Pure and Applied Functional Analysis
Volume 10, Number 5, 2025, 1089–1132

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

NON-ARCHIMEDEAN ANALOGUE OF THE SPACE OF VALUATIONS ON CONVEX SETS

SEMYON ALESKER

Dedicated to the memory of Professor Nicole Tomczak-Jaegermann

Abstract. In the last two decades a number of structures on the classical space of translation invariant valuations on convex bodies were discovered, e.g. product, convolution, a Fourier type transform. In this paper a non-Archimedean analogue of the space of such (even) valuations with similar structures is constructed. It is shown that, like in the classical case, the new space equipped with either product or convolution satisfies Poincaré duality and hard Lefschetz theorem.

1. Introduction

1 Theory of valuations on convex bodies (or convex valuations, for brevity) is a classical area of convex geometry, see e.g. [9] and [42], Ch. 6. By definition, valuations are finitely additive measures on the class of all convex compact subsets of a finite dimensional *real* vector space. In the last two decades a number of new structures on the space of smooth translation invariant convex valuations have been discovered.

The goal of this paper is to imitate this space of (even) valuations and the known structures on it in the new context of vector spaces over non-Archimedean local fields. Note that in this case we do not have an interpretation of valuations as measures on something. We just construct a new space carrying structures with formally similar properties. Nevertheless most of the constructions in this paper are motivated by the usual valuations theory.

2 For these reasons let us say briefly a few words on the valuations theory on convex subsets of real vector spaces and structures on them. Here we emphasize mostly formal aspects of the theory relevant to this paper. Let V be a finite dimensional *real* vector space of dimension n. Let $Val^\infty(V)$ denote the space of smooth translation invariant valuations on convex compact subsets of V (see e.g. [4] or the lecture notes [8] for the definitions). This space has a natural grading, called McMullen's decomposition

$$Val^\infty(V)=\oplus_{i=0}^n Val_i^\infty(V).$$

$Val_0^\infty(V)$ and $Val_n^\infty(V)$ are 1-dimensional, while all other summands are infinite dimensional.

Partially supported by ISF grant 743/22 and US- Israel BSF grant 2018115.

The author discovered a product on $Val^\infty(V)$ [4] and a Fourier type transform ([3], [7].) In the recent preprint [26] Faifman and Wannerer found a simpler approach to the Fourier type transform on valuations.

Bernig and Fu [20] discovered a convolution on convex valuations. The Fourier type transform intertwines product and convolution [7].

The algebra $Val^\infty(V)$ equipped either with product or convolution is a commutative associative graded algebra with a unit. The Fourier transform establishes an isomorphism between these two algebras.

These two algebras satisfy a hard Lefschetz type theorem. In some special cases it was proved by the author [3], [5], [7] and Bernig and Bröcker [15]. In the very recent preprint [23] Bernig, Kotrbatý, and Wannerer proved a rather general hard Lefschetz type theorem on the language of convolution previously conjectured by Kotrbatý [34].

Furthermore recently Kotrbatý [34] formulated a general conjecture on Hodge-Riemann bilinear relations for valuations on the language of convolution. He proved it in a special case. In the above mentioned recent preprint [23] by Bernig, Kotrbatý, and Wannerer this conjecture was fully proved; this preprint is based on another important special case of the conjecture proved by Kotrbatý and Wannerer [36].

3 Valuations and these structures on them found a number of non-trivial applications. Theory of valuations has traditionally strong connections to integral geometry, see e.g. [32]. The above mentioned structures on convex valuations greatly enriched these connections and started to play a central role in various problems of integral geometry [16], [21], [22], [27].

The Hodge-Riemann bilinear relations in valuations formulated on the language of convolution imply the classical Alexandrov-Fenchel inequality for convex bodies [34]. In some cases the Hodge-Riemann bilinear relations can also be formulated on the language of product on valuations [34], [35], and they imply new inequalities for mixed volumes [10], [35].

Some of the recent developments in valuations theory found deep connections to pseudo-Riemannian geometry [12], [17], [18], [19].

4 While valuations were originally introduced in convex geometry partly motivated by the needs of integral geometry, in the last two decades there were attempts to generalize the valuations theory beyond convexity. Thus the space of valuations was introduced by the author on arbitrary smooth manifold as the space of finitely additive measures of a special form on sufficiently 'nice' subsets of a manifold, see [6] and references therein. This space retains several properties of the classical space of valuations on convex sets, in particular the product on valuations makes sense in this generality [13]

In a different direction, the notion of a valuation on various classes of functions was introduced and investigated [24], [25], [33], [37].

This paper can be considered as another attempt to extend the valuations theory beyond convexity. A description of its main results will be given in the next section.

5 Acknowledgements. I am very grateful to J. Bernstein who suggested in 2001 the basic idea to study non-Archimedean analogues of the space of convex valuations, before all the structures on valuations were discovered. The construction of the space of valuations in this context is essentially contained in our joint paper [11], Section 2, and it is used in the present paper. I thank also A. Aizenbud and D. Gourevitch for useful correspondences.

2. Main results

1 Let $\mathbb{F}$ be a non-Achimedean local field. Let V be an n-dimensional vector space over $\mathbb{F}$ (see Section 3 for a reminder on definitions and basic properties). In Section 8 we introduce the main objects of this paper: a complex vector space $Val^\infty(V)$ whose elements are called smooth valuations, and a complex vector space $Val(V)$ whose elements are called continuous valuations. The former space has no topology, while the latter is a Banach space. Both spaces are infinite dimensional provided $n > 1$, and $Val^\infty(V)$ is a dense subspace of $Val(V)$.

2.1. **Remark.** For the reader familiar with the classical theory of convex valuations, the notations $Val^\infty(V)$ and $Val(V)$ might be misleading. Both are analogous to even valuations rather than arbitrary ones. Moreover the definition of $Val(V)$ is closer not to the definition of continuous even convex valuations, but rather to Klain continuous even convex valuations in the sense of Bernig and Faifman [17].

Nevertheless we will keep this notion for the sake of simplicity.

2 By definition both above spaces are graded:

$$Val^\infty(V) = \oplus_{i=0}^n Val_i^\infty(V),$$

and similarly

$$Val(V) = \oplus_{i=0}^n Val_i(V).$$

Here $Val_0^\infty(V) = Val_0(V) = \mathbb{C}$, while $Val_n^\infty(V) = Val_n(V)$ is the 1-dimensional complex vector space of $\mathbb{C}$-valued Lebesgue measures on V. All other spaces $Val_i^\infty(V), Val_i(V)$ are infinite dimensional (for $n > 1$).

The space $Val_i^\infty(V)$ was first defined in [11], Section 2, (see also Section 8 below) as the image of certain intertwining integral between spaces of smooth sections of certain $GL(V)$-equivariant complex line bundles over Grassmannians over $\mathbb{F}$; in the convex case this intertwining integral coincides with the well known cosine transform. It was shown in [11] that this image is an irreducible $GL(V)$-module. This definition is motivated by the analogy with the result from the same paper [11], Section 1, in the convex case that the image of the corresponding intertwining integral can naturally be identified with the space of even smooth i-homogeneous convex valuations via the Klain imbedding ([31], Theorem 3.1; see also [9]). Note that in the case of convex valuations the irreducibility of the space of valuations of given degree of homogeneity and parity (even or odd) was previously proved by the author [2].

The space $Val_i(V)$ is defined in Section 8 as the closure of $Val^\infty(V)$ in the space of continuous sections of the appropriate complex line bundle over a Grassmannian.

3 Let X, Y be finite dimensional vector spaces over $\mathbb{F}$. In Section 12 we define the exterior product as a bilinear map

$$\boxtimes\colon Val(X) \times Val^{\infty}(Y) \longrightarrow Val(X \times Y)$$

which is continuous with respect to the first argument. The exterior product on valuations on convex sets was defined in [4] by the author.

Note that the exterior product of smooth valuations does not have to be smooth.

2.2. **Example.** 1) Let $1 \in Val_0(V) = \mathbb{C}$. We will denote this element also by χ or χ_X to keep the analogy with the classical (convex) case where it corresponds to the Euler characteristic. Then

$$\chi_X \boxtimes \chi_Y = \chi_{X \times Y}.$$

2) Let μ_X, μ_Y be Lebesgue measures on X, Y respectively. Then $\mu_X \boxtimes \mu_Y$ is the product measure in the usual sense.

4 Let $F\colon X \longrightarrow Y$ be a linear map of finite dimensional vector spaces over $\mathbb{F}$. We construct the pull-back map

$$F^*\colon Val(Y) \longrightarrow Val(X)$$

which is a continuous linear map of Banach spaces. The pull-back map on convex valuations was introduced by the author in [7].

2.3. **Example.** $F^*(\chi_Y) = \chi_X$.

If F is injective then F^* preserves the class of smooth valuations, see Theorem 9.1(4). In general it is not true. The main properties of the pull-back are summarized in Theorem 9.1.

5 Let $D(V) = Val_n(V)$ denote the 1-dimensional complex vector space of $\mathbb{C}$-valued Lebesgue measures on V, $n = \dim V$. In Section 10 we construct an analogue of the Fourier transform

$$\mathbb{F}\colon Val(V) \longrightarrow Val(V^\vee) \otimes D(V),$$

where $V^\vee$ is the dual space of V. $\mathbb{F}$ is an isomorphism of Banach spaces commuting with the action of $GL(V)$. It induces an isomorphism on the spaces of smooth valuations. Our construction is a straightforward generalization of the construction in [3] for even convex valuations.

We show (Theorem 18.1) that for smooth valuations ϕ, ψ one has

$$\mathbb{F}\phi \boxtimes \mathbb{F}\psi = \mathbb{F}(\phi \boxtimes \psi).$$

For convex valuations this formula was recently proved in [26].

6 For a linear map $F\colon X \longrightarrow Y$ we define in Section 11 the push-forward map

$$F_*\colon Val(X) \otimes D(X)^* \longrightarrow Val(Y) \otimes D(Y)^*$$

which is a continuous linear map of Banach spaces. By the definition

$$F_* = \mathbb{F} \circ (F^\vee)^* \circ \mathbb{F}^{-1},$$

where $F^\vee\colon Y^\vee \longrightarrow X^\vee$ is the dual map. For convex valuations the push-forward was defined in [7].

7 In Section 14 we define product on the space of smooth valuations $Val^\infty(V)$ as follows

$$\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi),$$

where $\Delta\colon V \hookrightarrow V \times V$ is the diagonal imbedding, i.e. $\Delta(v) = (v, v)$. It is shown that equipped with this product, $Val^\infty(V)$ is a commutative associative algebra with a unit (equal to χ_V). It is graded:

$$Val_i^\infty(V) \cdot Val_j^\infty(V) \subset Val_{i+j}^\infty(V).$$

Denote $n := \dim V$. $Val^\infty(V)$ satisfies Poincaré duality: the bilinear map given by the product

$$Val_i^\infty(V) \times Val_{n-i}^\infty(V) \longrightarrow Val_n^\infty(V) = D(V)$$

is a perfect pairing, i.e. for any $0 \neq \phi \in Val_i^\infty(V)$ there exists $\psi \in Val_{n-i}^\infty(V)$ such that $\phi \cdot \psi \neq 0$.

The product on convex valuations was introduced by the author [4].

8 In Section 17 it is shown that the algebra of smooth valuations $Val^\infty(V)$ satisfies a version of hard Lefschetz theorem. To state it, let us denote by $V_1 \in Val_1^\infty(V)$ the only (up to a proportionality) element invariant under a maximal compact subgroup $GL_n(\mathcal{O})$ of $GL_n(\mathbb{F}) \simeq GL(V)$. Let $0 \leq i < n/2$. Then the map $Val_i^\infty(V) \longrightarrow Val_{n-i}^\infty(V)$ given by

$$\phi \mapsto \phi \cdot (V_1)^{n-2i}$$

is an isomorphism.

The proof of this theorem uses properties of the Radon transform on Grassmannians over $\mathbb{F}$ due to Petrov and Chernov [38].

9 In Section 19 we introduce a convolution

$$*\colon (Val^\infty(V) \otimes D(V)^*) \times (Val^\infty(V) \otimes D(V)^*) \longrightarrow Val^\infty(V) \otimes D(V)^*$$

by $\phi * \psi = a_*(\phi \boxtimes \psi)$, where $a\colon V \times V \longrightarrow V$ is the addition map, i.e. $a(x, y) = x + y$.

By Proposition 19.2 the convolution is related to the product and the Fourier transform by the formula

$$\mathbb{F}\phi * \mathbb{F}\psi = \mathbb{F}(\phi \cdot \psi).$$

Convolution also satisfies Poincaré duality and hard Lefschetz type theorem (Theorem 19.3).

On convex valuations the convolution was introduced by Bernig and Fu [20].

10 An interesting open question is to establish for $Val^\infty(V)$ the Hodge-Riemann bilinear relations similar to [34] (see also [36], [23]).

3. Reminder on local fields

1 In this section we collect a few basic well known facts on local fields sufficient for this paper. We refer to [43], Ch. 1, for details.

By definition, a local field is a topological locally compact non-discrete field. There is a classification of such fields: they are precisely $\mathbb{R}, \mathbb{C}, \mathbb{F}_q((t))$, and finite extensions of the fields of p-adic numbers $\mathbb{Q}_p$. Here $\mathbb{F}_q$ denotes the finite field with q elements, and $\mathbb{F}_q((t))$ denotes the field of formal Laurent power series. The first two examples, namely $\mathbb{R}, \mathbb{C}$, are called Archimedean, while all others are called non-Archimedean local fields.

2 Let $\mathbb{F}$ be a non-Archimedean local field. It has a unique maximal compact subring $\mathcal{O} \subset \mathbb{F}$. For example if $\mathbb{F} = \mathbb{F}_q((t))$ then $\mathcal{O} = \mathbb{F}[[t]]$ is the ring of all Taylor power series. If $\mathbb{F} = \mathbb{Q}_p$ then $\mathcal{O} = \mathbb{Z}_p$ is the ring p-adic integers.

The field of fractions of $\mathcal{O}$ equals $\mathbb{F}$.

3 $\mathcal{O}$ has a unique maximal ideal $\mathfrak{m} \subset \mathcal{O}$. For example for $\mathbb{F} = \mathbb{F}_p((t))$ the ideal $\mathfrak{m}$ is generated by t, while for $\mathbb{F} = \mathbb{Q}_p$ the ideal $\mathfrak{m}$ is generated by p.

The quotient $k := \mathcal{O}/\mathfrak{m}$ is necessarily a finite field; it is called the residue field of $\mathbb{F}$.

4 There exists a unique multiplicative norm

$$|\cdot| \colon \mathbb{F} \longrightarrow \mathbb{R}_{\geq 0}$$

such that

$$|x| = 1 \;\; \forall x \in \mathcal{O} \backslash \mathfrak{m},$$
$$|x| = \frac{1}{|k|^i} \;\; \forall x \in \mathfrak{m}^i \backslash \mathfrak{m}^{i+1}, \text{where } i \geq 1.$$

where $|k|$ denotes the cardinality of the residue field. Multiplicativity means that $|x \cdot y| = |x| \cdot |y|$ for any $x, y \in \mathbb{F}$.

This norm satisfies the strengthened triangle inequality

$$|x + y| \leq \max\{|x|, |y|\}.$$

5 The norm $|\cdot|$ has the following property. Let μ be a Lebesgue measure on $\mathbb{F}$ (μ exists and is unique up to a proportionality). Let $x \in \mathbb{F}$. Then

$$\mu(x \cdot A) = |x| \mu(A)$$

for any compact subset $A \subset \mathbb{F}$.

4. Lattices over non-Archimedean local fields

1 In this section $\mathbb{F}$ denotes a non-Archimedean local field, and $\mathcal{O} \subset \mathbb{F}$ its ring of integers. In this section we review, mostly following [43], a few well known facts on finite dimensional $\mathbb{F}$-vector spaces and lattices in them.

A proof of the following result can be found in [40], Thm. 3.2, Ch. 1.

4.1. **Theorem.** *Let V be an n-dimensional Hausdorff topological vector space over the local field $\mathbb{F}$. Let $v_1, \dots, v_n$ be its basis. Then the map $\mathbb{F}^n \longrightarrow V$ given by*

$$(x_1, \dots, x_n) \mapsto x_1 v_1 + \dots + x_n v_n$$

is an isomorphism of topological vector spaces when the source space is equipped with the product topology.

2 Let V be an n-dimensional Hausdorff topological vector space over the local field $\mathbb{F}$.

4.2. **Definition.** A lattice L in V is a compact open $\mathcal{O}$-submodule of V.

4.3. **Lemma.** *Let $L \subset V$ be a lattice. Let $E \subset V$ be a vector subspace. Then $E \cap L$ is a lattice in E, and $L/E \cap L$ is a lattice in V/E.*

Proof. This immediately follows from Definition 4.2. $\square$

4.4. **Theorem** ([43], Ch. II, §2, Thm. 1)**.** *Let $L \subset V$ be a lattice.*
(i) *Then V has a basis $v_1, \dots, v_n$ such that $L = \mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_n$. In particular L is a free module of rank n.*
(ii) *Moreover if $\{0\} = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V$ be a sequence of linear subspaces such that $\dim V_i = i$. Then the above vectors $v_1, \dots, v_n$ can be chosen so that $v_1, \dots, v_i$ is a basis of V_i for any i.*

4.5. **Remark.** In the assumptions of part (ii) of the last theorem, one clearly has for each i

$$L \cap V_i = \mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_i.$$

3 Given a lattice $L \subset V$. Denote by $GL(L)$ the subgroup

$$GL(L) := \{T \in GL(V) | \, T(L) = L\}.$$

4.6. **Proposition.** *Let $L \subset V$ be a lattice. The natural action of the group $GL(L) \simeq GL_n(\mathcal{O})$ on the Grassmannian Gr_i^V is transitive.*

Proof. We may and will assume that $V = \mathbb{F}^n$, $L = \mathcal{O}^n$. Let $e_1, \dots, e_n \in \mathbb{F}^n$ be the standard basis. Let $E_0 := span\{e_1, \dots, e_i\} \in Gr_i^V$. Let $E \in Gr_i^V$. We have to show that there exists $T \in GL_n(\mathcal{O})$ such that $T(E_0) = E$.

By Lemma 4.3 and Theorem 4.4 there exists a basis $v_1, \dots, v_i$ of E such that

$$E \cap \mathcal{O}^n = \mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_i.$$

Similarly there exists a basis $\bar{v}_{i+1}, \dots, \bar{v}_n$ of V/E such that

$$L/L \cap E = \mathcal{O}\bar{v}_{i+1} \oplus \dots \oplus \mathcal{O}\bar{v}_n.$$

Let us choose $v_i \in L$, $j = i+1, \dots, n$, such that $v_j \equiv \bar{v}_j mod(E)$. It is easy to see that $\mathcal{O}^n = \mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_n$.

Then define $T \colon \mathbb{F}^n \longrightarrow \mathbb{F}^n$ by $T(e_j) = v_j$, $j = 1, \dots, n$. Clearly $T(E_0) = E$ and $T(\mathcal{O}^n) = \mathcal{O}^n$, i.e. $T \in GL_n(\mathcal{O})$. $\square$

4

4.7. **Proposition.** *Let $L \subset V$ be a lattice. Let $F_0 \colon W \hookrightarrow V$ be an injective linear map of vector spaces. There exists a neighborhood U of F_0 in $Hom(W, V)$ such that any $F \in U$ is also injective and*

$$F^{-1}(Im(F) \cap L) = F_0^{-1}(Im(F_0) \cap L).$$

Proof. We may and will assume that $V = \mathbb{F}^n$ and $L = \mathcal{O}^n$. Let us define a norm on $\mathbb{F}^n$ by

$$||(x_1, \dots, x_n)|| = \max_i |x_i|.$$

Clearly

$$||x + y|| \leq \max\{||x||, ||y||\},$$
$$||\lambda \cdot x|| = |\lambda| \cdot ||x||.$$

Then

$$\mathcal{O}^n = \{x \in \mathbb{F}^n | \; ||x|| \leq 1\}$$

is the unit ball of this norm.

By Lemma 4.3 we may also assume that

$$Im(F_0) = span\{e_1, \dots, e_i\},$$

where $e_1, \dots, e_n \in \mathbb{F}^n$ is the standard basis. Let $w_1, \dots, w_i \in W$ be the basis such that $F_0(w_j) = e_j$ for $1 \le j \le i$.

Then clearly

$$F_0^{-1}(Im(F_0) \cap L) = \{\sum_{j=1}^{i} x_j w_j | \max_{1 \le j \le i} |x_j| \le 1\}.$$

Define $U := \{F \in Hom(W, \mathbb{F}^n) | \; ||F(w_j) - e_j|| < 1 \; \forall 1 \le j \le i\}$. Let $F \in U$. The vectors $F(w_1), \dots, F(w_i)$ are linearly independent since these are vectors with coordinates from $\mathcal{O}$, and their reduction modulo the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ are the first i vectors of the standard basis of $(\mathcal{O}/\mathfrak{m})^n$. It follows that F is injective.

Now it remains to show that

$$F^{-1}(Im(F) \cap L) = \{\sum_{j=1}^{i} x_j w_j | \max_{1 \le j \le i} |x_j| \le 1\}.$$

Equivalently, one has to show that for $x_1, \dots, x_i \in \mathbb{F}$ the inequality

$$||F(x_1 w_1 + \dots + x_i w_i)|| \le 1 \tag{4.1}$$

holds if and only if $\max_{1 \le j \le i}\{|x_j|\} \le 1$.

The 'if' part follows since $||Fw_j|| \le \max\{||Fw_j - e_j||, ||e_j||\} = 1$ for any $1 \le j \le i$.

Conversely, let us assume that (4.1) holds. Without loss of generality we may assume that $|x_1| = \max_{1 \le j \le i}\{|x_j|\}$. Let us denote $\theta_j = F(w_j) - e_j$, $1 \le j \le i$. Then $||\theta_j|| < 1$. We have

$$1 \ge |x_1| \cdot ||(e_1 + \theta_1) + \sum_{j=1}^{i} \frac{x_j}{x_1}(e_j + \theta_j)|| =$$

$$|x_1| \cdot ||(1, \frac{x_2}{x_1}, \dots, \frac{x_i}{x_1}, 0, \dots, 0) + (\theta_1 + \sum_{j=2}^{i} \frac{x_j}{x_1}\theta_j)||.$$

Since $|\frac{x_j}{x_1}| \le 1$ the norm of the first summand equals 1, while the norm of the second summand (in the parenthesis) is strictly less then 1. Hence the norm of their sum is equal to 1. Thus we get $1 \ge |x_1| \cdot 1 = |x_1|$. $\square$

5. Lebesgue measures on vector spaces

1 In this section we assume that all vector spaces are finite dimensional over a local field $\mathbb{F}$, either Archimedean or not. For such a vector space V we denote by $D(V)$ throughout the article the (one-dimensional) complex vector space of $\mathbb{C}$-valued Lebesgue measures on V.

Let $\sigma\colon V \longrightarrow W$ be a surjective linear map between such vector spaces. Let $K = Ker(\sigma)$. We are going to construct a linear isomorphism

$$\tilde{\sigma}\colon D(K) \otimes D(W) \tilde{\longrightarrow} D(V). \tag{5.1}$$

Let $\mu_K \in D(K), \mu_W \in D(W)$. Since measures are linear functionals on compactly supported continuous functions, define for any $\phi \in C_c(V)$

$$\int \phi(v) d\tilde{\sigma}(\mu_K \otimes \mu_W) := \int_{w\in W} d\mu_W(w) \int_{k\in\sigma^{-1}(w)} \phi(k) d\mu_K(k), \tag{5.2}$$

where we identify the measure μ_K on K with its (arbitrary) translate to the parallel affine subspace $\sigma^{-1}(w)$.

An equivalent description is as follows. Let us choose a splitting $V = K \oplus L$. Then

$$\sigma|_L : L \longrightarrow W$$

is an isomorphism. Set $\mu_L := (\sigma|_L^{-1})_* \mu_W$. Then it is easy to see that

$$\sigma(\mu_K \otimes \mu_W) = \mu_K \boxtimes \mu_L,$$

where $\boxtimes$ denotes the usual product measure.

2 Let $X^\vee$ denote the dual space of a vector space X. The goal of this paragraph is to construct a canonical isomorphism

$$D(X)^* \tilde{\longrightarrow} D(X^\vee). \tag{5.3}$$

Since

$$Hom_{\mathbb{C}}(D(X)^*, D(X^\vee)) \simeq D(X) \otimes D(X^\vee),$$

it suffices to construct a canonical non-zero element in $D(X) \otimes D(X^\vee)$.

For a lattice $\Lambda \subset X$ let us define the dual lattice in $X^\vee$

$$\Lambda^\vee := \{f \in X^\vee | \, f(\Lambda) \subset \mathcal{O}\}.$$

It is easy to see that $\Lambda^\vee$ is a lattice in $X^\vee$.

5.1. **Lemma.** *Let $\mu \in D(X)$ be a non-vanishing ($\mathbb{C}$-valued) Lebesgue measure on X. Then there is a unique Lebesgue measure on $X^\vee$ denoted by μ^{-1} such that for any lattice $\Lambda \subset X$ one has*

$$\mu^{-1}(\Lambda^\vee) = \frac{1}{\mu(\Lambda)}. \tag{5.4}$$

Proof. Let us fix a lattice $\Lambda \subset X$. We can obviously construct a unique μ^{-1} such that (5.4) is satisfied for this Λ. It remains to show that (5.4) is satisfied for any other lattice $\tilde{\Lambda}$. For there exists $g \in GL(X)$ such that $\tilde{\Lambda} = g(\Lambda)$. Then $\tilde{\Lambda}^\vee = (g^\vee)^{-1}\Lambda^\vee$, where $g^\vee$ is the dual map of g. Then we have

$$\mu^{-1}(\tilde{\Lambda}^\vee) = \mu^{-1}((g^\vee)^{-1}\Lambda^\vee) =$$
$$|\det g^\vee|^{-1}\mu^{-1}(\Lambda^\vee) = \frac{1}{|\det g|\mu(\Lambda)} = \frac{1}{\mu(\tilde{\Lambda})}$$

□

Let us now construct the promised non-zero element of $D(X) \otimes D(X^\vee)$. Fix an arbitrary $\mathbb{C}$-valued non-vanishing Lebesgue measure $\mu \in D(X)$. Define the element to be

$$\mu \otimes \mu^{-1} \in D(X) \otimes D(X^\vee).$$

The following claim is now obvious.

5.2. **Claim.** *The element $\mu \otimes \mu^{-1}$ is independent of μ. It is $GL(X)$-invariant under the natural action of this group on $D(X) \otimes D(X^\vee)$.*

Now we can explicitly describe the isomorphism (5.3). Fix a non-vanishing $\mu \in D(X)$. There is a unique element $\mu^\vee \in D(X)^*$ such that $\mu^\vee(\mu) = 1$. The the isomorphism (5.3) maps $\mu^\vee \mapsto \mu^{-1}$. It is easy to see that this map is independent of μ.

Very often, by the abuse of notation, we will write in this paper

$$D(X)^* = D(X^\vee)$$

meaning the isomorphism (5.3).

6. Reminder on analytic manifolds over local fields

1 The goal of this section is to review very briefly the notion of analytic manifold over a non-Archimedean local field $\mathbb{F}$. For more details we refer to [41], part II.

The theory of analytic manifolds over such a field is similar to the theory of real analytic manifolds, at least at the basic level needed for this paper. The main examples of analytic manifolds to keep in mind for the purposes of this paper are the Grassmannians of linear i-dimensional subspaces in $\mathbb{F}^n$ and, more generally, the manifolds of partial flags in $\mathbb{F}^n$. All the material of this section is well known.

2 Let $U \subset \mathbb{F}^n$ be an open subset. A function $f \colon U \longrightarrow \mathbb{F}$ is called analytic if any point $a \in U$ has a ball centered at a in which f can be represented by a series which absolutely converges in this ball.

Let $F = (F_1, \dots, F_m)$ be a map $F \colon U \subset \mathbb{F}^n \longrightarrow \mathbb{F}^m$. F is called analytic if every F_i is analytic.

Composition of analytic maps is analytic ([41], part II, Ch. II, Theorem 2).

For an analytic map F its Jacobian is defined as usual

$$J(F) = \left(\frac{\partial F_i}{\partial x_j}\right).$$

Thus $J(F)$ is an $m \times n$ matrix whose entries are analytic functions.

3 For an analytic map

$$F \colon U \subset \mathbb{F}^n \longrightarrow \mathbb{F}^n$$

a version of the inverse function theorem holds (see [41], part II, Ch. III, §9). Namely assume that at a point $a \in U$

$$\det(J(F)_a) \neq 0.$$

Then there exists an open neighborhood V of a such that $F(V) \subset \mathbb{F}^n$ is open, $F|_V \colon V \longrightarrow F(V)$ is a homeomorphism, and $F^{-1} \colon F(V) \longrightarrow V$ is an analytic map.

It is easy to see that the inverse function theorem implies in the usual way the implicit function theorem for analytic functions. It is formulated in the usual way and we leave it to the reader. The implicit function theorem will be used in this paper in the proof of Lemma 17.2.

4 A topological space X is an analytic manifold if it admits an open covering $\{U_\alpha\}$, homeomorphisms

$$\phi_\alpha \colon U_\alpha \longrightarrow V_\alpha,$$

where $V_\alpha \subset \mathbb{F}^n$ are open subsets such that the transition maps

$$\phi_\alpha|_{U\alpha\cap U_\beta} \circ \phi_\beta^{-1}|_{\phi_\beta(U\alpha\cap U_\beta)}$$

are analytic for any α, β.

One can define analytic maps between analytic manifolds in the obvious way.

5 A topological group G which is also an analytic manifold is called a Lie group if
(a) the product map $G \times G \longrightarrow G$ given by $(x, y) \mapsto x \cdot y$ is analytic;
(b) the inverse map $G \longrightarrow G$ given by $x \mapsto x^{-1}$ is analytic.

Basic examples of Lie groups are $GL_n(\mathbb{F}), GL_n(\mathcal{O})$. Another example is the subgroup of $GL_n(\mathbb{F})$ stabilizing the given partial flag:

$$G = \left\{ \left[\begin{array}{c|c|c|c} A_1 & * & \dots & * \\ \hline 0 & A_2 & \dots & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & \dots & A_s \end{array} \right] \right\},$$

where $A_1, A_2, \dots, A_s$ are invertible square matrices. Thus G is the subgroup of invertible block upper triangular matrices.

6 Let G be a Lie group. Let $H \subset G$ be a subgroup which is an analytic submanifold (this notion is naturally defined). Then H is also a Lie group which is called a Lie subgroup of G (see [41], part II, Ch. IV, §2.3).

6.1. **Theorem** ([41], part II, Ch. IV, §5). *Let G be a Lie group. Let $H \subset G$ be its Lie subgroup. Then G/H has a unique structure of analytic manifold such that the natural map $G \longrightarrow G/H$ is analytic and has surjective differential at every point.*

This theorem implies immediately that the Grassmannians and, more generally, partial flag spaces are analytic manifolds.

7 Let us discuss now integration over analytic manifolds over a non-Archimedean local field $\mathbb{F}$. Let us start with integration in $\mathbb{F}^n$.

Let dx denote the Lebesgue measure on $\mathbb{F}^n$ normalized so that its value on $\mathcal{O}^n$ equals 1. Let $U, V \subset \mathbb{F}^n$ be compact open subsets. Let

$$f\colon V \longrightarrow \mathbb{C}$$

be a continuous function. Let $F\colon U \tilde{\longrightarrow} V$ be an analytic homeomorphism such that F^{-1} is also analytic. Then there is the following change of variables formula (see [30], §7.4)

$$\int_V f(y)dy = \int_U f(F(x))|\det J(F)_x|dx. \tag{6.1}$$

8 Let us define the complex line bundle $|\omega_X|$ over X called the line bundle of densities. We will see that its continuous sections can be integrated over X. Let us fix charts $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of charts on X. Consider the transition functions

$$F_{\alpha\beta} := \phi_\alpha|_{U_\alpha\cap U_\beta} \circ \phi_\beta^{-1}|_{U_\alpha\cap U_\beta} \colon \phi_\beta(U_\alpha \cap U_\beta) \tilde{\longrightarrow} \phi_\alpha(U_\alpha \cap U_\beta).$$

Choose the trivial line bundle $U_\alpha \times \mathbb{C}$ over each U_α and identify them over pairwise intersections $U_\alpha \cap U_\beta$ as follows

$$(x, z) \sim (x, |J(F_{\alpha\beta})_{\phi_\beta(x)}| \cdot z).$$

9 Let us assume in addition that X is compact. Then continuous sections of $|\omega_X|$ can be integrated over X by patching together local integrations over subsets of $\mathbb{F}^n$. More precisely let us fix a finite atlas of charts $\{(U_\alpha, \phi_\alpha)\}$ on X. There exits a subordinate partition of unity $\{\psi_\alpha\}$ (see e.g. [39], Theorem 2.13), namely there exist continuous functions $\psi_\alpha \colon U_\alpha \longrightarrow \mathbb{R}_{\geq 0}$ with $supp(\psi_\alpha) \subset U_\alpha$ and such that

$$\sum_\alpha \psi_\alpha = 1.$$

Let ω be a continuous section over X of the line bundle $|\omega_X|$. Then

$$\omega = \sum_\alpha \psi_\alpha \cdot \omega.$$

Clearly $supp(\psi_\alpha \cdot \omega) \subset U_\alpha$. Then $\int_{U_\alpha} \psi_\alpha \cdot \omega$ is well defined since (U_α, ϕ_α) is a chart. Then one defines

$$\int_X \omega := \sum_\alpha \int_\alpha \psi_\alpha \cdot \omega.$$

This number is independent of the atlas of charts and the subordinate partition of unity.

7. Representations of the group $GL_n(\mathcal{O})$

1 In this section we summarize a few known results on representations of the group $GL_n(\mathcal{O})$ in the space of complex valued functions on Grassmannians $Gr_k^{\mathbb{F}^n}$. No result of this section is novel.

Let $\mathbb{F}$ be non-Archimedean local field. Let $\mathcal{O} \subset \mathbb{F}$ be its ring of integers. The group $GL_n(\mathcal{O})$ is compact and acts transitively on the Grassmannian $Gr_k^{\mathbb{F}^n}$. Hence there is a unique probability (Haar) measure μ_{Haar} on $Gr_k^{\mathbb{F}^n}$ invariant under this group. By the general representation theory of compact groups the representation of $GL_n(\mathcal{O})$ in $L^2(Gr_k^{\mathbb{F}^n})$ is unitary and has a dense subspace which is an orthogonal countable direct sum of irreducible representations. The irreducible representations are necessarily finite dimensional.

2 Each irreducible representation of $GL_n(\mathcal{O})$ enters $L^2(Gr_k^{\mathbb{F}^n})$ with multiplicity at most 1. For $char(\mathbb{F}) = 0$ this was proven first in [29], Corollary 3.2, in general in [14].

3 The linear subspace $C^\infty(Gr_k^{\mathbb{F}^n})$ of locally constant (called smooth) functions is $GL_n(\mathcal{O})$-invariant and dense in $L^2(Gr_k^{\mathbb{F}^n})$.

Every finite dimensional $GL_n(\mathcal{O})$-invariant subspace of $L^2(Gr_k^{\mathbb{F}^n})$ is contained in $C^\infty(Gr_k^{\mathbb{F}^n})$; moreover the representation of $GL_n(\mathcal{O})$ in this subspace factorized via a quotient of $GL_n(\mathcal{O})$ by finite index subgroup. The last two statements follow from the fact that, as a topological group, $GL_n(\mathcal{O})$ is a pro-finite group (i.e. inverse limit of finite groups).

8. Space of valuations

1 Let V be an n-dimensional vector space over a local field $\mathbb{F}$. We introduce the main object of study $Val(V) = \oplus_{k=0}^{n} Val_k(V)$ which is a vector space over $\mathbb{C}$ and is an analogue of *even and Klain continuous* valuations on convex sets in the terminology of [17], Section 3.

We define $Val_0^\infty(V) := \mathbb{C}$ and $Val_n^\infty(V) := D(V)$, the latter is the (1-dimensional) space of complex valued Lebesgue measures on V. Let now $1 \leq k \leq n-1$. We are going to define in this section $Val_k^\infty(V)$.

2 Let us start with an elementary construction. Given two vector spaces X and Y of equal (finite) dimension over the local field $\mathbb{F}$. Define a map linear with respect to the second variable

$$\mathcal{T}\colon Hom_{\mathbb{F}}(X,Y) \times D(Y) \longrightarrow D(X). \tag{8.1}$$

Let $F \in Hom_{\mathbb{F}}(X,Y), \mu \in D(Y)$. There exists a unique Lebesgue measure ν on X such that for some (equivalently, any) compact subset $A \subset X$ with non-empty interior one has $\nu(A) = \mu(F(A))$. Define $\mathcal{T}(F,\mu) := \nu$.

8.1. **Claim.** (1) *The map $\mathcal{T}$ is linear with respect to the second argument.*
(2) *If F is invertible then $\mathcal{T}(F,\mu) = (F^{-1})_*(\mu)$, where G_* denotes the push-forward of measures under a map G. Otherwise $\mathcal{T}(F,\mu) = 0$.*
(3) *The map $\mathcal{T}$ is jointly continuous.*

A proof is left to the reader.

3 Let $\mathcal{L}_k^V \longrightarrow Gr_k^V$ be the complex line bundle whose fiber over $E \in Gr_k^V$ is equal to the 1-dimensional space of complex valued Lebesgue measures on E.

Let $|\omega_{n-k}^V| \longrightarrow Gr_{n-k}^V$ denote the line bundle of densities on Gr_{n-k}^V as defined in Section 6. Its continuous global sections can be integrated over the Grassmannian Gr_{n-k}^V.

Let us denote by $\mathcal{M}'_{n-k} \longrightarrow Gr_{n-k}^V$ the line bundle whose fiber over $F \in Gr_{n-k}^V$ is equal to the space of complex valued Lebesgue measures on V/F. Set finally $\mathcal{M}_{n-k}^V := \mathcal{M}'_{n-k} \otimes |\omega_{n-k}^V|$. All the line bundles $\mathcal{L}_{n-k}^V, |\omega_{n-k}^V|, \mathcal{M}'_{n-k}, \mathcal{M}_{n-k}^V$ are $GL(V)$-equivariant in a natural way.

A section of any of the above vector bundles is called smooth if its stabilizer in $GL(V)$ is an open subgroup. More generally a vector in a continuous representation of $GL(V)$ is called smooth if its stabilizer is open. It is easy to see that any function on Gr_i^V is smooth in this sense if and only if it is locally constant.

4 Let us define the $GL_n(V)$-equivariant operator between spaces of smooth sections

$$\mathcal{D}\colon C^\infty(Gr_{n-k}^V, \mathcal{M}_{n-k}^V) \longrightarrow C^\infty(Gr_k^V, \mathcal{L}_k^V) \tag{8.2}$$

as follows. For $F \in Gr_{n-k}^V$ and $E \in Gr_k^V$ define $p_{E,F}\colon E \longrightarrow V/F$ the natural map. Given a section $\xi \in C^\infty(Gr_{n-k}^V, \mathcal{M}_{n-k}^V)$, define for any $E \in Gr_k^V$

$$\mathcal{D}(\xi)(E) = \int_{F \in Gr_{n-k}^V} \mathcal{T}(p_{E,F}, \xi(F)), \tag{8.3}$$

where $\mathcal{T}$ is the map from paragraph **2**. Note that

$$\mathcal{T}(p_{E,F}, \xi(F)) \in D(E) \otimes |\omega_{n-k}^V|\big|_F.$$

Hence the integral belongs to $D(E)$.

Let us reformulate the definition of $\mathcal{D}$. Given a k-dimensional subspace E. The set of $F \in Gr_{n-k}^V$ intersecting E non-transversally, or equivalently $p_{E,F}$ is non-invertible, has zero measure in the Grassmannian. Ignoring this subset we have

$$\mathcal{D}(\xi)(E) = \int_{F \in Gr_{n-k}^V} (p_{E,F})_*^{-1} \xi(F). \tag{8.4}$$

8.2. **Proposition.** (1) *The operator $\mathcal{D}$ is $GL(V)$-equivariant. It extends uniquely by continuity to the space of continuous sections. Then it maps continuous sections to continuous, smooth to smooth.*
(2) *The image of $\mathcal{D}$ on the space of smooth sections is an irreducible subspace.*

Part (1) follows from Claim 8.1(3). Part (2) was proved in [11], Theorem 2.1; see also [28], Corollary 1.3, for a more general statement.

8.3. **Definition.** Let us denote by $Val_k^\infty(V)$ the image of $\mathcal{D}$ on smooth vectors.

5 Let us define the space of continuous valuations.

8.4. **Definition.** The space of continuous valuations $Val_k(V)$ is the closure of $Val_k^\infty(V)$ in $C^\infty(Gr_k^V, \mathcal{L}_k^V)$.

$Val_k(V)$ is a $GL(V)$-invariant subspace and is (topologically) irreducible, i.e. has no invariant closed proper subspaces.

6 Let us fix a lattice $\Lambda \subset V$. Since the action of $GL(\Lambda)$ on Gr_k^V is transitive, there exist unique (up to a proportionality) non-zero $GL(\Lambda)$-invariant continuous sections of $\mathcal{M}_{n-k}^V$ and of $\mathcal{L}_k^V$ (which are obviously smooth). It is easy to see that $\mathcal{D}$ applied to the former is a non-zero multiple of the latter. We call such a $GL(\Lambda)$-invariant section a spherical vector. We have the following easy characterization of $Val_k^\infty(V), Val_k(V)$.

8.5. **Lemma.** *$Val_k^\infty(V)$ (resp. $Val_k(V)$) is the only irreducible $GL(V)$-submodule of $C^\infty(Gr_k^V, \mathcal{L}_k^V)$ (resp. $C(Gr_k^V, \mathcal{L}_k^V)$) containing a spherical vector.*

Proof. Let us prove the non-smooth case, the smooth one is very similar. Assume that $T \subset C(Gr_k^V, \mathcal{L}_k^V)$ is another closed irreducible $GL(V)$-submodule containing the spherical vector. Then $T \cap Val_k(V)$ also has these properties. It is non-zero since contains the spherical vector. But $T \cap Val_k(V) \subset Val_k(V)$. Since $Val_k(V)$ is $GL(V)$-irreducible, it follows that $T \cap Val_k(V) = Val_k(V)$. Hence $Val_k(V) \subset T$. Hence $Val_k(V) = T$. □

8.6. **Remark.** In fact a stronger characterization holds: $Val_k^\infty(V)$ (resp. $Val_k(V)$) is the only irreducible $GL(V)$-submodule of $C^\infty(Gr_k^V, \mathcal{L}_k^V)$ (resp. $C(Gr_k^V, \mathcal{L}_k^V)$). This statement is a special case of [28], Theorem 1.2.

9. Pull-back on valuations

1 Let $\mathbb{F}$ be a non-Archimedean local field. The goal of this section is to construct an operation of pull-back on valuations. More precisely we will prove

9.1. **Theorem.** *Let X and Y be finite dimensional vector spaces over $\mathbb{F}$. For any linear map $F\colon X \longrightarrow Y$ there exists a canonical continuous linear map*

$$F^*\colon Val(Y) \longrightarrow Val(X)$$

which is called the pull-back map and satisfies the following properties:
(1) F^* preserves degree of homogeneity, i.e. $F^*(Val_k(Y)) \subset Val_k(X)$;
(2) $(F \circ G)^* = G^* \circ F^*$;
(3) $Id^* = Id$;
(4) If F is injective then F^* preserves the subspace of smooth valuations, i.e. $F^*(Val^\infty(Y)) \subset Val^\infty(X)$.

Proof of this theorem occupies the rest of this section.

2 The map F^* on valuations of degree 0 is just the identity map of $\mathbb{C}$.

3 Let $F\colon X \longrightarrow Y$ be a linear map and $\dim X = \dim Y = n$. The the pullback $F^*\colon Val_n(Y) \longrightarrow Val_n(X)$ is the map $F^*\colon D(Y) \longrightarrow D(X)$ given, by definition, by

$$F^*(\mu) = \mathcal{T}(F, \mu),$$

where $\mathcal{T}$ is the map from Claim 8.1.

4 Let $F\colon X^n \longrightarrow Y^m$ be a linear map. Let us define first the pull-back map

$$F^*\colon D(Y) \longrightarrow C(Gr_m^X, \mathcal{L}_m^X).$$

If $m > n$ then $F^* = 0$ by the definition. Assume that $m \le n$. For $\mu \in D(Y)$ and any $E \in Gr_m^X$ set

$$(F^*\mu)(E) := \mathcal{T}(F|_E, \mu).$$

The continuity of $\mathcal{T}$ with respect to the first variable implies that $F^*\mu$ is continuous.

5

9.2. **Lemma.** (1) *One has $F^*(D(Y)) \subset Val_m(X)$.*
(2) *The linear span of valuations of the form $F^*\mu$ over all possible linear maps $F\colon X \longrightarrow Y$ and $\mu \in D(Y)$ is dense in $Val_m(X)$.*

Proof. Proof of (1). If $F(X) \neq Y$ then $F^*(D(Y)) = 0$. Thus let us assume that $F(X) = Y$. We will identify Y with $X/Ker(F)$ in the natural way.

Let us fix $\mu \in D(X/Ker(F))$. Recall that $m = \dim(X/Ker(F))$. Let us chose a smooth section $\tilde{\mu}$ of the line bundle $\mathcal{M}'_{n-m}$ over Gr_{n-m}^X such that its value at $Ker(F)$ is equal to μ. (Recall that the fiber of $\mathcal{M}'_{n-m}$ over any subspace E is equal to $D(X/E)$.) Let us choose a sequence of smooth measures $\{\rho_a\}$ on Gr_{n-m}^X which weakly converges to the δ-measure supported at $\{Ker(F)\}$. Then $\rho_a \otimes \tilde{\mu} \in C^\infty(Gr_{n-m}^X, \mathcal{M}_{n-m}^X)$. Applying to it operator $\mathcal{D}$ from (8.2) we get

$$\mathcal{D}(\rho_a \otimes \tilde{\mu}) \in Val_m^\infty(X).$$

When $a \longrightarrow \infty$ the latter section converges to $F^*\mu$ in $C(Gr_k^X, \mathcal{L}_k^X)$, hence $F^*\mu \in Val_m(X)$.

Proof of (2). It is clear that the linear span of valuations of the form $F^*\mu$ is a $GL(X)$-invariant subspace of $Val_m(X)$. Since the latter space is $GL(X)$-irreducible by Proposition 8.2, the result follows. $\square$

The constructed map $F^*\colon D(Y) \longrightarrow Val_{\dim Y}(X)$ is called pull back on densities.

6 The following corollary will be used later.

9.3. **Corollary.** *Let $\mathcal{X}$ be a compact metrizable space. Let m be a complex valued Borel measure on $\mathcal{X}$. Let*

$$T\colon \mathcal{X} \longrightarrow Hom(X,Y)$$

be a continuous map when X, Y are finite dimensional vector spaces over $\mathbb{F}$. Let $\mu \in D(Y)$. Then

$$\int_{\mathcal{X}} [T(x)]^*(\mu) dm(x)$$

belongs to $Val_{\dim Y}(X)$.

Proof. By Claim 8.1 the map $\mathcal{X} \longrightarrow C(Gr^X_{\dim Y}, \mathcal{L}^X_{\dim Y})$ given by the expression under the last integral $x \mapsto [T(x)]^*(\mu)$ is continuous. By Lemma 9.2 the expression under the integral $[T(x)]^*(\mu)$ belongs to $Val(X)$. Since $Val(X)$ is complete (it is a Banach space), the integral is well defined as a limit of Riemann sums. $\square$

7 Given a linear map $F\colon X \longrightarrow Y$. First let us define an auxiliary linear map, also denoted by F^* by the abuse of notation,

$$F^*\colon C(Gr^Y_k, \mathcal{L}^Y_k) \longrightarrow C(Gr^X_k, \mathcal{L}^X_k) \tag{9.1}$$

as follows. Let $f \in C(Gr^Y_k, \mathcal{L}^Y_k)$. For any subspace $E \in Gr^X_k$ let us define

$$(F^*f)(E) = \begin{cases} \mathcal{T}(F|_E, f(F(E))) & \text{if} \quad \dim F(E) = k \\ 0 & \text{otherwise} \end{cases}. \tag{9.2}$$

9.4. **Proposition.** *F^*f is a continuous section of $\mathcal{L}^X_k$.*

We will need the following elementary lemma whose proof is easy and is left to the reader. For vector spaces K, X we denote by $Inj(K, X)$ the space of linear imbeddings $K \hookrightarrow X$.

9.5. **Lemma.** *Let K be a k-dimensional vector space. Let f be a not necessarily continuous section of $\mathcal{L}^X_k$ over Gr^X_k. Then f is continuous if and only if the map $Inj(K, X) \longrightarrow D(K)$ given by $h \mapsto h^*f$ is continuous.*

Proof of Proposition 9.4. Fix a k-dimensional vector space K. By Lemma 9.5 we have to show that $h \mapsto h^*(F^*f)$ is a continuous map $Inj(K, X) \longrightarrow D(K)$. It easily follows from the definition that $h^*(F^*f) = (F \circ h)^*f$. The continuity of $h \mapsto (F \circ h)^*f$ follows from the continuity of f and Lemma 9.5. $\square$

8

9.6. **Proposition.** *The map F^* in (9.1) is continuous.*

Proof. Let us fix an open bounded subset $\mathcal{C} \subset X$ which contains the origin. The topology on $C(Gr^X_k, \mathcal{L}^X_k)$ is given by the norm

$$||g|| := \sup_{E \in Gr^X_k} |\int_{E \cap \mathcal{C}} g(E)|.$$

Fix an open bounded subset $\mathcal{K} \subset Y$ such that $F(\mathcal{C}) \subset \mathcal{K}$. Then we have

$$||F^* f|| = \sup_{E \in Gr_k^X,\, \dim F(E)=k} |\int_{F(E \cap \mathcal{C})} f(F(E))| \leq$$

$$\sup_{E \in Gr_k^X,\, \dim F(E)=k} |\int_{F(E) \cap \mathcal{K}} f(F(E))| \leq ||f||.$$

□

9 Let $W \xrightarrow{G} X \xrightarrow{F} Y$ be linear maps. The equality $(F \circ G)^* = G^* \circ F^*$ follows directly from the definition of the pull-back.

10

9.7. **Proposition.** *One has*

$$F^*(Val_k(Y)) \subset Val_k(X).$$

9.8. **Definition.** The restriction of F^* to $Val(Y)$ is called the pull-back map on valuations.

Proof of Proposition 9.7. By the continuity of F^* it suffices to show that $F^*(Val_k^\infty(Y)) \subset Val_k(X)$. One has to show that for any $\xi \in C^\infty(Gr_{n-k}^Y, \mathcal{M}_{n-k}^Y)$ one has $F^*(\mathcal{D}\xi) \in Val_k(X)$ where $\mathcal{D}$ was defined in (8.3). Let us choose a finite open covering of Gr_{n-k}^Y with a trivialization of the bundle $\mathcal{M}_{n-k}^Y$ over each of its subsets. Let us choose a partition of unity subordinate to this covering (see Section 6, paragraph **9**). These choices reduce the problem to the following one. Given a complex valued Borel measure m on Gr_{n-k}^Y, a continuous map $T\colon Gr_{n-k}^Y \longrightarrow Hom(Y, \mathbb{F}^k)$, and a density $\mu \in D(\mathbb{F}^k)$, then

$$\mathcal{D}(\xi) = \int_{Gr_{n-k}^Y} [T(x)]^*(\mu) dm(x).$$

Then $F^*(\mathcal{D}(\xi)) = \int_{Gr_{n-k}^Y} (T(x) \circ F)^*(\mu) dm(x)$. The latter expression belongs to $Val_k(X)$ by Corollary 9.3. □

This completes the construction of the pull-back F^* on valuations and finishes the proof of Theorem 9.1. In the next paragraph we prove a continuity property of the pull-back map.

11 Let us prove now part (4) of Theorem 9.1. Namely let us assume that $F\colon X \longrightarrow Y$ is injective. Let us show that $F^*(Val^\infty(Y)) \subset Val^\infty(X)$. We may and will assume that $X \subset Y$. Let us choose a splitting $Y = X \oplus Z$. It induces a groups imbedding $GL(X) \hookrightarrow GL(Y)$ given by $g \mapsto (g, Id_Z)$. Clearly any $GL(Y)$-smooth vector is $GL(X)$-smooth. The result follows.

12

9.9. **Proposition.** *The map $Hom(X,Y) \times C(Gr_i^Y, \mathcal{L}_i^Y) \longrightarrow C(Gr_i^X, \mathcal{L}_i^X)$ given by $(F, f) \mapsto F^* f$ is jointly continuous.*

Proof. It it well known (and easily follows from the Banach-Steinhauss theorem) that any separately continuous map

$$T \times A \longrightarrow B,$$

where T is a metric space and A, B are Banach spaces, and the operator is linear with respect to the second argument, is jointly continuous.

Hence is suffices to show that our map is separately continuous. By Claim 9.6 it suffices to show that if $\xi \in C(Gr_i^Y, \mathcal{L}_i^Y)$ is fixed then the map $Hom(X,Y) \longrightarrow C(Gr_i^X, \mathcal{L}_i^X)$ given by $F \mapsto F^*\xi$ is continuous. Let us prove the continuity of this map at certain fixed $F_0 \in Hom(X,Y)$.

Let us fix a lattice $L \subset X$. It suffices to show that for any $\varepsilon > 0$ there is a neighborhood of F_0 in $Hom(X,Y)$ such that for any F from this neighborhood

$$\sup_{E \in Gr_i^X} \left| \int_{E \cap L} (F^*\xi)(E) - \int_{E \cap L} (F_0^*\xi)(E) \right| < \varepsilon. \tag{9.3}$$

It suffices to prove this statement locally, i.e. to prove that any $E_0 \in Gr_i^X$ has a neighborhood $\mathcal{U}_{E_0} \subset Gr_i^X$, and there is a neighborhood $\mathcal{V}_{E_0}$ of F_0 in $Hom(X,Y)$ such that

$$\sup_{E \in \mathcal{U}_{E_0}} \left| \int_{E \cap L} (F^*\xi)(E) - \int_{E \cap L} (F_0^*\xi)(E) \right| < \varepsilon \text{ for any } F \in \mathcal{V}_{E_0}. \tag{9.4}$$

Indeed then we could choose a finite subcovering $\{\mathcal{U}_{E_\alpha}\}$ of Gr_i^X. Then for any $F \in \cap_\alpha \mathcal{V}_\alpha$ one had (9.3).

Thus let us fix $E_0 \in Gr_i^X$.

<u>Case 1.</u> Let us assume that $\dim F_0(E_0) = i$. By Theorem 4.4(i) there exists a linear isomorphism $H_0 \colon \mathbb{F}^i \tilde{\longrightarrow} E_0$ such that $H_0(\mathcal{O}^i) = E_0 \cap L$. It suffices to show that for any $\varepsilon > 0$ there exists a neighborhood $\mathcal{X}$ of H_0 in $Hom(\mathbb{F}^i, X)$ and a neighborhood $\mathcal{V}$ of F_0 in $Hom(X,Y)$ such that for any $H \in \mathcal{X}$ and any $F \in \mathcal{V}$ one has

$$\left| \int_{H^{-1}(Im(H) \cap L)} (F \circ H)^*\xi - \int_{H^{-1}(Im(H) \cap L)} (F_0 \circ H)^*\xi \right| < \varepsilon.$$

Note that $H_0^{-1}(Im(H_0) \cap L) = \mathcal{O}^i$. Then we can choose the neighborhood $\mathcal{X}$ of H_0 as in Proposition 4.7, i.e. so that

$$H^{-1}(Im(H) \cap L) = \mathcal{O}^i \text{ for all } H \in \mathcal{X}.$$

Hence one has to show that

$$\left| \int_{\mathcal{O}^i} (F \circ H)^*\xi - \int_{\mathcal{O}^i} (F_0 \circ H)^*\xi \right| < \varepsilon.$$

But this is clear from the definition of topology on $\mathcal{L}_i^X$.

<u>Case 2.</u> Let us assume that $\dim F_0(E_0) < i$. By Theorem 4.4 and Remark 4.5 there is an isomorphism $H_0 \colon \mathbb{F}^i \tilde{\longrightarrow} E_0$ such that $H_0(\mathcal{O}^i) = E_0 \cap L$ and

$$H_0(\mathcal{O}^k \times \{0_{i-k}\}) = (Ker F_0) \cap E_0 \cap L,$$

where $k := \dim((Ker F_0) \cap E_0)$. It suffices to show that for any $\varepsilon > 0$ there exists a neighborhood $\mathcal{X}$ of H_0 in $Hom(\mathbb{F}^i, X)$ and a neighborhood $\mathcal{V}$ of F_0 in $Hom(X,Y)$

such that for any $H \in \mathcal{X}$ and any $F \in \mathcal{V}$ one has

$$|\int_{H^{-1}(Im(H)\cap L)} (F \circ H)^* \xi| < \varepsilon. \tag{9.5}$$

Note that $H_0^{-1}(Im(H_0) \cap L) = \mathcal{O}^i$. By Proposition 4.7 H_0 has a neighborhood $\mathcal{X} \subset Hom(\mathbb{F}^i, X)$ so that

$$H^{-1}(Im(H) \cap L) = \mathcal{O}^i \text{ for all } H \in \mathcal{X},$$

Hence (9.5) is equivalent to say that there exists a neighbourhood $\mathcal{X}$ of H_0 and $\mathcal{V}$ of F_0 such that

$$|\int_{\mathcal{O}^i} (F \circ H)^* \xi| < \varepsilon \text{ for any } H \in \mathcal{X},\ F \in \mathcal{V}. \tag{9.6}$$

It suffices to show that given the linear map $g_0 \colon \mathbb{F}^i \longrightarrow Y$ such that $rk(g) < i$ then there is a neighborhood $\mathcal{U} \subset Hom(\mathbb{F}^i, Y)$ of g_0 such that

$$|\int_{\mathcal{O}^i} g^* \xi| < \varepsilon \ \ \forall g \in \mathcal{U}.$$

We leave to the reader this simple and elementary estimate. $\square$

10. Construction of Fourier transform on valuations

1 Let $\mathbb{F}$ be a local field. In this section we construct a $GL(V)$-equivariant isomorphism between spaces of valuations on a vector space V and its dual.

We denote by $V^\vee$ (rather than V^*) the dual space of V. Nevertheless we will keep $*$ to denote duals of one dimensional spaces (thus $D(V)^*$ denotes the dual space of the space $D(V)$ of $\mathbb{C}$-valued Lebesgue measures on V).

For a vector subspace $E \subset V$ let us denote by $E^\perp \subset V^\vee$ its annihilator defined by

$$E^\perp := \{f \in V^\vee |\ f(E) = 0\}.$$

This induces a $GL(V)$-equivariant homeomorphism

$$Gr_i^V \overset{\perp}{\underset{\sim}{\longrightarrow}} Gr_{n-i}^{V^\vee}.$$

2 Recall that the fiber of the line bundle $\mathcal{L}_i^V$ over $E \in Gr_i^V$ is equal, by the definition, to the space $D(E)$ of complex valued Lebesgue measures on E. We are going to construct a $GL(V)$-equivariant homeomorphism $a_i \colon \mathcal{L}_i^V \longrightarrow \mathcal{L}_{n-i}^{V^\vee} \otimes D(V^\vee)^*$ such that the diagram

$$\begin{array}{ccc} \mathcal{L}_i^V & \xrightarrow{a_i} & \mathcal{L}_{n-i}^{V^\vee} \otimes D(V^\vee)^* \\ \downarrow & & \downarrow \\ Gr_i^V & \xrightarrow{\perp} & Gr_{n-i}^{V^\vee} \end{array}$$

is commutative and the map a is linear on fibers of the bundles (the vertical arrows are the obvious bundle projections).

Fix a linear subspace $E \subset V$. Note that canonically the dual space $E^\vee = V^\vee / E^\perp$. We have the isomorphisms from Section 5:

(10.1)

$$D(E) \tilde{\longrightarrow} D(E^\vee)^* \tilde{\longrightarrow} D(V^\vee/E^\perp)^* \tilde{\longrightarrow} (D(V^\vee) \otimes D(E^\perp)^*)^* \tilde{\longrightarrow} D(E^\perp) \otimes D(V^\vee)^*.$$

This defines the required map a on the fiber $D(E)$ over E.

3 By taking sections of the bundles in the diagram from paragraph **2** we get a $GL(V)$-equivariant isomorphism of Banach spaces

$$\mathbb{F}\colon C(Gr_i^V,\mathcal{L}_i)\tilde{\longrightarrow}C(Gr_{n-i}^{V^\vee},\mathcal{L}_{n-i}\otimes D(V^\vee)^*). \tag{10.2}$$

10.1. **Remark.** For $i=0$ the Fourier transform is the obvious isomorphism

$$\mathbb{F}\colon \mathbb{C}\longrightarrow D(V^\vee)\otimes D(V^\vee)^*.$$

For $i=n=\dim V$ the Fourier transform is the obvious isomorphism

$$\mathbb{F}\colon D(V)\longrightarrow \underbrace{Val_0(V^\vee)}_{\mathbb{C}}\otimes D(V^\vee)^*.$$

By Lemma 8.5 both the target and the source spaces of the map $\mathbb{F}$ in (10.2) contain a unique irreducible subspace which contains a non-zero $GL(\Lambda)$-invariant vector for some (equivalently, any) lattice $\Lambda\subset V$. Hence $\mathbb{F}$ induces $GL(V)$-equivariant isomorphisms

$$\mathbb{F}\colon Val_i(V)\tilde{\longrightarrow}Val_{n-i}(V^\vee)\otimes D(V^\vee)^*, \tag{10.3}$$

$$\mathbb{F}\colon Val_i^\infty(V)\tilde{\longrightarrow}Val_{n-i}^\infty(V^\vee)\otimes D(V^\vee)^*, \tag{10.4}$$

when the first one is an isomorphism of Banach spaces.

4 We have the following Plancherel type inversion formula.

10.2. **Theorem.** *The composition*

$$C(Gr_i^V,\mathcal{L}_i^V)\xrightarrow{\mathbb{F}_V}C(Gr_{n-i}^{V^\vee},\mathcal{L}_{n-i}^{V^\vee})\otimes D(V^\vee)^*\xrightarrow{\mathbb{F}_{V^\vee}\otimes Id}$$
$$\big(C(Gr_i^V,\mathcal{L}_i^V)\otimes D(V)^*\big)\otimes D(V^\vee)^*=C(Gr_i^V,\mathcal{L}_i^V)$$

is the identity map.

Proof. First, taking the annihilator $\perp$ twice is the identity map on Gr_i^V. Next, for $E\in Gr_i^V$ the composition of the following isomorphisms is the identity map of $D(E)$

$$D(E)\tilde{\longrightarrow}D(E^\vee)^*\tilde{\longrightarrow}(D(V^\vee/E^\perp))^*\tilde{\longrightarrow}D(E^\perp)\otimes D(V^\vee)^*\tilde{\longrightarrow}$$
$$D((E^\perp)^\vee)^*\otimes D(V^\vee)^*\tilde{\longrightarrow}D(V/E)^*\otimes D(V^\vee)^*\tilde{\longrightarrow}(D(E)\otimes D(V)^*)\otimes D(V^\vee)^*\tilde{\longrightarrow}$$
$$D(E).$$

Hence $a_{n-i}\circ a_i=Id$.

The two simplest examples of computation of the Fourier transform of valuations are as follows. Let $\mu\in D(X)=Val_{\dim X}(V)$. Then

$$\mathbb{F}(\mu)=\chi_{X^\vee}\otimes\mu. \tag{10.5}$$

The Fourier transform of the Euler characteristic $\chi_X\in Val_0(X)$ can also be easily computed. Let us fix a non-vanishing Lebesgue measure $vol_X\in D(X)$. Let $vol_X^{-1}\in D(X)^*\simeq D(X^\vee)$ be the inverse Lebesgue measure. Then

$$\mathbb{F}(\chi_X)=vol_X^{-1}\otimes vol(X). \tag{10.6}$$

$\square$

11. Push-forward on valuations

1 Let $F\colon X \longrightarrow Y$ be a linear map. We will define a continuous linear map

$$F_*\colon Val(X)\otimes D(X)^* \longrightarrow Val(Y)\otimes D(Y)^* \tag{11.1}$$

called the push-forward map and describe it more explicitly in the two cases of injective and surjective maps.

Let us define first the push-forward

$$F_*\colon C(Gr_k^X, \mathcal{L}_k^X\otimes D(X)^*) \longrightarrow C(Gr^Y_{k-\dim X+\dim Y}, \mathcal{L}_k^Y\otimes D(Y)^*)$$

as follows. We have the dual map $F^\vee\colon Y^\vee \longrightarrow X^\vee$. Define

$$F_* := \mathbb{F}_Y\circ (F^\vee)^*\circ \mathbb{F}_X^{-1}, \tag{11.2}$$

where $\mathbb{F}_X, \mathbb{F}_Y$ are the Fourier transforms on X, Y respectively.

11.1. **Lemma.** *F_* is a continuous map*

$$C(Gr_k^X, \mathcal{L}_k^X\otimes D(X)^*) \longrightarrow C(Gr^Y_{k-\dim X+\dim Y}, \mathcal{L}^Y_{k-\dim X+\dim Y}\otimes D(Y)^*).$$

This easily follows from the properties of pull-back and Fourier transform. It is clear that

$$F_*(Val_k(X)\otimes D(X)^*)\subset Val_{k-\dim X+\dim Y}(Y)\otimes D(Y)^*.$$

Hence F_* can also be considered as the map (11.1).

2 Let us describe the push-forward map when $F\colon X\longrightarrow Y$ is an imbedding. The description is contained in Propositions 11.2 and 11.3 below.

We will identify X with its image in Y. Denote $c := \dim Y - \dim X$ the codimension of X.

Let $\xi \in C(Gr_k^X, \mathcal{L}_k^X \otimes D(X)^*)$. Let $E\in Gr^Y_{k+c}$. We are going to describe $(F_*\xi)(E)$. We will consider two cases: $Y\neq E+X$ and $Y = E+X$.

11.2. **Proposition.** *Assume $Y\neq E+X$. Then $(F_*\xi)(E) = 0$.*

Proof. Let us introduce a notation. For a linear subspace $L\subset V$ consider the isomorphism

$$\alpha_{L,V}\colon D(L)\tilde{\longrightarrow} D(L^\perp)\otimes D(V^\vee)^* \tag{11.3}$$

which is the composition of natural isomorphisms (10.1) with E replaced with L:

$$D(L)\tilde{\longrightarrow} D(L^\vee)^*\tilde{\longrightarrow} D(V^\vee/L^\perp)^*\tilde{\longrightarrow}(D(V^\vee)\otimes D(L^\perp)^*)^*\tilde{\longrightarrow} D(L^\perp)\otimes D(V^\vee)^*.$$

We have

$$(F_*\xi)(E) = \left((\mathbb{F}_Y\circ F^{\vee *}\circ \mathbb{F}_X^{-1})(\xi)\right)(E) = \alpha_{E^\perp, Y^\vee}\left(((F^{\vee *}\circ \mathbb{F}_X^{-1})(\xi))(E^\perp)\right). \tag{11.4}$$

We have to show that the last expression vanishes. It suffices to show that $Ker(F^\vee\colon E^\perp\longrightarrow X^\vee)\neq 0$. By duality this is equivalent to

$$Ker(X\longrightarrow (E^\perp)^\vee))\neq 0.$$

But $(E^\perp)^\vee = Y/(E^\perp)^\perp = Y/E$. Hence equivalently we have

$$Ker(X\longrightarrow Y/E)\neq 0,$$

where the map is the composition of the natural maps $X\longrightarrow Y\longrightarrow Y/E$. This is equivalent to our assumption $Y\neq E+X$. □

3 Assume now that $Y = E + X$. Denote $E_0 = E \cap X$. Then $\dim E_0 = k$. The natural map $E \longrightarrow Y/X$ induces the isomorphism

$$E/E_0 \tilde{\longrightarrow} Y/X. \tag{11.5}$$

11.3. **Proposition.** *Assume that $Y = E + X$. Then $(F_*\xi)(E)$ equals the image of $\xi(E_0) \in D(E_0) \otimes D(X)^*$ under the composition of natural isomorphisms from Section 5*

$$D(E_0) \otimes D(X)^* \tilde{\longrightarrow} D(E_0) \otimes D(Y/X) \otimes D(Y)^* \tilde{\longrightarrow}$$
$$D(E_0) \otimes D(E/E_0) \otimes D(Y)^* \tilde{\longrightarrow} D(E) \otimes D(Y)^*,$$

where in the second isomorphism we used the isomorphism (11.5).

Proof. We can choose a decomposition

$$Y = Z \oplus E_0 \oplus E_1 \tag{11.6}$$

such that

$$E = E_0 \oplus E_1, \ X = Z \oplus E_0.$$

By (11.4) we have

$$(F_*\xi)(E) = \alpha_{E^\perp, Y^\vee} \left(((F^{\vee *} \circ \mathbb{F}_X^{-1})(\xi))(E^\perp) \right). \tag{11.7}$$

Recall that $\alpha_{E^\perp, Y^\vee} \colon D(E^\perp) \tilde{\longrightarrow} D(E) \otimes D(Y)^*$. Using decomposition (11.6) we clearly have $E^\perp = Z^\vee$. Under this identification we have

$$\alpha_{E^\perp, Y^\vee} \colon D(Z^\vee) \longrightarrow D(E) \otimes D(Y)^*.$$

The dual map $F^\vee$ is the natural projection $Z^\vee \oplus E_0^\vee \oplus E_1^\vee \longrightarrow Z^\vee \oplus E_0^\vee$. The subspace $E^\perp = Z^\vee$ is mapped identically under these identifications. Hence it follows

$$((F^{\vee *} \circ \mathbb{F}_X^{-1})(\xi))(E^\perp) = \left(F^{\vee *}(\mathbb{F}_X^{-1}\xi)\right)(Z^\vee) = (\mathbb{F}_X^{-1}\xi)(Z^\vee) = \alpha_{Z^\vee, X^\vee}^{-1}(\xi(E_0)),$$

where $\alpha_{Z^\vee, X^\vee} \colon D(Z^\vee) \longrightarrow D(E_0) \otimes D(X)^*$. Substituting this into (11.7) we get

$$(F_*\xi)(E) = \left(\alpha_{E^\perp, Y^\vee} \circ \alpha_{Z^\vee, X^\vee}^{-1}\right)(\xi(E_0)).$$

The map $\alpha_{E^\perp, Y^\vee} \circ \alpha_{Z^\vee, X^\vee}^{-1} \colon D(E_0) \otimes D(X)^* \longrightarrow D(E) \otimes D(Y)^*$ coincides with the claimed one. $\square$

4

11.4. **Proposition.** *Let us assume that a linear map $F \colon X \longrightarrow Y$ is onto. Then push-forward of a smooth valuation is smooth:*

$$F_*(Val^\infty(X) \otimes D(X)^*) \subset Val^\infty(Y) \otimes D(Y)^*.$$

Proof. Recall that $F_* = \mathbb{F}_Y \circ (F^\vee)^* \circ \mathbb{F}_X^{-1}$. Since $F^\vee$ is an imbedding, $(F^\vee)^*$ maps smooth valuations to smooth ones by Theorem 9.1(4). By Section **3**, paragraph **3**, $\mathbb{F}$ is an isomorphism between spaces of smooth valuations. The result follows. $\square$

5 In this paragraph we will describe explicitly the push-forward map when $F \colon X \longrightarrow Y$ is onto. Denote $K := Ker(F)$. Clearly $X/K \simeq Y$.

11.5. **Proposition.** *Let $F\colon X\longrightarrow Y$ is onto. Let $\xi\in C(Gr_k^X,\mathcal{L}_k^X\otimes D(X)^*)$. Let $E\in Gr^X_{k-\dim X+\dim Y}$ with $k\geq \dim X-\dim Y$. Denote $\tilde E:=F^{-1}(E)$. Then $(F_*\xi)(E)$ is equal to the image of $\xi(\tilde E)$ under the composition of natural isomorphisms*

$$D(\tilde E)\otimes D(X)^*\tilde{\longrightarrow} D(E)\otimes D(K)\otimes D(X)^*$$
$$\tilde{\longrightarrow} D(E)\otimes D(K)\otimes D(K)^*\otimes D(Y)^*\tilde{\longrightarrow} D(E)\otimes D(Y)^*,$$

where we used isomorphism $\tilde E/K\simeq E$.

Proof. Let us choose a splitting $X=K\oplus Y$. Then $F^\vee\colon Y^\vee\longrightarrow K^\vee\oplus Y^\vee$ is the obvious imbedding $y\mapsto(0,y)$. Also $\tilde E=K\oplus E$. We will identify the annihilator $E^\perp$ of E in $Y^\vee$ with its image in $X^\vee$ and denote in the same way by the abuse of notation.

We have

$$(F_*\xi)(E)=\left((\mathbb{F}_Y\circ F^{\vee *}\circ\mathbb{F}_X^{-1})(\xi)\right)(E)=\alpha_{E^\perp,Y^\vee}\left(((F^{\vee *}\circ\mathbb{F}_X^{-1})(\xi))(E^\perp)\right)=$$
$$\alpha_{E^\perp,Y^\vee}\left((\mathbb{F}_X^{-1}\xi)(E^\perp)\right)=\alpha_{E^\perp,Y^\vee}\left(\alpha^{-1}_{E^\perp,X^\vee}(\xi(E))\right)=$$
$$(\alpha_{E^\perp,Y^\vee}\circ\alpha^{-1}_{E^\perp,X^\vee})(\xi(E)),$$

where we recall that

$$\alpha_{E^\perp,X^\vee}\colon D(E^\perp)\tilde{\longrightarrow} D(\tilde E)\otimes D(X)^*,$$
$$\alpha_{E^\perp,Y^\vee}\colon D(E^\perp)\tilde{\longrightarrow} D(E)\otimes D(Y)^*.$$

It is easy to see that the map $\alpha_{E^\perp,Y^\vee}\circ\alpha^{-1}_{E^\perp,X^\vee}$ coincides with the map from the proposition. $\square$

12. Exterior product on valuations

1 The goal of this section is to construct a bilinear map for any two finite dimensional vector spaces X and Y over $\mathbb{F}$

$$\boxtimes\colon Val_i(X)\times Val_j^\infty(Y)\longrightarrow Val_{i+j}(X\times Y)$$

which is $GL(X)\times GL(Y)$-equivariant. The map is continuous with respect to the first variable in the case of non-Archimedean $\mathbb{F}$.

This map will be constructed as a restriction to valuations of a bilinear map

$$\boxtimes\colon C(Gr_i^X,\mathcal{L}_i)\times Val_j^\infty(Y)\longrightarrow C(Gr_{i+j}^{X\times Y},\mathcal{L}_{i+j}^{X\times Y}).\tag{12.1}$$

2 First let us construct the map (12.1) in the special case $Val_j^\infty(Y)=D(Y)$. In this case we need to construct a continuous bilinear map

$$\boxtimes\colon C(Gr_i^X,\mathcal{L}_i)\otimes D(Y)\longrightarrow C(Gr_{i+\dim Y}^{X\times Y},\mathcal{L}_{i+\dim Y}^{X\times Y}).\tag{12.2}$$

This map is essentially the push-forward map for the obvious imbedding

$$F\colon X\hookrightarrow X\times Y$$

given by $F(x)=(x,0)$. Indeed the push-forward map for this imbedding, as defined in Section 11, is a continuous linear map

$$F_*\colon C(Gr_i^X,\mathcal{L}_i^X)\otimes D(X)^*\longrightarrow C(Gr_{i+\dim Y}^{X\times Y},\mathcal{L}_{i+\dim Y}^{X\times Y})\otimes D(X\times Y)^*.$$

Since $D(X\times Y)^* = D(X)^*\otimes D(Y)^*$ we get the required map (12.2) by twisting all spaces by $D(X\times Y)$.

Exterior product of $\phi\in C(Gr_i^X,\mathcal{L}_i), \mu\in D(Y)$ is denoted by $\phi\boxtimes\mu$. Thus

$$\phi\boxtimes\mu = (F_*\otimes Id_{D(X\times Y)})(\phi\otimes\mu). \tag{12.3}$$

3

12.1. **Lemma.** *Let $T\colon X\longrightarrow Z$ be a linear map. Let $\mu\in D(Z)$, $\nu\in D(W)$. Then*

$$(T^*\mu)\boxtimes\nu = (T\times Id_W)^*(\mu\boxtimes\nu).$$

In particular both sides belong to $Val(X\times W)$

Proof. If T is not onto then both sides vanish. Thus let us assume that T is onto. Fix a linear subspace $E\subset X\times W$ with $\dim E = \dim Z + \dim W$. We have to show that

$$\left((T^*\mu)\boxtimes\nu\right)(E) = \left((T\times Id)^*(\mu\boxtimes\nu)\right)(E).$$

Both sides are continuous in E. Hence it suffices to assume that E is generic. Denote $E_0 := E\cap X$. For generic E the maps $(T\times Id_W)|_E\colon E\longrightarrow Z\times W$ and $T_{E_0}\colon E_0\longrightarrow Z$ are isomorphisms. The lemma follows from the commutativity of the following diagram

$$G = \left\{ \left[\begin{array}{c|c|c|c} A_1 & * & \dots & * \\ \hline 0 & A_2 & \dots & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & \dots & A_s \end{array} \right] \right\},$$

The last statement follows from Proposition 9.7. $\square$

4 To construct the map (12.1) in general, first we will construct,using the construction from paragraph **2**, a bilinear map

$$\tilde{\boxtimes}\colon C(Gr_i^X,\mathcal{L}_i^X)\times C^\infty(Gr_{n-j}^Y,\mathcal{M}_{n-j}^Y)\longrightarrow C(Gr_{i+j}^{X\times Y},\mathcal{L}_{i+j}^{X\times Y}), \tag{12.4}$$

which is continuous with respect to the first variable, where the bundle $\mathcal{M}_{n-j}^Y$ was defined in Section 8.1, paragraph **3**.

Let ξ be a section of $\mathcal{M}_{n-j}^Y$. Recall that for any $F\in Gr_j^Y$ one has $\xi(F)\in D(Y/F)\otimes|\omega_{n-j}^Y|\big|_F$, where $|\omega_{n-j}^Y|\big|_F$ is the line bundle of densities over Gr_{n-j}^Y. Then by the construction of paragraph **2**

$$\phi\boxtimes\xi(F)\in C(Gr_{i+j}^{X\times(Y/F)},\mathcal{L}_{i+j}^{X\times(Y/F)})\otimes|\omega_{n-j}^Y|\big|_F.$$

Let us define

$$\phi\tilde{\boxtimes}\xi := \int_{F\in Gr_{n-j}^Y}(Id_X\times p_F)^*(\phi\boxtimes\xi(F))\in C(Gr_{i+j}^{X\times Y},\mathcal{L}_{i+j}^{X\times Y}), \tag{12.5}$$

where $p_F\colon Y\longrightarrow Y/F$ is the canonical projection. Note that for the expression under the integral one has

$$(Id_X\times p_F)^*(\phi\boxtimes\xi(F))\in C(Gr_{i+j}^{X\times Y},\mathcal{L}_{i+j}^{X\times Y})\otimes|\omega_{n-j}^Y|\big|_F.$$

We will prove the following

12.2. **Claim.** *The integral in (12.5) is*
(1) *well defined;*
(2) *linear with respect to ϕ and ξ;*
(3) *continuous with respect to ϕ.*

5 *Proof of Claim 12.2.* Let us fix a smooth positive measure ν_0 on Gr^Y_{n-j}. Then we can write uniquely $\xi = \tilde{\xi} \otimes \nu_0$ where $\tilde{\xi}(F) \in D(Y/F)$, and $\tilde{\xi}$ is a continuous section. Then

$$(Id_X \times p_F)^*(\phi \boxtimes \xi(F)) = (Id_X \times p_F)^*(\phi \boxtimes \tilde{\xi}(F)) \cdot \nu_0,$$

and $\phi \tilde{\boxtimes} \xi = \int_{Gr^Y_{n-j}} (Id_X \times p_F)^*(\phi \boxtimes \tilde{\xi}(F)) d\nu_0(F)$, where

$$(Id_X \times p_F)^*(\phi \boxtimes \tilde{\xi}(F)) \in C(Gr^{X\times Y}_{i+j}, \mathcal{L}^{X\times Y}_{i+j}).$$

It suffices to show that the map $C(Gr^X_i, \mathcal{L}^X_i) \times Gr^Y_{n-j} \longrightarrow C(Gr^{X\times Y}_{i+j}, \mathcal{L}^{X\times Y}_{i+j})$ given by

$$(\phi, F) \mapsto (Id_X \times p_F)^*(\phi \boxtimes \tilde{\xi}(F)) \tag{12.6}$$

is continuous. For let us fix $F_0 \in Gr^Y_{n-j}$. It has a neighborhood $\mathcal{U} \subset Gr^Y_{n-j}$ over which there is a trivialization of the vector bundle whose fiber over F is Y/F, thus this bundle is isomorphic to $\mathcal{U} \times \mathbb{F}^j$. Under this identification $\tilde{\xi} \colon \mathcal{U} \longrightarrow D(\mathbb{F}^j)$ is a continuous map. Hence the map $(\phi, F) \mapsto \phi \boxtimes \tilde{\xi}(F)$ is a continuous map $C(Gr^X_i, \mathcal{L}^X_i) \times \mathcal{U} \longrightarrow C(Gr^{X\times\mathbb{F}^j}_{i+j}, \mathcal{L}^{X\times\mathbb{F}^j}_{i+j})$ by paragraph **2**.

Under the above trivialization the map p_F becomes a linear map $Y \longrightarrow \mathbb{F}^j$ which we denote in the same way; it depends continuously on F. By Proposition 9.9 the expression under the integral (12.5) is jointly continuous with respect to (ϕ, F). Hence parts (1), (3) follow. Part (2) is obvious. *square*

6

12.3. **Lemma.**
Let us restrict the map $\tilde{\boxtimes}$ given by (12.5) to $Val_i(X) \times C^\infty(Gr^Y_j, \mathcal{M}^Y_{n-j})$.
(1) *This restriction takes values in $Val_{i+j}(X \times Y)$. Hence we get a bilinear map*

$$Val_i(X) \times C^\infty(Gr^Y_j, \mathcal{M}^Y_{n-j}) \longrightarrow Val_{i+j}(X \times Y) \tag{12.7}$$

which is continuous with respect to the first variable.
(2) *The map (12.7) uniquely factorizes via $Val_i(X) \times Val^\infty_j(Y)$. Thus we get a bilinear map*

$$\boxtimes \colon Val_i(X) \times Val^\infty_j(Y) \longrightarrow Val_{i+j}(X \times Y)$$

which is continuous with respect to the first variable; it is called the exterior product on valuations and is denoted by $(\phi_1, \phi_2) \mapsto \phi_1 \boxtimes \phi_2$.

Proof. Part (1) follows from Lemma 12.1 and the construction of the map (12.4).

Let us prove part (2). For any linear map $T \colon X \longrightarrow Z$, $\dim Z = i$, and any $\mu \in D(Z)$ the pull-back $T^*\mu \in Val_i(X)$ by Lemma 9.2(1). The linear span of valuations of this form is dense in $Val_i(X)$ by Lemma 9.2(2). Hence it suffices to show that for any $\xi \in C(Gr^Y_{n-j}, \mathcal{M}^Y_{n-j})$ the expression $T^*\mu \tilde{\boxtimes} \xi$ depends only on $\mathcal{D}(\xi)$, where $\mathcal{D}$ was defined in (8.3). Let us fix a positive smooth measure ν_0 on

Gr^Y_{n-j}. Then we can write uniquely $\xi = \tilde{\xi} \cdot \nu_0$ where $\tilde{\xi}$ is a smooth section of the line bundle $\mathcal{M}'_{n-j}$ whose fiber over $F \in Gr^Y_{n-j}$ is equal to $D(Y/F)$.

We have

$$T^*\mu\tilde{\boxtimes}\xi = \int_{Gr^Y_{n-j}} (Id_X \times p_F)^*(T^*\mu \boxtimes \xi(F)) =$$
$$\int_{Gr^Y_{n-j}} (Id_X \times p_F)^*(T^*\mu \boxtimes \tilde{\xi}(F))d\nu_0 \overset{Lemma\ 12.1}{=}$$
$$\int_{Gr^Y_{n-j}} (Id_X \times p_F)^* \circ (T \times Id)^*(\mu \boxtimes \tilde{\xi}(F) f\nu_0) \overset{Thm\ 9.1(2)}{=}$$
$$\int_{Gr^Y_{n-j}} (T \times p_F)^*(\mu \boxtimes \tilde{\xi}(F))d\nu_0.$$

Since $\mu \boxtimes \tilde{\xi}(F) \in D(X \times (Y/F))$ then by Lemma 9.2 the last expression belongs to $Val_{i+j}(X \times Y)$. $\square$

7 Let us generalize Lemma 12.1 as follows.

12.4. **Proposition.** *Let $T\colon X \longrightarrow Z$ be a linear map. Let W be another vector space. Let $\omega \in Val_i(Z)$, $\xi \in Val_j^\infty(W)$. Then*

$$T^*\omega \boxtimes \xi = (T \times Id_W)^*(\omega \boxtimes \xi).$$

Proof. Since the pull-back $T^*\colon Val_i(Z) \longrightarrow Val_i(X)$ is continuous by Theorem 9.1 and the exterior product is continuous by Lemma 12.3(2), Lemma 9.2(2) implies that we may assume that

$$\omega = S^*\mu,$$

where $S\colon Z \longrightarrow \mathbb{F}^i$ is a linear map and $\mu \in D(\mathbb{F}^i)$. We can represent ξ in the form

$$\xi = \int_{F \in Gr} p_F^*\zeta(F),$$

where $p_F\colon W \longrightarrow W/F$ is the quotient map to F, ζ is a smooth section of the line bundle $\mathcal{M}'_{n-j}$ over Grassmannian Gr^W_{n-j} whose fiber over F is $D(W/F)$ tensorized with the fiber of the line bundle of densities on Gr^W_{n-j}.

We have

$$
\begin{aligned}
&T^*\omega\boxtimes\xi \overset{(12.5)}{=}\\
&\int_{F\in Gr}(Id\times p_F)^*(T^*\omega\boxtimes\zeta(F))=\int_{F\in Gr}(Id\times p_F)^*(T^*S^*\mu\boxtimes\zeta(F)) \overset{Lemma\,12.1}{=}\\
&\int_{F\in Gr}(Id\times p_F)^*(ST\times Id)^*(\mu\boxtimes\zeta(F)) \overset{Thm\,9.1(2)}{=}\\
&\int_{F\in Gr}(ST\times p_F)^*(\mu\boxtimes\zeta(F)) \overset{Thm\,9.1}{=}\\
&\int_{F\in Gr}(T\times Id)^*(Id\times p_F)^*(S\times Id)^*(\mu\boxtimes\zeta(F)) \overset{Lemma\,12.1}{=}\\
&(T\times Id)^*\int_{F\in Gr}(Id\times p_F)^*(\omega\boxtimes\zeta(F)) \overset{(12.5)}{=}\\
&(T\times Id)^*(\omega\boxtimes\xi).
\end{aligned}
$$

□

8

12.5. **Lemma.** *Let X,Y,Z be finite dimensional $\mathbb{F}$-vector spaces. Then the two maps*

$$Val^\infty(X)\times Val(Y)\times Val^\infty(Z)\longrightarrow Val(X\times Y\times Z)$$

given respectively by

$$(\phi,\psi,\xi)\mapsto(\phi\boxtimes\psi)\boxtimes\xi,$$
$$(\phi,\psi,\xi)\mapsto\phi\boxtimes(\psi\boxtimes\xi)$$

coincide with each other.

Proof. Step 1. Both maps are 3-linear and continuous with respect to ψ. By Lemma 9.2(2) linear combinations of valuations of the form $T^*\mu$, where μ is a Lebesgue measure, are dense in $Val(Y)$. Thus it suffices to assume that $\psi=T^*\mu$ where $T\colon Y\longrightarrow W$, $\mu\in D(W)$.

Step 2. Let us assume that $\psi=T^*\mu$. Applying twice Proposition 12.4 we have

$$
\begin{aligned}
(\phi\boxtimes T^*\mu)\boxtimes\xi &= ((Id\times T)^*(\phi\boxtimes\mu))\boxtimes\xi=\\
&(Id\times T\times Id)^*((\phi\boxtimes\mu)\boxtimes\xi).
\end{aligned}
$$

Similarly

$$\phi\boxtimes(T^*\mu\boxtimes\xi)=(Id\times T\times Id)^*(\phi\boxtimes(\mu\boxtimes\xi)).$$

Hence it remains to show that $(\phi\boxtimes\mu)\boxtimes\xi=\phi\boxtimes(\mu\boxtimes\xi)$.

Step 3. Thus let us assume that $\psi=\mu$ is a density, in particular is a smooth valuation. Let us denote $x:=\dim X$, $z:=\dim Z$. Since ϕ and ξ are smooth valuations they can be presented as

$$\phi=\int_{Gr^X_{x-i}}p_F^*(\psi(F)),\ \xi=\int_{Gr^Z_{z-j}}q_E^*\nu(E),$$

where $p_F\colon X \longrightarrow X/F$, $q_E\colon Z \longrightarrow Z/E$ are the natural quotient maps, and ψ and ν are smooth sections of the line bundles $\mathcal{M}^X_{x-i}$ over Gr^X_{x-i} and $\mathcal{M}^Z_{z-i}$ over Gr^Z_{z-i} respectively. Then using Proposition 12.4 several times we get

$$\phi \boxtimes (\mu \boxtimes \xi) = \int_{F\in Gr^X_{x-i}} p_F^*(\psi(F)) \boxtimes \left(\mu \boxtimes \int_{E\in Gr^Z_{z-i}} q_E^*(\nu(E))\right) =$$

$$\int_{F\in Gr^X_{x-i}} p_F^*(\psi(F)) \boxtimes \left(\int_{E\in Gr^Z_{z-i}} (\mu \boxtimes q_E^*(\nu(E)))\right) \overset{Prop.\ 12.4}{=}$$

$$\int_{F\in Gr^X_{x-i}} p_F^*(\psi(F)) \boxtimes \left(\int_{E\in Gr^Z_{z-i}} (Id_Y \boxtimes q_E)^* (\mu \boxtimes \nu(E))\right) \overset{Prop.\ 12.4}{=}$$

$$\int_{E\in Gr^Z_{z-i}} (Id_X \times (Id_Y \times q_E))^* \left(\int_{F\in Gr^X_{x-i}} p_F^*(\psi(F)) \boxtimes (\mu \boxtimes \nu(E))\right) \overset{Prop.\ 12.4}{=}$$

$$\int_{E\in Gr^Z_{z-i}} \int_{F\in Gr^X_{x-i}} (Id_X \times Id_Y \times q_E)^*$$

$$\circ (p_F \times (Id_Y \times Id_Z))^* (\psi(F) \boxtimes (\mu \boxtimes \nu(E))) \overset{Thm\ 9.1}{=}$$

$$\int_{E\in Gr^Z_{z-i}} \int_{F\in Gr^X_{x-i}} (p_F \times Id_Y \times q_E)^* (\psi(F) \boxtimes (\mu \boxtimes \nu(E))).$$

Since the exterior product on Lebesgue measures is associative we have

$$\phi \boxtimes (\mu \boxtimes \xi) =$$

$$\int_{E\in Gr^Z_{z-i}} \int_{F\in Gr^X_{x-i}} (p_F \times Id_Y \times q_E)^* ((\psi(F) \boxtimes \mu) \boxtimes \nu(E)) \overset{Thm\ 9.1}{=}$$

$$\int_{E\in Gr^Z_{z-i}} \int_{F\in Gr^X_{x-i}} (p_F \times Id_Y \times Id_Z)^*$$

$$\circ((Id_X \times Id_Y) \times q_E)^* ((\psi(F) \boxtimes \mu) \boxtimes \nu(E)) \overset{Prop.\ 12.4}{=}$$

$$\int_{E\in Gr^Z_{z-i}} \int_{F\in Gr^X_{x-i}} (p_F \times Id_Y \times Id_Z)^* ((\psi(F) \boxtimes \mu) \boxtimes q_E^*\nu(E)) =$$

$$\int_{F\in Gr^X_{x-i}} (p_F \times Id_Y \times Id_Z)^* \left((\psi(F) \boxtimes \mu) \boxtimes \int_{E\in Gr^Z_{z-i}} q_E^*\nu(E)\right) =$$

$$\int_{F\in Gr^X_{x-i}} ((p_F \times Id_Y) \times Id_Z)^* ((\psi(F) \boxtimes \mu) \boxtimes \xi) \overset{Prop.\ 12.4}{=}$$

$$\left(\int_{F\in Gr^X_{x-i}} (p_F \times Id_Y)^*(\psi(F) \boxtimes \mu)\right) \boxtimes \xi \overset{Prop.\ 12.4}{=}$$

$$\left(\int_{F\in Gr^X_{x-i}} p_F^*\psi(F) \boxtimes \mu\right) \boxtimes \xi = (\phi \boxtimes \mu) \boxtimes \xi.$$

□

9 Let us prove the following identity.

12.6. **Proposition.** *Let $\chi_X \in Val_0(X)$, $\chi_Y \in Val_0(Y)$ be the unit valuations (Euler characteristics). Then $\chi_X \boxtimes \chi_Y = \chi_{X\oplus Y}$.*

Proof. Let $p_X \colon X \longrightarrow \{0\}, p_Y \colon Y \longrightarrow \{0\}$ be the obvious maps to the zero-space. Then formally $\chi_X = p_X^*\chi_0$, $\chi_Y = p_Y^*\chi_0$, where $\chi_0 \in Val_0(\{0\})$ is the Euler characteristic on $\{0\}$. Then

$$\chi_X \boxtimes \chi_Y = p_X^*\chi_0 \boxtimes p_Y^*\chi_0 \overset{\text{Lemma 12.1}}{=}$$
$$(Id_X \times p_Y)^*(p_X^*\chi_0 \boxtimes \chi_0) \overset{\text{Lemma 12.1}}{=}$$
$$(Id_X \times p_Y)^* \circ (p_X \times Id_Y)^*(\chi_0 \boxtimes \chi_0) \overset{\text{Thm 9.1(2)}}{=}$$
$$(p_X \times p_Y)^*(\chi_0 \boxtimes \chi_0) = (p_X \times p_Y)^*(\chi_0) = \chi_{X\oplus Y}.$$

□

13. Product on smooth valuations

Let $\Delta \colon V \longrightarrow V \times V$ denote the diagonal imbedding, i.e. $\Delta(v) = (v,v)$. Let $\phi \in Val(V), \psi \in Val^\infty(V)$ be smooth valuations. Define their product

$$Val(V) \times Val^\infty(V) \longrightarrow Val(V)$$

by $\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi)$. The product is a bilinear map continuous with respect to the first valuation when the second one is fixed.

13.1. **Lemma.** *Product of smooth valuations is smooth.*

Proof. The product map $Val(V) \otimes Val^\infty(V) \longrightarrow Val(V)$ is $GL(V)$-equivariant. Hence it maps smooth vectors to smooth ones. □

Thus we got a product on smooth valuations $Val^\infty(V) \times Val^\infty(V) \longrightarrow Val^\infty(V)$.

13.2. **Theorem.** *Equipped with the above product, $Val^\infty(V)$ is a commutative associative algebra with unit $1 \in Val_0(V) = \mathbb{C}$. It is graded, namely $Val_i^\infty(V) \cdot Val_j^\infty(V) \subset Val_{i+j}^\infty(V)$.*

Proof. Let us prove commutativity. Let us denote by $\sigma \colon V \times V \longrightarrow V \times V$ the involution $\sigma(x,y) = (y,x)$. Then we have

$$\psi \cdot \phi = \Delta^*(\psi \boxtimes \phi) = \Delta^*(\sigma^*(\phi \boxtimes \psi)) =$$
$$(\Delta^* \circ \sigma^*)(\phi \boxtimes \psi) \overset{Thm\ 9.1(2)}{=} (\sigma \circ \Delta)^*(\phi \boxtimes \psi) = \Delta^*(\phi \boxtimes \psi),$$

where in the last equality we used that $\sigma \circ \Delta = \Delta$.

Let us prove associativity. We have

$$(\phi \cdot \psi) \cdot \xi = \Delta^*((\phi \cdot \psi) \boxtimes \xi) =$$
$$\Delta^*(\Delta^*(\phi \boxtimes \psi) \boxtimes \xi) \overset{Prop.\ 12.4}{=}$$
$$\Delta^*(\Delta \times Id_V)^*((\phi \boxtimes \psi) \boxtimes \xi) \overset{Thm\ 9.1(2)}{=}$$
$$((\Delta \times Id_V) \circ \Delta)^*((\phi \boxtimes \psi) \boxtimes \xi).$$

It is easy to see that $(\Delta \times Id_V) \circ \Delta = \Delta_3$ where $\Delta_3 \colon V \longrightarrow V \times V \times V$ is given by $\Delta_3(v) = (v, v, v)$. Thus

$$(\phi \cdot \psi) \cdot \xi = \Delta_3((\phi \boxtimes \psi) \boxtimes \xi).$$

Similarly

$$\phi \cdot (\psi \cdot \xi) = \Delta_3(\phi \boxtimes (\psi \boxtimes \xi)).$$

But $(\phi \boxtimes \psi) \boxtimes \xi = \phi \boxtimes (\psi \boxtimes \xi)$ by Lemma 12.5.

The rest of properties are trivial. □

14. Product on smooth valuations

Let $\Delta \colon V \longrightarrow V \times V$ denote the diagonal imbedding, i.e. $\Delta(v) = (v, v)$. Let $\phi \in Val(V), \psi \in Val^\infty(V)$ be smooth valuations. Define their product

$$Val(V) \times Val^\infty(V) \longrightarrow Val(V)$$

by $\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi)$. The product is a bilinear map continuous with respect to the first valuation when the second one is fixed.

14.1. **Lemma.** *Product of smooth valuations is smooth.*

Proof. The product map $Val(V) \otimes Val^\infty(V) \longrightarrow Val(V)$ is $GL(V)$-equivariant. Hence it maps smooth vectors to smooth ones. □

Thus we got a product on smooth valuations $Val^\infty(V) \times Val^\infty(V) \longrightarrow Val^\infty(V)$.

14.2. **Theorem.** *Equipped with the above product, $Val^\infty(V)$ is a commutative associative algebra with unit $1 \in Val_0(V) = \mathbb{C}$. It is graded, namely $Val_i^\infty(V) \cdot Val_j^\infty(V) \subset Val_{i+j}^\infty(V)$.*

Proof. Let us prove commutativity. Let us denote by $\sigma \colon V \times V \longrightarrow V \times V$ the involution $\sigma(x, y) = (y, x)$. Then we have

$$\psi \cdot \phi = \Delta^*(\psi \boxtimes \phi) = \Delta^*(\sigma^*(\phi \boxtimes \psi)) =$$
$$(\Delta^* \circ \sigma^*)(\phi \boxtimes \psi) \overset{Thm\ 9.1(2)}{=} (\sigma \circ \Delta)^*(\phi \boxtimes \psi) = \Delta^*(\phi \boxtimes \psi),$$

where in the last equality we used that $\sigma \circ \Delta = \Delta$.

Let us prove associativity. We have

$$(\phi \cdot \psi) \cdot \xi = \Delta^*((\phi \cdot \psi) \boxtimes \xi) =$$
$$\Delta^*(\Delta^*(\phi \boxtimes \psi) \boxtimes \xi) \overset{Prop.\ 12.4}{=}$$
$$\Delta^*(\Delta \times Id_V)^*((\phi \boxtimes \psi) \boxtimes \xi) \overset{Thm\ 9.1(2)}{=}$$
$$((\Delta \times Id_V) \circ \Delta)^*((\phi \boxtimes \psi) \boxtimes \xi).$$

It is easy to see that $(\Delta \times Id_V) \circ \Delta = \Delta_3$ where $\Delta_3 \colon V \longrightarrow V \times V \times V$ is given by $\Delta_3(v) = (v, v, v)$. Thus

$$(\phi \cdot \psi) \cdot \xi = \Delta_3((\phi \boxtimes \psi) \boxtimes \xi).$$

Similarly

$$\phi \cdot (\psi \cdot \xi) = \Delta_3(\phi \boxtimes (\psi \boxtimes \xi)).$$

But $(\phi \boxtimes \psi) \boxtimes \xi = \phi \boxtimes (\psi \boxtimes \xi)$ by Lemma 12.5.

The rest of properties are trivial. $\square$

15. Poincaré duality for valuations

1 The main result of this section is

15.1. **Theorem.** *For any $0 \leq i \leq n$ the bilinear form given by the product on valuations*

$$Val_i(V) \times Val^{\infty}_{n-i}(V) \longrightarrow Val_n(V)$$

is a non-degenerate pairing, i.e. for any $\phi \in Val_i(V)$ there exists $\psi \in Val^{\infty}_{n-i}(V)$ such that $\phi \cdot \psi \neq 0$.

2 To prove the theorem, let us observe that the subspace

$$\{\phi \in Val_i(V) |\ \phi \cdot \psi = 0\, \forall \psi \in Val^{\infty}_{n-i}(V)\}$$

is a $GL(V)$-invariant closed subspace. By the irreducibility property (Proposition 8.2(2)) it suffices to show that it is non-zero. This follows from the following slightly more general lemma which also will be needed later on.

15.2. **Lemma.** *Let $i, j \geq 0$ be such that $i+j \leq n$. Let $\phi = \mathcal{D}(f)$, $\psi = \mathcal{D}(g)$ where $f \in C^{\infty}(Gr^V_{n-i}, \mathcal{M}^V_{n-i})$ and $g \in C^{\infty}(Gr^V_{n-j}, \mathcal{M}^V_{n-j})$ (see Section 8, paragraph 4 for the definition of the operator $\mathcal{D}$) be non-negative sections, both not identically 0. Then $\phi \cdot \psi \neq 0$ and for any $E \in Gr^V_{i+j}$ one has $(\phi \cdot \psi)(E) \geq 0$.*

Proof. For $F \in Gr^V_{n-i}$ we denote by $p_F \colon V \longrightarrow V/E$ the quotient map, and similarly for $L \in Gr^V_{n-j}$ we denote by $q_L \colon V \longrightarrow V/L$ the quotient map. We have

$$\phi \boxtimes \psi = \int_{F \in Gr^V_{n-i}} p_F^* f(F) \boxtimes \int_{L \in Gr^V_{n-j}} q_L^* g(F) =$$

$$\int_{F \in Gr^V_{n-i}} \left(p_F^* f(F) \boxtimes \int_{L \in Gr^V_{n-j}} q_L^* g(F) \right) \overset{Prop.\ 12.4}{=}$$

$$\int_{F \in Gr^V_{n-i}} (p_F \times Id_V)^* \left(f(F) \boxtimes \int_{L \in Gr^V_{n-j}} q_L^* g(F) \right) \overset{Prop.\ 12.4}{=}$$

$$\int_{F \in Gr^V_{n-i}} \int_{L \in Gr^V_{n-j}} (p_F \times Id_V)^* \circ (Id_V \times q_L)^* (f(F) \boxtimes g(F)) \overset{Thm\ 9.1(2)}{=}$$

$$\int_{F \in Gr^V_{n-i}} \int_{L \in Gr^V_{n-j}} (p_F \times q_L)^* (f(F) \boxtimes g(F)).$$

Let $\Delta \colon V \longrightarrow V \times V$ be the diagonal imbedding, i.e. $\Delta(v) = (v, v)$. Let $E \in Gr^V_{i+j}$. Since $\phi \cdot \psi = \Delta^*(\phi \boxtimes \psi)$ we have

$$(\phi \cdot \psi)(E) = \int_{F \in Gr^V_{n-i}} \int_{L \in Gr^V_{n-j}} (p_{E, F \times L})^* (f(F) \boxtimes g(L)) \in D(E), \tag{15.1}$$

where $p_{E,F \times L} \colon E \longrightarrow V/F \times V/L$ is the map $v \mapsto (p_F(v), q_L(v))$ with p_F, p_L being the natural quotient maps. The integrand is non-negative. There is E

such that the integrand is strictly positive for a pair (F,L). Hence the integral is positive. $\square$

3 Below we will use the last lemma to show that powers of certain valuation from $Val_1^\infty(V)$ do not vanish.

Let V be a finite dimensional vector space over a non-Archimedean local field $\mathbb{F}$. Let us fix a lattice $\Lambda \subset V$. Let $GL(\Lambda)$ denote the subgroup of $GL(V)$ consisting of such transformations T such that $T(\Lambda)=\Lambda$. Thus $GL(\Lambda)\simeq GL_n(\mathcal{O})$.

Let $V_1 \in Val_1^\infty(V)$ be a $GL(\Lambda)$-invariant non-zero valuation. Such a valuation is unique up to a proportionality. We normalize V_1 so that its restriction to any line $l \subset V$ is the only Lebesgue measure whose value on the set $l\cap\Lambda$ is equal to 1. This characterizes V_1 uniquely.

15.3. **Proposition.** *For any $1\le k\le n$ and any $E\in Gr_k^V$ one has $(V_1^k)(E)>0$. In particular $V_1^k\neq 0$ for any $1\le k\le n$.*

Proof. Let us prove it by induction in k. For $k=1$ this is clear. Assume $(V_1^{k-1})(L)>0$ for any $L\in Gr_{k-1}^V$. This valuation is $GL_n(\mathcal{O})$-invariant and hence can be presented as $V_1^{k-1}=\mathcal{D}(g)$ where g is $GL_n(\mathcal{O})$-invariant section of $\mathcal{M}_{n-k+1}^V$. g must be positive since $(V_1^{k-1})(L)>0$ for any $L\in Gr_{k-1}^V$. Also $V_1=\mathcal{D}(f)$ where f is $GL_n(\mathcal{O})$-invariant. Hence $f>0$. Hence by Lemma 15.2 and $GL_n(\mathcal{O})$-invariance the valuation $V_1\cdot V_1^{k-1}=V_1^k$ satisfies the conclusion. $\square$

16. Integral transforms on Grassmannians

1 First let us discuss the Radon transform. Let us fix a lattice $\Lambda\subset V$. Let $0\le p<q\le n-1$. Define the Radon transform

$$R_{pq}\colon C^\infty(Gr_q^V)\longrightarrow C^\infty(Gr_p^V)$$

as follows

$$(R_{pq}f)(E)=\int_{Gr_p^E} f(F)d\nu_E(F),$$

where the integration is over the manifold of p-subspaces F contained in E with respect to the only probability Haar measure ν_F which is invariant under all $GL(\Lambda)$-transformations preserving E. Clearly the operator R_{pq} is $GL(\Lambda)$-equivariant.

16.1. **Theorem** ([38])**.** *If $p+q=n$ then $R_{pq}\colon C^\infty(Gr_q^V)\longrightarrow C^\infty(Gr_p^V)$ is an isomorphism.*

We immediately get

16.2. **Corollary.** *If $p+q=n$ then the representations of $GL_n(\mathcal{O})$ in $C^\infty(Gr_p^V)$ and in $C^\infty(Gr_q^V)$ are isomorphic.*

2 The goal of this paragraph is to explicitly describe the operator $\mathcal{D}$ from Section 8, paragraph **4**. For any linear subspace $M\subset V$ let us denote by μ^M the only Lebesgue measure on M whose value on $M\cap\Lambda$ is equal to 1. In particular μ^V is the Lebesgue measure on V such that $\mu(\Lambda)=1$. Let us denote by μ_M the only Lebesgue measure on V/M whose value on $\Lambda/(M\cap\Lambda)\subset V/M$ is equal to 1.

This induces a $GL(\Lambda)$-invariant trivialization of the bundle $\mathcal{L}_i^V$. The $GL(\Lambda)$-invariant Haar probability measure on Grassmannians induces $GL(\Lambda)$-invariant trivialization of the line bundle of densities. Hence we get $GL(\Lambda)$-invariant trivialization of the line bundles $\mathcal{M}^V_{n-i}$. Under these identifications the operator $\mathcal{D}$ is an operator on functions

$$\mathcal{D}_i\colon C^\infty(Gr^V_{n-i}) \longrightarrow C^\infty(Gr^V_i), \tag{16.1}$$

where we put subscript i in $\mathcal{D}_i$ for convenience. Clearly $\mathcal{D}_i$ commutes with $GL(\Lambda)$.

To describe $\mathcal{D}_i$ more explicitly let us introduce a notation.

Let $M, N \subset V$ be linear subspaces of complementary dimension. Let us define

$$s(M,N) := \mu^V((M\cap\Lambda)+(N\cap\Lambda)). \tag{16.2}$$

Clearly

$$0 \leq s(M,N) \leq 1.$$

It is easy to see that

$$s(M,N) = \mu_M(N\cap\Lambda) = \mu_N(M\cap\Lambda). \tag{16.3}$$

Also obviously for any $T \in GL(\Lambda)$ one has

$$s(TM,TN) = s(M,N). \tag{16.4}$$

The next claim easily follows by unwinding the definitions.

16.3. **Claim.** *The operator $\mathcal{D}_i$ from (16.1) is equal to*

$$(\mathcal{D}_i f)(M) = \int_{Gr^V_{n-i}} s(M,N) f(N) d\mu_{n-i,Haar}(N),$$

where $\mu_{n-i,Haar}$ is the $GL(\Lambda)$-invariant Haar probability measure on Gr^V_{n-i}.

3 Consider the Hermitian product of functions on Grassmannians using the Haar measures on them. Using the symmentry in the definition of $s(M,N)$ we easily have the following adjointness property

16.4. **Claim.** *For any $f \in C^\infty(Gr^V_{n-i}), g \in C^\infty(Gr^V_i)$ one has*

$$(\mathcal{D}_i f, g) = (f, \mathcal{D}_{n-i} g).$$

Proof. This immediately follows from the definition (16.2) of $s(M,N)$ which is clearly symmetric with respect to M and N. ☐

4 Below in Section 17 we will need the following

16.5. **Proposition.** *One has*

$$Ker(\mathcal{D}_{n-i}\circ R_{i,n-i}) = Ker(\mathcal{D}_i),$$

where all the operators are considered between spaces of C^∞-smooth functions on appropriate Grassmannians.

Proof. Claim 16.4 and the discussion of representations of $GL_n(\mathcal{O})$ in the space of functions on Grassmannians (see Section 7) imply that

$$Ker(\mathcal{D}_i) = (Im\,\mathcal{D}_{n-i})^\perp,$$

and hence

$$C^\infty(Gr^V_{n-i}) = Ker(\mathcal{D}_i) \oplus Im(\mathcal{D}_{n-i}).$$

It follows that $\mathcal{D}_i$ maps $Im(\mathcal{D}_{n-i})$ isomorphically to $Im(\mathcal{D}_i)$. By Corollary 16.2 $C^\infty(Gr_i^V)$ is isomorphic to $C^\infty(Gr_{n-i}^V)$ as $GL_n(\mathcal{O})$-modules. The last two facts and the multiplicity freeness of $C^\infty(Gr_p^V)$ as $GL_n(\mathcal{O})$-module (see Section 7, paragraph **2**) imply that $Ker(\mathcal{D}_i) \simeq Ker(\mathcal{D}_{n-i})$ as $GL_n(\mathcal{O})$-modules. This and Theorem 16.1 imply that $Ker(\mathcal{D}_{n-i} \circ R_{i,n-i}) = Ker(\mathcal{D}_i)$. $\square$

17. Hard Lefschetz theorem

1 Let V be a finite dimensional vector space over a non-Archimedean local field $\mathbb{F}$. Let us fix a lattice $\Lambda \subset V$. Let $GL(\Lambda)$ denote the subgroup of $GL(V)$ consisting of such transformations T such that $T(\Lambda) = \Lambda$. Thus $GL(\Lambda) \simeq GL_n(\mathcal{O})$.

Let us denote by $V_1 \in Val_1^\infty(V)$ the $GL(\Lambda)$-invariant non-zero valuation from Section 15. The main result of this section is

17.1. **Theorem.** *Let $0 \leq i < n/2$. The operator of multiplication by V_1^{n-2i}*

$$Val_i^\infty(V) \longrightarrow Val_{n-i}^\infty(V)$$

is an isomorphism of vector spaces.

2 The proof of the last theorem occupies the rest of this section. Let $\phi \in Val_i^\infty(V)$ take the form

$$\phi(E) = \int_{Gr_{n-i}^V} p_{E,F}^*(f(F)) \ \text{ for all } E \in Gr_i(V), \tag{17.1}$$

where $f \in C^\infty(Gr_{n-i}^V, \mathcal{M}_{n-i}^V)$, and $p_{E,F}\colon E \longrightarrow V/F$ is the natural map.

Let $k \leq n-i$. By Proposition 15.3 $V_1^k \neq 0$ and is clearly $GL(\Lambda)$-invariant. Hence it can be presented as

$$V_1^k(E) = \int_{Gr_{n-k}^V} p_{E,L}^*(g(L)),$$

where $g \in C^\infty(Gr_{n-k}^V, \mathcal{M}_{n-k}^V)$ is $GL(\Lambda)$-invariant.

It will be helpful to make some identifications induced by the lattice $\Lambda \subset V$. The choice of Λ induces a probability Haar measure $\mu_{p,Haar}$ on each Grassmannian Gr_p^V and hence a trivialization of the linear bundle of densities. It also induces trivialization of the line bundles $\mathcal{M}_p^V, \mathcal{L}_q^V$ as follows. By the definition, the fiber of $\mathcal{M}_p^V$ over a subspace $H \in Gr_p^V$ is the space of Lebesgue measures $D(V/H)$. Let us choose the (only) Lebesgue measure μ_H on V/H such that its value on the lattice $\Lambda/(\Lambda \cap H) \subset V/H$ is equal to 1. Then the sections f, g get identified with continuous functions on appropriate Grassmannians which will be denoted by $\hat{f}, \hat{g}$.

Similarly we trivialize the line bundle $\mathcal{L}_q^V$ over Gr_q^V by choosing in each fiber $\mathcal{L}_q|_G$ the Lebesgue measure μ^G on G which is equal to 1 on the lattice $\Lambda \cap G \subset G$.

For a linear subspace $E \subset V$ let us denote by vol_E the Lebesgue measure on E such that $vol_E(E \cap \Lambda) = 1$.

17.2. **Lemma.** *Assume that a valuation ϕ is given by (17.1). Then one has*

$$\phi \cdot V_1^k = c_{n,i,k}(\mathcal{D}_{i+k} \circ R_{n-i-k,n-i})(\hat{f}),$$

where $c_{n,i,k} > 0$ is a constant.

Proof. By (15.1) we have for any $E\in Gr_{i+k}^V$

$$(17.2)\ (\phi\cdot V_1^k)(E)=\left(\int_{F\in Gr_{n-i}^V}\int_{L\in Gr_{n-k}^V}((p_F\times p_L)^*(f\boxtimes g))\,(E)\right)vol_E,$$

where $p_F\colon V\longrightarrow V/F$ and $p_L\colon V\longrightarrow V/L$ are the natural maps, and here we denote by $p_F\times p_L$ the obvious map $V\longrightarrow V/F\times V/L$ (this is different from our previous convention according to which it would be the map $V\times V\longrightarrow V/F\times V/L$).

Let us rewrite the integrand in the right hand side of (17.2) in terms of $\hat f,\hat g$. We can factorize uniquely the map $p_F\times p_L\colon V\longrightarrow V/F\times V/L$ as

$$V\overset{p_{F\cap L}}{\longrightarrow}V/(F\cap L)\overset{\overline{p_F\times p_L}}{\longrightarrow}V/F\times V/L.$$

The pairs $(F,L)\in Gr_{n-i}^V\times Gr_{n-k}^V$ which are transversal to each other form an open subset of full measure. Hence below we will consider only pair of transversal subspaces. In this case the map

$$\overline{p_F\times p_L}\colon V/(F\cap L)\longrightarrow V/F\times V/L$$

is an isomorphism.

With the above identifications we have

$$\begin{aligned}((p_F\times p_L)^*(f\boxtimes g))\,(E)&=\\ \hat f(F)\cdot(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda\cap E))\cdot vol_E&=\\ \hat f(F)\cdot\frac{(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda\cap E))}{(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda))}\left[(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda))\right]\cdot vol_E&=\\ \hat f(F)\cdot\frac{\mu_{F\cap L}(p_{F\cap L}(\Lambda\cap E))}{\mu_{F\cap L}(p_{F\cap L}(\Lambda))}\left[(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda))\right]\cdot vol_E&=\\ \hat f(F)\cdot\mu_{F\cap L}(p_{F\cap L}(\Lambda\cap E))\left[(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda))\right]\cdot vol_E&,\end{aligned}$$

where the third equality follows from the fact that linear isomorphisms ($\overline{p_F\times p_L}$ in this case) preserve the ratio of Lebesgue measures of sets; the last equality follows from the definition of $\mu_{F\cap L}$. To abbreviate, let us denote

$$(17.3)\qquad c(F,L):=(\mu_F\boxtimes\mu_L)((p_F\times p_L)(\Lambda)),$$

$$(17.4)\qquad s(F\cap L,E):=\mu_{F\cap L}(p_{F\cap L}(\Lambda\cap E)).$$

Clearly

$$0\le c(F,L),\ s(F\cap L,E)\le 1$$

and $c(F,L)>0$ if F and L are transversal. Also obviously $s(F\cap L,E)$ depends only on $F\cap L$ and not on F and L separately.

Let us consider the compact metrizable topological space

$$\mathcal{W}:=\{(F,L,W)\in Gr_{n-i}^V\times Gr_{n-k}^V\times Gr_{n-i-k}^V|\ W\subset F\cap L\}.$$

17.3. **Claim.** *$\mathcal{W}$ has a structure of an analytic manifold (see Section 6) such that the natural projections $\mathcal{W}\longrightarrow Gr_{n-i}^V,Gr_{n-k}^V,Gr_{n-i-k}^V$ are analytic, and the natural action $GL(V)\times\mathcal{W}\longrightarrow\mathcal{W}$ is an analytic map.*

Proof. Consider the partial flag manifolds

$$Z_1 = \{(F,W) \in Gr^V_{n-i} \times Gr^V_{n-i-k} |\, W \subset F\},$$
$$Z_2 := \{(L,W) \in Gr^V_{n-k} \times Gr^V_{n-i-k} |\, W \subset L\}.$$

Z_1, Z_2 are analytic manifolds by Section 6, paragraph **5**. We have the natural analytic map

$$T \colon Z_1 \times Z_2 \longrightarrow (Gr^V_{n-i-k})^2$$

given by $((F,W_1),(L,W_2)) \overset{T}{\mapsto} (W_1,W_2)$. Its differential is onto at every point. Let $\Delta \subset (Gr^V_{n-i-k})^2$ be the diagonal submanifold. The implicit function theorem (see Section 6, paragraph **3**) implies that $\mathcal{W} = T^{-1}(\Delta)$ is an analytic submanifold of $Z_1 \times Z_2$. The claimed properties of $\mathcal{W}$ follow easily. □

Let us continue proving Lemma 17.2. Clearly if $(F,L,W) \in \mathcal{W}$ and if F, L are transversal then $W = F \cap L$. Hence the natural map $\mathcal{W} \longrightarrow Gr^V_{n-i} \times Gr^V_{n-k}$ forgetting W is an isomorphism between open subsets of full measure where E and F are transversal.

Let m be the push-forward under the inverse map of the product of the Haar probability measures (with respect to the natural action of $GL(\Lambda)$) on Gr^V_{n-i} and Gr^V_{n-k} with subsequent extension by 0 to the whole $\mathcal{W}$. Clearly m is a $GL(\Lambda)$-invariant probability measure whose value on any open subset of $\mathcal{W}$ is positive (the action of $GL(\Lambda)$ is diagonal). Then we can rewrite

$$(\phi \cdot V_1^k)(E) = \int_{(F,L,W)\in\mathcal{W}} \hat{f}(F)c(F,L)s(W,E)dm$$

Let us consider the analytic manifold

$$\mathcal{X} := \{(F,W) \in Gr^V_{n-i} \times Gr^V_{n-i-k} |\ W \subset F\}$$

and the natural $GL(\Lambda)$-equivariant map $\tau \colon \mathcal{W} \longrightarrow \mathcal{X}$ given by $\tau(F,L,W) = (F,W)$, where $GL(\Lambda)$ acts diagonally. Let $\tilde{m} := \tau_*(c \cdot m)$ denote the push-forward of the measure $c \cdot m$. Clearly $\tilde{m}$ is $GL(\Lambda)$-invariant and positive (Haar) measure on the $GL(\Lambda)$-homogeneous space $\mathcal{X}$. Then

$$(\phi \cdot V_1^k)(E) = \left(\int_{(F,W)\in\mathcal{X}} \hat{f}(F)s(W,E)d\tilde{m}\right) vol_E =$$
$$c_{n,i,k}\left(\int_{W\in Gr^V_{n-i-k}} dW s(W,E) \int_{F\supset W} dF \hat{f}(F)\right) vol_E,$$

where $c_{n,i,k} > 0$ is a constant, the inner integral is taken with respect to the set of all subspaces F containing W which can be identified with the Grassmannian $Gr^{V/W}_k$. The measures dF, dW on the corresponding Grassmannians are the $GL(\Lambda)$-invariant probability Haar measures.

The inner integral $\int_{F\supset W} dF \hat{f}(F)$ is the Radon transform $(R_{n-i-k,n-i}\hat{f})(W)$. Thus we get in this notation

$$(\phi \cdot V_1^k)(E) = c_{n,i,k}\left(\int_{W\in Gr^V_{n-i-k}} (R_{n-i-k,n-i}\hat{f})(W) \cdot s(W,E)dW\right) vol_E. \tag{17.5}$$

By Claim 16.3 the last expression is equal to $c_{n,i,k}(\mathcal{D}_{i+k}(R_{n-i-k,n-i}\hat{f}))(E)$ where $c_{n,i,k}>0$. Lemma 17.2 is proven.
Let us finish the proof of Theorem 17.1. Let us assume now that $i<n/2$. Let $k=n-2i$. Choice of lattice $\Lambda\subset V$ induced a trivialization of the line bundle $\mathcal{L}_i$ over Gr_i^V. Indeed recall that the fiber of $\mathcal{L}_i$ over $E\in\mathcal{L}_i$ is $\mathcal{L}_i|_E=D(E)$. But $D(E)=\mathbb{C}\cdot vol_E$. Hence we will identify in this paragraph the space $Val_i^\infty(V)$ with a subspace of locally constant $\mathbb{C}$-valued functions on Gr_i^V.

Let $\phi\in Val_i^\infty(V)$. By Lemma 17.2 we have

$$\phi\cdot V_1^{n-2i}=c_{n,i}(\mathcal{D}_{n-i}\circ R_{i,n-i})(\hat{f}),\ \ c_{n,i}>0.$$

By the definition, $Val_{n-i}^\infty(V)=Im(\mathcal{D}_{n-i})$. But since $R_{i,n-i}\colon C^\infty(Gr_{n-i}^V)\longrightarrow C^\infty(Gr_i^V)$ is an isomorphism by Theorem 16.1, it follows that $Im(\mathcal{D}_{n-i}\circ R_{i,n-i})=Im(\mathcal{D}_{n-i})=Val_{n-i}^\infty(V)$. Hence the operator of multiplication by V_1^{n-2i} is onto on smooth valuations.

It remains to show that the later operator is injective on smooth valuations. Let us assume that

$$\phi\cdot V_1^{n-2i}=0, \tag{17.6}$$

and let $\phi=\mathcal{D}_i(\hat{f})$. It suffices to show that $\hat{f}\in Ker(\mathcal{D}_i)$.

Assumption (17.6) implies that $\hat{f}\in Ker(\mathcal{D}_{n-i}\circ R_{i,n-i})$. But by Proposition 16.5 the latter kernel is equal to $Ker(\mathcal{D}_i)$. Theorem is proved. □

18. Fourier transform commutes with exterior product

1 In this section we fix a non-Archimedean local field. All vector spaces $X,Y,Z,...$ will be over this field. The main result if this section is

18.1. **Theorem.** *Let X,Y be finite dimensional vector spaces. Let $\phi\in Val(X)$, $\psi\in Val^\infty(Y)$. Then*

$$\mathbb{F}(\phi\boxtimes\psi)=\mathbb{F}(\phi)\boxtimes\mathbb{F}(\psi).$$

18.2. **Remark.** For convex valuations an analogue of this result was conjectured by the author in [8] and proved recently by Faifman and Wannerer [26].

We will need two lemmas to prove Theorem 18.1.

2 Let X and $Y=X\oplus Z$ be finite dimensional vector spaces over the given non-Archimedean local field. Fix a positive Lebesgue measure $vol_Z\in D(Z)$. Let $vol_Z^{-1}\in D(Z)^*=D(Z^\vee)$ be the corresponding Lebesgue measure on $Z^\vee$ as defined in Lemma 5.1.

18.3. **Lemma.** *Let $F\colon X\hookrightarrow X\oplus Z$ be the obvious imbedding given by $F(x)=(x,0)$. Let $\phi_0\in Val(X)$, $\mu\in D(X)^*$. Then*

$$F_*(\phi_0\otimes\mu)=(\phi_0\boxtimes vol_Z)\otimes(\mu\otimes vol_Z^{-1}).$$

Proof. We may assume that $\mu=vol_X^{-1}$, where vol_X is a non-vanishing Lebesgue measure on X. By 12.3 we have

$$\phi_0\boxtimes vol_Z=(F_*\otimes Id_{D(X\oplus Z)})((\phi_0\otimes vol_X^{-1})\otimes(vol_X\otimes vol_Z))=$$
$$F_*\phi\otimes(vol_X\otimes vol_Z).$$

This is equivalent to the required equality. □

3

18.4. **Lemma.** *Let a vector space X be a direct sum $X = Z \oplus Z_1$. Let $p\colon X \longrightarrow Z$ be the obvious projection. Let $\mu_Z \in D(Z)$. Then*

$$\mathbb{F}(p^*\mu_Z) = (\chi_{Z^\vee} \boxtimes vol_{Z_1}^{-1}) \otimes (\mu_Z \otimes vol_{Z_1}),$$

where vol_{Z_1} is an arbitrary non-vanishing Lebesgue measure on Z_1, and $vol_{Z_1}^{-1} \in D(Z_1^\vee)$ is the inverse Lebesgue measure on $Z_1^\vee$ defined in Lemma 5.1.

Proof. We have

$$\mathbb{F}(p^*\mu_Z) = p_*^\vee(\mathbb{F}\mu_Z) \overset{(10.5)}{=}$$
$$p_*^\vee(\chi_{Z^\vee} \otimes \mu_Z) \overset{\text{Lemma } 18.3}{=}$$
$$(\chi_{Z^\vee} \boxtimes vol_{Z_1}^{-1}) \otimes (\mu_Z \otimes vol_{Z_1}).$$

□

4 Let us prove a special case of Theorem 18.1.

18.5. **Lemma.** *Let $\mu_X \in D(X)$, $\psi \in Val(Y)$. Then*

$$\mathbb{F}(\mu_X \boxtimes \psi) = \mathbb{F}(\mu_X) \boxtimes \mathbb{F}(\psi).$$

Proof. Since both sides are linear and continuous with respect to $\psi \in Val(Y)$, we may and will assume that

$$\psi = p^*\mu_M,$$

where $Y = M \oplus L$, $p\colon Y \longrightarrow M$ is the obvious projection, $\mu_M \in D(M)$. Then we have

$$\mathbb{F}(\mu_X \boxtimes p^*\mu_M) \overset{\text{Lemma } 12.1}{=} \mathbb{F}((Id_X \times p)^*(\mu_X \boxtimes \mu_M)) =$$
$$(Id_{X^\vee} \times p^\vee)_*(\mathbb{F}(\mu_X \boxtimes \mu_M)) \overset{(10.5)}{=}$$
$$(Id_{X^\vee} \times p^\vee)_*(\chi_{X^\vee \oplus M^\vee} \otimes (\mu_X \boxtimes \mu_M)).$$

By Lemma 18.3 we can continue

$$\mathbb{F}(\mu_X \boxtimes p^*\mu_M) = (\chi_{X^\vee \oplus M^\vee} \otimes vol_L^{-1}) \otimes ((\mu_X \boxtimes \mu_M) \otimes vol_L). \tag{18.1}$$

One the other hand we have

$$\mathbb{F}(\mu_X) \boxtimes \mathbb{F}(p^*\mu_M) \overset{(10.5)}{=}$$
$$(\chi_{X^\vee} \otimes \mu_X) \boxtimes \mathbb{F}(p^*\mu_M) \overset{\textit{Lemma} 18.4}{=}$$
$$(\chi_{X^\vee} \otimes \mu_X) \boxtimes \big((\chi_{M^\vee} \boxtimes vol_L^{-1}) \otimes (\mu_M \otimes vol_L)\big) =$$
$$(\chi_{X^\vee} \boxtimes \chi_{M^\vee} \boxtimes vol_L^{-1}) \otimes (\mu_X \otimes \mu_M \otimes vol_L) \overset{\text{Prop. } 12.6}{=}$$
$$(\chi_{X^\vee \oplus M^\vee} \otimes vol_L^{-1}) \otimes ((\mu_X \boxtimes \mu_M) \otimes vol_L) \overset{(18.1)}{=}$$
$$\mathbb{F}(\mu_X \boxtimes p^*\mu_M).$$

□

5 *Proof of Theorem 18.1.* It suffices to assume that $\phi = p_M^*\mu$ where $p_M\colon X \longrightarrow X/M$, $\mu \in D(X/M)$. Since ψ is smooth, it can be presented in the form $\psi = \int_{N\in Gr^Y_{n-j}} p_N^*\nu(N)$, where $p_N\colon Y \longrightarrow Y/N$ is the quotient map. Then we have

$$\mathbb{F}(\phi\boxtimes\psi) = \mathbb{F}(p_M^*\mu\boxtimes\int_{Gr^Y_{n-j}} p_N^*\nu(N)) \overset{\text{Lemma 12.1}}{=}$$

$$\mathbb{F}\left((p_M\times Id_Y)^*(\mu\boxtimes\int_{Gr^Y_{n-j}} p_N^*\nu(N))\right) =$$

$$\int_{Gr^Y_{n-j}} \mathbb{F}\left((p_M\times Id_Y)^*(\mu\boxtimes p_N^*\nu(N))\right) \overset{\text{Lemma 12.1}}{=}$$

$$\int_{Gr^Y_{n-j}} \mathbb{F}\left(((p_M\times Id_Y)^*\circ(Id_X\times p_N)^*)\,(\mu\boxtimes\nu(N))\right) \overset{\text{Thm 9.1(2)}}{=}$$

$$\int_{Gr^Y_{n-j}} \mathbb{F}\left((p_M\times p_N)^*(\mu\boxtimes\nu(N))\right) =$$

$$\int_{Gr^Y_{n-j}} (p_M^\vee\times p_N^\vee)_*(\mathbb{F}(\mu\boxtimes\nu(M))) \overset{\text{Lemma 18.5}}{=}$$

$$\int_{Gr^Y_{n-j}} (p_M^\vee\times p_N^\vee)_*(\mathbb{F}(\mu)\boxtimes\mathbb{F}(\nu(N))) \overset{(10.5)}{=}$$

$$\int_{Gr^Y_{n-j}} (p_M^\vee\times p_N^\vee)_*\left((\chi_{(X/M)^\vee}\otimes\mu)\boxtimes(\chi_{(Y/N)^\vee}\otimes\nu(N))\right) \overset{\text{Prop. 12.6}}{=}$$

Let us choose splittings

$$X = M\oplus M_1,\ Y = N\oplus N_1.$$

Then by Lemma 18.3 the expression under the integral in the last expression is equal to:

$$(\chi_{M_1^\vee\times N_1^\vee}\boxtimes vol^{-1}_{M\times N})\otimes(\mu\otimes\nu(N))\otimes vol_{M\times N} \overset{\text{Prop. 12.6}}{=}$$

$$(\chi_{M_1^\vee}\boxtimes vol_M^{-1})\boxtimes(\chi_{N_1^\vee}\boxtimes vol_N^{-1})\otimes(\mu\otimes vol_M)\otimes(\nu(N)\otimes vol_N) \overset{\text{Lemma 18.4}}{=}$$

$$\mathbb{F}\phi\boxtimes\left((\chi_{N_1^\vee}\boxtimes vol_N^{-1})\otimes(\nu(N)\otimes vol_N)\right) \overset{\text{Lemma 18.4}}{=}$$

$$\mathbb{F}(\phi)\boxtimes\mathbb{F}(p_N^*\nu(N)).$$

Hence after integrating we get

$$\mathbb{F}(\phi\boxtimes\psi) = \int_{N\in Gr^Y_{n-j}} \mathbb{F}(\phi)\boxtimes\mathbb{F}(p_N^*\nu(N)) = \mathbb{F}(\phi)\boxtimes\mathbb{F}(\psi).$$

$\square$

19. Convolution on smooth valuations

1 The goal of this section is to define a convolution on valuations and prove its basic properties. First not that if

$$\phi \in Val^{\infty}(X) \otimes D(X)^*, \;\; \psi \in Val^{\infty}(Y) \otimes D(Y)^*$$

then $\phi \boxtimes \psi$ is well defined as an element of $Val(X \times Y) \otimes D(X \times Y)^* = Val(X \times Y) \otimes D(X)^* \otimes D(Y)^*$.

Let $a\colon V \times V \longrightarrow V$ be the addition map, i.e. $a(x, y) = x + y$. Let us define the convolution

$$*\colon (Val^{\infty}(V) \otimes D(V)^*) \times (Val^{\infty}(V) \otimes D(V)^*) \longrightarrow Val^{\infty}(V) \otimes D(V)^*$$

by

$$\phi * \psi := a_*(\phi \boxtimes \psi). \tag{19.1}$$

19.1. **Proposition.** *Convolution of smooth valuations is smooth*

Proof. Indeed the convolution can be considered as a linear map

$$*\colon (Val^{\infty}(V) \otimes D(V)^*) \otimes (Val^{\infty}(V) \otimes D(V)^*) \longrightarrow Val^{\infty}(V) \otimes D(V)^*.$$

Evidently is commutes with the natural action of the group $GL(V)$. Hence smooth vectors are mapped into smooth ones. $\square$

2

19.2. **Proposition.** *Let $\phi, \psi \in Val^{\infty}(V)$. Then*

$$\mathbb{F}\phi * \mathbb{F}\psi = \mathbb{F}(\phi \cdot \psi).$$

Proof. Let $a\colon V^{\vee} \times V^{\vee} \longrightarrow V^{\vee}$ denote the addition map. Then the dual map

$$a^{\vee}\colon V \longrightarrow V \times V$$

is the diagonal map, i.e. $a(v) = (v, v)$. Then by definition of the convolution, push-forward, and the product we have

$$\mathbb{F}\phi * \mathbb{F}\psi = a_*(\mathbb{F}\phi \boxtimes \mathbb{F}\psi) \overset{\text{Thm 18.1}}{=} a_*(\mathbb{F}(\phi \boxtimes \psi)) = \mathbb{F}\left((a^{\vee})^*(\phi \boxtimes \psi)\right) = \mathbb{F}(\phi \cdot \psi).$$

$\square$

3 Let $vol_V \in D(V)$ be a non-zero Lebesgue measure on V. We denote by $vol_V^{-1} \in D(V)^*$ the element of the dual space of $D(V)$ whose value on vol_V is equal to 1. Note that $vol_V \otimes vol_V^{-1} \in D(V) \otimes D(V)^*$ is independent of a choice of vol_V.

Denote $n := \dim V$.

19.3. **Theorem.** *1) $Val^{\infty}(V) \otimes D(V)^*$ equipped with the convolution is a commutative associative algebra with the unit element $vol_V \otimes vol_V^{-1}$.*
2)

$$(Val^{\infty}_{n-i}(V) \otimes D(V)^*) * (Val^{\infty}_{n-j}(V) \otimes D(V)^*) \subset Val^{\infty}_{n-i-j}(V) \otimes D(V)^*.$$

3) The Poincaré duality is satisfied: the bilinear map

$$Val^{\infty}_{i}(V) \otimes D(V)^* \times Val^{\infty}_{n-i}(V) \otimes D(V)^* \longrightarrow Val_0(V) \otimes D(V)^* = D(V)^*$$

given by $(\phi,\psi)\mapsto\phi*\psi$ *is a perfect pairing, i.e. for any for non-zero* $\phi\in Val_i^\infty(V)\otimes D(V)^*$ *there is* $\phi\in Val_{n-i}^\infty(V)\otimes D(V)^*$ *such that* $\phi*\psi\neq 0$.
4) The hard Lefschetz type result is true: Let us fix a lattice $\Lambda\subset V$. *This induces an isomorphism* $D(V)^*\simeq\mathbb{C}$. *Let* $V_{n-1}\in Val_{n-1}(V)$ *be the only (up to a proportionality)* $GL(\Lambda)$*-invariant element. Let* $n/2<i\leq n$. *Then the linear map*

$$Val_i^\infty(V)\otimes D(V)^*\longrightarrow Val_{n-i}^\infty\otimes D(V)^*$$

given by $\phi\mapsto\phi*\underbrace{V_{n-1}*\cdots*V_{n-1}}_{2i-n}$ *is an isomorphism.*

Proof. Immediately follows from Proposition 19.2, the corresponding properties of the product, and the obvious fact that $\mathbb{F}(V_1)$ is proportional to V_{n-1} (after the appropriate identifications induced by the choice of lattice $\Lambda\subset V$ are applied). □

20. Valuations invariant under a subgroup

1 Convex valuations invariant under various subgroups of $GL_n(\mathbb{R})$ have a number of interesting properties and found applications in integral geometry, see e.g. [3], [16], [17], [21], [27]. For a compact subgroup $G\subset GL_n(\mathbb{R})$ the author showed [1] that the space of G-invariant convex valuations is finite dimensional if and only if G acts transitively on the unit sphere, and in this case all G-invariant valuations are smooth [4]. Furthermore in this case the algebra $Val^G(\mathbb{R}^n)$ of G-invariant valuations equipped either with product or convolution satisfies Poincaré duality, hard Lefschetz theorem, and Hodge-Riemann bilinear relations inherited from $Val^\infty(\mathbb{R}^n)$.

It is well known that any compact subgroup $G\subset GL_n(\mathbb{F})$ is conjugated to a subgroup of $GL_n(\mathcal{O})$. In particular any maximal compact subgroup of $GL_n(\mathbb{F})$ is conjugated to $GL_n(\mathcal{O})$.

We will see below that in the non-Archimedean case there are a lot of compact subgroups $G\subset GL(V)$ such that the subspace of G-invariant valuations is finite dimensional and all such valuations are smooth.

2 Let V be an n-dimensional vector space over a non-Archimedean local field $\mathbb{F}$. The group $GL(V)$ has many subgroups which are simultaneously open and compact, more precisely such subgroups form a basis of neighborhoods of $I_n\in GL(V)$.

20.1. **Proposition.** *Let* $G\subset GL(V)$ *be an open and compact subgroup. Then the space* $Val^G(V)$ *of* G*-invariant valuations is finite dimensional and all its elements are smooth, i.e.*

$$Val^G(V)\subset Val^\infty(V).$$

Proof. Orbits of any open subgroup of $GL(V)$ on Gr_i^V are open. Indeed the map $GL(V)\longrightarrow Gr_i^V$ given by $g\mapsto g(E_0)$ is a submersion for any $E_0\in Gr_i^V$. In particular G-orbits are open. Since Gr_i^V is compact and different orbits are disjoint, there are finitely many of them. By definition of a smooth valuation

$$Val^G(V)\subset Val^\infty(V).$$

□

3 Obviously
$$Val^G(V)=\oplus_{i=0}^n Val_i^G(V).$$
It is easy to see that $(Val^G(V),\cdot)$ is a finite dimensional graded subalgebra of $(Val^\infty(V),\cdot)$ satisfying Poincaré duality and hard Lefschetz theorem (with respect to $V_1\in Val_1^G(V)$ which is invariant under a maximal compact subgroup containing G).

Similarly $(Val^G(V)\otimes D(V),*)$ is a finite dimensional graded subalgebra of $(Val^\infty(V)\otimes D(V),*)$ satisfying Poincaré duality and hard Lefschetz (with respect to V_{n-1} which is invariant under a maximal compact subgroup containing G). (Note that the compatibility with the grading is given by Theorem 19.3(2).)

The Fourier transform establishes an isomorphism of algebras
$$\mathbb{F}\colon (Val^G(V),\cdot)\tilde{\longrightarrow}(Val^G(V^\vee)\otimes D(V),*).$$

References

[1] S. Alesker, *On P. McMullen's conjecture on translation invariant valuations*, Adv. Math. **155** (2000), 239–263.

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), 244–272.

[3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95.

[4] S. Alesker, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26.

[5] S. Alesker, *Hard Lefschetz theorem for valuations and related questions of integral geometry*, Geometric Aspects of Functional Analysis, Lecture Notes in Math., 1850, Springer, Berlin, 2004, pp. 9–20.

[6] S. Alesker, *Theory of valuations on manifolds: a survey. Geom*, Funct. Anal. **17** (2007), 1321–1341.

[7] S. Alesker, *A Fourier-type transform on translation-invariant valuations on convex sets*, Israel J. Math. **181** (2011), 189–294.

[8] S. Alesker, *New structures on valuations and applications*, Integral geometry and valuations, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2014, pp. 1–45.

[9] S. Alesker, *Introduction to the Theory of Valuations*, CBMS Regional Conference Series in Mathematics, 126. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2018.

[10] S. Alesker, *Kotrbatý's theorem on valuations and geometric inequalities for convex bodies*, Israel J. Math. **247** (2022), 361–378.

[11] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Adv. Math. **184** (2004), 367–379.

[12] S. Alesker and D. Faifman, *Convex valuations invariant under the Lorentz group*, J. Differential Geom. **98** (2014), 183–236.

[13] Alesker, Semyon; Fu, Joseph H. G.; Theory of valuations on manifolds. III. Multiplicative structure in the general case. Trans. Amer. Math. Soc. 360 (2008), no. 4, 1951–1981.

[14] U. Bader and U. Onn, *On some geometric representations of $GL_N(\mathcal{O})$*, Comm. Algebra **40** (2012), 3169–3191.

[15] A. Bernig and L. Bröcker, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. **75** (2007), 433–457.

[16] A. Bernig, *Algebraic integral geometry*, Global Differential Geometry, Springer Proc. Math., 17, Springer, Heidelberg, 2012, pp. 107–145.

[17] A. Bernig and D. Faifman, *Valuation theory of indefinite orthogonal groups*, J. Funct. Anal. **273** (2017), 2167–2247.

[18] A. Bernig, D. Faifman and G. Solanes, *Uniqueness of curvature measures in pseudo-Riemannian geometry*, J. Geom. Anal. **31** (2021), 11819–11848.
[19] A. Bernig, D. Faifman and G. Solanes, *Curvature measures of pseudo-Riemannian manifolds*, J. Reine Angew. Math. **788** (2022), 77–127.
[20] A. Bernig, J. Fu and H. G. Joseph, *Convolution of convex valuations*, Geom. Dedicata **123** (2006), 153–169.
[21] A. Bernig, J. Fu and H. G. Joseph, *Hermitian integral geometry*, Ann. of Math. (2) **173** (2011), 907–945.
[22] A. Bernig, J. Fu, H. G. Joseph and G. Solanes, *Integral geometry of complex space forms*, Geom. Funct. Anal. **24** (2014), 403–492.
[23] A. Bernig, J. Kotrbatý and T. Wannerer, *Hard Lefschetz theorem and Hodge-Riemann relations for convex valuations*, arXiv:2312.12294
[24] A. Colesantim, M. Ludwig and F. Mussnig, *Valuations on convex functions*, Int. Math. Res. Not. IMRN **2019** (2019), 2384–2410.
[25] A. Colesanti and M. Ludwig, *Mussnig, Fabian; Hessian valuations*, Indiana Univ. Math. J. **69** (2020), 1275–1315.
[26] D. Faifman and T. Wannerer, *The Fourier transform on valuations is the Fourier transform*, arXiv:2309.07048
[27] J. Fu and H. G. Joseph, *Algebraic integral geometry. Integral geometry and valuations*, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2014, pp. 47–112.
[28] D. Gourevitch, *Composition series for degenerate principal series of $GL(n)$*, C. R. Math. Acad. Sci. Soc. R. Can. **39** (2017), 1–12.
[29] G. Hill, *On the nilpotent representations of $GL_n(\mathcal{O})$*, Manuscripta Math. **82** (1994), 293–311.
[30] J. Igusa, *An Introduction to the Theory of Local Zeta Functions*, AMS/IP Studies in Advanced Mathematics, 14. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000.
[31] D. A. Klain, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), 71–93.
[32] D. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Lezioni Lincee [Lincei Lectures] Cambridge University Press, Cambridge, 1997.
[33] J. Knoerr, *The support of dually epi-translation invariant valuations on convex functions*, J. Funct. Anal. **281** (2021): Paper No. 109059, 52 pp.
[34] J. Kotrbatý, *On Hodge-Riemann relations for translation-invariant valuations*, Adv. Math. **390** (2021): Paper No. 107914, 28 pp.
[35] J. Kotrbatý and T. Wannerer, *On mixed Hodge-Riemann relations for translation-invariant valuations and Aleksandrov-Fenchel inequalities*, Commun. Contemp. Math. **24** (2022): Paper No. 2150049, 24 pp.
[36] J. Kotrbatý, T. Wannerer, J. Kotrbatý and T. Wannerer, *From harmonic analysis of translation-invariant valuations to geometric inequalities for convex bodies*, Geom. Funct. Anal. **33** (2023), 541–592.
[37] M. Ludwig, *Valuations on function spaces*, Adv. Geom. **11** (2011), 745–756.
[38] E. E. Petrov and V. G. Chernov, *The Radon transform in a Grassmann manifold over a local field. (Russian) Functional analysis*, No. 13 (Russian), Ulyanovsk. Gos. Ped. Inst., Ulyanovsk, 1979, pp. 107–120.
[39] W. Rudin, *Real and Complex Analysis*, Third edition. McGraw-Hill Book Co., New York, 1987.
[40] H. H. Schaefer, *Topological Vector Spaces*, The Macmillan Company, New York; Collier Macmillan Ltd., London 1966.
[41] J.-P. Serre, *Lie Algebras and Lie Groups*, 1964 lectures given at Harvard University. Second edition. Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 1992.
[42] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014.
[43] A. Weil, *Basic Number Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144 Springer-Verlag New York, Inc., New York 1967

Manuscript received February 6 2024
revised May 18 2024

Semyon Alesker
Department of Mathematics, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel
E-mail address: alesker.semyon75@gmail.com