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A SINGULAR NONCONVEX OPTIMAL CONTROL PROBLEM

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ABSTRACT. We address a nonconvex optimal control problem governed by a nonlinear elliptic equation with a discontinuous diffusion term. This optimal control problem may be viewed as singular in the sense of Lions (see [7]).

1. Problem presentation

In this paper we study a nonconvex optimal control problem governed by a singular nonlinear elliptic equation.

Let β be the subdifferential of a convex function $j : \mathbb{R}^N \to \mathbb{R}$ which has the property

$$C_1 |r|_N^p + C_1^0 \leq j(r) \leq C_2 |r|_N^p + C_2^0, \tag{1.1}$$

where $p > 1$, C_1, $C_2 > 0$, C_1^0, $C_2^0 \in \mathbb{R}$, and $|\cdot|_N$ denotes the norm in the Euclidian space $\mathbb{R}^N$ ($N \geq 1$). In particular, f is continuous (see [5], p. 74, Proposition 2.17). We introduce the minimization problem

$$\min_{(u,y)} \left(J(u,y) = \int_\Omega \left(\frac{|u(x)|^{q'}}{q'} + \frac{\sigma}{2}(y(x) - y_o(x))^2 \right) dx \right) \tag{P}$$

for all pairs $(u, y) \in U$ satisfying the boundary value problem

$$\begin{aligned} -\nabla \cdot \beta(\nabla y) &\ni u \text{ in } \Omega, \\ \beta(\nabla y) \cdot \nu &\ni 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.2}$$

and

$$\int_\Omega u(x)dx = 0. \tag{1.3}$$

Here, (1.2) is written in the sense of distributions (this being rigorously explained in Definition 2.2 later), y_o is given and

$$U = \{(u, y);\ u \in L^{q'}(\Omega),\ y \in L^2(\Omega),\ \nabla y \in (L^p(\Omega))^N\}, \tag{1.4}$$

where $p > 1$ and $q' \geq 2$. There is a relation between p and q' which will be described a little later. The domain Ω is an open bounded subset of $\mathbb{R}^N$, having the boundary

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$\partial\Omega$ of class C^1, ν is the unit outer normal to the boundary and σ is a positive real value. The notation $|\cdot|$ stands for the norm in $\mathbb{R}$. For a later use we denote

$$\mathcal{U} = \{(u, y) \in U;\ (u, y) \text{ satisfies (1.2), (1.3)}\}, \tag{1.5}$$

so that we deal with minimizing $J(u, y)$ for all pairs $(u, y) \in \mathcal{U}$. Since

$$\beta(r) = \partial j(r) \text{ for all } r \in \mathbb{R}^N \tag{1.6}$$

it follows that $\beta : \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone (and possibly multivalued), that is,

$$(\eta - \overline{\eta}) \cdot (r - \overline{r}) \geq 0, \text{ for any } r, \overline{r} \in \mathbb{R}^N, \text{ where } \eta \in \beta(r),\ \overline{\eta} \in \beta(\overline{r}), \tag{1.7}$$

and

$$R(I + \beta) = \mathbb{R}^N \tag{1.8}$$

where I is the identity operator and R is the range. Here, $r \cdot \omega$ is the scalar product in $\mathbb{R}^N$.

It should be remarked that the state equation (1.2) may be viewed as an elliptic equation in divergence form with a discontinuous diffusion term β, in which the jumps have been filled in, actually providing the multivalued function β. This is the reason for which we see the equation as a singular one.

On the other hand, the state equation (1.2) may have not a solution for each u, such that the optimal control problem can be considered as being singular in the sense of Lions (see [7]).

By $j^* : \mathbb{R}^N \to \mathbb{R}$ we denote the conjugate of j defined by

$$j^*(\omega) = \sup_{r \in \mathbb{R}^N} (r \cdot \omega - j(r)), \tag{1.9}$$

which, in virtue of (1.1), is convex and continuous. Without loss of generality we may assume that

$$j(r) \geq 0 \text{ for all } r \in \mathbb{R}, \quad j(0) = 0, \tag{1.10}$$

because one can redefine $j(r)$ as $j(r) - j(0) - \eta \cdot r$, with $\eta \in \beta(0)$. So, by (1.10) we deduce that

$$j^*(\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^N. \tag{1.11}$$

The functions j and j^* satisfy the Legendre-Fenchel relations,

$$j(r) + j^*(\omega) - r \cdot \omega \geq 0 \text{ for all } r, \omega \in \mathbb{R}^N, \tag{1.12}$$

$$j(r) + j^*(\omega) - r \cdot \omega = 0 \text{ if and only if } \omega \in \partial j(r). \tag{1.13}$$

Also, we recall that

$$\partial j^*(\omega) = (\partial j)^{-1}(\omega) \text{ for all } \omega \in \mathbb{R}^N, \tag{1.14}$$

(see e.g., [4], p. 8).

This kind of minimization problem is relevant in several problems occurring in applied sciences and physics, since particular β can model various physical phenomena. Steady state nonlinear diffusion processes in fluids, heat transfer or population dynamics can be controlled by manipulating sources u in order to obtain certain concentrations, temperatures or densities y. Besides stationary diffusion processes, we refer to phenomena related to the equilibrium of continuous media, material science, strength of materials. A domain where this problem has a particular relevance is the image denoising variational technique. In the latter case, y_o represents the blurred image and y is the denoised image obtained by the action of the control u. The less p is, the better denoising result is obtained, because a smaller value of p serves to the aim of a better image edges preserving of (see e.g., [2]). This justifies the interest in problem (P) for $p > 1$, but close to 1. The special case $p = 1$ which corresponds to a bounded variation flow technique in image denoising is ruled out by the hypothesis (1.1). This case was investigated in [3], but the results obtained in the current paper cannot be deduced from those exposed in [3].

The paper is structured as follows. After providing some properties of the functions j, j^* and of the graph β, the proof of a least a solution to problem (P) is given in Section 2.2. The calculus of the first order conditions of optimality directly in problem (P), and consequently a gradient-type algorithm for the numerical computation of the optimality conditions, would formally involve the directional derivative of β, which in our case does not exist. A possibility to avoid this inconvenient is to work instead with the subdifferentials of j and j^*. This suggests to introduce an approximating control problem (P_ε) involving these functions, in Section 2.3. A result guaranteeing the convergence of the approximating problem to the exact one is proved further and the computation of the approximating optimality conditions in Section 2.4 will end the theoretical results of the paper. Finally, a gradient-type algorithm to compute the the approximating optimality conditions is sketched.

2. Main results

2.1. **Preliminaries.** Let $(u, y) \in U$. Then, $y \in W^{1,p}(\Omega)$ since the norm $\|\nabla y\|_{L^p(\Omega)} + \|y\|_{L^2(\Omega)}$ is equivalent with the usual norm $\|y\|_{W^{1,p}(\Omega)}$ for all $p > 1$ (see [6], p. 286, Remark 15). By the Rellich–Kondrachov embedding inequalities ([6], p. 285) we have

$$\begin{aligned} W^{1,p}(\Omega) &\subset L^q(\Omega), \quad q \in [1, q^*),\ q^* = \frac{pN}{N-p}, && \text{for } N > p \\ W^{1,p}(\Omega) &\subset L^q(\Omega), \quad q \in [p, +\infty), && \text{for } N = p \\ W^{1,p}(\Omega) &\subset C(\overline{\Omega}), && \text{for } N < p, \end{aligned} \tag{2.1}$$

with compact injections. We define q' in the following way

$$q' = 2 \text{ if } q \geq 2 \text{ and } q' = \left(1 - \frac{1}{q}\right)^{-1} \text{ if } 1 < q < 2 \tag{2.2}$$

and this choice will be explained later.

Lemma 2.1. *Let us assume (1.1) and (1.10). Then we have*

$$C_3 \, |\omega|_N^{p'} + C_3^0 \leq j^*(\omega) \leq C_4 \, |\omega|_N^{p'} + C_4^0, \ \textit{for all } \omega \in \mathbb{R}^N, \ C_3 > 0, \tag{2.3}$$

$$|\eta|_N \leq C_5 \, |r|_N^{p-1} + C_5^0, \ \eta \in \partial j(r), \ \textit{for all } r \in \mathbb{R}^N, \tag{2.4}$$

$$|\chi|_N \leq C_6 \, |\omega|_N^{p'-1} + C_6^0, \ \chi \in \partial j^*(\omega) \ \textit{for all } \omega \in \mathbb{R}^N. \tag{2.5}$$

Proof. We recall that $j(r) \leq C_2 |r|_N^p + C_2^0$ and write

$$j^*(\omega) \geq r \cdot \omega - j(r), \text{ for any } r \in \mathbb{R}^N.$$

We apply this relation for $r = \rho |\omega|_N^{\frac{p'}{p}} \operatorname{sign} \omega$, and $\omega \neq 0$, with ρ a constant such that $0 < \rho < C_2^{-\frac{1}{p-1}}$. Here, sign $\omega = \frac{\omega}{|\omega|_N}$ for $\omega \neq 0$ and sign $0 = \{\omega \in \mathbb{R}^N;\ |\omega|_N \leq 1\}$. We get

$$\begin{aligned} j^*(\omega) &\geq \rho |\omega|_N^{p'} - C_2 \rho^p |\omega|_N^{p'} - C_2^0 = \rho(1 - \rho^{p-1} C_2) |\omega|_N^{p'} - C_2^0 \\ &= C_3 |\omega|_N^{p'} + C_3^0, \ C_3 > 0. \end{aligned}$$

Similarly, we have for any $\omega \in \mathbb{R}^N$,

$$\begin{aligned} j^*(\omega) &= \sup_{r \in \mathbb{R}^N} (\omega \cdot r - j(r)) \leq \sup_{r \in \mathbb{R}^N} (\omega \cdot r - C_1 |r|_N^p - C_1^0) \\ &= \frac{1}{C_1^{\frac{1}{p-1}}} |\omega|_N^{p'} \left(\frac{1}{p^{\frac{1}{p-1}}} - \frac{1}{p^{\frac{p}{p-1}}} \right) - C_1^0 = C_4 |\omega|_N^{p'} + C_4^0. \end{aligned}$$

We took into account that the supremum is reached at $|r|_N = |\omega|_N^{\frac{1}{p-1}} (pC_1)^{-\frac{1}{p-1}}$.

In order to prove (2.4), we write that

$$j(r) - j(\theta) \leq \eta \cdot (r - \theta), \text{ for any } \theta \in \mathbb{R}^N, \ \eta \in \beta(r)$$

and set $\theta = \lambda \frac{\eta}{|\eta|_N}$. We get

$$j(r) + \lambda |\eta|_N \leq \eta \cdot r + C_2 \lambda^p + C_2^0,$$

whence

$$\lambda |\eta|_N \leq |\eta|_N |r|_N + C_2 \lambda^p + \left|C_2^0\right| + \left|C_1 |r|_N^p + C_1^0\right|. \tag{2.6}$$

Let $|r|_N > 1$ and consider (2.6) for $\lambda = 2 |r|_N$. It follows that

$$|\eta|_N \leq (C_2 + C_1) |r|_N^{p-1} + \frac{\left|C_2^0\right| + \left|C_1^0\right|}{|r|_N} \leq C_5 |r|_N^{p-1} + C_5^0.$$

If $|r|_N \leq 1$ we consider (2.6) for $\lambda = 2$. We get $|\eta|_N \leq C_5^1$. Finally, (2.5) is deduced by applying the above arguments for j^* and recalling that $\partial j^*(\omega) = \beta^{-1}(\omega)$. □

We introduce the space

$$V = \{z \in H^1(\Omega);\ \int_\Omega z(x) dx = 0\} \tag{2.7}$$

with the norm

$$|||z||| = \|\nabla z\|_{(L^2(\Omega))^N}, \tag{2.8}$$

which is equivalent with the norm $\|z\|_{H^1(\Omega)}$ (see [6], p. 286).

Now, let us consider the boundary value problem

$$\begin{aligned} -\Delta z &= u \text{ in } \Omega, \\ \nabla z \cdot \nu &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.9}$$

where $u \in L^{q'}(\Omega)$ and $\int_\Omega u(x)dx = 0$, $q' \geq 2$. One can easily see that (2.9) has a unique solution $z \in V$, which satisfies the estimate

$$c \|z\|_{H^1(\Omega)} \leq |||z||| \leq \|u\|_{L^2(\Omega)}. \tag{2.10}$$

2.2. **Existence for problem (P).** Let $y \in L^2(\Omega)$, such that $\nabla y \in (L^p(\Omega))^N$. Then, by Lemma 2.1, it follows that $\xi \in (L^{p'}(\Omega))^N$, if $\xi(x) \in \beta(\nabla y(x))$, a.e. $x \in \Omega$.

Definition 2.2. Let q' defined as in (2.2) and let $u \in L^{q'}(\Omega)$. We call a *solution* to (1.2) a function $y \in L^2(\Omega)$, such that $\nabla y \in (L^p(\Omega))^N$, which satisfies

$$\int_\Omega \xi(x) \cdot \nabla\psi(x)dx = \int_\Omega u(x)\psi(x)dx, \tag{2.11}$$

for all $\psi \in L^2(\Omega)$ with $\nabla\psi \in (L^p(\Omega))^N$, and some $\xi(x) \in \beta(\nabla y(x))$, a.e. $x \in \Omega$.

It is obvious by Lemma 2.1 that $\xi \in (L^{p'}(\Omega))^N$ and so the left-hand side in (2.11) makes sense. Also, by (1.1) it follows that $j(\nabla y) \in L^1(\Omega)$ and by (1.13) we get that $j^*(\xi) \in L^1(\Omega)$.

We note that ψ belonging to the spaces indicated in (2.11) is in fact in U, hence it is also in $L^q(\Omega)$, with q defined in (2.1).

We explain now the motivation of the choice of q'. We have $\psi \in L^q(\Omega)$ and $u \in L^{q'}(\Omega)$ so that the integral on the right-hand side in (2.11) makes sense if q' is the conjugate of q. However, in order to prove existence in (P) it will be required that u be necessarily in $L^2(\Omega)$ (see the estimate (2.16) in the next theorem). This is verified for $q \leq 2$ when the conjugate $q' \geq 2$. But, if $q > 2$ it is not sufficient to consider exactly its conjugate as exponent of u, and so we have to choose $q' \geq 2$.

Later, for simplicity, we shall not indicate the function arguments in the integrals.

Theorem 2.3. *Let $y_o \in L^2(\Omega)$, q and q' given in (2.1)-(2.2). Then, problem (P) has at least a solution (u^*, y^*).*

Proof. It is clear that $d = \inf_{(u,y)\in\mathcal{U}} J(u,y)$ exists and it is nonnegative because $J(u,y) \geq 0$. We take a minimizing sequence, $\{u_n, y_n\}_n \in \mathcal{U}$, that is $u_n \in L^{q'}(\Omega)$, $y_n \in L^2(\Omega)$, $\nabla y_n \in (L^p(\Omega))^N$, such that

$$\begin{aligned} -\nabla \cdot \xi_n &= u_n \text{ in } \Omega, \\ \xi_n \cdot \nu &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.12}$$

where $\xi_n(x) \in \beta(\nabla y_n(x))$ a.e. $x \in \Omega$, and

$$\int_\Omega u_n dx = 0. \tag{2.13}$$

The minimizing sequence satisfies for all $n \geq 1$ the inequalities

$$d \leq J(u_n, y_n) = \int_\Omega \left(\frac{|u_n(x)|^{q'}}{q'} + \frac{\sigma}{2}(y_n(x) - y_o(x))^2 \right) dx \leq d + \frac{1}{n}, \tag{2.14}$$

and it is clear that $\{u_n\}_{n\geq1}$ and $\{y_n\}_{n\geq1}$ lie in bounded subsets of $L^{q'}(\Omega)$ and $L^2(\Omega)$, respectively. In particular, $u_n \in L^2(\Omega)$, since $q' \geq 2$.

Since y_n is a solution to (2.12) it also follows by Definition 2.2 that $\xi_n \in (L^{p'}(\Omega))^N$, $\xi_n \in \beta(\nabla y_n)$ a.e., $j(\nabla y_n) \in L^1(\Omega)$, $j^*(\xi_n) \in L^1(\Omega)$, while by (2.11)

$$\int_\Omega \xi_n \cdot \nabla\psi dx = \int_\Omega u_n \psi dx, \text{ for all } \psi \in L^2(\Omega), \text{ with } \nabla\psi \in (L^p(\Omega))^N. \tag{2.15}$$

Thus, we can take $\psi = y_n$ in (2.15), getting

$$\int_\Omega \xi_n \cdot \nabla y_n dx = \int_\Omega u_n y_n dx \leq \|u_n\|_{L^2(\Omega)} \|y_n\|_{L^2(\Omega)} \leq C. \tag{2.16}$$

By C we denote constants independent of n. By (1.13) we can write

$$\int_\Omega (j(\nabla y_n) + j^*(\xi_n) dx = \int_\Omega \xi_n \cdot \nabla y_n dx \leq C. \tag{2.17}$$

Using (2.17) and hypotheses (1.1) and (2.3), we obtain that $\{\nabla y_n\}_n$ lies in a bounded subset of $(L^p(\Omega))^N$ and $\{\xi_n\}_n$ is in a bounded subset of $(L^{p'}(\Omega))^N$. Thus, we can select a subsequence of $\{u_n, y_n\}_n$, denoted still by the subscript n, and get, as $n \to \infty$, that

$$\begin{array}{rcll} u_n & \to & u^* & \text{weakly in } L^{q'}(\Omega), \\ y_n & \to & y^* & \text{weakly in } L^2(\Omega), \\ \nabla y_n & \to & \nabla y^* & \text{weakly in } (L^p(\Omega))^N, \\ \xi_n & \to & \xi^* & \text{weakly in } (L^{p'}(\Omega))^N, \\ -\nabla \cdot \xi_n & \to & \zeta^* & \text{weakly in } L^{q'}(\Omega). \end{array}$$

Hence, $(u^*, y^*) \in U$. Since $y_n \in W^{1,\overline{p}}(\Omega)$ we get by (2.1) that

$$y_n \to y^* \text{ strongly in } L^q(\Omega),$$

and, therefore,

$$u_n y_n \to u^* y^* \text{ weakly in } L^1(\Omega), \quad \int_\Omega u_n dx \to \int_\Omega u^* dx = 0.$$

Passing to the limit in (2.17) as $n \to \infty$, we obtain on the basis of the lower semicontinuity of convex integrands and positiveness of j and j^* that

$$j(\nabla y^*) \in L^1(\Omega), \quad j^*(\xi^*) \in L^1(\Omega).$$

We pass to the limit in (2.15) and get that

$$\int_\Omega \xi^* \cdot \nabla\psi dx = \int_\Omega u^* \psi dx, \text{ for all } \psi \in U,$$

which, in particular, is true also for $\psi = y^* \in U$. This means that y^* is a solution in the sense of distributions (that is in the sense of Definition 2.2) to

$$\begin{array}{rcl} -\nabla \cdot \xi^* & = & u^* \text{ in } \Omega, \\ \xi^* \cdot \nu & = & 0 \text{ on } \partial\Omega, \end{array}$$

and these equations are satisfied also a.e. because $\xi^* \in (L^{p'}(\Omega))^N$. Going back to (2.16), we see that

$$\limsup_{n\to\infty} \int_\Omega \xi_n \cdot \nabla y_n dx = \int_\Omega u^* y^* dx = \int_\Omega \xi^* \cdot \nabla y^* dx.$$

It follows (see [4], p. 39) that $\xi^* \in \beta(\nabla y^*)$ a.e. on Ω, and so $\zeta^* \in \nabla \cdot \beta(\nabla y^*)$ a.e. on Ω and $\zeta^* \in L^{q'}(\Omega)$. Thus, $(u^*, y^*) \in \mathcal{U}$. Finally, we can pass to the limit in (2.14) and obtain that $J(u^*, y^*) = d$, which proves that (P) has at least a solution. □

We give next an equivalent result to Theorem 2.3.

Corollary 2.4. *Under the hypotheses of Theorem 2.3, the minimization problem*

$$(\widetilde{P}) \qquad \min_{(u,y,z)\in\widetilde{U}} \left(\int_\Omega \left(\frac{|u(x)|^{q'}}{q'} + \frac{\sigma}{2}(y(x) - y_o(x))^2 \right) dx \right),$$

where

$$(\widetilde{U}) \qquad \widetilde{U} = \{(u, y, z);\ (u, y) \in \mathcal{U},\ z \in V,\ \beta(\nabla y(x)) \ni \nabla z(x) \ \text{a.e. on } \Omega\},$$

has at least a solution (u^*, y^*, z^*).

Proof. The proof is led on the basis of all arguments in Theorem 2.3. The minimizing sequence should satisfy in addition $\beta(\nabla y_n) \ni \nabla z_n$ a.e on Ω, where $z_n \in V$ turns out to be the unique solution to

$$(2.18) \qquad \begin{aligned} -\Delta z_n &= u_n \text{ in } \Omega, \\ \nabla z_n \cdot \nu &= 0 \text{ on } \partial\Omega, \end{aligned}$$

satisfying $c\,\|z_n\|_{H^1(\Omega)} \le \|u_n\|_{L^2(\Omega)}$. Since, in Theorem 2.3, in particular, $u_n \to u^*$ weakly in $L^2(\Omega)$, then we get that $z_n \to z^*$ weakly in $H^1(\Omega)$, as $n \to \infty$. Writing the weak form of (2.18) and passing to the limit we obtain that (u^*, z^*) satisfy (2.9). Moreover, $\int_\Omega z_n dx \to \int_\Omega z^* dx$, and so $(u^*, y^*, z^*) \in \widetilde{U}$. □

2.3. **The approximating problem.** As we explained at the beginning, in order to compute the optimality conditions in a rigorous way we shall consider the following approximating control problem involving a differentiable cost functional,

$$(\widetilde{P_\varepsilon}) \qquad \min_{(u,y,z)\in U_1} \left\{ \int_\Omega \left(\frac{|u|^{q'}}{q'} + \frac{\sigma}{2}(y - y_o)^2 \right) dx + \frac{1}{\varepsilon} \int_\Omega (j(\nabla y) + j^*(\nabla z) - \nabla y \cdot \nabla z) dx \right\}$$

where $\varepsilon > 0$ and

$$(2.19) \qquad \begin{aligned} U_1 = \Big\{ &(u, y, z);\ (u, y) \in U,\ \int_\Omega u dx = 0,\ z \in H^1(\Omega), \\ &j(\nabla y) \in L^1(\Omega),\ j^*(\nabla z) \in L^1(\Omega),\ \nabla y \cdot \nabla z \in L^1(\Omega), \\ &-\Delta z = u \text{ on } \Omega,\ \nabla z \cdot \nu = 0 \text{ on } \partial\Omega,\ \int_\Omega z dx = 0 \Big\}. \end{aligned}$$

This problem has also the main advantage of involving a linear state system instead of that nonlinear in problem (P). The motivation of the choice of the second integral term in the cost functional relies on the Legendre-Fenchel relations. First, this term is nonnegative due to (1.12) and secondly if this is (close to) 0, then it would imply that $\nabla z \in \beta(\nabla y)$ a.e. Loosely speaking it means that $(\widetilde{P_\varepsilon})$ approximates (P) in an appropriate way. This will be rigorously proved in Theorem 2.6.

Now, if $(u,y,z)\in U_1$ we have

$$\int_\Omega \nabla y\cdot\nabla z dx=\int_{\partial\Omega} y\nabla z\cdot\nu d\sigma-\int_\Omega y\Delta z dx=\int_\Omega uy dx. \tag{2.20}$$

Therefore, problem $(\widetilde{P_\varepsilon})$ is equivalent with the following one

$$\min_{(u,y,z)\in U_1} J_\varepsilon(u,y,z), \tag{P_ε}$$

with

$$\begin{aligned} J_\varepsilon(u,y,z) &= \int_\Omega \left(\frac{|u|^{q'}}{q'}+\frac{\sigma}{2}(y-y_o)^2\right)dx \\ &\quad +\frac{1}{\varepsilon}\int_\Omega (j(\nabla y)+j^*(\nabla z)-uy)dx. \end{aligned} \tag{2.21}$$

For each $u\in L^2(\Omega)$, the function $z\in V$ is the unique solution to (2.9), as explained before.

Theorem 2.5. *Let $y_o\in L^2(\Omega)$ and let q and q' be as in (2.1), (2.2). Then, problem (P_ε) has at least a solution $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$.*

Proof. The functional $J_\varepsilon(u,y,z)$ is nonnegative, because the second integral in $(\widetilde{P_\varepsilon})$ is nonnegative by the first Legendre-Fenchel relation (1.12). Then, there exists the infimum d_ε and taking a minimizing sequence $\{u_n,y_n,z_n\}_n$ we have

$$d_\varepsilon\le J_\varepsilon(u_n,y_n,z_n)\le d_\varepsilon+\frac{1}{n},\ \text{for } n\ge 1, \tag{2.22}$$

where $u_n\in L^{q'}(\Omega)$, $\int_\Omega u_n dx=0$, $y_n\in L^2(\Omega)$, $\nabla y_n\in (L^p(\Omega))^N$, $z_n\in H^1(\Omega)$, $\int_\Omega z_n dx=0$, $j(\nabla y_n)\in L^1(\Omega)$, $j(\nabla z_n)\in L^1(\Omega)$, $\nabla y_n\cdot\nabla z_n\in L^1(\Omega)$, and z_n is the solution to

$$\begin{aligned} -\Delta z_n &= u_n \text{ in } \Omega,\\ \nabla z_n\cdot\nu &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.23}$$

Moreover,

$$\int_\Omega\left(\frac{|u_n|^{q'}}{q'}+\frac{\sigma}{2}(y_n-y_o)^2\right)dx\le J_\varepsilon(u_n,y_n,z_n)\le d_\varepsilon+1$$

and recalling that $q'\ge 2$ it follows that $\{u_n\}_n$ and $\{y_n\}_n$ are bounded in $L^{q'}(\Omega)$ and $L^2(\Omega)$, respectively. Since the first integral in (P_ε) is nonnegative we can write

$$\begin{aligned} \frac{1}{\varepsilon}\int_\Omega (j(\nabla y_n)\ +\ j^*(\nabla z_n))dx &\le \frac{1}{\varepsilon}\int_\Omega u_n y_n dx+d_\varepsilon+1\\ &\le \frac{1}{\varepsilon}\|u_n\|_{L^2(\Omega)}\|y_n\|_{L^2(\Omega)}+d_\varepsilon+1\le C_\varepsilon, \end{aligned} \tag{2.24}$$

with C_ε independent of n. Hence we can select a subsequence $n \to \infty$, such that

$$\begin{array}{rcll} u_n & \to & u_\varepsilon^* & \text{weakly in } L^{q'}(\Omega), \\ y_n & \to & y_\varepsilon^* & \text{weakly in } L^2(\Omega) \text{ and strongly in } L^q(\Omega), \\ \nabla y_n & \to & \nabla y_\varepsilon^* & \text{weakly in } (L^p(\Omega))^N, \\ u_n y_n & \to & u_\varepsilon^* y_\varepsilon^* & \text{weakly in } L^1(\Omega), \\ \int_\Omega u_n dx & \to & \int_\Omega u_\varepsilon^* dx = 0. & \end{array}$$

By (2.23) and (2.10) it follows that $\{z_n\}_n$ is bounded in $H^1(\Omega)$ and so

$$z_n \to z_\varepsilon^* \text{ weakly in } H^1(\Omega), \text{ and } \int_\Omega z_\varepsilon^* dx = 0.$$

Moreover, by (2.3) it follows that $\{\nabla z_n\}_n$ is in a bounded subset of $(L^{p'}(\Omega))^N$ and so on a subsequence

$$\nabla z_n \to \nabla z_\varepsilon^* \text{ weakly in } (L^{p'}(\Omega))^N.$$

By (2.23)

$$\int_\Omega z_n \cdot \nabla \psi dx = \int_\Omega u_n \psi dx, \text{ for all } \psi \in L^2(\Omega), \text{ with } \nabla\psi \in (L^p(\Omega))^N, \tag{2.25}$$

we get at limit

$$\int_\Omega z_\varepsilon^* \cdot \nabla \psi dx = \int_\Omega u_\varepsilon^* \psi dx, \text{ for all } \psi \in L^2(\Omega), \text{ with } \nabla\psi \in (L^p(\Omega))^N,$$

which can be equivalently written in the sense of distributions as

$$\begin{array}{rcl} -\Delta z_\varepsilon^* & = & u_\varepsilon^* \text{ in } \Omega, \\ \nabla z_\varepsilon^* \cdot \nu & = & 0 \text{ on } \partial\Omega. \end{array} \tag{2.26}$$

Using (2.20) we can write

$$\int_\Omega \nabla y_n \cdot \nabla z_n dx = \int_\Omega u_n y_n dx \to \int_\Omega u_\varepsilon^* y_\varepsilon^* dx = \int_\Omega \nabla y_\varepsilon^* \cdot \nabla z_\varepsilon^* dx.$$

By passing to the limit in (2.24) and (2.22) we obtain that $j(\nabla y_\varepsilon^*)$ and $j^*(\nabla z_\varepsilon^*)$ are in $L^1(\Omega)$, and that $J_\varepsilon(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) = d_\varepsilon$, respectively. Thus, we have proved that $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) \in U_1$ and that it is a solution for (P_ε). □

Theorem 2.6. *Let and q and q' be as in (2.1), (2.2) and let $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$ be optimal for problem (P_ε). Then,*

$$\begin{array}{rcll} u_\varepsilon^* & \to & \widetilde{u} & \textit{weakly in } L^{q'}(\Omega), \\ y_\varepsilon^* & \to & \widetilde{y} & \textit{weakly in } L^2(\Omega) \textit{ and strongly in } L^q(\Omega), \\ \nabla y_\varepsilon^* & \to & \nabla\widetilde{y} & \textit{weakly in } (L^p(\Omega))^N, \\ z_\varepsilon^* & \to & \widetilde{z} & \textit{weakly in } V. \end{array}$$

Moreover, $(\widetilde{u}, \widetilde{y}, \widetilde{z})$ is optimal in $(\widetilde{P})$ and $(\widetilde{u}, \widetilde{y})$ is optimal in (P).

Proof. Let $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$ be optimal in (P_ε), that is

$$J_\varepsilon(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) \leq J_\varepsilon(u, y, z) \text{ for all } (y, y, z) \in U_1.$$

Then, using (2.21), we write

$$\begin{aligned}(2.27)\qquad &\int_\Omega \left(\frac{|u_\varepsilon^*|^{q'}}{q'} + \frac{\sigma}{2}(y_\varepsilon^* - y_o)^2 \right) dx + \frac{1}{\varepsilon}\int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - u_\varepsilon^* y_\varepsilon^*) dx \\ \leq &\int_\Omega \left(\frac{|u|^{q'}}{q'} + \frac{\sigma}{2}(y - y_o)^2 \right) dx + \frac{1}{\varepsilon}\int_\Omega (j(\nabla y) + j^*(\nabla z) - \nabla y \cdot \nabla z) dx,\end{aligned}$$

for all $(u, y, z) \in U_1$. For the last term on the right-hand side, on the second line, we have already used (2.20).

Let (u^*, y^*, z^*) be an optimal pair in $(\widetilde{P})$, that is $u^* \in L^{q'}(\Omega)$, $y^* \in L^2(\Omega)$, $\nabla y^* \in (L^p(\Omega))^N$, (u^*, y^*) satisfies (1.2) and $\int_\Omega u^* dx = 0$. Also, $\nabla z^* \in \beta(\nabla y^*)$ a.e. on Ω and so $z^* \in V$ is the unique solution to (2.9) corresponding to u^*. Then, $\nabla z^* \in (L^{p'}(\Omega))^N$, $j(\nabla y^*)$, $j^*(\nabla z^*) \in L^1(\Omega)$. Let us set in (2.27) $u = u^*$, $y = y^*$, and $z = z^*$. This triplet is in U_1.

It follows that the last integral on the right-hand side in (2.27) vanishes due to (2.20) and the remainder is bounded by a constant independent of ε,

$$\begin{aligned}(2.28)\qquad &\int_\Omega \left(\frac{|u_\varepsilon^*|^{q'}}{q'} + \frac{\sigma}{2}(y_\varepsilon^* - y_o)^2 \right) dx + \frac{1}{\varepsilon}\int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - u_\varepsilon^* y_\varepsilon^*) dx \\ &\qquad\leq \int_\Omega \left(\frac{|u^*|^{q'}}{q'} + \frac{\sigma}{2}(y^* - y_o)^2 \right) dx.\end{aligned}$$

This allows us to use similar arguments as in Theorem 2.5 to deduce that on a subsequence $\{\varepsilon \to 0\}$ we have

$$\begin{aligned}
u_\varepsilon^* &\to \widetilde{u} && \text{weakly in } L^{q'}(\Omega),\\
y_\varepsilon^* &\to \widetilde{y} && \text{weakly in } L^2(\Omega) \text{ and strongly in } L^q(\Omega),\\
\nabla y_\varepsilon^* &\to \nabla\widetilde{y} && \text{weakly in } (L^p(\Omega))^N,\\
z_\varepsilon^* &\to \widetilde{z} && \text{weakly in } H^1(\Omega),\\
\nabla z_\varepsilon^* &\to \nabla\widetilde{z} && \text{weakly in } (L^{p'}(\Omega))^N,\\
u_\varepsilon^* y_\varepsilon^* &\to \widetilde{u}\widetilde{y} && \text{weakly in } L^1(\Omega),\\
\textstyle\int_\Omega u_\varepsilon^* dx &\to \int_\Omega \widetilde{u} dx = 0, && \int_\Omega z_\varepsilon^* dx \to \int_\Omega \widetilde{z} dx = 0,
\end{aligned}$$

and $(\widetilde{u}, \widetilde{z})$ satisfies (2.9). Moreover, by (2.28), the sequence

$$\zeta_\varepsilon = \frac{1}{\varepsilon}\int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - u_\varepsilon^* y_\varepsilon^*) dx$$

is bounded by a constant, hence, using (2.20), we can write

$$0 \leq \int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - \nabla y_\varepsilon^* \cdot \nabla z_\varepsilon^*) dx = \varepsilon\zeta_\varepsilon.$$

By passing to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned} 0 &\leq \int_\Omega (j(\nabla\widetilde{y}) + j^*(\nabla\widetilde{z}) - \widetilde{u}\widetilde{y})dx \leq \lim_{\varepsilon\to 0}\int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - u_\varepsilon^* y_\varepsilon^*)dx \\ &= \lim_{\varepsilon\to 0}\int_\Omega (j(\nabla y_\varepsilon^*) + j^*(\nabla z_\varepsilon^*) - \nabla y_\varepsilon^* \cdot \nabla z_\varepsilon^*)dx = 0. \end{aligned}$$

This implies that

$$\int_\Omega (j(\nabla\widetilde{y}) + j^*(\nabla\widetilde{z}) - \widetilde{u}\widetilde{y})dx = \int_\Omega (j(\nabla\widetilde{y}) + j^*(\nabla\widetilde{z}) - \nabla\widetilde{y}\cdot\nabla\widetilde{z})dx = 0.$$

Since the integrand in the last integral is nonnegative it follows that $j(\nabla\widetilde{y})+j^*(\nabla\widetilde{z})-\nabla\widetilde{y}\cdot\nabla\widetilde{z} = 0$ a.e. on Ω, leading to the conclusion that $\nabla\widetilde{z} \in \beta(\nabla\widetilde{y})$ a.e. on Ω. The equality of the two integrals above was based on (2.20) because by passing to the limit in (2.9) written for $(z_\varepsilon^*, u_\varepsilon^*)$ it follows that the limit $\widetilde{z}$ is the solution to

$$-\Delta\widetilde{z} = \widetilde{u} \text{ in } \Omega,\ \nabla\widetilde{z}\cdot\nu = 0 \text{ on } \partial\Omega.$$

Hence, it turns out that $(\widetilde{u},\widetilde{y})$ satisfies (1.2) and $(\widetilde{u},\widetilde{y},\widetilde{z}) \in U_1$. Returning to (2.28),

$$\int_\Omega\left(\frac{|u_\varepsilon^*|^{q'}}{q'} + \frac{\sigma}{2}(y_\varepsilon^* - y_o)^2\right)dx \leq J_\varepsilon(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) \leq \int_\Omega\left(\frac{|u^*|^{q'}}{q'} + \frac{\sigma}{2}(y^* - y_o)^2\right)dx$$

we pass to the limit as $\varepsilon \to 0$ and obtain

$$\int_\Omega\left(\frac{|\widetilde{u}|^{q'}}{q'} + \frac{\sigma}{2}(\widetilde{y} - y_o)^2\right)dx \leq \int_\Omega\left(\frac{|u^*|^{q'}}{q'} + \frac{\sigma}{2}(y^* - y_o)^2\right)dx,$$

which proves that $(\widetilde{u},\widetilde{y},\widetilde{z})$ is an optimal pair in $(\widetilde{P})$. Since $(\widetilde{u},\widetilde{y})$ satisfies (1.2) too, it follows that $(\widetilde{u},\widetilde{y})$ is optimal in (P). □

2.4. **Optimality conditions for the approximating problem.**

Let $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) \in U_1$ be an optimal pair in (P_ε) and let λ be a real value. We take v and Y as follows

$$v \in L^{q'}(\Omega),\ \int_\Omega v dx = 0,\ Y \in L^2(\Omega),\ \nabla Y \in (L^p(\Omega))^N, \tag{2.29}$$

and define

$$u_\varepsilon^\lambda = u_\varepsilon^* + \lambda v,\ y_\varepsilon^\lambda = y_\varepsilon^* + \lambda Y.$$

Then, problem

$$\begin{aligned} -\Delta z_\varepsilon^\lambda &= u_\varepsilon^\lambda \text{ in } \Omega, \\ \nabla z_\varepsilon^\lambda\cdot\nu &= 0 \text{ on } \partial\Omega \end{aligned}$$

has a unique solution and we deduce that

$$Z = \lim_{\lambda\to 0}\frac{z_\varepsilon^\lambda - z_\varepsilon^*}{\lambda} \text{ weakly in } H^1(\Omega)$$

is the unique solution to the problem

$$\begin{aligned} -\Delta Z &= v \text{ in } \Omega, \\ \nabla Z\cdot\nu &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.30}$$

Proposition 2.7. *Let $q' \geq 2$. Let $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$ be an optimal pair for problem (P_ε), and assume that there exists $\chi_\varepsilon^* \in (L^2(\Omega))^N$, such that $\chi_\varepsilon^* \in \beta^{-1}(\nabla z_\varepsilon^*)$ a.e. in Ω. Then, there is a constant C_ε^* such that*

$$|u_\varepsilon^*|^{q'-2}\, u_\varepsilon^* = \frac{1}{\varepsilon} y_\varepsilon^* - p_\varepsilon + C_\varepsilon^*, \tag{2.31}$$

where p_ε, y_ε^* and z_ε^* are the solutions to

$$\begin{aligned} \Delta p_\varepsilon &\in \frac{1}{\varepsilon}\nabla\cdot\beta^{-1}(\nabla z_\varepsilon^*) \text{ in } \Omega, \\ \nabla p_\varepsilon\cdot\nu &\in \frac{1}{\varepsilon}\beta^{-1}(\nabla z_\varepsilon^*)\cdot\nu \text{ on } \partial\Omega, \end{aligned} \tag{2.32}$$

$$\begin{aligned} -\nabla\cdot\beta(\nabla y_\varepsilon^*) + \lambda\varepsilon(y_\varepsilon^* - y_o) &\ni u_\varepsilon^* \text{ in } \Omega, \\ \beta(\nabla y_\varepsilon^*)\cdot\nu &\ni 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.33}$$

$$\begin{aligned} -\Delta z_\varepsilon^* &= u_\varepsilon^* \text{ in } \Omega, \\ \nabla z_\varepsilon^*\cdot\nu &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.34}$$

Proof. Assume that $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$ is optimal in (P_ε). Then, we have

$$J_\varepsilon(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*) \leq J_\varepsilon(u_\varepsilon^\lambda, y_\varepsilon^\lambda, z_\varepsilon^\lambda).$$

Using the form (2.21) for J_ε, and making some calculations in the above inequality, that is, subtracting the left-hand side term, dividing by λ and passing to the limit as $\lambda \to 0$ we obtain

$$\begin{aligned} &\frac{1}{\varepsilon}\int_\Omega \left(j'(\nabla y_\varepsilon^*; \nabla Y) + (j^*)'(\nabla z_\varepsilon^*; \nabla Z) - u_\varepsilon^* Y - v y_\varepsilon^*\right) dx \\ &\qquad + \int_\Omega \left(|u_\varepsilon^*|^{q'-2}\, u_\varepsilon^* v + \lambda(y_\varepsilon^* - y_o)Y\right) \geq 0, \end{aligned} \tag{2.35}$$

where

$$j'(w; W) = \lim_{\lambda\to 0} \frac{j(w+\lambda W) - j(w)}{\lambda}$$

is the directional derivative of j at w in direction W. A similar definition is for $(j^*)'$. Passing from λ to $-\lambda$ and repeating all computations we obtain that the left-hand side in (2.35) is less or equal to zero, so that the final relation is

$$\begin{aligned} &\frac{1}{\varepsilon}\int_\Omega \left(j'(\nabla y_\varepsilon^*; \nabla Y) + (j^*)'(\nabla z_\varepsilon^*; \nabla Z) - u_\varepsilon^* Y - v y_\varepsilon^*\right) dx \\ &\qquad + \int_\Omega \left(|u_\varepsilon^*|^{q'-2}\, u_\varepsilon^* v + \lambda(y_\varepsilon^* - y_o)Y\right) = 0. \end{aligned} \tag{2.36}$$

Now, we recall that

$$\int_\Omega j'(\nabla y_\varepsilon^*; \nabla Y) dx \geq \int_\Omega \xi_\varepsilon^* \cdot \nabla Y\, dx$$

for all $\xi_\varepsilon^* \in \partial j(\nabla y_\varepsilon^*)$ a.e. on Ω, and similarly for $(j^*)'$ (see e.g., [4], p. 53), and so the previous equality (2.36) implies

$$\int_\Omega \left(\frac{1}{\varepsilon}(\xi_\varepsilon^*\cdot\nabla Y + \chi_\varepsilon^*\cdot\nabla Z - u_\varepsilon^* Y - v y_\varepsilon^*) + |u_\varepsilon^*|^{q'-2}\, u_\varepsilon^* v + \lambda(y_\varepsilon^* - y_o)Y\right) dx \leq 0, \tag{2.37}$$

where $\chi_\varepsilon^* \in \partial j^*(\nabla z_\varepsilon^*) = \beta^{-1}(\nabla z_\varepsilon^*)$ a.e. on Ω, by (1.14). Considering now the pair $(-v, -Y)$ we get the converse inequality, so that, finally, we get

$$\int_\Omega \left(\frac{1}{\varepsilon}(\xi_\varepsilon^* \cdot \nabla Y + \chi_\varepsilon^* \cdot \nabla Z - u_\varepsilon^* Y - v y_\varepsilon^*) + |u_\varepsilon^*|^{q'-2} u_\varepsilon^* v + \lambda(y_\varepsilon^* - y_o) Y \right) dx = 0, \tag{2.38}$$

for any v and Y taken as in (2.29).

We introduce the adjoint problem (2.32) and assert that it has a unique solution $p_\varepsilon \in V$, which satisfies the weak form

$$\int_\Omega \nabla p_\varepsilon \cdot \nabla \psi dx = \frac{1}{\varepsilon} \int_\Omega \chi_\varepsilon^* \cdot \nabla \psi d\sigma \text{ for all } \psi \in H^1(\Omega), \tag{2.39}$$

and some $\chi_\varepsilon^* \in \beta^{-1}(\nabla z_\varepsilon^*)$ a.e. on Ω.

By (2.30) we get

$$\int_\Omega \nabla p_\varepsilon \cdot \nabla Z dx = \int_\Omega v p_\varepsilon dx \tag{2.40}$$

and by (2.39), setting $\psi = Z$ we obtain that

$$\frac{1}{\varepsilon} \int_\Omega \chi_\varepsilon^* \cdot \nabla Z dx = \int_\Omega \nabla p_\varepsilon \cdot \nabla Z dx = \int_\Omega v p_\varepsilon dx. \tag{2.41}$$

Therefore (2.38), (2.41) and (2.40) yield

$$\int_\Omega \left(\lambda(y_\varepsilon^* - y_o) - \frac{1}{\varepsilon} \nabla \cdot \xi_\varepsilon^* - \frac{1}{\varepsilon} u_\varepsilon^* \right) Y dx + \int_\Omega v \left(|u_\varepsilon^*|^{q-2} u_\varepsilon^* - \frac{1}{\varepsilon} y_\varepsilon^* + p_\varepsilon \right) dx = 0 \tag{2.42}$$

which is true for all Y and v. In particular, for $v = 0$ one has

$$\int_\Omega \left(\lambda(y_\varepsilon^* - y_o) - \frac{1}{\varepsilon} \nabla \cdot \xi_\varepsilon^* - \frac{1}{\varepsilon} u_\varepsilon^* \right) Y dx = 0, \text{ for all } Y \in L^2(\Omega).$$

It turns out that y_ε^* is the solution in the sense of distributions to the problem

$$\begin{aligned} -\nabla \cdot \xi_\varepsilon^* + \lambda \varepsilon (y_\varepsilon^* - y_o) &\ni u_\varepsilon^*, \text{ in } \Omega, \\ \xi_\varepsilon^* \cdot \nu &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.43}$$

where $\xi_\varepsilon^* \in \beta(\nabla y_\varepsilon^*)$ a.e. on Ω, which means in fact that y_ε^* is the solution to (2.33).

Next, $Y = 0$ yields

$$\int_\Omega v \left(|u_\varepsilon^*|^{q-2} u_\varepsilon^* - \frac{1}{\varepsilon} y_\varepsilon^* + p_\varepsilon \right) dx = 0, \text{ for all } v \in L^{q'}(\Omega), \text{ with } \int_\Omega v dx = 0.$$

This implies that $|u_\varepsilon^*|^{q'-2} u_\varepsilon^* - \frac{1}{\varepsilon} y_\varepsilon^* + p_\varepsilon = C_\varepsilon^*$, with C_ε^* a constant which may be deduced from the condition $\int_\Omega u_\varepsilon^* dx = 0$.

It remains to discuss the existence for the solutions to (2.32) and (2.33). In (2.32) z_ε^* is fixed in $H^1(\Omega)$ (by Theorem 2.5) and by hypothesis the section χ_ε^* of $\beta^{-1}(\nabla z_\varepsilon^*)$ is in $(L^2(\Omega))^N$. Eq. (2.32) is equivalent with the following minimization problem

$$\min_{p \in U_2} \Phi(p) = \int_\Omega \left(\frac{1}{2} |\nabla p|^2 - \frac{1}{\varepsilon} \chi_\varepsilon^* \cdot \nabla p \right) dx, \tag{2.44}$$

where

$$U_2 = \left\{ p \in H^1(\Omega); \int_\Omega p dx = 0 \right\} = V. \tag{2.45}$$

Note that the functional $\Phi(p)$ has an infimum d since

$$-\frac{4}{\varepsilon^2}\left\|\chi_\varepsilon^*\right\|_{(L^2(\Omega))^N}^2 \leq \Phi(p)$$

and so a minimizing sequence satisfies

$$\|p_n\|_{H^1(\Omega)}^2 \leq d+1+\frac{4}{\varepsilon^2}\left\|\chi_\varepsilon^*\right\|_{(L^2(\Omega))^N}^2.$$

Hence $p_n \to p^*$ weakly in V and so at limit $\Phi(p^*) = d$.

As regards equation (2.33) where u_ε^* is fixed in $L^2(\Omega)$ we consider the minimization problem

$$\min_{y\in U_3} \Psi(y) = \int_\Omega \left(j(\nabla y) + \lambda\varepsilon \left(\frac{1}{2}y^2 - y_o y \right) - u_\varepsilon^* y \right) dx, \tag{2.46}$$

where

$$U_3 = \left\{ y \in L^2(\Omega);\ \nabla y \in L^p(\Omega) \right\}. \tag{2.47}$$

It is obvious that a minimizing sequence $\{y_n\}_n$ in bounded in $L^2(\Omega)$, $\{\nabla y_n\}_n$ is bounded in $(L^p(\Omega))^N$, so that $y_n \to y^*$ strongly in $L^q(\Omega)$. Also, $\{j(\nabla y_n)\}_n$ is bounded in $L^1(\Omega)$ and the weakly lower semicontinuity of j ensures the existence of the minimum y^*. Then, taking a variation of y^* along a direction $\psi \in L^2(\Omega)$ with $\nabla\psi \in (L^p(\Omega))^N$ we obtain, by a similar calculus as before, the equation

$$\int_\Omega (\xi^* \cdot \nabla\psi + (\lambda\varepsilon(y^* - y_o) - u_\varepsilon^*)\psi)\, dx = 0,$$

where $\xi^* \in \beta(\nabla y^*)$ a.e. on Ω, which is in fact the definition of the weak solution to (2.33). □

We note that constant C_ε^* in (2.31) can be easier determined in the case when $q' = 2$, from the condition $\int_\Omega u_\varepsilon^* dx = 0$, that is

$$C_\varepsilon^* = \frac{1}{\text{meas}(\Omega)} \int_\Omega \left(p_\varepsilon - \frac{1}{\varepsilon} y_\varepsilon^* \right) dx. \tag{2.48}$$

In the case when $q' > 2$, C_ε^* could be determined numerically.

Remark 2.8. Similarly, one can solve the problem

$$\min_{(u,y)\in\mathcal{U}} \left(J(u,y) = \int_\Omega \left(\frac{|u(x)|^{q'}}{q'} + \frac{\sigma}{m} |y(x) - y_o(x)|^m \right) dx \right) \tag{P}$$

subject to (1.2), where

$$\mathcal{U} = \{(u,y);\ u \in L^{q'}(\Omega),\ \int_\Omega u(x)dx = 0,\ y \in L^m(\Omega),\ \nabla y \in (L^p(\Omega))^N\},$$

for $q' \geq 2$ and $m \geq 2$. In this case, if $(u,y) \in U$, then $y \in W^{1,\underline{m}}(\Omega)$, with $\underline{m} = \min\{m,p\}$. A modification occurs in the first equation in (2.33) which becomes

$$-\nabla \cdot \beta(\nabla y_\varepsilon^*) + \lambda\varepsilon \left| y_\varepsilon^* - y_o \right|^{m-2} (y_\varepsilon^* - y_o) \ni u_\varepsilon^* \text{ in } \Omega.$$

Remark 2.9. We mention that relations (2.31)-(2.34) can be effectively used to compute an approximating optimal state $(u_\varepsilon^*, y_\varepsilon^*, z_\varepsilon^*)$ in (P_ε) by an iterative steepest descent gradient algorithm, similar with that one provided in [3] for another case. According to the steepest descent formula (see [1]) we have iteratively (e.g., for single-valued β and β^{-1})

$$p^{n+1} = p^n + \rho^n \left(\Delta p^n - \frac{1}{\varepsilon}\beta^{-1}(\nabla z^n) \right), \tag{2.49}$$

$$y^{n+1} = y^n + \rho^n(\nabla \cdot \beta(\nabla y^n) - \varepsilon\lambda(y^n - y^{obs}) + u^n)(x_1, x_2), \tag{2.50}$$

where $n \in \{0, ..., \overline{N}\}$ is the number of iterations. Relations (2.49) and (2.50) are written relying on the fact that p and y are the solutions to problems (2.44) and (2.46), respectively. A possible choice for the step ρ^n in (2.49) is given by

$$\Phi(p^n + \rho^n w^n) = \min \{\Phi(p^n + \rho w^n);\ \rho \geq 0\} \tag{2.51}$$

and similarly for ρ^n in (2.50) by replacing Φ by Ψ. Also, we add the relations

$$u^n = \frac{1}{\varepsilon} y^n - p^n + C^n, \tag{2.52}$$

$$z^{n+1} = z^n + \rho^n(\Delta z^n + u^n) \tag{2.53}$$

implied by (2.31) and (2.34). (We did not indicate the superscript * and the subscript $_\varepsilon$ for the control and states.) For all equations, finite differences can be written and the algorithm can be constructed in an iterative way, imposing finally an appropriate stop criterion.

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