ such that $x_m \to x$. As $P_m \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ we can write $x_m = (1-\lambda)y_m + \lambda v_m$ with $y_m \in P_m$ and $v_m \in V(P_m)$. Since v_m and y_m are bounded, there exists sub-sequences v_{m_i} y_{m_i} which converge. We call their limits v, y respectively, and note that $x = (1-\lambda)y + \lambda v$. Using again that $P_m \to P$ in the Hausdorff distance and $V(P_m) \to V(P)$ in the Hausdorff's distance, it follows that $v \in V(P)$ and $y \in P$. We thus see $P = (1-\lambda)P + \lambda V(P)$, as claimed. □

4.3. **Generalized vertex generated polytopes.** We start with the result that any polytope is a so-called "generalized vertex generated polytope" namely that for any polytope one can find some zonotope such that their Minkowski sum is vertex generated. The name is inspired by the notion of a "generalized zonoid" from convexity (see [11, Section 3.5]) which is defined to be a convex body K such that there exist two zonoids Z_1, Z_2 with $K + Z_1 = Z_2$ (a zonoid is defined to be a limit of zonotopes in the Hausdorff metric, for definition and many interesting facts see [11, Section 3.5]). For zonoids, however, if a polytope is a generalized zonoid then it must be a zonotope to begin with (see Corollary 3.5.7 in [11]) whereas for our question, any polytope can be summed with a vertex generated polytope to obtain a vertex generated polytope. On the other hand, Theorem 1.4 is similar in spirit to the fact that generalized zonoids are dense within the class of centrally symmetric convex bodies in the Hausdorff metric (see [11, Corollary 3.5.7]).

Theorem 1.4. Let $n \geq 1$. For any polytope $P \subseteq \mathbb{R}^n$ there exists a zonotope $Z \subseteq \mathbb{R}^n$ such that $P + Z$ is vertex generated.

Proof. We prove the claim by induction on the dimension n. For $n = 1$ there is nothing to prove as a segment is vertex generated.

Next, assume the statement of the theorem holds for any dimension $k < n$, and let $P \subseteq \mathbb{R}^n$ be a polytope. Its boundary is the union of a finite number of faces F_i of dimension $k < n$, and so by the inductive assumption we may, for each i, find a zonotope Z_i in the affine hull of F_i (or, more precisely, the linear subspace which is a translation of this affine hull) such that $F_i + Z_i$ is vertex generated. Denote $Z = \sum Z_i$, and by (3.2) we have for any θ

$$F_\theta(P + Z) = F_\theta(P) + \sum F_\theta(Z_i).$$

We do not claim that $P' = P + Z$ is vertex generated, but only that it satisfies the condition $\partial P' \subseteq (P' + V(P'))/2$ (note that for $n = 2$ this condition is satisfied automatically as all facets are segments). To show this containment, consider some vector $\theta \in S^{n-1}$, and let $i = i(\theta)$ denote the index of the corresponding face of P, namely $F_\theta(P) = F_i$. Because $Z_i \subseteq \theta^\perp$, clearly $F_\theta(Z_i) = Z_i$, which means in particular that

$$F_\theta(P + Z) = F_i + Z_i + \sum_{j \neq i} F_\theta(Z_j).$$

But $F_\theta(Z_j)$ are zonotopes and $F_i + Z_i \in \mathrm{VG}(\mathbb{R}^n)$, and so by Theorem 3.1 also their sum is vertex generated. This shows that the face of P' in direction θ is vertex generated. Since θ was arbitrary, this shows that every face of P' is vertex generated, and we get that $\partial P' \subseteq (P' + V(P'))/2$.

We next find a zonotope Z' such that $P' + Z'$ is vertex generated. Choose $\eta \in S^{n-1}$ such that

$$F_\eta(P') = \{u_1\},\ F_{-\eta}(P') = \{u_2\} \tag{4.1}$$

for some $u_1, u_2 \in V(P')$. (namely $\eta, -\eta$ are each in the interior of the normal cone of some vertex, this is true for almost every η).

Note that $P' \subseteq [u_1, u_2] + \eta^\perp$, so there exists a zonotope $Z' \subseteq \eta^\perp$ large enough such that $P' \subseteq [u_1, u_2] + Z'$. Since a zonotope is a sum of line segments, we see that, by inductively using Theorem 3.1 (the conditions of which are satisfied by our construction of P') we have

$$(P' + Z') \setminus \frac{1}{2}(P' + Z' + V(P' + Z')) \subseteq P' \setminus \frac{1}{2}(P' + V(P')) \subseteq P'. \tag{4.2}$$

Note that by construction $F_\eta(Z') = F_{-\eta}(Z') = Z'$ and hence, by (4.1) and (3.2), we have $F_\eta(P' + Z') = u_1 + Z'$ and $F_{-\eta}(P' + Z') = u_2 + Z'$. Since vertices of a polytope's face are vertices of the polytope itself we get

$$V([u_1, u_2] + Z') \subseteq V([u_1, u_2]) + V(Z') = V(u_1 + Z') \cup V(u_2 + Z') \subseteq V(P' + Z').$$

Finally from the fact $[u_1, u_2] + Z'$ is a zonotope, Corollary 1.3 implies that it is vertex generated, and therefore

$$P' \subseteq [u_1, u_2] + Z' \subseteq \frac{1}{2}([u_1, u_2] + Z' + V([u_1, u_2] + Z')) \subseteq \frac{1}{2}(P' + Z' + V(P' + Z')).$$

Joining this with (4.2), we see that $(P' + Z') \setminus (P' + Z' + V(P' + Z'))/2$ must be empty, meaning that $P' + Z'$ is vertex generated, and the proof is complete. □

Remark 4.5. Let us remark on the degrees of freedom in our final choice of Z in the proof of Theorem 1.4. For a two dimensional polytope $P \subseteq \mathbb{R}^2$, our choice of η was only limited to the condition that $F_\eta(P) = \{u_1\}$, $F_{-\eta}(P) = \{u_2\}$ for some $u_1, u_2 \in V(P)$, which is true for almost any $\eta \in S^1$. More generally, for any polytope $P \subseteq \mathbb{R}^n$ for which

$$\partial P \subseteq \frac{1}{2}(P + V(P)),$$

our choice of η in the proof requires only that $F_\eta(P) = \{u_1\}$, $F_{-\eta}(P) = \{u_2\}$ for some $u_1, u_2 \in V(P)$, which is true for almost any $\eta \in S^{n-1}$, and the choice of the zonotope Z is only limited by $P \subseteq [u_1, u_2] + Z$.

Note that if one could choose, in Theorem 1.4, a very small Z, namely if for every $\varepsilon > 0$ one could find such a Z contained in εB_2^n, then this would imply denseness of vertex generated polytopes, with respect to the Hausdorff distance, within the class of all polytopes (and hence within the class of all convex bodies). However, this cannot hold true. Indeed, denote the Minkowski subtraction of two sets $A, B \subseteq \mathbb{R}^n$ by

$$A \ominus B = \{x \in \mathbb{R}^n \,:\, x - B \subseteq \operatorname{int}(A)\}.$$

Our next proposition implies that if $P \notin \mathrm{VG}(\mathbb{R}^n, \lambda)$ then its Minkowski sum with any body in a small enough ball cannot be in $\mathrm{VG}(\mathbb{R}^n, \lambda)$.

Proposition 4.6. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Let $P, Q \subseteq \mathbb{R}^n$ be polytopes. Then*

$$(P \setminus ((1-\lambda)P + \lambda V(P))) \ominus Q \subseteq (P+Q) \setminus ((1-\lambda)(P+Q) + \lambda V(P+Q)).$$

In particular, if $P + Q \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ then no translate of Q can fit into $P \setminus ((1-\lambda)P + \lambda V(P))$.

Proof. It is clear that $(P \setminus ((1-\lambda)P + \lambda V(P))) \ominus Q \subseteq P \ominus Q \subseteq P + Q$, so we need only show that the left hand side includes no points in $((1-\lambda)(P+Q) + \lambda V(P+Q))$. The latter is included in $Q + (1-\lambda)P + \lambda V(P)$, so in particular for any point $x \in ((1-\lambda)(P+Q) + \lambda V(P+Q))$ there exists some $y \in Q$ such that $x - y \in (1-\lambda)P + \lambda V(P)$, namely $x - Q \not\subseteq P \setminus ((1-\lambda)P + \lambda V(P))$. This means that $x \notin (P \setminus ((1-\lambda)P + \lambda V(P))) \ominus Q$, as claimed. □

5. Vertex generated symmetric polytopes

Restricting to the class of centrally-symmetric polytopes enables us to prove additional properties regarding λ-vertex generated polytopes.

We first show that for a centrally symmetric polytope, being λ-vertex generated is equivalent to all of its facets being λ-vertex generated, namely that for centrally symmetric P, the converse of Lemma 2.2 holds. This gives an easy inductive proof for the fact that zonotopes are vertex generated, reproving Corollary 1.3. After establishing this fact, we show that any n-dimensional λ-vertex generated polytope can be realized as a facet of a centrally symmetric $(n+1)$-dimensional λ-vertex generated polytope. In particular this means that if one knows that centrally symmetric λ-vertex generated polytopes in dimension $n+1$ are closed under Minkowski addition, then so is the class of all λ-vertex generated polytopes in $\mathbb{R}^n$. While this fact is yet to be proven, we do show that if two centrally symmetric

λ-vertex generated polytopes are in generic position then their Minkowski sum is also λ-vertex generated (a generic pair of polytopes was defined in Definition 1.6). Finally we prove a statement regarding the sum of the vertices $V(P)$ of a polytope with some linear image of P, a special case of which is the following curious fact: If $P - V(P)$ is a convex set, then P is centrally symmetric and vertex generated, namely $P - V(P) = P + V(P) = 2P$.

We start by showing that for centrally symmetric polytopes, the converse of Lemma 2.2 holds.

Lemma 5.1. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric polytope and assume that $\partial P \subseteq (1-\lambda)P + \lambda V(P)$. Then P is λ-vertex generated.*

Proof. Given $x \in P$ let $c \geq 1$ such that $cx \in \partial P$. By our assumption there exists $v \in V(P)$ such that $cx \in \lambda v + (1-\lambda)P$. However, since P is centrally symmetric, also $0 \in \lambda v + (1-\lambda)P$ (since $0 = \lambda v + (1-\lambda)(-\frac{\lambda}{1-\lambda}v)$ and $-\frac{\lambda}{1-\lambda}v \in P$). The set $\lambda v + (1-\lambda)P$ is convex, and so together with 0 and cx it includes $x = (1-1/c)0 + (1/c)cx$, and the proof is complete. □

Remark 5.2. In fact we see that the "central symmetry" condition is an overshoot for $\lambda < 1/2$, and we can ask for less, $-\lambda P \subseteq (1-\lambda)P$. Note that we always have this for $\lambda = 1/(n+1)$ since $-P \subseteq nP$. This observation implies, however, that if for example, $-P \subseteq (n-1)P$, then (using that the facets are $1/n$-vertex generated) the whole polytope is $1/n$-vertex generated.

Using Lemma 5.1 we can provide a simple proof for Corollary 1.3.

Another proof of Corollary 1.3. Recall that a zonotope is centrally symmetric, and all of its faces are centrally symmetric zonotopes of lower dimension (in fact, a polytope is a zonotope if and only of all of its two-dimensional faces are centrally symmetric, see [11, Theorem 3.5.2]). We prove the corollary by induction, where clearly one dimensional zonotopes are vertex generated since these are simply segments. If we know that $(n-1)$-dimensional zonotopes are vertex generated, and we are given an n-dimensional zonotope Z then it is centrally symmetric, and its boundary ∂Z is the union of finitely many zonotopes which are vertex generated by induction, so that $\partial Z \subseteq (\partial Z + V(\partial Z))/2 \subseteq (Z + V(Z))/2$. Applying Lemma 5.1, Z is vertex generated as well. □

Remark 5.3. It is well-known that all centrally symmetric polytopes in the plane $\mathbb{R}^2$ are zonotopes, see e.g., [11, Corollary 3.5.7] and hence vertex generated. However, in higher dimensions there exist centrally symmetric polytopes which are not zonotopes. One such example is the cross polytope for $n \geq 3$, which has faces which are simplices and hence is not vertex generated. On the other hand, there also exist centrally symmetric vertex generated polytopes which are not zonotopes, and one such example is the sum of the cross polytope with a suitable chosen zonotope. Indeed, one may choose a suitable zonotope by Theorem 1.4, and the fact that the cross polytope summed with a zonotope is not a zonotope follows e.g. from [11, Corollary 3.5.7].

Next we show that every vertex generated polytope in $\mathbb{R}^n$ can be realized as a facet of a centrally symmetric vertex generated polytope in dimension $(n+1)$.

Proposition 5.4. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$, and let $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$. Then there exists a centrally-symmetric $Q \in \mathrm{VG}(\mathbb{R}^{n+1}, \lambda)$ such that $P = F_u(Q)$ for some $u \in S^n$.*

Proof. Given $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$, define $P' = \mathrm{conv}(P \times \{1\}, -P \times \{-1\}) \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. The polytope P' is centrally symmetric and its facets in directions e_{n+1} and $-e_{n+1}$ are translates of P and $-P$ respectively (where $(e_i)_{i=1}^{n+1}$ is the standard vector basis in $\mathbb{R}^n \times \mathbb{R}$.) Moreover, for any $(u_j)_{j=1}^m \subseteq S^n \setminus e_n^\perp$ and $(c_j)_{j=1}^m \subseteq \mathbb{R}$, the polytope $Q = P' + \sum_{j=1}^m c_j[-u_j, u_j]$ satisfies that its facets in directions e_{n+1} and $-e_{n+1}$ are translates of P and $-P$ respectively (again, by using (3.2), say). It is also important to note that all faces of P' which are *not* in the hyperplanes $H_1 = \{x_{n+1} = 1\}$ $H_{-1} = \{x_{n+1} = -1\}$, are not parallel to these hyperplanes (equivalently, are not orthogonal to e_{n+1}). Indeed, a face of P' is a convex hull of some subset of its vertices, and these vertices belong to $H_1 \cup H_{-1}$, so if the subset includes at least one element from each of the hyperplanes, then the corresponding face includes the edge between these two vertices, which is not orthogonal to e_{n+1}.

Our task is to choose $(u_j), (c_j)$ such that Q is λ-vertex generated. To this end, we will make sure that all of its faces are λ-vertex generated and use Lemma 5.1. We will go over all the faces of P' which are not included in the hyperplanes H_1 and H_2 (as these are automatically λ-vertex generated) and use, for each one, the construction in the proof of Theorem 1.4. In other words, for each such face F of P' we find a zonotope Z_F in the affine hull of F such that $F + Z_F \in \mathrm{VG}(\mathbb{R}^{n+1}, \lambda)$. (If F was λ-vertex generated, we pick $Z_F = \{0\}$). Moreover, we can, using Remark 4.5, choose all the vectors ξ_i participating in the construction of $Z_F = \sum_{i=1}^{m_F}[-\xi_i, \xi_i]$ so that they are not in $e_{n+1}^\perp$ (here it is essential that F is not orthogonal to e_{n+1}. We let $Q = P' + \sum_F Z_F$ where F runs over all of the faces of P' which are not orthogonal to e_{n+1}. We claim that all faces of Q are λ-vertex generated. Indeed, let $u \in S^n$, and denote F^* the face of P' in direction u. Employing (3.2), as usual, we have

$$F_u(Q) = F^* + \sum_F F_u(Z_F) = F^* + F_u(Z_{F^*}) + \sum_{F \neq F^*} F_u(Z_F)$$

The set $F^* + F_u(Z_{F^*}) = F^* + Z_{F^*}$ is λ-vertex generated and it is summed with zonotopes, so by Theorem 3.1 the face $F_u(Q)$ is λ-vertex generated as well. This completes the proof. □

We proceed with proving Proposition 1.7, namely that the sum of a generic pair (see Definition 1.6) of centrally symmetric λ-vertex generated polytopes is λ-vertex generated as well.

We remark that we can relax the condition that the polytopes are centrally symmetric and assume instead that the polytopes satisfy the conditions asserted in Remark 5.2.

Proof of Proposition 1.7. Since $P + Q$ is centrally symmetric, by Lemma 5.1 it suffices to show that its facets are λ-vertex generated, which is equivalent to $\partial(P+Q) \subseteq$

$(1-\lambda)(P+Q)+\lambda V(P+Q)$. Given $x \in \partial(P+Q)$, there exists a unit vector $u \in S^{n-1}$ such that $x \in F_u(P+Q) = F_u(P)+F_u(Q)$ (by (3.2), as usual). Let $x = x_1 + x_2$ where $x_1 \in F_u(P)$ and $x_2 \in F_u(Q)$. As P, Q are λ-vertex generated, so are their faces, (see Lemma 2.2) namely there exist $v_1 \in V(F_u(P))$, $y_1 \in F_u(P)$, $v_2 \in V(F_u(Q))$ and $y_2 \in F_u(Q)$ such that $x_1 = (1-\lambda)y_1 + \lambda v_1$ and $x_2 = (1-\lambda)y_2 + \lambda v_2$. Clearly $u \in N_P(v_1) \cap N_Q(v_2)$, meaning in particular that $N_P(v_1) \cap N_Q(v_2) \neq \emptyset$. Since P, Q are assumed to be a generic pair, this implies that $\mathrm{int}(N_P(v_1)) \cap \mathrm{int}(N_Q(v_2)) \neq \emptyset$. In such a case it is easy to check (see e.g. [4, Chapter 6, Lemma 1.3]) that $v_1 + v_2 \in V(P+Q)$ (as they are the Minkowski sum of the faces of P and Q in direction $w \in \mathrm{int}(N_P(v_1)) \cap \mathrm{int}(N_Q(v_2))$, say). We see thus that

$$x = (1-\lambda)(y_1+y_2) + \lambda(v_1+v_2) \in (1-\lambda)(P+Q) + \lambda V(P+Q).$$

This shows that $\partial(P+Q) \subseteq (1-\lambda)(P+Q)+\lambda V(P+Q)$, and by Lemma 5.1 $P+Q$ is λ-vertex generated. □

In the remainder of this section we focus on $\lambda = 1/2$. The idea behind our next theorem is based on the following attempt to generalize the notion of vertex generated polytopes. What would happen if we asked for a polytope to satisfy, instead of $P + V(P) = 2P$, the relation

$$P - V(P) = P - P.$$

Clearly there is an inclusion of the left hand side in the right hand side, and in the special case of centrally symmetric polytopes, this is yet again the definition of a vertex generated polytope. It turns out that in fact there is no other instance where this equality can hold. In fact, much more can be said. The mere requirement that $P - V(P)$ is a convex set, already implies that P is centrally symmetric and vertex generated. Moreover, the operation $K \mapsto -K$ can be replaced in this claim by any $K \mapsto AK$ for any $A \in GL_n(\mathbb{R})$ such that $A^k = Id$ for some positive integer k.

Theorem 5.5. *Let $P \subseteq \mathbb{R}^n$ be a polytope and let $A \in GL_n(\mathbb{R})$ satisfy $A^k = Id$ for some natural number $k > 0$. Then the following statements are equivalent:*

(1) $P + V(AP)$ is convex
(2) P is vertex generated and $P = AP + x$ for some $x \in \mathbb{R}^n$.

In particular, if $P - P = P - V(P)$ then P is a centrally symmetric vertex generated polytope.

To prove Theorem 5.5, we need the following simple lemma.

Lemma 5.6. *Let $P, Q \subseteq \mathbb{R}^n$ be two polytopes and let $u \in S^{n-1}$. Suppose that $P + V(Q) = P + Q$. Then*

$$F_u(P+Q) = F_u(P) + V(F_u(Q))$$

and, in particular, $\dim(F_u(P+Q)) = \dim(F_u(P))$.

Proof. First note that by our assumption and (3.2),

$$F_u(P+V(Q)) = F_u(P+Q) = F_u(P) + F_u(Q) \supseteq F_u(P) + V(F_u(Q)).$$

To prove the reverse inclusion, let $x \in F_u(P+V(Q))$. By definition, there exist $y \in P$ and $v \in V(Q)$ such that $x = y + v$ and $\langle u, x\rangle = h_{P+Q}(u)$. By the additivity

of the support function with respect to Minkowski addition (see e.g., [11, Theorem 1.7.5]), we therefore have

$$h_P(u) + h_Q(u) = h_{P+Q}(u) = \langle u, x\rangle = \langle u, y\rangle + \langle u, v\rangle.$$

Since $\langle u, y\rangle \le h_P(u)$ and $\langle u, v\rangle \le h_Q(u)$, it follows that $\langle u, y\rangle = h_P(u)$ and $\langle u, v\rangle = h_Q(u)$. We thus conclude that, $y \in F_u(P)$ and $v \in V(F_u(Q))$, which completes our proof. □

Proof of Theorem 5.5. The fact that (2) implies (1) is immediate. We proceed to prove that (1) implies (2). Since $P + V(AP)$ includes all the extremal points of $P+AP$, it follows that $\mathrm{conv}(P+V(AP)) = P+AP$, and so convexity of $P+V(AP)$ implies that $P + V(AP) = P + AP$.

We next show that for any $u \in S^{n-1}$, $\dim(F_{Au}(P)) = \dim(F_u(P))$. We use that $P + AP = P + V(AP)$, that $F_{Au}(AP) = A^{-T}F_u(P)$ and Lemma 5.6, to conclude

$$\begin{aligned} \dim(A^{-T}F_u(P)) &\le \dim(F_{Au}(P) + A^{-T}F_u(P)) \\ &= \dim(F_{Au}(P + AP)) = \dim(F_{Au}(P)). \end{aligned} \tag{5.1}$$

Therefore, $\dim(F_u(P)) \le \dim(F_{Au}(P))$. Since (5.1) holds for any direction u, using our assumption that $A^k = I$, we see that

$$\dim(F_u(P)) \le \dim(F_{Au}(P)) \le \cdots \le \dim(F_{A^k u}(P)) = \dim(F_u(P)), \tag{5.2}$$

and so $\dim(F_u(P)) = \dim(F_{Au}(P))$, as claimed, and the inequality in (5.1) is in fact an equality.

Note that we can also infer that the affine hull of $F_{Au}(P)$, denoted by $\mathrm{aff}(F_{Au}(P))$, is a translate of the affine hull of $A^{-T}F_u(P)$ (which we denote here by $\mathrm{aff}(A^{-T}F_u(P))$) since the equality $\dim(F_{Au}(P) + A^{-T}F_u(P)) = \dim(A^{-T}F_u(P))$ implies that $\mathrm{aff}(F_{Au}(P))$ is a subset of a translate of the set $\mathrm{aff}(A^{-T}F_u(P))$ and hence the equality $\dim(A^{-T}F_u(P)) = \dim(F_{Au}(P))$ implies that their affine hulls coincide.

Next we show that if $\dim(F_u(P)) = 1$ then we have $\mathrm{vol}_1(F_{Au}(P)) = \mathrm{vol}_1(A^{-T}F_u(P))$. Indeed, if $\dim(F_u(P)) = 1$ then, by (5.1), $\dim(F_{Au}(P)) = \dim(F_{Au}(P + AP)) = 1$. In particular, since $F_{Au}(P + AP) = F_{Au}(P) + A^{-T}F_u(P)$, it follows that

$$\mathrm{vol}_1(F_{Au}(P + AP)) = \mathrm{vol}_1(F_{Au}(P)) + \mathrm{vol}_1(A^{-T}F_u(P)).$$

Moreover, by Lemma 5.6, we have (since $F_{Au}(AP)$ is one-dimensional)

$$\mathrm{vol}_1(F_{Au}(P + AP)) = \mathrm{vol}_1(F_{Au}(P) + V(F_{Au}(AP))) \le 2\mathrm{vol}_1(F_{Au}(P))$$

and hence $\mathrm{vol}_1(A^{-T}F_u(P)) = \mathrm{vol}_1(F_{Au}(AP)) \le \mathrm{vol}_1(F_{Au}(P))$ for any u. We can iterate this inequality to get the chain of inequalities $\mathrm{vol}_1(F_{Au}(P)) = \mathrm{vol}_1(F_{Au}(A^kP)) \le \cdots \le \mathrm{vol}_1(F_{Au}(A^2P)) \le \mathrm{vol}_1(F_{Au}(AP)) \le \mathrm{vol}_1(F_{Au}(P))$. We conclude that $\mathrm{vol}_1(F_{Au}(P)) = \mathrm{vol}_1(A^{-T}F_u(P))$.

So far, we have established that for any $u \in S^{n-1}$ such that $\dim(F_u(P)) = 1$, $A^{-T}F_u(P)$ and $F_{Au}(P)$ are translates of one another. Next, we prove the following claim:

$$\forall F \in \mathcal{F}(P)\ \exists G \in \mathcal{F}(P) \text{ such that } An_P(F) = n_P(G). \tag{5.3}$$

Indeed, first note that for every $u \in \operatorname{relint}(n_P(F))$, we have $F = F_u(P)$ and that, by definition, the finite set $\{F_{Au} : u \in \operatorname{relint}(n_P(F))\} := \{F_1, \dots, F_k\}$ satisfies that

$$A \operatorname{relint}(n_P(F)) \subseteq \bigcup_{i=1}^{k} \operatorname{relint}(n_P(F_i)).$$

As shown in (5.2) we know that $\dim(F) = \dim(F_i)$ for all i, which means $\operatorname{span}(An_P(F)) = \operatorname{span}(n_P(F_i))$. However, in this spanned subspace (which is simply $(AF)^\perp$) the sets given by $\operatorname{relint}(n_P(F_1)), \dots, \operatorname{relint}(n_P(F_k))$ are pairwise disjoint and open (relative to the subspace). Since $A \operatorname{relint}(n_P(F))$ is a connected set, it cannot be covered by disjoint open sets. Therefore $F_1 = \cdots = F_k$. In other words, $A \operatorname{relint}(n_P(F)) \subseteq \operatorname{relint}(F_{Au})$ for every $u \in \operatorname{relint}(n_P(F))$. Applying A and using this inclusion repeatedly, we obtain

$$A^{k+1} \operatorname{relint}(n_P(F)) \subseteq A^k \operatorname{relint}(n_P(F_{Au}(P))) \subseteq A \operatorname{relint}(n_P(F_{A^k u}(P))).$$

Since $A^k = I$ and $F = F_u(P)$, we have $A \operatorname{relint} n_P(F) = \operatorname{relint} n_P(F_{Au}(P))$ and hence (as normal cones are closed) $An_P(F) = n_P(F_{Au}(P))$, as claimed.

Next, we claim that

$$F_w(P) \subseteq F_u(P) \implies F_{Aw}(P) \subseteq F_{Au}(P). \tag{5.4}$$

Indeed, suppose that $u \in n_P(F_u(P)) \subseteq n_P(F_w(P))$. By applying A on both sides, we get

$$Au \in An_P(F_u(P)) \subseteq An_P(F_w(P)).$$

By (5.3), $An_P(F_u(P)) = n_P(G)$ and $An_P(F_w(P)) = n_P(G')$ for some faces G, G' of P with $G' \subseteq G$. Therefore , it follows that $n_P(F_{Au}(P)) \subseteq n_P(G)$ and $n_P(F_{Aw}(P)) \subseteq n_P(G')$. Since, by (5.2), $\dim G = \dim F_u(P) = \dim F_{Au}$, we have $F_{Au}(P) = G$, and similarly $F_{Aw}(P) = G'$. Thus, we obtain that $F_{Aw}(P) = G' \subseteq G = F_{Au}(P)$, as claimed.

Let F_u be a 1-dimensional face of P and let F_v one of its vertices. By (5.2) and (5.4), F_{Au} is also a 1-dimensional face of P with F_{Av} as one of its vertices. Moreover, as we already established, $F_{Au} = A^{-T}F_u + x_u$ for some $x_u \in \mathbb{R}^n$, and (trivially) $F_{Av} = A^{-T}F_v + x_v$ for some $x_v \in \mathbb{R}^n$. Our goal is to show that $x_u = x_v$, from which it readily follows that $V(AP) = V(P) + x$ for some $x \in \mathbb{R}^n$ (as all vertices are connected via 1-dimensional faces). Indeed, denote $E = F_u(P)$ and $V = F_v(P)$. Note that $A^{-T}E + x_u$ and $A^{-T}E + x_v$ are parallel line segments of the same length and with a common vertex $A^{-T}V + x_v$ (as F_{Av} is a vertex of F_{Au}). Therefore, these segments are either identical, namely $x_u = x_v$ or consecutive so that their union T is a segment and their intersection is $A^{-T}theV + x_v$. Assume the latter. In particular, we have $A^{-T}V + x_v \in \operatorname{relint}(T)$. Denote $H = (Av)^\perp$ and $H^- = \{x \in \mathbb{R}^n : \langle x, Av\rangle \le 0\}$. Since Av is a normal of P at the vertex $A^{-T}V + x_v$ and $A^{-T}E + x_u$ is a face of P, we have $A^{-T}E + x_u \subseteq H^- + A^{-T}V + x_v$. On the other hand, clearly $F_{Av}(AP + x_v) = A^{-T}V + x_v$ and $A^{-T}E + x_v$ is a face of $AP + x_v$, which means that $A^{-T}E + x_v \subseteq H^- + A^{-T}V + x_v$ and so $T \subseteq H^- + A^{-T}V + x_v$. However, since $A^{-T}V + x_v \in \operatorname{relint}(T)$, it follows that Av must be orthogonal to T, which contradicts the fact that $\dim(F_{Av}) = 0$. Thus, we have $x_u = x_v$.

Concluding the above, we have $V(AP) = V(P)+x$ for some $x \in \mathbb{R}^n$, and so clearly $AP = P + x$. Since $P + V(AP)$ is assumed to be convex and $V(P) = V(AP)$, P must also be vertex generated, which completes our proof. □

6. A series expansion and covering estimates

In this section, we discuss two more properties of λ-vertex generated polytopes which are straightforward from the definition, and are the reason for our choice of name for this class.

We begin with the property, explained in the introduction, characterizing members of the class $\mathrm{VG}(\mathbb{R}^n)$ as polytopes P that can be written as the closure of a certain infinite sum involving the vertices of P.

Proposition 6.1. *Let $n \geq 1$ and $\lambda \in [1/(n+1), 1/2]$. Let $P \subseteq \mathbb{R}^n$ be a polytope. Then $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ if and only if*

$$P = \mathrm{cl}\left(\sum_{i=0}^{\infty}(1-\lambda)^i \lambda V(P)\right).$$

Proof. For any polytope, as P is closed and convex and $\sum_{i=0}^{\infty}(1-\lambda)^i\lambda = 1$, we have

$$\sum_{i=0}^{\infty}(1-\lambda)^i \lambda V(P) \subseteq P \tag{6.1}$$

so that the inclusion $P \supseteq \mathrm{cl}\big(\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)\big)$ holds without any assumptions. We thus need to show that the opposite inclusion holds if and only if $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$.

Assume $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$. Let $R > 0$ be such that $P \subseteq RB_2^n$. Using that $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$ we see inductively that for every $k \in \mathbb{N}$ it satisfies $P = \sum_{i=0}^{k}(1-\lambda)^i\lambda V(P) + (1-\lambda)^{k+1}P$. Given $x \in P$ and $\varepsilon > 0$, we choose $k \in \mathbb{N}$ such that $R(1-\lambda)^{k+1} < \varepsilon/2$. Then

$$P = \sum_{i=0}^{k}(1-\lambda)^i\lambda V(P) + (1-\lambda)^{k+1}P \subseteq \sum_{i=0}^{k}(1-\lambda)^i\lambda V(P) + \frac{\varepsilon}{2}B_2^n. \tag{6.2}$$

Therefore there exists $v_1 \in \sum_{i=0}^{k}(1-\lambda)^i\lambda V(P)$ such that $|v_1 - x| \leq \varepsilon/2$. From (6.1) we know that

$$\sum_{i=k+1}^{\infty}(1-\lambda)^i\lambda V(P) = (1-\lambda)^{k+1}\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P) \subseteq (1-\lambda)^{k+1}P \subseteq \frac{\varepsilon}{2}B_2^n. \tag{6.3}$$

Therefore for any $v_2 \in \sum_{i=k+1}^{\infty}(1-\lambda)^i\lambda V(P)$ we have that $|v_2| < \frac{\varepsilon}{2}$. Thus we may conclude that

$$d(x, \sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)) \leq |(v_1+v_2) - x| \leq |v_1 - x| + |v_2| < \varepsilon, \tag{6.4}$$

proving the inclusion $P \subseteq \mathrm{cl}\big(\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)\big)$.

For the other direction, assume $P = \mathrm{cl}\big(\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)\big)$. Write

$$\begin{aligned} P &= \mathrm{cl}\left(\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)\right) = \mathrm{cl}\left(\lambda V(P) + (1-\lambda)\sum_{i=0}^{\infty}(1-\lambda)^i\lambda V(P)\right) \\ &\subseteq \mathrm{cl}(\lambda V(P) + (1-\lambda)P) = \lambda V(P) + (1-\lambda)P, \end{aligned}$$

where the last equality is a result of $\lambda V(P)+(1-\lambda)P$ being a finite union of closed sets. Since the opposite inclusion $\lambda V(P) + (1-\lambda)P \subseteq P$, holds trivially (for any polytope) it follows that $P \in \mathrm{VG}(\mathbb{R}^n, \lambda)$. □

Remark 6.2. It is instructive to note that when P is the standard simplex in $\mathbb{R}^2$, the set $\mathrm{cl}\big(\sum_{i=1}^{\infty} 2^{-i}V(P)\big) = \bigcap_k(\sum_{i=1}^{k} 2^{-i}V(P) + 2^{-k}P)$ is the Sierpinski triangle. For other $P \notin \mathrm{VG}(\mathbb{R}^n)$ one may obtain other fractal-like objects.

Another useful fact about the class of λ-vertex generated polytopes is that it has good covering properties. This should not come as a surprise, as the definition of the class itself is that $|V(P)|$ copies of $(1-\lambda)P$ form a cover for P. Recall the notion of covering numbers.

Definition 6.3 (Covering numbers)**.** Let K and T be convex bodies in $\mathbb{R}^n$, The Covering number $N(K,T)$ of K by T is defined as follows:

$$N(K,T) := \min\left\{ N \in \mathbb{N} \mid \exists x_1,\dots,x_N \in \mathbb{R}^n : K \subseteq \bigcup_{i=1}^{N} x_i + T \right\}.$$

The results of this paper imply a simple covering estimate for the rich class of vertex generated polytopes, which is in the spirit of Maurey's lemma. Below $T_p(X)$ refers to the type-p constant of the Banach space X, for definitions see e.g. [2, Chapter 5].

Lemma 6.4 (Maurey's lemma, [9, Lemma 2])**.** *Let X be a space with type p, and unit ball T. Let m be an integer and assume that P is the convex hull of m points in T. Then for any integer k*

$$N(P, 2k^{-1/q}T_p(X)T) \le m^k$$

where q is the conjugate of p, that is $1/p + 1/q = 1$.

The proof of Maurey's lemma uses averages of the vertices of the polytope. In the case of vertex generated-polytopes, we can similarly provide a net using weighted averages (indeed, this is almost the definition of vertex generated).

Recall that $\lambda(P) = \sup\{\lambda : P = (1-\lambda)P + \lambda V(P)\}$, and denote $a(P) = 1-\lambda(P)$.

Proposition 6.5. *If $P \subseteq \mathbb{R}^n$ is a polytope with m vertices then for any $k \in \mathbb{N}$,*

$$N\Big(P, a(P)^k P\Big) \le m^k.$$

Proof. Set $a := a(P)$. In the same spirit of the proof of Proposition 6.1, we note that for any $k \in \mathbb{N}$, we have

$$P = aP + (1-a)V(P) = a^k P + (1-a)\sum_{i=0}^{k-1} a^i V(P).$$

As the number of points in $\sum_{i=0}^{k-1} a^i V(P)$ is at most m^k, we get the desired result. □

Note that on the one hand, by (1.1), it holds for any polytope $P \subseteq \mathbb{R}^n$ with m vertices that

$$N\left(P, \left(\frac{n}{n+1}\right)^k P\right) \leq m^k.$$

On the other hand, for a vertex generated polytope $P \in \mathrm{VG}(\mathbb{R}^n)$ (in particular, for any zonotope P), we get the superior estimates $N(P, 2^{-k}P) \leq m^k$.

Also note that we have the volume lower bound

$$N(P, 2^{-k}P) \geq \frac{\mathrm{vol}(P)}{\mathrm{vol}(2^{-k}P)} = 2^{nk} = (2^n)^k.$$

Since, by Lemma 2.1, for a vertex generated polytope we have $m \geq 2^n$, with equality precisely when P is a parallelopiped, the proposition can be interpreted as the fact that when the number of vertices of the vertex generated polytope is not much larger that 2^n, the volume lower bound is close to being an equality.

7. Some concluding remarks

Let us describe yet another angle from which to approach vertex generated-polytopes. Motivated by studying Brunn-Minkowski type inequalities for sums of boundaries [1], the following theorem for n-dimensional polytopes was shown in [7]). We recently discovered that a more general result was proved in [6].

Theorem 7.1. *Let $n \geq 1$. For any polytope $P \subseteq \mathbb{R}^n$ it holds that*

$$P = \frac{\partial^{\lceil \frac{n}{2} \rceil} P + \partial^{\lfloor \frac{n}{2} \rfloor} P}{2}.$$

Here for a polytope $P \subseteq \mathbb{R}^n$, we denoted the union of its k-dimensional faces by $\partial^k P$. Moreover, we showed that if $2P = \partial^k P + \partial^{n-k} P$ for some $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$ then $2P = \partial^m P + \partial^{n-m} P$ for all $k \leq m \leq \lfloor n/2 \rfloor$.

This allows to define, for every polytope, its "critical dimension" $k^*(P)$ which is the smallest k for which $2P = \partial^k P + \partial^{n-k} P$. Theorem 7.1 guarantees that k^* exists and is at most $\lfloor n/2 \rfloor$, see again [7]. A simplex $\Delta_n \subset \mathbb{R}^n$ is an example of a polytope with largest possible critical dimension, $k^*(\Delta) = \lfloor n/2 \rfloor$. The class of vertex generated-polytopes is precisely the class of polytopes for which the critical dimension is $k^* = 0$.

Remark 7.2. It is worth mentioning that the λ-parameter we introduced to measure closeness of a polytope to being vertex generated does not have a similar straightforward analogue for the class of polytopes for which $P = \frac{1}{2}(\partial^k P + \partial^{n-k} P)$, $0 < k < n$. Indeed, asking for an identity of the form $P = \lambda \partial^k P + (1-\lambda)\partial^{n-k} P$ to hold, for some $1/2 \neq \lambda$, already fails for say $P = [-1, 1]^n$ the unit cube. In other words, for the cube P, we have that $P = \lambda \partial^k P + (1-\lambda)\partial^{n-k} P$ for a given $0 < k < n$ if and only if $\lambda = \frac{1}{2}$. Indeed, assume without loss of generality that $\lambda > \frac{1}{2}$, and let us show that $0 \notin \lambda \partial^k P + (1-\lambda)\partial^{n-k} P$. Assume this statement is false, so there exist $x \in \partial^k P$ and $y \in \partial^{n-k} P$ such that $\lambda x + (1-\lambda)y = 0$, because $x \in \partial P$ there exist $1 \leq i \leq n$ such that $|\langle x, e_i \rangle| = 1$, so without loss of generality assume that $\langle x, e_1 \rangle = 1$, this means that

$$(1-\lambda)\langle y, e_1 \rangle = \langle 0, e_1 \rangle - \lambda \langle x, e_1 \rangle$$

and hence

$$\langle y, e_1\rangle = 0 - \frac{\lambda}{1-\lambda}\langle x, e_1\rangle < -1,$$

which is a contradiction as $y \in [-1,1]^n$.

When considering higher dimensional boundary parts instead of vertices, one can improve the factor $1/(n+1)$ from (1.1) significantly. The fact that ∂P is connected, together with a classical result of Fenchel, it follows that for every polytope $P \subseteq \mathbb{R}^n$

$$P = \frac{n-1}{n}P + \frac{1}{n}\partial^1 P.$$

Indeed, Fenchel [8] showed that if a set A cannot be separated into two disconnected parts by a hyperplane (which does not intersect A), then any point $x \in \mathrm{conv}(A)$ can be written as the convex hull of n points from A. The rest of the proof is just as we showed (1.1).

More generally, Bárány and Karasev showed [3, Corollary 2.4],

Proposition 7.3. *Let $P \subseteq \mathbb{R}^n$ be a polytope. Then*

$$P = \frac{n-k}{n-k+1}P + \frac{1}{n-k+1}\partial^k P.$$

It is worth mentioning that the Shapley-Folkman Lemma is also a generalization of (1.1), by plugging in $A_i = V(P)$ and $m = n+1$.

Theorem 7.4 (Shapley-Folkman Lemma)**.** *Let $A_1, \dots A_m \subset \mathbb{R}^n$, $m \geq n$ and let $x \in \sum_{i=1}^m \mathrm{conv}(A_i)$. Then there is some subset $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, m\}$ such that*

$$x \in \sum_{i\in I} \mathrm{conv}(A_i) + \sum_{i\notin I} A_i.$$

We end this section with a proof of the fact that the only polytope in $\mathbb{R}^n$ for which $\lambda(P) = 1/(n+1)$ is the simplex. This was first proved by Schneider in [10].

Proposition 7.5. *For a polytope $P \subseteq \mathbb{R}^n$, if $\lambda(P) = 1/(n+1)$ then P is an n-dimensional simplex.*

Proof. Assume $\lambda(P) = 1/(n+1)$. Using (1.1), which implies that if the affine dimension of P is d then $\lambda(P) \geq 1/(d+1)$, we see that P is not contained in any affine hyperplane of $\mathbb{R}^n$. For an n-dimensional simplex S with center of mass at the origin, the set $S \setminus ((1-\lambda)S + \lambda V(S))$ is relatively easy to analyze. Indeed, it is given by the intersection of S with $(n+1)$ half-spaces. More precisely, assume that

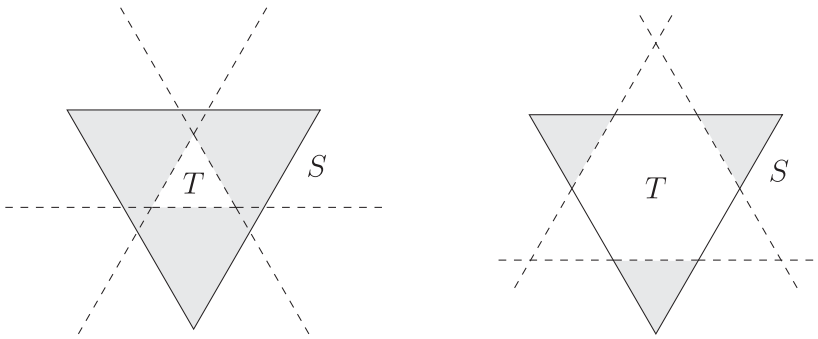
$$S = \bigcap_{i=1}^{n+1} \{x : \langle x, \theta_i\rangle \leq r_i\}$$

with $\theta_i \in S^{n-1}$. Then, since the center of mass is assumed to be the origin, for the vertex v_i opposite to the facet $\{x \in S : \langle x, \theta_i\rangle = r_i\}$ we have $\langle v_i, \theta_i\rangle = -nr_i$. Therefore if $\lambda > 1/(n+1)$ we have

$$(1-\lambda)S + \lambda v_i = S \cap \{x : \langle x, \theta_i\rangle \leq r_i((1-\lambda)(1+n) - n)\}.$$

Letting $r_i' = r_i(n - (1-\lambda)(1+n)) \geq 0$ we thus have (see Figure 2)

$$T := S \setminus \cup_i((1-\lambda)S + \lambda v_i) \subseteq \cap\{x : \langle x, -\theta_i\rangle \leq r_i'\} = -(n-(1-\lambda)(1+n))S.$$



(A) for $\lambda = 1/2$, T is the white triangle (B) for $\lambda = 2/3$, T is the white hexagon

FIGURE 2. Illustrating T, S and $-\lambda S$ (dashed) for two values of λ.

.

In particular, as $\lambda \to 1/(n+1)$, this set converges to the point $\{0\}$. It is easy to check (by translating S) that for a simplex with center of mass $m(S)$ we have, similarly,

$$S \setminus \cup_i((1-\lambda)S + \lambda v_i) \subseteq -(n-(1-\lambda)(1+n))S + m(S)(1-n+(1-\lambda)(1+n))$$

Again, as $\lambda \to 1/(n+1)$, this set converges to a point, namely $m(S)$.

Given some polytope P that is not a simplex, consider the set of all simplices with vertices that are a subset of $V(P)$. We pick $\lambda > 1/(n+1)$ close enough to $1/(n+1)$ so that the sets

$$\begin{aligned} T(S) &:= S \setminus ((1-\lambda)S + \lambda V(S)) \\ &\subseteq -(n-(1-\lambda)(1+n))S + m(S)(1-n+(1-\lambda)(1+n)) \end{aligned}$$

do not intersect for any pair of simplices S_1 and S_2 which differ by only one vertex (and so, have in particular different centers of mass). This is clearly possible by the convergence of $T(S)$ to $m(S)$ which we have demonstrated.

Let $x \in P$. Then x belongs to two simplices with vertices in $V(P)$ that differ by only one vertex. Indeed, x belongs to some simplex S (if it belongs to a lower dimensional simplex, then the assertion is trivial). Consider any vertex v of P which is not participating in S, and the ray emanating from v in direction x. This ray intersects S at two points, one of which, x', satisfies that $x \in [x', v]$. Since $x' \in \partial S$ it is in the convex hull of $(n-1)$ vertices of S, and along with v these span another simplex S' which includes x and differs from S by only one vertex. Therefore, x cannot belong to both $T(S)$ and $T(S')$. If $x \notin T(S)$, say, then

$$x \in (1-\lambda)S + \lambda V(S) \subseteq (1-\lambda)P + \lambda V(P),$$

as claimed (and similarly for S'). □

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