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MINIMUM RESTRAINT FUNCTIONS FOR UNBOUNDED DYNAMICS: GENERAL AND CONTROL-POLYNOMIAL SYSTEMS

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ABSTRACT. We consider an exit-time minimum problem with a running cost $l \geq 0$ and unbounded controls. The occurrence of points where $l = 0$ can be regarded as a transversality loss. Furthermore, since controls range over unbounded sets, the family of admissible trajectories may lack important compactness properties. In the first part of the paper we show that the existence of a p_0-*Minimum Restraint Function* provides not only global asymptotic controllability (despite non-transversality) but also a state-dependent upper bound for the value function (provided $p_0 > 0$). This extends to unbounded dynamics a former result which heavily relied on the compactness of the control set.

In the second part of the paper we apply the general result to the case when the system is *polynomial* in the control variable. Some elementary, algebraic, properties of the convex hull of vector-valued polynomials' ranges allow some simplifications of the main result, in terms of either near-affine-control systems or reduction to weak subsystems for the original dynamics.

1. Introduction

Mainly motivated by the case when the dynamics is polynomial in the control, we deal with optimal control problems of the form

$$\dot{x} = f(x,u), \qquad x(0) = z, \tag{1.1}$$

$$(x(t), u(t)) \in (\Omega \backslash \mathbf{C}) \times U, \qquad \lim_{t \to T_x^-} \mathbf{d}(x(t), \mathbf{C}) = 0, \tag{1.2}$$

$$\mathcal{I}(x,u) := \int_0^{T_x} l(x(t), u(t))\, dt, \qquad V(z) := \inf_{(x,u)} \mathcal{I}(x,u), \tag{1.3}$$

where: i) for given positive integers n, m, the *state space* Ω is an open subset of $I\!R^n$, the *controls* u range over a (possibly unbounded) subset of $U \subseteq I\!R^m$, and $\mathbf{C} \subset \Omega$ is a closed *target* with compact boundary; ii) the *current cost* $l(x,u)$ is ≥ 0 for all $(x,u) \in (\Omega \backslash \mathbf{C}) \times U$; iii) $T_x \in [0, +\infty]$ is the infimum of times needed for

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the trajectory $x(\cdot)$ to approach the target $\mathbf{C}$; and iv) $\mathbf{d}(x, \mathbf{C})$ denotes the usual (Euclidean) distance of the point x from the subeset $\mathbf{C}$.

We focus on a particular kind of Lyapunov function, called p_0-*Minimum Restraint Function* ($p_0 \geq 0$). This notion has been introduced in [14] under the extra-hypothesis that the controls range over a bounded set. The existence of a p_0-Minimum Restraint Function, besides implying global asymptotic controllability to $\mathbf{C}$, was shown to provide a continuous upper estimate for the value function V. Such an estimate is not trivial, in that the problem (here and in [14] as well) lacks what in first order PDE's is called *transversality*, which would correspond to the assumption $l(x,u) \neq 0$ for all (x,u) (as in the minimal time problem, where $l = 1$)[1]. Here, we extend the concept of p_0-Minimum Restraint Function to unbounded dynamics f. Notice that the unboundedness of f (and l) cannnot be neglected, for no *coercivity* hypotheses –roughly speaking, the fact that $u \mapsto l(x,u)$ grows suitably faster than $u \mapsto f(x,u)$– rule out the need of larger and larger velocities in a minimizing sequence.

Precisely, for a $p_0 \geq 0$ we call p_0-*Minimum Restraint Function* every continuous function

$$W : \Omega \setminus \overset{\circ}{\mathbf{C}} \to [0, +\infty[$$

whose restriction to $\Omega \setminus \mathbf{C}$ (is locally semiconcave, positive definite and proper[2], and) verifies

$$H_{l,f}(x, p_0, D^*W(x)) < 0 \quad \forall x \in \Omega \setminus \mathbf{C}, \tag{1.4}$$

where the Hamiltonian $H_{l,f}$ is defined by

$$H_{l,f}(x, p_0, p) := \inf_{u \in U} \Big\{ \langle p, f(x,u) \rangle + p_0 \, l(x,u) \Big\}. \tag{1.5}$$

The inequality (1.4) has to be interpreted as $H_{l,f}(x, p_0, p) < 0$ $\forall p \in D^*W(x)$— which includes the case $H_{l,f}(x, p_0, p) = -\infty$. The following hypothesis will be crucial:

Hypothesis A: *For every compact subset* $\mathcal{K} \subset \Omega \backslash \mathbf{C}$ *the function*

$$(\bar{l}, \bar{f})(x,u) := \frac{(l,f)}{1 + |(l,f)(x,u)|}(x,u) \tag{1.6}$$

is uniformly continuous on $\mathcal{K} \times U$.

Observe that Hypothesis **A** allows for a vast class of cost-dynamic pairs $(l,f)(x,u)$[3], including (x-dependent) polynomials in $u_1, \cdots, u_m$, $|u_1|, \cdots, |u_m|$, $|u|$, and compositions of polynomials with exponential and Lipschitz continuous functions. Let us bring forward the statement of our main result:

Theorem 1.1. *Assume Hypothesis* **A** *and let* W *be a* p_0-*Minimum Restraint Function for the problem* $(l, f, \mathbf{C})$, *for some* $p_0 \geq 0$. *Then*

(i) *system* (1.1) *is globally asymptotically controllable to* $\mathbf{C}$.

[1] But here the exit time can well be infinite.

[2] See Definition 2.2, where, as soon as $\Omega \subsetneq I\!R^n$, one also posits $W_0 \in I\!R \cup \{+\infty\}$ such that $W(\Omega \setminus \mathbf{C}) < W_0$ and $\lim_{x \to x_0} W(x) = W_0$, for every $x_0 \in \partial\Omega$.

[3] See Remark 2.1, for a bit stronger hypothesis.

Furthermore,

(ii) *if* $p_0 > 0$, *then*

$$V(z) \leq \frac{W(z)}{p_0} \qquad \forall z \in \Omega \setminus \mathbf{C}. \tag{1.7}$$

The proof of the theorem relies on a state-based time rescaling of the problem, which in turn is made possible by Hypothesis **A**. The controls of the rescaled problem (see Section 2) still range in the (possibly unbounded) set U. Yet, some compactness properties of the rescaled dynamics are of crucial importance in the construction of trajectories reaching the target at least asymptotically.

An application to the gyroscope (see Subsection 2.2) concludes Section 2: an explicit p_0-Minimum Restraint Function is provided for a minimum problem where the control is identified with the pair made by the precession and spin velocities, while the state corresponds to pair made by the nutation angle and its time-derivative.

The remaining part of the paper is devoted to problems whose dynamics can be parameterized by a u-polynomial:

$$\dot{x} = f(x,u) := f_0(x) + \sum_{i=1}^{d} \left(\sum_{\alpha \in \mathbb{N}^m,\, \alpha_1 + \cdots + \alpha_m = i} u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1,\ldots,\alpha_m}(x) \right). \tag{1.8}$$

Among applications for which the polynomial dependence is relevant let us mention Lagrangian mechanical systems, possibly with friction forces, in which inputs are identified with the derivatives of some Lagrangian coordinates. In this case $d = 2$[4]. We point out also that, in connection with the investigation of uniqueness and regularity of solutions for Hamilton-Jacobi equations, dynamics and current costs with unbounded controls and polynomial growth have been already addressed in [13], [15], by embedding the problem in a space-time problem through techniques of graph's reparameterization – see e.g. [3, 4, 5, 9, 12, 19, 18, 21]. With similar arguments (see also [11]) necessary conditions for the existence of (possibly impulsive) minima of input-polynomial optimal control problems have been studied in [8]. Furthermore, the interplay between convexity and polynomial dependence of both the dynamics and the running cost has been investigated also in [17], in connection with problems of existence of optimal solutions.

A careful investigation of elementary, algebraic properties of the convex hull $co\ f(x, I\!R^m)$ proves essential for the application of Theorem 1.1 to the polynomial case (1.8). For instance, we consider *near-control-affine* control systems, a class of control-polynomial systems where the convex hull of the dynamics can be parameterized as a control-affine system with controls in a neighborhood of the origin[5]. For instance, this is clearly false for the system $\dot{x} = f_0(x) + uf_1(x) + u^2 f_2(x)$, $u \in I\!R$, – because the origin $(0,0)$ does not belong to the the convex hull's interior of the

[4]This is clearly a consequence of the fact that the kinetic energy is a quadratic form of the velocity (see, besides Subsection 2.2, [2] and [4]).

[5]Once the convex hull of the dynamics is so nicely parameterized, relaxation arguments allow applying several well-established results for control-affine systems.

curve (u, u^2). Instead, in view of Theorem 4.3, the convex hull of the image of

$$f_0(x) + u_1 f_{1,0,0,0,0,0,0}(x) + u_1 u_3^5 f_{1,0,5,0,0,0,0}(x) + u_2^3 u_6^3 f_{0,3,0,0,0,3,0}(x) \\ + u_1 u_3^5 u_7^9 f_{1,0,5,0,0,0,9}(x),$$

$(u_1, \dots, u_7) \in I\!R^7$ does coincide with the range of

$$f_0(x) + w_1 f_{1,0,0,0,0,0,0}(x) + w_2 f_{1,0,5,0,0,0,0}(x) + w_3 f_{0,3,0,0,0,3,0}(x) + w_4 f_{1,0,5,0,0,0,9}(x),$$

$(w_1, w_2, w_3, w_4) \in I\!R^4$.

When the system is not near-control-affine (and $U = I\!R^m$), one can try to exploit *weak subsystems*: the latter are selections of the set-valued function $x \mapsto co\ f(x, I\!R^m)$. In particular, we consider the *maximal degree subsystem* and, for any λ in the m-dimensional simplex, the λ-*diagonal subsystems* (see Definition 4.9 and Subsection 4.2, respectively). The idea of utilizing *subsystems* might look counterproductive with respect to the task of finding a p_0-Minimum Restraint Function: indeed, for such a purpose, having a sufficiently large amount of available directions plays crucial. However, from a practical perspective, a diminished complexity in the dynamics might ease the guess of a p_0-Minimum Restraint Function, which would automatically be a p_0-Minimum Restraint Function for the original polynomial problem. To give the flavour of this viewpoint, let us anticipate a result (see Theorem 4.7 for details) concerning maximal degree subsystems.

Theorem 1.2. *Let the growth assumption specified in Hypothesis* $\mathbf{A}_{max}$ *below (Section 4.2) be verified. If* W *is a* p_0*-Minimum Restraint Function for the* maximal degree subsystem

$$f^{max}(x, u) := f_0(x) + \sum_{\alpha \in \mathbb{N}^m,\, \alpha_1 + \dots + \alpha_m = d} u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1 \dots \alpha_m}(x),$$

then W *is also a* p_0*-Minimum Restraint Function for the original control polynomial system*

$$f(x, u) := f_0(x) + \sum_{i=1}^{d} \left(\sum_{\alpha \in \mathbb{N}^m,\, \alpha_1 + \dots + \alpha_m = i} u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1, \dots, \alpha_m}(x) \right).$$

The paper is organized as follows. In the remaining part of the present section we provide some preliminary definitions and notation. In Section 2 we prove Theorem 1.1 and exhibit a p_0-Minimum Restraint Function for the gyroscope (see Subsection 2.2). Section 3 is entirely devoted to the proof of Theorem 3.1 which deals with a suitably rescaled problem. In Section 4 we focus on the case when the system is polynomial in the control variable. An Appendix with a technical proof concludes the paper.

1.1. Preliminary concepts and notation.

Let us gather some notational conventions as well as some basic concepts and results which will be used throughout the paper.

We are given an open set $\Omega \subset I\!R^n$ and a *target* $\mathbf{C} \subset \Omega$, which we assume to have compact boundary $\partial\mathbf{C}$. For brevity, let us use the notation $\mathbf{d}(x)$ in place of $\mathbf{d}(x, \mathbf{C})$.

Definition 1.3. We say that a path $x : [0, T_x[\to \Omega$ is admissible if

i) $0 < T_x \leq +\infty$,
ii) $x \in AC_{loc}([0, T_x[, \Omega)$,
iii) $x([0, T_x[) \subset \Omega \backslash \mathbf{C}$,
iv) $\lim_{t \to T_x^-} \mathbf{d}(x(t)) = 0$.

We call T_x the exit time of x from $\Omega \setminus \mathbf{C}$.

Notice that the limit of $x(\cdot)$ for $t \to T_x^-$ need not exist, even when $T_x < +\infty$. Of course, if the limit exists, then it belongs to the target $\mathbf{C}$.

Definition 1.4. Let $g : \Omega \times U \to I\!R^n$ be a continuous function. For every $z \in \Omega \backslash \mathbf{C}$, we will say that (x, u) is an admissible trajectory-control pair from z for the control system

$$\dot{x} = g(x, u), \quad x(0) = z \tag{1.9}$$

if

i) $x : [0, T_x[\to \Omega \setminus \mathbf{C}$ is an admissible path,
ii) $u(\cdot) \in L^\infty_{loc}([0, T_x[, U)$,
iii) $x(\cdot)$ is a Carathéodory solution[6] of (1.9) corresponding to the input u.

We shall use $\mathcal{A}_g(z)$ to denote the family of admissible trajectory-control pairs from z for the control system (1.9).

As customary, we shall use $\mathcal{K}L$ to denote the set of all continuous functions

$$\beta : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$$

such that: (1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r, \cdot)$ is decreasing for each $r \geq 0$; (3) $\beta(r, t) \to 0$ as $t \to +\infty$ for each $r \geq 0$.

Definition 1.5. The system (1.9) is *globally asymptotically controllable to* $\mathbf{C}$ – shortly, (1.9) is *GAC to* $\mathbf{C}$ – provided there is a function $\beta \in \mathcal{K}L$ such that, for each initial state $z \in \Omega \setminus \mathbf{C}$, there exists an admissible trajectory-control pair $(x, u) \in \mathcal{A}_g(z)$ that verifies

$$\mathbf{d}(x(t)) \leq \beta(\mathbf{d}(z), t) \qquad \forall t \in [0, +\infty[. \tag{1.10}$$ [7]

Definition 1.6 (Positive definite and proper functions)**.** Let $\mathbf{E}$, $\Theta \subset I\!R^n$ be, respectively, a closed and an open set with $\mathbf{E} \subset \Theta$ and let $F : \Theta \setminus \overset{\circ}{\mathbf{E}} \to I\!R$ be a continuous function. Then F is *positive definite on* $\Theta \setminus \mathbf{E}$ if $F(x) > 0$ for all $x \in \Theta \setminus \mathbf{E}$ and $F(x) = 0$ for all $x \in \partial\mathbf{E}$.

[6] Notice that such a solution might be not unique.

[7] By convention, we fix an arbitrary $\bar{z} \in \partial\mathbf{C}$ and formally establish that, if $T_x < +\infty$, the trajectory $x(\cdot)$ is prolonged to $[0, +\infty[$, by setting $x(t) = \bar{z}$ for all $t \geq T_x$.

The function F is called *proper on* $\Theta \setminus \mathbf{E}$ if the pre-image $F^{-1}(K)$ of any compact set $K \subset [0, +\infty)$ is compact.

Definition 1.7 (Semiconcave functions)**.** Let $\Theta \subseteq I\!R^n$. A continuous function $F : \Theta \to I\!R$ is said to be semiconcave on Θ if

$$F(z_1) + F(z_2) - 2F\left(\frac{z_1 + z_2}{2}\right) \leq \rho |z_1 - z_2|^2,$$

for all z_1, $z_2 \in \Theta$ such that $[z_1, z_2] \subseteq \Theta$. F is said to be locally semiconcave on Θ if it semiconcave on every compact subset of Θ.

We remind that locally semiconcave functions are locally Lipschitz continuous.

Definition 1.8 (Limiting gradient)**.** Let $\Theta \subset I\!R^n$ be an open set and let $F : \Theta \to I\!R$ be a locally Lipschitz function. For every $x \in \Theta$ we set

$$D^*F(x) := \Big\{ w \in I\!R^n \mid \quad w = \lim_k \nabla F(x_k), \quad x_k \in DIFF(F) \setminus \{x\}, \quad \lim_k x_k = x \Big\}$$

where ∇ denotes the classical gradient operator and $DIFF(F)$ is the set of differentiability points of F. $D^*F(x)$ is called the *set of limiting gradients* of F at x.

Remark 1.9. The set-valued map $x \mapsto D^*F(x)$ is upper semicontinuous on Θ, with non-empty, compact values. Notice that $D^*F(x)$ is not convex. When F is a locally semiconcave function, D^*F coincides with the limiting subdifferential $\partial_L F$, namely,

$$D^*F(x) = \partial_L F(x) := \{\lim p_i : \ p_i \in \partial_P F(x_i), \ \lim x_i = x\} \quad \forall x \in \Theta,$$

where $\partial_P F$ denotes the proximal subdifferential, largely used in the literature on Lyapunov functions.

Basic properties of the semiconcave functions imply the following fact:

Lemma 1.10. *Let $\Theta \subset I\!R^n$ be an open set and let $F : \Theta \to I\!R$ be a locally semiconcave function. Then for any compact set $\mathcal{K} \subset \Theta$ there exist some positive constants L and ρ such that, for any $x \in \mathcal{K}$*[8]*,*

$$\begin{aligned} & F(\hat{x}) - F(x) \leq \langle p, \hat{x} - x \rangle + \rho |\hat{x} - x|^2, \\ & |p| \leq L \quad \forall p \in D^*F(x), \end{aligned} \tag{1.11}$$

for any point $\hat{x} \in \mathcal{K}$ such that $[x, \hat{x}] \subset \mathcal{K}$.

[8]The inequality (1.11) is usually formulated with the proximal superdifferential $\partial^P F$. However, this does not make a difference here since $\partial^P F = \partial_C F = coD^*F$ as soon as F is locally semiconcave. Hence (1.11) is true in particular for D^*F.

2. p_0-Minimum restraint functions

2.1. **The main result.**

Let us begin with a precise formulation of the minimum problem. For every initial condition $z \in \Omega \setminus \mathbf{C}$, we consider the control system

$$\dot{x} = f(x,u), \quad x(0) = z, \tag{2.1}$$

and, for any admissible trajectory-control pair $(x,u) \in \mathcal{A}_f(z)$ (see Definition 1.4), let us introduce the payoff

$$\mathcal{I}(x,u) := \int_0^{T_x} l(x(t),u(t))\,dt \qquad (T \in]0,+\infty]). \tag{2.2}$$

The corresponding *value function* is given by

$$V(z) \;=\; \inf_{(x,u)\in\mathcal{A}_f(z)} \mathcal{I}(x,u) \quad (\leq +\infty). \tag{2.3}$$

Recall our principal hypothesis:

Hypothesis A: *For every compact subset $\mathcal{K} \subset \Omega\backslash\mathbf{C}$ the function*

$$(\bar{l},\bar{f})(x,u) := \frac{(l,f)}{1+|(l,f)(x,u)|}(x,u) \tag{2.4}$$

is uniformly continuous on $\mathcal{K} \times U$.

Remark 2.1. As observed in the Introduction, this hypothesis allows for a wide set of *unbounded* dynamics and running costs. Furthermore, it is easy to check that the following condition *is sufficient for Hypothesis* **A** to hold true:

> *The map (l,f) is continuous with respect to the state variable x and locally Lipschitz with respect to the control variable u, and*
>
> $$\left|\frac{D_u(l,f)}{(1+|(l,f)|)^2}\right|(x,u) \leq \eta(x) \qquad \textit{for a.e. } (x,u) \in (\Omega\backslash\mathbf{C}) \times U,$$
>
> *for some continuous function $\eta:\ \Omega\backslash\mathbf{C} \to\ [0,+\infty[$.*

Let us extend the definition of p_0-Minimum Restraint Function ([14]) to the case of unbounded control sets.

Definition 2.2. Let $W : \Omega \setminus \overset{\circ}{\mathbf{C}} \to [0,+\infty[$ be a continuous function, and let us assume that W is locally semiconcave, positive definite, and proper on $\Omega \setminus \mathbf{C}$. We say that W is a *p_0-Minimum Restraint Function –in short, p_0-MRF– for $(l,f,\mathbf{C})$ in Ω for some $p_0 \geq 0$* if

$$H_{l,f}(x,p_0,D^*W(x)) < 0 \quad \forall x \in \Omega \setminus \mathbf{C} \quad [9] \tag{2.5}$$

[9] This means that $H_{l,f}(x,p_0,p) < 0$ for every $p \in D^*W(x)$.

and, moreover, there exists $W_0 \in [0, +\infty]$, such that

$$W(\Omega \setminus \mathbf{C}) < W_0 \quad \text{and} \quad \lim_{x \to x_0,\ x \in \Omega} W(x) = W_0$$

for every $x_0 \in \partial\Omega$.

We can now state our main result:

Theorem 1.1. *Assume Hypothesis* $\mathbf{A}$ *and let* W *be a* p_0*-Minimum Restraint Function for the problem* $(l, f, \mathbf{C})$*, for some* $p_0 \geq 0$*. Then:*

(i) *system* (2.1) *is globally asymptotically controllable to* $\mathbf{C}$*;*

(ii) *if* $p_0 > 0$*, then*

$$V(z) \leq \frac{W(z)}{p_0} \qquad \forall z \in \Omega \setminus \mathbf{C}. \tag{2.6}$$

Proof. We begin with a state-based rescaling procedure. Precisely, we consider the optimal control problem

$$\begin{aligned} &y'(s) = \bar{f}(y, v) \quad y(0) = z; \\ &\mathcal{I}_{\bar{l},\bar{f}}(y, v) := \int_0^{S_y} \bar{l}(y(s), v(s))ds, \qquad \bar{V}(z) := \inf_{(y,v) \in \mathcal{A}_{\bar{f}}(z)} \mathcal{I}_{\bar{l},\bar{f}}(y, v), \end{aligned} \tag{2.7}$$

where $\bar{l}$, $\bar{f}$ are defined in (2.4), the apex denotes differentiation with respect to the parameter s, and $S_y \leq +\infty$ is the exit time of the admissible trajectory $y(\cdot)$ (in the time parameter s).

The connection between the original optimal control problem and the rescaled one is established by the following result.

Claim 2.1. The path (y, v) is an admissible trajectory-control pair for (2.7) if and only if, setting

$$\begin{aligned} &t(s) := \int_0^s (1 + |(l, f)(y(\eta), v(\eta))|)^{-1} d\eta \qquad \forall s \in [0, S_y[\\ &x(t) := y \circ s(t) \qquad u(t) := v \circ s(t) \qquad \forall t \in [0, T_x[, \quad T_x := t(S_y), \end{aligned}$$

the path (x, u) is an admissible trajectory-control pair for (2.1)–(2.3). Furthermore,

$$\int_0^{S_y} \bar{l}(y(s), v(s))ds = \int_0^{T_x} l(x(t), u(t))dt.$$

In particular, one has

$$V(z) = \bar{V}(z)$$

for all $z \in \Omega \setminus \mathbf{C}$.

Indeed, since $t = t(s)$ is absolutely continuous and $t'(s) > 0$ almost everywhere, the inverse map $s(\cdot) = t^{-1}(\cdot)$ is absolutely continuous (see e.g. [16, Theorem 4, page 253] or, for a more general statement, [7, Theorem 2.10.13, page 177]). In particular, $x = y \circ s$ is absolutely continuous, and $u = v \circ s$ turns out to be Borel measurable as well. Hence the claim follows by a standard application of the chain rule[10].

The Hamiltonian $H_{\bar{l},\bar{f}}$ associated to $\bar{l}$, $\bar{f}$,

$$H_{\bar{l},\bar{f}}(x,p_0,p) := \inf_{u\in U}\left\{\langle p,\bar{f}(x,u)\rangle + p_0\,\bar{l}(x,u)\right\}$$

for all $(x,p_0,p)\in(\Omega\backslash\mathbf{C})\times I\!R^{1+n}$, is continuous and sublinear in (p_0,p), uniformly with respect to x. Furthermore, it is also trivial to check that, for every $(x,p_0,p)\in(\Omega\setminus\mathbf{C})\times I\!R^{1+n}$,

$$H_{\bar{l},\bar{f}}(x,p_0,p)<0 \quad\Longleftrightarrow\quad H_{l,f}(x,p_0,p)<0. \tag{2.8}$$

In particular, for every $p_0\geq 0$ W is a p_0-MRF for $(l,f,\mathbf{C})$ if and only if W is a p_0-MRF for $(\bar{l},\bar{f},\mathbf{C})$. Moreover, because of Hypothesis **A**, the problem $(\bar{l},\bar{f},\mathbf{C})$ meets the hypotheses of Theorem 3.1 below. Therefore:

(i) *if there exists a p_0-MRF W for $(l,f,\mathbf{C})$, then the rescaled system in* (2.7) *is GAC to $\mathbf{C}$, i.e. there exists a function $\beta\in\mathcal{KL}$ such that for any $z\in\Omega\setminus\mathbf{C}$ there is an admissible trajectory-control pair $(y,v)\in\mathcal{A}_{\bar{f}}(z)$ that verifies*

$$\mathbf{d}(y(s))\leq\beta(\mathbf{d}(z),s) \qquad \forall s\in[0,+\infty[; \tag{2.9}$$

(ii) *moreover, if $p_0>0$, then*

$$\bar{V}(z)\leq\frac{W(z)}{p_0}. \tag{2.10}$$

If $x(\cdot)$ is the trajectory defined in Claim 2.1, one then obtains

$$\mathbf{d}(x(t))\leq\beta(\mathbf{d}(z),s(t)) \qquad \forall t\in[0,+\infty[\tag{2.11}$$

and, if $p_0>0$,

$$V(z)\leq\frac{W(z)}{p_0} \qquad \forall z\in\Omega\setminus\mathbf{C}. \tag{2.12}$$

Notice that $t(s)\leq s$ for all s, so that $t\leq s(t)$ for all t. Since the map $\beta(z,\cdot)$ is decreasing, one gets

$$\beta(z,s(t))\leq\beta(z,t)$$

for all t. It follows by (2.9) that

$$\mathbf{d}(x(t))\leq\beta(\mathbf{d}(z),t) \qquad \forall t\in[0,+\infty[, \tag{2.13}$$

so the theorem is proved. □

We conclude this section with an application of Theorem 1.1 to Mechanics.

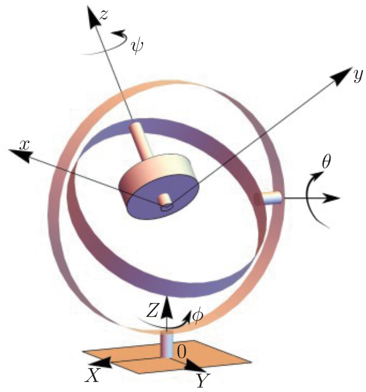

FIGURE 1

2.2. The gyroscope: controlling the nutation through precession and spin.

A gyroscope can be represented as a mechanism composed by a rotor –in our setting a spinning disk– and two gimbals. The spin axis of the rotor is fixed to the inner gimbal, whose spin axis is fixed to the outer gimbal (see Figure 1).

Besides an inertial reference frame $OXYZ$ we consider a reference frame $oxyz$ fixed to the rotor. In particular, we choose the latter reference so that the centre of mass of the rotor has coordinates $(0, 0, z_G)$. The motion of the rotor can be parametrized by Euler angles as depicted in Figure 1: the outer gimbal's position is represented by the *precession* angle ϕ, the inner gimbal's position is given by the *nutation* angle θ, and the rotor's position is measured by the *spin* angle ψ. The kinetic energy (in the inertial frame) is so given by

$$\mathcal{T} = \frac{1}{2} I_0(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I(\dot{\phi} \cos \theta + \dot{\psi})^2,$$

where I_0 is the moment of inertia of the rotor with respect to any axis through o and orthogonal to z[11] and I is the moment of inertia of the rotor about its spin axis oz. We have tacitly assumed that the rotor's mass M is the only non-negligible mass of the system. For simplicity, we also suppose $I_0 = I$. If g denotes the gravitational acceleration, the potential energy $\mathcal{V}$ is given by

$$\mathcal{V}(\theta) := M g z_G \cos \theta \quad \forall \theta \in [-\pi/2, \pi/2].$$

We will regard the precession velocity $\dot{\phi}$ and the spin velocity $\dot{\psi}$ as *controls* belonging to $U = I\!R^2$. Considering the predetermination of $\phi(\cdot)$ and $\psi(\cdot)$ as a holonomic constraint, we assume the classical D'Alembert hypothesis (see [2]).

[10]Notice that the solutions to $\dot{x} = f$ or $\dot{y} = \bar{f}$ are not necessarily unique.

[11]All these moments coincide because of the symmetry of the rotor.

The resulting control mechanical system is

$$\begin{cases} \dot{\theta} = \dfrac{1}{I}\pi_\theta \\ \dot{\pi}_\theta = Mgz_G \sin\theta - I\sin\theta\dot{\phi}\dot{\psi}, \end{cases} \tag{2.14}$$

where π_θ is the conjugate momentum $\pi_\theta := \frac{\partial(\mathcal{T}+\mathcal{V})}{\partial\dot{\theta}} = I\,\dot{\theta}$.

If we set $u := (\dot{\phi}, \dot{\psi})$, $x = (x_1, x_2)^{tr} := (\theta, \pi_\theta)$, $f_0(x) = (I^{-1}x_2, Mgz_G \sin x_1)^{tr}$, and $f_{11}(x) = (0, -I\sin x_1)^{tr}$ we obtain the control-quadratic control system

$$\dot{x} = f(x,u) := f_0(x) + u_1u_2 f_{11}(x), \tag{2.15}$$

with $(u_1, u_2) \in I\!R^2$. The state space of the control system (2.15) is the open set $\Omega =]-\pi/2, \pi/2[\times I\!R$ and we choose $\mathbf{C} = \{(0,0)\}$ as a target and $l(x_1, x_2) = x_2^2$ as a running cost .

Let us set

$$W(x_1,x_2) := W_1(x_1,x_2)(2 - |W_2(x_1,x_2)|),$$

where

$$W_1(x_1,x_2) := \tan^2 x_1 + x_2^2,$$

$$W_2(x_1,x_2) := \begin{cases} \sin\left(2\text{arctan}\left(\frac{-\tan x_1 + \sqrt{3}x_2}{\sqrt{3}\tan x_1 + x_2}\right)\right) & \text{if } x_2 \neq -\sqrt{3}\tan x_1 \\ 0 & \text{otherwise .} \end{cases}$$

With some computation, one proves that

Claim 2.2. For any $p_0 < \min\{1/I, 8\sqrt{3}/3\}$, the function W is p_0-MRF for the problem $(f, l, \mathbf{C})$.

Therefore, by Theorem 1.1 we can conclude that the control system for the nutation θ and its conjugate moment π_θ is *GAC* to the origin. In addition, the optimal value V of the minimum problem with running cost equal to π_θ^2 $(= I^2\dot{\theta}^2)$ verifies

$$V(\bar{\theta}, \bar{\pi}_\theta) \leq \frac{W(\bar{\theta}, \bar{\pi}_\theta)}{p_0}$$

for all initial data $(\bar{\theta}, \bar{\pi}_\theta)$ and $p_0 < \min\{1/I, 8\sqrt{3}/3\}$. Notice that, as it might be expected, the larger the moment of inertia I is, the larger is the provided bound for V.

3. The rescaled problem

The main step of the proof of Theorem 1.1 is based on Theorem 3.1 below, which concerns GAC and optimization for a cost-dynamics pair $(\mathbf{l}, \mathbf{f})$ verifying the following boundedness and uniform continuity hypothesis:

Hypothesis $\mathbf{A}_{UC}$ *The vector field* $(\mathbf{l}, \mathbf{f})$ *is continuous on* $(\Omega\backslash\mathbf{C}) \times U$ *and, for every compact subset* $\mathcal{K} \subset \Omega\backslash\mathbf{C}$*, it is bounded and uniformly continuous on* $\mathcal{K} \times U$*.*

We point out that the control set U is still allowed to be unbounded.

Let us consider the exit time optimal control problem

$$y' = \mathbf{f}(y, v), \quad y(0) = z, \tag{3.1}$$

$$\mathbf{V}(z) := \inf_{(y,v)\in\mathcal{A}_{\mathbf{f}}(z)} \int_0^{T_y} \mathbf{l}(y(t), v(t))dt. \tag{3.2}$$

Theorem 3.1. *Let us assume Hypothesis $\mathbf{A}_{UC}$, and let W be a p_0-Minimum Restraint Function for the problem $(\mathbf{l}, \mathbf{f}, \mathbf{C})$. Then:*

(i) *system* (3.1) *is GAC to* $\mathbf{C}$*;*

(ii) *moreover, if* $p_0 > 0$,

$$\mathbf{V}(z) \le \frac{W(z)}{p_0} \qquad \forall z \in \Omega \setminus \mathbf{C}. \tag{3.3}$$

3.1. **Preliminary results.**

The proof of Theorem 3.1 relies on Propositions 3.2, 3.3, and 3.5 below. Hypothesis $\mathbf{A}_{UC}$ is used throughout the whole subsection.

Proposition 3.2. *For every $\sigma > 0$ there exists a continuous, increasing map γ : $]0, 2\sigma] \to]0, +\infty[$ such that, for every $r \in]0, 2\sigma]$,*

$$H_{\mathbf{l},\mathbf{f}}(x, p_0, D^*W(x)) < -\gamma(r) \qquad \forall x \in W^{-1}([r, 2\sigma]) \text{ and } p \in D^*W(x). \tag{3.4}$$

This result is a consequence of the upper semicontinuity of the set-valued map $x \to D^*W(x)$ together with the continuity of $(x, p) \mapsto H_{\mathbf{l},\mathbf{f}}$, when the latter is restricted to the sets $W^{-1}([r, 2\sigma]) \times I\!R^n$ (for the details, see [14, Proposition 3.1]).

Proposition 3.3. *For a given $\sigma > 0$, let $\gamma(\cdot)$ be a map as in Proposition 3.2. Then there exists a continuous, decreasing function N :$]0, 2\sigma] \to]0, +\infty[$ such that, setting*

$$H_{\mathbf{l},\mathbf{f},N(r)}(x, p_0, p) := \min_{u\in U\cap B(0,N(r))} \Big\{ \langle p, \mathbf{f}(x, u)\rangle + p_0\mathbf{l}(x, u) \Big\} \quad \forall r \in]0, 2\sigma],$$

we get

$$H_{\mathbf{l},\mathbf{f},N(W(x))}(x, p_0, D^*W(x)) < -\gamma(W(x)) \qquad \forall x \in W^{-1}(]0, 2\sigma]). \tag{3.5}$$

Proof. Given $r \in]0, 2\sigma]$, let us first show that there exists some $N(r)$ such that

$$H_{\mathbf{l},\mathbf{f},N(r)}(x, p_0, D^*W(x)) < -\gamma(r) < 0 \qquad \forall x \in W^{-1}([r, 2\sigma]) \text{ and } p \in D^*W(x). \tag{3.6}$$

Assume by contradiction that for any integer k there is some pair (x_k, p_k) with $x_k \in W^{-1}([r, 2\sigma])$ and $p_k \in D^*W(x_k)$ such that,

$$\Big(u \in U : \quad \langle p_k, \mathbf{f}(x_k, u)\rangle + p_0\mathbf{l}(x_k, u) < -\gamma(r) < 0\Big) \implies |u| > k \tag{3.7}$$

(by Proposition 3.2, controls verifying the inequality surely exist). Because of the compactness of $W^{-1}([r, 2\sigma])$ and of the upper semicontinuity of the set-valued map $D^*W(\cdot)$, there is a subsequence, which we still denote (x_k, p_k), converging to some $(\bar{x}, \bar{p})$ such that $\bar{x} \in W^{-1}([r, 2\sigma])$ and $\bar{p} \in D^*W(\bar{x})$. Since W verifies (3.4), there is some $\bar{u} \in U$ such that

$$\alpha := \langle \bar{p}, \mathbf{f}(\bar{x}, \bar{u})\rangle + p_0\mathbf{l}(\bar{x}, \bar{u}) < -\gamma(r) < 0.$$

Thus, the uniform continuity of the maps $\mathbf{l}$, $\mathbf{f}$ on $W^{-1}([r,2\sigma])\times U$ implies that

$$\langle(p_k,\mathbf{f}(x_k,\bar u)\rangle + p_0\mathbf{l}(x_k,\bar u)+\gamma(r) < \frac{\alpha}{2} < 0 \qquad \forall k\geq \bar k,$$

some integer $\bar k$, which contradicts (3.7) as soon as $k>|\bar u|$.
Moreover, for every $r_1, r_2 \in]0,2\sigma]$, $r_1<r_2$, one clearly has $N(r_1)\geq N(r_2)$ and, enlarging $N(r)$ if necessary, one can assume the map $r\mapsto N(r)$ continuous. Therefore, for any $x\in W^{-1}(]0,2\sigma])$, the thesis (3.5) follows from (3.6) as soon as $r=W(x)$. □

Let us introduce the following definition, useful in the sequel.

Definition 3.4. Let $\sigma>0$ and fix a selection $p(x)\in D^*W(x)$ for any $x\in W^{-1}(]0,2\sigma])$. Let $\gamma(\cdot)$, $N(\cdot)$ be the same as in Proposition 3.3. We call a feedback on $W^{-1}(]0,2\sigma])$ a map

$$x\mapsto \mathbf{u}(x)\in U\cap B(0,N(W(x))$$

verifying

$$\langle p(x),\mathbf{f}(x,\mathbf{u}(x))\rangle + p_0\mathbf{l}(x,\mathbf{u}(x)) < -\gamma(W(x)) \tag{3.8}$$

for every $x\in W^{-1}(]0,2\sigma])$.

Moreover, for any $\mu>0$ and any continuous path $\tilde y:[\tau,+\infty[\to I\!R^n$ such that $W(\tilde y(\tau))>\mu$, we define the time to reach the enlarged target $W^{-1}([0,\mu])$ as

$$\mathcal{T}^{\mu}_{\tilde y} := \inf\{r\geq\tau:\ W(\tilde y(r))\leq\mu\} \tag{3.9}$$

(in particular, $\mathcal{T}^{\mu}_{\tilde y}=+\infty$ if $W(\tilde y(r))>\mu$ for all $r\geq\tau$).

Proposition 3.5. *Fix $\sigma\in]0,W_0[$, and let $\gamma(\cdot)$, $N(\cdot)$ be as in Propositions 3.2, 3.3. Moreover, let ε, $\bar\mu$, $\hat\mu$ verify $\varepsilon>0$ and $0<\hat\mu<\bar\mu\leq\sigma$. Then there exists some $\delta>0$ such that, for every partition $\pi=(t^j)$ of $[0,+\infty[$ with $diam(\pi)\leq\delta$[12] and for each $x\in\Omega\setminus\mathbf{C}$ satisfying $W(x)=\bar\mu$, there are a piecewise constant control $v:[0,\hat t]\to U\cap B(0,N(\hat\mu))$ and a solution $y:[0,\hat t]\to W^{-1}([\hat\mu,\bar\mu])$ to the Cauchy problem*

$$y'=\mathbf{f}(y,v),\qquad y(0)=x,$$

enjoying following properties:

- **(a)** *$\hat t:=\mathcal{T}^{\hat\mu}_{y}<+\infty$ and $\bar n:=\sup\{j\geq1: t^{j-1}<\mathcal{T}^{\hat\mu}_{y}\}<+\infty$.*
- **(b)** *for every $t\in[0,\hat t[$ and $j\geq1$ such that $t\in[t^{j-1},t^j[$,*

$$W(y(t))-W(y(t^{j-1}))+p_0\int_{t^{j-1}}^{t}\mathbf{l}(y(\tau),v(\tau))\,d\tau \leq -\frac{\gamma(W(y(t^{j-1})))}{\varepsilon+1}(t-t^{j-1}). \tag{3.10}$$

Proof. Let $p(\cdot)$ be a selection of D^*W on $W^{-1}([\hat\mu/4,2\sigma])$ and let us consider a feedback $\mathbf{u}$ as in Definition 3.4. Let M denote the sup-norm of $\mathbf{f}$ on $W^{-1}([\hat\mu/4,2\sigma])\times U$, and let $\omega_{\mathbf{l}}(\cdot)$ be the modulus of continuity of $\mathbf{l}$ on $W^{-1}([\hat\mu/4,2\sigma])\times U$. By the local semiconcavity and the properness of W, Lemma 1.10 implies that there exist

[12] A *partition* of $[0,+\infty[$ is a sequence $\pi=(t^j)$ such that $t^0=0$, $t^{j-1}<t^j$ $\forall j\geq1$, and $\lim_{j\to+\infty}t^j=+\infty$. The number $\mathrm{diam}(\pi)\doteq\sup(t^j-t^{j-1})$ is called the diameter of the sequence π.

ρ, $L > 0$ such that, for any x belonging to the compact set $W^{-1}([\hat{\mu}/4, 2\sigma])$, one has[13]

$$W(\hat{x}) - W(x) \leq \langle p, \hat{x} - x \rangle + \rho\, |\hat{x} - x|^2 \qquad \forall p \in D^* W(x), \tag{3.11}$$

for every $\hat{x}$ such that the segment $[x, \hat{x}] \subset W^{-1}([\hat{\mu}/4, 2\sigma])$, and

$$|p| \leq L \qquad \forall p \in D^* W(x). \tag{3.12}$$

Let $\psi : I\!\!R^n \to [0,1]$ be a C^∞ (cut-off) map such that

$$\psi = 1 \quad \text{on} \quad W^{-1}([\hat{\mu}/2, \sigma]), \qquad \psi = 0 \quad \text{on} \quad I\!\!R^n \backslash W^{-1}([\hat{\mu}/4, 2\sigma])\,. \tag{3.13}$$

Let ω denote the modulus of continuity of the product $(\psi\, \mathbf{f})$ on $I\!\!R^n \times U$.
We set

$$\delta := \min\left\{ \frac{\hat{\mu}}{2LM}, \delta_2 \right\}, \tag{3.14}$$

where $\delta_2 > 0$ verifies

$$\frac{L\, \omega\,(M\, \delta_2) + \rho\, M^2\, \delta_2 + p_0\, \omega_{\mathbf{l}}\,(M\, \delta_2)}{\gamma(\hat{\mu}/4)} = \frac{\varepsilon}{\varepsilon + 1}. \tag{3.15}$$

Let $\pi = (t^j)$ be an arbitrary partition of $[0, +\infty[$ such that $\operatorname{diam}(\pi) \leq \delta$. For each $x \in \Omega \setminus \mathbf{C}$ verifying $U(x) = \bar{\mu}$, define recursively a sequence of trajectory-control pairs $(y^j, v^j) : [t^{j-1}, t^j] \to \Omega \times U$, $j \geq 1$, as follows:

- $y^1(t^0) := x^1 := x\,,\quad v^1 := \mathbf{u}(x^1)$;
- for every $j > 1$,
$$y^j(t^{j-1}) := y^{j-1}(t^{j-1}) := x^j\,, \quad v^j := \mathbf{u}(x^j);$$
- for every $j \geq 1$, $y^j : [t^{j-1}, t^j] \to I\!\!R^n$ is a solution of the Cauchy problem
$$y'(t) = \psi(y)\, \mathbf{f}(y, v^j) \quad y(t^{j-1}) = x^j.$$

Notice that, by the continuity of the vector field and because of the cut-off factor ψ, any trajectory $y^j(\cdot)$ exists globally and cannot exit the compact subset $W^{-1}([\hat{\mu}/4, 2\sigma])$. Let us set

$$(y(t), v(t)) := (y^j(t), v^j) \ \ \forall t \in [t^{j-1}, t^j[, \quad \text{for every } j \geq 1.$$

In view of the L-Lipschitz continuity of W on $W^{-1}([\hat{\mu}/4, 2\sigma])$, the condition $\delta \leq \hat{\mu}/2LM$ in (3.14), implies that $|W(y^j(t)) - W(x^j)| \leq L|y^j(t) - x^j| \leq \hat{\mu}/2$, so that

$$W(y^j(t)) \geq \hat{\mu}/2 \quad \forall t \in [t^{j-1}, t^j], \quad \text{for every } j \geq 1,$$

as soon as $W(x^j) \geq \hat{\mu}$.
Recalling that $|\psi| \leq 1$ and $\psi(x^j) = 1$ when $x^j \in W^{-1}([\hat{\mu}/2, 2\sigma])$, (3.8) and (3.11) and imply that, for every $j \geq 1$ such that $t^{j-1} < \mathcal{T}_y^{\hat{\mu}}$ (see Definition 3.9), one has,

[13] The inequality (3.11) is usually formulated with the proximal superdifferential $\partial^P F$ instead of $\partial_C F$. However, this does not make a difference here since $\partial^P F = \partial_C F$ as soon as F is locally semiconcave.

$\forall t \in [t^{j-1}, t^j]$,

$$\begin{aligned}
&W(y^j(t)) - W(x^j) + p_0 \int_{t^{j-1}}^{t} \mathbf{l}(y^j(\tau), v^j)\, d\tau \\
&\le \langle p(x^j), y^j(t) - x^j\rangle + \rho |y^j(t) - x^j|^2 \\
&\quad + p_0 \int_{t^{j-1}}^{t} \left[\mathbf{l}(y^j(\tau), v^j) - \mathbf{l}(x^j, v^j)\right]\, d\tau + p_0\, \mathbf{l}(x^j, v^j)(t - t^{j-1}) \\
&\le \left\langle p(x^j), \int_{t^{j-1}}^{t} \left[\psi(y^j(\tau))\, \mathbf{f}(y^j(\tau), v^j) - \mathbf{f}(x^j, v^j)\right]\, d\tau \right\rangle \\
&\quad + \rho \left(\int_{t^{j-1}}^{t} \left|\psi(y^j(\tau))\mathbf{f}(y^j(\tau), v^j)\right|\, d\tau\right)^2 + p_0\, \omega_{\mathbf{l}} \left(M\, (t^j - t^{j-1})\right)\, (t - t^{j-1}) \\
&\quad + \left\langle p(x^j), \mathbf{f}(x^j, v^j)\right\rangle\, (t - t^{j-1}) + p_0\, \mathbf{l}(x^j, v^j)(t - t^{j-1}) \\
&\le L\, \omega \left(M\, (t^j - t^{j-1})\right)\, (t - t^{j-1}) + \rho\, M^2\, (t - t^{j-1})^2 \\
&\quad + p_0\, \omega_{\mathbf{l}} \left(M\, (t^j - t^{j-1})\right)\, (t - t^{j-1}) - \gamma(W(x^j))(t - t^{j-1}) \\
&\le \left[\frac{L\, \omega \left(M\, (t^j - t^{j-1})\right) + \rho\, M^2\, (t^j - t^{j-1}) + p_0\, \omega_{\mathbf{l}} \left(M\, (t^j - t^{j-1})\right)}{\gamma(W(x^j))} - 1\right] \\
&\quad \cdot \gamma(W(x^j))(t - t^{j-1}).
\end{aligned}$$

Since $\forall t \in [t^{j-1}, t^j]$, $t - t^{j-1} \le \delta \le \delta_2$, by (3.15) it follows that

$$W(y^j(t)) - W(x^j) + p_0 \int_{t^{j-1}}^{t} \mathbf{l}(y^j(\tau), v^j)\, d\tau \le -\frac{\gamma(W(x^j))}{\varepsilon + 1}(t - t^{j-1}), \tag{3.16}$$

which implies, also recalling the definition $x^j = y^{j-1}(t^{j-1})$,

$$\begin{aligned}
&W(y(t)) - W(x) + p_0 \int_{0}^{t} \mathbf{l}(y(\tau), v(\tau))\, d\tau \\
&\qquad = [W(y^j(t)) - W(x^j)] + \cdots + [W(y^1(t^1)) - W(x)] \\
&\qquad\quad + p_0 \int_{t^{j-1}}^{t} \mathbf{l}(y^j(\tau), v^j)\, d\tau + \cdots + p_0 \int_{0}^{t^1} \mathbf{l}(y^1 j(\tau), v^1)\, d\tau \\
&\qquad \le -\frac{\gamma(W(x^j))(t - t^{j-1}) + \sum_{i=1}^{j-1} \gamma(W(x^i))(t^i - t^{i-1})}{\varepsilon + 1}.
\end{aligned} \tag{3.17}$$

In particular, (3.17) yields that $W(y(t)) \le W(x) = \bar{\mu}$ for all $t \in [0, t^j]$.

Notice that $\mathcal{T}_y^{\hat{\mu}} < +\infty$. Indeed, if by contradiction $\mathcal{T}_y^{\hat{\mu}} = +\infty$, (3.17) held true for all $t \in [0, t^j]$ with j arbitrarily large, i.e. (since (t^j) is a partition of $[0, +\infty[$), for all $t \ge 0$. Therefore, recalling that $\gamma(W(x^i)) \ge \gamma(\hat{\mu}/4) > 0$ for all $i = 1, \ldots, j$, one would have $\lim_{t \to +\infty} W(y(t)) = 0$, which is not allowed, since, by the definition of $\mathcal{T}_y^{\hat{\mu}}$,

$$W(y(t)) > \hat{\mu} \qquad \forall t \in [0, \mathcal{T}_y^{\hat{\mu}}[. \tag{3.18}$$

Let us set

$$\hat{t} := \mathcal{T}_y^{\hat{\mu}} (< +\infty),$$

so that $\bar{n}$ reads
$$\bar{n} = \sup\{j \geq 1 : t^{j-1} < \hat{t}\}.$$
Let us observe that $\bar{n} < +\infty$. Finally, notice that, because of (3.18), $\psi(y(t)) = 1$ for every $t \in [0, t^{\bar{n}}]$. Hence, for any $j \in \{1, \ldots, \bar{n}\}$, $y^j(\cdot)$ is a solution of
$$\frac{dy}{dt} = \mathbf{f}(y, v^j) \quad \forall t \in [t^{j-1}, t^j], \quad y(t^{j-1}) = x^j.$$
It follows that conditions **(a)**–**(b)** are satisfied. $\square$

3.2. **Proof of Theorem 3.1.**

Let $\sigma \in]0, W_0[$ and let $\gamma(\cdot)$, $N(\cdot)$ be defined as in Proposition 3.3. Fix $\varepsilon > 0$ and let $(\nu_k) \subset]0, 1]$ be a sequence such that $1 = \nu_0 > \nu_1 > \nu_2 > \ldots$ and $\lim_{k\to\infty} \nu_k = 0$. Assume that $z \in W^{-1}(]0, \sigma])$ and set
$$\mu_k := \nu_k W(z) \quad \forall k \geq 0.$$
We are going to exploit Proposition 3.5 in order to build a trajectory-control pair
$$(y, v) : [0, \bar{t}[\to (\Omega \setminus \mathbf{C}) \times U$$
by concatenation
$$(y(t), v(t)) = (y_k(t), v_k(t)) \quad \forall t \in [t_{k-1}, t_k[, \quad \forall k \geq 1,$$
where the pairs $(y_k(t), v_k(t))$ are described by induction as follows.

The case $k = 1$. Let us begin by constructing (y_1, v_1). Let us set $\bar{\mu} = \mu_0$, $\hat{\mu} = \mu_1$, and let us build a trajectory-control pair
$$(y_1, v_1) : [0, \hat{t}] \to W^{-1}([\mu_1, \mu_0]) \times U \cap B(0, N(\mu_1)), \qquad y_1(0) = z,$$
according to Proposition 3.5. We set $t_0 := 0$ and $t_1 := \hat{t}$ and observe that, in view of **(a)** in Proposition 3.5, $t_1 = \mathcal{T}_{y_1}^{\mu_1}$.

The case $k > 1$. Let us define (y_k, v_k) for $k > 1$. Let us set $\bar{\mu} = \mu_{k-1}$, $\hat{\mu} = \mu_k$, and construct
$$(\hat{y}_k, \hat{v}_k) : [0, \hat{t}] \to W^{-1}([\mu_k, \mu_{k-1}]) \times U \cap B(0, N(\mu_k)), \qquad \hat{y}_k(0) = y_{k-1}(t_{k-1}),$$
still according to Proposition 3.5. We set $t_k := t_{k-1} + \hat{t}$ and $(y_k, v_k)(t) = (\hat{y}_k, \hat{v}_k)(t - t_{k-1})$ $\forall t \in [t_{k-1}, t_k]$. We observe that $t_k = \mathcal{T}_{y_k}^{\mu_k}$.

The concatenation procedure is concluded as soon as we set $\bar{t} := \lim_{k\to\infty} t_k$. Notice that it may well happen that $\bar{t} = +\infty$.

We claim that
$$\lim_{t \to \bar{t}^-} \mathbf{d}(y(t)) = 0. \tag{3.19}$$
Indeed, for every $k \geq 1$, Proposition 3.5 yields the existence of a finite partition $\pi_k = \{\hat{t}_k^0, \ldots, \hat{t}_k^{\bar{n}_k}\}$ of $[0, t_k - t_{k-1}]$ such that, setting,
$$t_k^j := t_{k-1} + \hat{t}_k^j \qquad \forall j \in \{0, \ldots, \bar{n}_k\},$$
one has $y(0)\ (= y_1(0)) = z$, and, for every $k \geq 1$:

(a)$_k$ $y_{k+1}(t_k) = y_k(t_k)$, $W(y_k(t_{k-1})) = \mu_{k-1}$; and
$W(y_k(t_k)) < W(y_k(t)) \le W(y_k(t_{k-1})) \le W(z) \quad \forall t \in [t_{k-1}, t_k[$;
(b)$_k$ for all $j \in \{1, \dots, \bar{n}_k\}$,
$W(y_k(t)) - W(y_k(t_k^{j-1})) + p_0 \int_{t_k^{j-1}}^t \mathbf{l}(y_k^j(\tau), v_k(\tau))\, d\tau \le$
$-\frac{1}{\varepsilon+1}\gamma(W(y_k(t_k^{j-1})))(t - t_k^{j-1}) \quad \forall t \in [t_k^{j-1}, t_k^j[$.

In particular, by **(a)**$_k$, claim (3.19) is equivalent to

$$\lim_{k\to\infty} \mathbf{d}(y_k(t_k)) = 0. \tag{3.20}$$

Since W is proper and positive definite, (3.20) is a straightforward consequence of

$$\lim_{k\to\infty} W(y_k(t_k)) = \lim_{k\to\infty} \nu_k\, W(z) = 0,$$

so (3.19) is verified as well.

We now need precise estimates of both the decreasing rate of W and the cost gain along (y, v).

Let us consider t, k, j such that $t < \bar{t}$ and $t \in [t_k^{j-1}, t_k^j[$. Notice that **(b)**$_k$ implies

$$W(y(t)) \le W(y_k(t_k^{j-1})) \le W(y(t_{k-1})) \le \cdots \le W(y(t_1)) \le W(z) \le \sigma, \tag{3.21}$$

and, in view of the definition of (y_k, v_k), also

$$\begin{aligned}
&W(y_k(t)) - W(y_k(t_{k-1})) + p_0 \int_{t_{k-1}}^t \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau \\
&= [W(y_k(t)) - W(y_k(t_k^{j-1}))] + [W(y_k(t_k^{j-1})) - W(y_k(t_k^{j-2}))] \\
&\quad + \cdots + [W(y_k(t_k^1)) - W(y_k(t_k^0))] \\
&\quad + p_0 \int_{t_k^{j-1}}^t \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau + \cdots + p_0 \int_{t_k^0}^{t_k^1} \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau \\
&\le -\frac{1}{\varepsilon+1}\left[\gamma(W(y_k(t_k^{j-1})))(t - t_k^{j-1}) + \sum_{i=1}^{j-1} \gamma(W(y_k(t_k^{i-1})))(t_k^i - t_k^{i-1})\right].
\end{aligned}$$

By the monotonicity of γ one has $\gamma(W(y_k(t_k^{j-1}))) \le \gamma(W(y_k(t_k^{i-1})))$ for any $i = 1, \dots, j-1$, which implies

$$W(y_k(t)) - W(y_k(t_{k-1})) + p_0 \int_{t_{k-1}}^t \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau \le -\frac{1}{\varepsilon+1}\gamma(W(y_k(t_k^{j-1})))(t - t_{k-1}).$$

Hence, recalling the definition of (y, v), we have

$$\begin{aligned}
&W(y(t)) - W(z) + p_0 \int_0^t \mathbf{l}(y(\tau), v(\tau))\, d\tau \\
&= [W(y(t)) - W(y(t_{k-1}))] + [W(y(t_{k-1})) - W(y(t_{k-2}))] \\
&\quad + \cdots + [W(y(t_1)) - W(y(0)] \\
&\quad + p_0 \int_{t_{k-1}}^t \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau + \cdots + p_0 \int_0^{t_1} \mathbf{l}(y_k(\tau), v_k(\tau))\, d\tau,
\end{aligned}$$

so, by using (3.21), we finally obtain

$$W(y(t)) - W(z) + p_0 \int_0^t \mathbf{l}(y(\tau), v(\tau))\, d\tau \leq -\frac{1}{\varepsilon + 1} \gamma(W(y_k(t_k^{j-1})))t. \tag{3.22}$$

This is the key inequality for proving both claim **(i)** and claim **(ii)** of the theorem.

As for claim **(i)** –stating that the system is (GAC) to $\mathbf{C}$–, we have to establish the existence of a $\mathcal{K}L$ function β as in Definition 1.5. Let t belong to $[0, \bar{t}[$. Then $t \in [t_k^{j-1}, t_k^j[$ for some $k \geq 1$ and some $j \in \{0, \dots, \bar{n}_k\}$. Since $l \geq 0$, by (3.22) we get

$$W(y(\tau)) + \frac{\gamma(W(y(t_k^{j-1}))\, \tau}{\varepsilon + 1} \leq W(z) \qquad \forall \tau \in [t_k^{j-1}, t_k^j]. \tag{3.23}$$

Observe that the function $\tilde{\gamma} : [0, +\infty[\to [0, +\infty[$ defined by $\tilde{\gamma}(r) := \min\{r, \gamma(r)\}$ for all $r \in [0, +\infty[$ is continuous, strictly increasing, and $\tilde{\gamma}(r) > 0 \quad \forall r > 0$, $\tilde{\gamma}(0) = 0$. Then, taking $\tau = t_k^{j-1}$ in (3.23), one has

$$\tilde{\gamma}(W(y(t_k^{j-1}))\left[1 + \frac{t_k^{j-1}}{\varepsilon + 1}\right] \leq W(z),$$

so that

$$W(y(t)) \leq W(y(t_k^{j-1})) \leq \tilde{\gamma}^{-1}\left(\frac{\varepsilon + 1}{\varepsilon + 1 + t_k^{j-1}}\, W(z)\right).$$

By Proposition 3.5 it is not restrictive to assume $diam(\pi_k) \leq 1/2$. Therefore we get

$$W(y(t)) \leq \tilde{\gamma}^{-1}\left(\frac{2(\varepsilon + 1)}{\varepsilon + 1 + t}\, W(z)\right).$$

Proceeding as usual in the construction of the function β, we set

$$\sigma_-(r) := \min\{r\,,\, \min\{\mathbf{d}(x) :\ W(x) \geq r\}\}, \quad \sigma^+(r) := \max\{\mathbf{d}(x) :\ W(x) \leq r\}. \tag{3.24}$$

Clearly, σ_-, $\sigma^+ : [0, +\infty[\to I\!R$ are continuous, strictly increasing, unbounded functions such that $\sigma_-(0) = \sigma^+(0) = 0$ and

$$\forall x \in W^{-1}([0, \sigma]) : \quad \sigma_-(W(x)) \leq \mathbf{d}(x) \leq \sigma^+(W(x)).$$

We now define $\beta : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$ by setting

$$\beta(r, t) := \sigma^+ \circ \tilde{\gamma}^{-1}\left(\sigma_-^{-1}(r)\, \frac{2(\varepsilon + 1)}{\varepsilon + 1 + t}\right), \tag{3.25}$$

so, by straightforward calculations, it follows that ($T_y = \bar{t}$ and)

$$\mathbf{d}(y(t)) \leq \beta(\mathbf{d}(z), t) \qquad \forall t \in [0, T_y[.$$

By the arbitrariness of $\sigma > 0$, this concludes the proof of claim **(i)** of the theorem.

As for claim **(ii)**, we now observe that inequality (3.22) implies also

$$\int_0^{\bar{t}} \mathbf{l}(y(t), v(t))\, dt = \lim_{k \to +\infty} \int_0^{t_k} \mathbf{l}(y(t), v(t))\, dt \leq \lim_{k \to +\infty} \frac{W(z) - W(y(t_k))}{p_0} = \frac{W(z)}{p_0},$$

from which (2.6) follows. □

4. Control-polynomial systems

in this section and in the next one we will assume the dynamics f to be a polynomial of degree $d \geq 0$ in the control variable u:
(4.1)

$$\dot{x} = f(x,u) := f_0(x) + \sum_{i=1}^{d} \left(\sum_{\alpha \in \mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m = i} u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x) \right), \quad x(0) = z,$$

$$V(z) := \inf_{(x,u) \in \mathcal{A}_f(z)} \int_0^{T_x} l(x(t), u(t))\, dt.$$

We assume the vector fields $f_0, f_{\alpha_1,\dots,\alpha_m}$ to be continuous and the controls to range on the set

$$U_r := [-r, r]^m,$$

for some r, $0 < r \leq +\infty$ (if $r = +\infty$ we mean $U_r := I\!R^m$).

On the one hand such polynomial structure is of obvious interest for applications. For instance, in the example of the gyroscope (Section 2.2) the dynamics is quadratic in the controls, namely the precession and rotation velocities. Also the impressive behaviour of the Kapitza pendulum –where a fast oscillation of the pivot turns an unstable (or even a non-equilibrium) point into a stable point– can be explained by saying that the square of the pivot velocity –regarded as a control– prevails on gravity. Many other mechanical systems, possibly non-holonomic, can be thought as control systems with quadratic dependence on the inputs, see e.g. [4].

On the other hand, it is natural to try to exploit the control polynomial dependence for a careful study of the vectogram's convex hull[14].

4.1. **Near-control-affine systems.**

In this subsection we address the task of representing a control-polynomial system – actually, its convexification – by means of a control-affine dynamics like

$$f_{\mathit{aff}}(x,w) := f_0(x) + \sum_{i=1}^{d} \left(\sum_{\alpha \in \mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m = i} w_{\alpha_1,\dots,\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x) \right).$$

Such a representation in general does not exist, as it is clear when $f(x,u) = uf_1(x) + u^2 f_2(x)$, $u \in I\!R$. However, an affine representation is achievable in the case of *near-control-affine* systems, where the only non-zero terms are those corresponding to control monomials such that each component u_i $(i = 1,\dots,m)$ has an exponent equal either 0 or a fixed odd positive number K_i. To state precisely the main result, let us give some definitions.

For every $\alpha \in \mathbb{N}^m$, let us set $c(\alpha) := \#\{\alpha_i \neq 0;\ i = 1,\dots,m\}$.

[14] In some classical literature, as well as in some recent papers, objects akin to the convex hull of the image of the vector valued function that maps $u \in I\!R^m$ into the (suitably ordered) sequence of all monomials of u up to the degree d, are referred to as *spaces of moments*, see e.g. [1, 6, 10, 17, 20].

Definition 4.1 (Near-control-affine systems)**.** We say that the control-polynomial dynamics $f(x,u)$ in (4.1) is *near-control-affine* if there exist an $m-$tuple $K = (K_1,\dots,K_m)$ of positive odd numbers and a positive integer $\bar{d} \le m$ such that

$$f(x,u) := f_0(x) + \sum_{i=1}^{\bar{d}} \left(\sum_{\alpha\in\mathbb{N}^m:\ c(\alpha)=i,\ \alpha_1\in\{0,K_1\},\dots,\alpha_m\in\{0,K_m\}} u_1^{\alpha_1}\cdots u_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x) \right).$$

Remark 4.2. If the near-control-affine system (4.1) is of degree d, one obviously has $\bar{d} \le d$. Moreover, when $\bar{d} = m$, the number M of non-drift terms of a near-control-affine system f verifies $M \le \sum_{k=1}^m \binom{m}{k} = 2^m - 1$. Indeed for every $k \le m$, the maximum number of non zero terms of the form $u_1^{\alpha_1}\cdots u_m^{\alpha_m} f_{\alpha_1\cdots\alpha_m}$ with k coefficients $\alpha_i \ne 0$ is equal to $\binom{m}{k}$.

For every $r \in]0,+\infty[$ we set

$$\bar{r} := \frac{1}{M}\min\{r^{jK_i} \mid i = 1,\dots,m;\ j = 1,\bar{d}\} \tag{4.2}$$

and

$$\bar{U}_r := [-\bar{r},\bar{r}]^M.$$

In addition, we set

$$\bar{U}_{+\infty} := I\!R^M.$$

Theorem 4.3, where we assume Hypothesis $\mathbf{A}_b$ below, establishes that near-control-affine systems can be regarded as control-affine systems with independent control variables.

Hypothesis $\mathbf{A}_b$:

(1) f is near-control-affine;
(2) for every $x \in \Omega\backslash\mathbf{C}$, the map $l(x,\cdot) : U_r \to I\!R$ is bounded;
(3) let us define the (non-negative, continuous) function

$$\ell(x) := \sup_{u\in U} l(x,u).$$

The control set for the minimum problems $(\ell, f_{aff}, \mathbf{C})$ coincides with $\bar{U}_r$.

Theorem 4.3. *Let us assume Hypothesis $\mathbf{A}_b$ and let W be a p_0-MRF for the affine problem $(\ell, f_{aff}, \mathbf{C})$ for some $p_0 \ge 0$. Then the map W is a p_0-MRF for the original (non-affine) problem $(l, f, \mathbf{C})$ as well. In particular, the control system in* (4.1) *is* GAC *to* $\mathbf{C}$ *and, if $p_0 > 0$,*

$$V(z) \le \frac{W(z)}{p_0} \qquad \forall z \in \Omega\backslash\mathbf{C}.$$

Proof. Let $x \in \Omega \setminus \mathbf{C}$. By assumption one has

$$\inf_{w\in\bar{U}_r}\left\{\Big\langle p\,,\ f_{aff}(x,w)\Big\rangle\right\} + p_0\ell(x) < 0 \qquad \text{for all } p \in D^*W(x).$$

By Lemma 4.4 below, $f_{aff}(x, \bar{U}_r) \subseteq co f(x, U_r)$, which implies

$$\inf_{u \in U_r} \left\{ \left\langle p\,,\, f(x,u) \right\rangle \right\} + p_0 \ell(x) < 0 \qquad \text{for all } p \in D^* W(x). \tag{4.3}$$

This concludes the proof, since (4.3) yields

$$\inf_{u \in U_r} \left\{ \left\langle p\,,\, f(x,u) \right\rangle + p_0 l(x,u) \right\} < 0 \qquad \text{for all } p \in D^* W(x).$$

□

Lemma 4.4. *For every $r \in [0, +\infty]$*

$$f_{aff}(x, \bar{U}_r) \subset co\; f(x, U_r) \quad \forall x \in \Omega \setminus \mathbf{C}. \tag{4.4}$$

This result will be proved in Appendix A.

Remark 4.5. Besides implying Theorem 4.3, Lemma 4.4 gives access to classical results on control-affine systems for the study of local controllability of near-control-affine systems. For instance, consider the driftless, near-control-affine system (with $d = 8$, $K = (1, 3, 5)$ and $\bar{d} = 2$)

$$\dot{x} = f(x,u) = u_1 u_2^3 f_{1,3,0}(x) + u_1 u_3^5 f_{1,0,5}(x) + u_2^3 u_3^5 f_{0,3,5}(x), \tag{4.5}$$

with $x = (x_1, x_2, x_3, x_4) \in I\!R^4$, $u = (u_1, u_2, u_3) \in I\!R^3$ and

$$f_{1,3,0}(x) = (1, 0, x_2, 0)^{tr}; \quad f_{1,0,5}(x) = (0, 1, -x_1, 0)^{tr}; \quad f_{0,3,5}(x) = (0, 0, 0, 1)^{tr}.$$

Notice that $\{(u_1 u_2^3, u_1 u_3^5, u_2^3 u_3^5) \mid (u_1, u_2, u_3) \in I\!R^3\} \subset I\!R^3$ and, for instance,

$$(0, 1, 1) \notin \{(u_1 u_2^3, u_1 u_3^5, u_2^3 u_3^5) \mid (u_1, u_2, u_3) \in I\!R^3\},$$

so f cannot be parameterized as control-linear vector field with controls in $I\!R^3$. However, by Lemma 4.4 the control-linear vector field

$$f_{aff}(x, w) = w_{1,3,0} f_{1,3,0}(x) + w_{1,0,5} f_{1,0,5}(x) + w_{0,3,5} f_{0,3,5}(x) \quad (w_{1,3,0}, w_{1,0,5}, w_{0,3,5}) \in I\!R^3$$

satisfies

$$f_{aff}(x, \bar{U}_r) \subset co(f(x, U_r)) \quad \forall x \in I\!R^4;\ \forall r > 0.$$

For example, we have that $f_{1,0,5}(x) + f_{0,3,5}(x) \notin f(x, U_r)$, while

$$f_{1,0,5}(x) + f_{0,3,5}(x) = \frac{1}{2} f(x, (1, 0, 2^{1/5})) + \frac{1}{2} f(x, (0, 1, 2^{1/5})).$$

Remark 4.6. Let us see a simple utilization of the affine representability of f_{aff} for system (4.5). Observe that the latter verifies the so-called Lie algebra rank condition,

$$Lie_x\{f_{1,3,0}, f_{1,0,5}, f_{0,3,5}\} = I\!R^4 \qquad \forall x \in I\!R^4.$$

Indeed the Lie bracket $[f_{1,3,0}, f_{1,0,5}]$ coincides with the vector field constantly equal to $(0, 0, 2, 0)^t$, so that

$$span\{f_{1,3,0}, f_{1,0,5}, f_{0,3,5}, [f_{1,3,0}, f_{1,0,5}]\} = I\!R^4$$

at every point. Therefore, by Chow-Rashevsky's Theorem the system $\dot{x} = f_{aff}(x, w)$ turns out to be small time locally controllable. Now, by Lemma 4.4

$$f_{aff}(x, \bar{U}_r) \subset co(f(x, U_r)) \quad \forall x \in \Omega \setminus \mathbf{C}.$$

Consequently, by a standard relaxation argument, we can deduce that the system $\dot{x} = f(x, u)$ is small time locally controllable as well.

4.2. **Maximal degree weak subsystems.**

In this subsection and the next one, we assume $r = +\infty$, i.e. $U_r = I\!R^m$ and look for *weak subsystems*, namely set-valued selections of the convex-valued multifunction $x \mapsto co\ f(x, I\!R^m)$.

We begin with a class of weak subsystems which we call *maximal degree* subsystems. Theorem 4.7 below extends in several directions a result contained in [4] and valid for the case $d = 2$. It states that in order to test if a function W is a p_0-MRF function for problem (4.1), it is sufficient to test W on the (simpler) *maximal degree* problem

$$\dot{x} = f^{max}(x, u), \qquad x(0) = z,$$

$$\inf_{(x,u)\in\mathcal{A}_{f^{max}}(z)} \int_0^{T_x} l(x(t), u(t))dt, \tag{4.6}$$

where the *maximal degree* control-polynomial vector field f^{max} is defined by

$$f^{max}(x, u) := f_0(x) + \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d} u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1,\ldots,\alpha_m}(x).$$

We shall assume the following additional hypothesis on the running cost:

Hypothesis $\mathbf{A}_{max}$: *There exist non negative continuous functions $M_0 = M_0(x)$, $M_1 = M_1(x, u)$ such that*

$$l(x, u) = M_0(x) + M_1(x, u), \tag{4.7}$$

with M_1 verifying

$$M_1(x, 0) = 0, \qquad M_1(x, ku) \le k^d M_1(x, u) \qquad \forall k \ge 1,\ x \in \Omega \setminus \mathbf{C},\ u \in I\!R^m.$$

Notice that running costs of the form

$$l(x, u) = l_0(x) + l_1(x)|u| + \cdots + l_d(x)|u|^d,$$

where the maps $l_i(\cdot)$ are continuous and non-negative, verify Hypothesis $\mathbf{A}_{max}$.

Theorem 4.7. *Let us assume Hypothesis $\mathbf{A}_{max}$, and let W be a p_0-MRF for the maximal degree problem $(l, f^{max}_\lambda, \mathbf{C})$, for some $p_0 \ge 0$. Then the map W is a p_0-MRF for the original problem $(l, f, \mathbf{C})$. In particular, the control system in* (4.1) *is* GAC *to* $\mathbf{C}$ *and, if* $p_0 > 0$,

$$V(z) \le \frac{W(z)}{p_0} \qquad \forall z \in \Omega\backslash\mathbf{C}.$$

Proof. Assume by contradiction that there exist $x \in \Omega\backslash\mathbf{C}$ and $p \in D^*W(x)$ such that

$$p_0 l(x, u) + \langle p, f_0(x)\rangle + \sum_{i=1}^{d} \left(\sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=i} \langle p, u_1^{\alpha_1} \cdots u_m^{\alpha_m} f_{\alpha_1,\ldots,\alpha_m}(x)\rangle \right) \ge 0 \tag{4.8}$$

for all $u \in I\!R^m$. By taking $u = 0$ we obtain

$$p_0 M_0(x) + \langle p, f_0(x)\rangle \ge 0. \tag{4.9}$$

By assumption, there exists $\tilde{u} \in I\!R^m$ and $\eta > 0$ such that

$$p_0\, l(x,\tilde{u}) + \langle p, f_0(x)\rangle + \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle = -\eta. \tag{4.10}$$

Moreover, (4.9)-(4.10) imply

$$p_0 k^d M_1(x,\tilde{u}) + k^d \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \le -\eta k^d \tag{4.11}$$

for any $k \geq 0$. Hence, for every $k \geq 1$

$$\begin{aligned}
p_0 l(x,k\tilde{u}) + \langle p, f_0(x)\rangle + k &\sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \\
&+ \cdots + k^{d-1} \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d-1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \\
&+ k^d \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \le \\
p_0 k^d M_1(x,\tilde{u}) + p_0 M_0(x) + \langle p, f_0(x)\rangle & \\
&+ k \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \\
&+ \cdots + k^{d-1} \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d-1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \\
&+ k^d \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \le \\
p_0 M_0(x) + \langle p, f_0(x)\rangle + k &\sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle \\
+ &\cdots + k^{d-1} \sum_{\alpha\in\mathbb{N}^m,\, \alpha_1+\cdots+\alpha_m=d-1} \langle p, \tilde{u}_1^{\alpha_1}\cdots\tilde{u}_m^{\alpha_m} f_{\alpha_1,\dots,\alpha_m}(x)\rangle - \eta k^d.
\end{aligned}$$

If k is sufficiently large the last term is negative, which contradicts (4.8). $\square$

Remark 4.8. The thesis of Theorem 4.7 cannot be extended to the case of bounded control sets. For instance, if $d = 3$, $n = m = 1$, $U = [-1, 1]$, $\mathbf{C} = \{0\}$, $l \equiv 0$, and $f(x, u) = (u^2 + u^3)x$, one has $\dot{x} = f(x, u) \geq 0$ for $x \geq 0$, so the system is not GAC to $\mathbf{C}$ and no control Lyapunov function[15] exists. However, $W(x) = x^2$ is a control Lyapunov function for (l, f^{max}), so that the system $\dot{x} = f^{max}(x, u)$ is GAC to $\mathbf{C}$. Nevertheless, some symmetry arguments may allow the extension of Theorem 4.7 to some special classes of polynomial control systems with *bounded* control sets. This might be the case when $d = 2$, U is a (compact) symmetric control set (i.e. $u \in U$ implies $-u \in U$) and, for all $x \in \Omega \setminus \mathbf{C}$, $l(x, \cdot)$ is an even function. For example, consider the system

$$\dot{x} = f(x,u), \quad x(0) = z, \qquad u \in U := [-1,1]^2$$

[15]When $l = 0$ the notion of p_0-MRF coincides with that of control Lyapunov function.

where

$$f(x,u) := f_0(x) + u_1 f_{1,0}(x) + u_2 f_{0,1}(x) + u_1^2 f_{2,0}(x) + u_2^2 f_{0,2}(x) + u_1 u_2 f_{1,1}(x),$$

together with the minimum problem

$$\inf_{(x,u)\in\mathcal{A}_f(z)} \int_0^{T_x} (|u| + x^2u^2)dt.$$

Notice that

$$(l, f^{max})(x,u) = \frac{1}{2}(l,f)(x,u) + \frac{1}{2}(l,f)(x,-u) \in co(l,f)(x,U) \qquad \forall x \in \Omega\backslash\mathbf{C},\ u \in U.$$

Therefore, for every $(x,(p_0,p)) \in (\Omega \setminus \mathbf{C}) \times I\!R^{1+n}$, one has

$$H_{l,f^{max}}(x,p_0,p) < 0 \quad \Rightarrow \quad H_{l,f}(x,p_0,p) < 0.$$

Consequently a map W is p_0-MRF for $(l, f^{max}, \mathbf{C})$ for some $p_0 \geq 0$ if and only if W is a p_0-MRF for $(l, f, \mathbf{C})$. Then Theorem 1.1 applies and, consequently, Theorem 4.7 can be extended to this case.

4.3. **Diagonal weak subsystems.** Another class of weak subsystems is given by the *diagonal subsystems* described below. We still assume $U = I\!R^m$.

Let us use $\mathbf{e}_1, \cdots, \mathbf{e}_m$ to denote the basis of $I\!R^m$ and let us set $\mathbf{e}_0 := 0$.

Definition 4.9. For every λ belonging to the simplex $\Lambda := \{\lambda \in I\!R^m \mid \sum_{i=1}^m \lambda_i \leq 1;\ \lambda_i \geq 0\}$,

$$f_\lambda^{diag}(x,u) := \sum_{i=0}^m \lambda_i f(x, \lambda_i{}^{-\frac{1}{d}} u_i \mathbf{e}_i), \tag{4.12}$$

where $\lambda_0 := 1 - \sum_{i=1}^m \lambda_i$, will be called the λ-*diagonal* control vector field corresponding to f and λ.

For instance, setting $f_{\alpha_1,\dots,\alpha_m} := f_\alpha$ for every $\alpha \in \mathbb{N}^m$, when $d = 2$, $d = 3$ one has

$$f_\lambda^{diag}(x,u) = f_0(x) + \sum_{i=1}^m \lambda_i^{\frac{1}{2}} u_i f_{\mathbf{e}_i}(x) + \sum_{i=1}^m u_i^2 f_{2\mathbf{e}_i}(x).$$

and

$$f_\lambda^{diag}(x,u) = f_0(x) + \sum_{i=1}^m \lambda_i^{\frac{2}{3}} u_i f_{\mathbf{e}_i}(x) + \sum_{i=1}^m \lambda_i^{\frac{1}{3}} u_i^2 f_{2\mathbf{e}_i}(x) + \sum_{i=1}^m u_i^3 f_{3\mathbf{e}_i}(x),$$

respectively.

Remark 4.10. Since $\sum_{i=0}^m \lambda_i = 1$, this implies that

$$f_\lambda^{diag}(x, I\!R^m) \subseteq co\ f(x, I\!R^m). \tag{4.13}$$

We shall assume the following hypothesis on the running cost:

Hypothesis $\mathbf{A}_{diag}$: There exists a real number $M_0 \geq 0$ such that, for every $\lambda \in \Lambda$ verifying $\lambda_i > 0$, $i = 1, \dots, m$, one has

$$l(x,0)+\sum_{i=1}^{m}\lambda_i l(x,\frac{u_i}{\sqrt[d]{\lambda_i}}\mathbf{e}_i)\leq M_0\, l(x,u) \qquad \forall u\in I\!R^m. \tag{4.14}$$

Remark 4.11. Notice that for every $q\geq 1$, the particular running cost

$$l(x,u):=l_0(x)+l_1(x)|u|+\cdots l_q(x)|u|^q \tag{4.15}$$

does verify Hypothesis $\mathbf{A}_{diag}$ (with $M_0=\sqrt{m}$)[16]. As a model, simple case, one could consider $l(x,u)=|u|^q$, $q\geq d$, so that the functional to be minimized would be nothing but the q-th power of the L^q-norm of u .

Theorem 4.12. *Assume that Hypothesis $\mathbf{A}_{diag}$ holds true for a suitable $M_0\geq 0$, and let W be a p_0-MRF for the λ-diagonal problem $(l,f_\lambda^{diag},\mathbf{C})$, for some $p_0\geq 0$. Then the map W is a $\bar{p}_0$-MRF for the original problem $(l,f,\mathbf{C})$, where $\bar{p}_0:=\frac{p_0}{M_0}$ if $M_0>0$, while, if $M_0=0$, $\bar{p}_0$ is allowed to be any positive real number.*
In particular, the control system in (4.1) *is* GAC *to* $\mathbf{C}$ *and, if* $p_0>0$,

$$V(z)\leq\frac{M_0W(z)}{p_0} \qquad \forall z\in\Omega\backslash\mathbf{C}. \tag{4.16}$$

Proof. Set $\lambda_0=1-\sum_{i=1}^m\lambda_i$ and $\mathbf{e}_0=0$. First assume $M_0>0$. Then for every $i=0,\dots,m$, every $(x,u)\in(\Omega\backslash\mathbf{C})\times I\!R^m$ and every $p\in D^*W(x)$, one has

$$\lambda_i H_{l,f}(x,\frac{p_0}{K},p)\leq\lambda_i\left\langle(\frac{p_0}{K},p)\ ,\ (l,f)(x,\lambda_i^{-\frac{1}{d}}u_i\mathbf{e}_i)\right\rangle$$

that, summing up for $i=0,\dots,m$, yields

$$\begin{aligned}&H_{l,f}(x,\frac{p_0}{K},p)\leq\sum_{i=0}^{m}\lambda_i\left\langle(\frac{p_0}{K},p)\ ,\ (l,f)(x,\lambda_i^{-\frac{1}{d}}u_i\mathbf{e}_i)\right\rangle\leq\\ &\frac{p_0}{M_0}M_0l(x,u)+\left\langle p\ ,\ f_\lambda^{diag}(x,u)\right\rangle=p_0l(x,u)+\left\langle p\ ,\ f_\lambda^{diag}(x,u)\right\rangle.\end{aligned} \tag{4.17}$$

Since by hypothesis $\max_{p\in D^*W(x)}H_{l,f_\lambda^{diag}}(x,p_0,p)<0$, then there exists $\tilde{u}$ such that

$$p_0l(x,\tilde{u})+\left\langle p,f_\lambda^{diag}(x,\tilde{u})\right\rangle<0 \quad \forall p\in D^*W(x),$$

this, together with by (4.17), implies

$$H_{l,f}(x,\frac{p_0}{M_0},p)<0 \quad \forall p\in D^*W(x)$$

which indeed is the thesis of the theorem. Assume otherwise $M_0=0$. Then $l\equiv 0$, consequently $W(z)\equiv 0$ and (4.16) is trivially verified. Since W is a p_0-MRF for $(l,f_\lambda^{diag},\mathbf{C})$ and since $l\equiv 0$, for every $x\in\Omega\setminus\mathbf{C}$ there exists $\tilde{u}\in I\!R^m$ such that

[16]This is due to the elementary inequalities

$$|u_1|+\cdots+|u_m|\leq\sqrt{m}|u| \qquad (|u_1|^q+\cdots+|u_m|^q)^{\frac{1}{q}}\leq|u| \qquad \forall q>1.$$

$\langle p, f_\lambda^{diag}(x,\tilde{u})\rangle < 0$ for all $p \in D^*W(x)$. Consequently, for every $\bar{p}_0 \in I\!R$ and for every $p \in D^*W(x)$

$$H_{l,f}(x,\bar{p}_0,p) = \inf_{u\in I\!R^m}\langle p,\ f(x,u)\rangle \leq \sum_{i=0}^m \lambda_i\langle p,\ f(x,\lambda_i^{-\frac{1}{d}}u_i\mathbf{e}_i)\rangle$$
$$= \langle p,\ f_\lambda^{diag}(x,u)\rangle < 0.$$

This gives the thesis in the case $M_0 = 0$ and completes the proof. □

Example 4.13. Let $\mathbf{C} := \{0\}$, $u \in I\!R^2$ and let us consider in $I\!R^2$ the exit-time problem

(4.18)
$$\dot{x} = f(x,u) := x + u_1u_2(|x|^{-1},1)^{tr} - u_1^2(1,0)^{tr} - u_2^2(0,1)^{tr} + 3u_1^2u_2^2x \quad x(0) = z;$$

$$V(z) := \inf_{(x,u)\in\mathcal{A}(z)}\int_0^{T_x} x^2|u|^2\,dt.$$

Let $\Phi : [0,+\infty[\to I\!R$ be a smooth convex function such that $\Phi(0) = 0$, $\Phi'(0) \geq 1$. In order to verify that a function of the form

$$W(x) = \Phi(|x|^2)$$

is a p_0-MRF function for some $p_0 > 0$, let us begin with observing that *the maximal degree subsystem*

$$\dot{x} = f^{max}(x,u) = x + 3u_1^2u_2^2x$$

does not give any useful information. Indeed

$$H_{l,f^{max}}(x,p_0,\nabla W(x)) = \inf_u\left\{\Big\langle\nabla W(x)\ ,\ f^{max}(x,u)\Big\rangle + p_0x^2|u|^2\right\}$$
$$= \inf_u\left\{2\Phi'(|x|^2)|x|^2(1+3u_1^2u_2^2) + p_{\mathcal{I}}x^2|u|^2\right\} \geq 0$$

for all $x \in I\!R^2\backslash\{0\}$ and $p_0 \geq 0$. On the other hand, by considering the diagonal subsystem

$$\dot{x} = f^{diag}_{(\frac{1}{2},\frac{1}{2})} = x - u_1^2(1/\sqrt{2},0)^{tr} - u_2^2(0,1/\sqrt{2})^{tr},$$

if $p_0 < 1 \quad (\leq \Phi'(|x|^2)$ for all $x \in I\!R^2)$, we get, for all $x \in I\!R^2 \setminus \{0\}$,

$$H_{l,f^{diag}_{(\frac{1}{2},\frac{1}{2})}}(x,p_0,\nabla W(x)) \leq \inf_u\left\{|x|^2\Big(\Phi'(|x|^2)(2-u^2) + p_0u^2\Big)\right\} = -\infty,$$

i.e., W is a p_0-MRF for the problem $(l, f^{diag}_{(\frac{1}{2},\frac{1}{2})})$. Therefore, in view of Theorem 4.12, W is a p_0-MRF for the problem (4.18) as well.

Appendix A. Proof of Lemma 4.4

For the reader convenience let us recall the statemen of Lemma 4.4:

For every $r \in [0,+\infty]$

(A.1) $$f_{\mathit{aff}}(x,\bar{U}_r) \subset co\ f(x,U_r) \quad \forall x \in \Omega \setminus \mathbf{C}.$$

We prove this result in the case all components of the m-tuple K are equal to 1, i.e., $K = (1,\dots,1)$ (this assumption implies $\bar{d} = m = d$, see Remark 4.2). Indeed, to prove the theorem when K is a general m-tuple of odd numbers it is sufficient to apply the result to the rescaled control-polynomial vector field

$$\hat{f}(x,u) := f(x, u_1^{\frac{1}{K}1}, \dots, u_m^{\frac{1}{K}m}).$$

Fix $k \in \mathbb{N}$ and denote by $\{1,-1\}^k$ the set of k-tuples $(s_1,\dots,s_k)$ with $s_j \in \{-1,1\}$. Denote by $P(S)$ the power set of a set S and consider the set-valued map $S_k : \{1,-1\} \to P(\{-1,1\}^k)$ defined by

$$S_k(s) = \left\{(s_1,\dots,s_k) \in \{-1,1\}^k \mid s_1 \cdots s_k = s\right\}.$$

Let us begin with a combinatorial result:

Claim A: Let $k, d \in \mathbb{N}$, $k < d$. For every $i_1,\dots,i_k \in \mathbb{N}$, $1 \le i_1 < \cdots < i_k \le d$, and for every $s \in \{-1,1\}$

$$\sum_{(s_1,\dots,s_d)\in S_d(s)} s_{i_1}\cdots s_{i_k} = 0. \tag{A.2}$$

To prove *Claim A*, notice that

$$\sum_{(s_1,\dots,s_k)\in\{-1,1\}^k} s_1 s_2 \cdots s_k = 0. \tag{A.3}$$

Now, fix $i_1,\dots,i_k \in \mathbb{N}$, $1 \le i_1 < \cdots < i_k \le d$ and an auxiliary k-uple $\bar{\mathbf{s}} = (\bar{s}_1,\dots,\bar{s}_k) \in \{-1,1\}^k$. One has

$$\#\left\{(s_1,\dots,s_d) \in \{-1,1\}^d \mid s_{i_h} = \bar{s}_h;\ h = 1,\dots,k\right\} = 2^{d-k}.$$

Therefore, by a symmetry argument,

$$\#\left\{(s_1,\dots,s_d) \in S_d(s) \mid s_{i_h} = \bar{s}_h;\ h = 1,\dots,k\right\} = 2^{d-k-1} \quad \forall s \in \{-1,1\}. \tag{A.4}$$

In view of (A.3) and of (A.4), for every $s \in \{-1,1\}$

$$\sum_{(s_1,\dots,s_d)\in S_d(s)} s_{i_1}\cdots s_{i_k} = 2^{d-k-1}\left(\sum_{(s_{i_1},\dots,s_{i_k})\in\{-1,1\}^k} s_{i_1}\cdots s_{i_k}\right) = 0.$$

This concludes the proof of *Claim A*.

We continue the proof of Lemma 4.4 by proving *Claim B* below, which concerns the convex hull $co\ f(x,U_r)$. For every integer $j \ge 1$, let us set

$$I_{r,j} := \begin{cases} [-r^j, r^j] & \text{if } r < +\infty \\ \mathbb{R} & \text{if } r = +\infty. \end{cases}$$

Claim B: Let $d \le m$. For every $k \le d$, $i_1,\dots,i_k \in \mathbb{N}$, $1 \le i_1 < \cdots < i_k \le d$, and $w \in I_{r,k}$, one has

$$f_0(x) + w f_{\alpha_1,\dots,\alpha_m}(x) \in co\ f(x,U_r), \tag{A.5}$$

where $\alpha_j = 1$ *for* $j \in \{i_1, \dots, i_k\}$ *and* $\alpha_j = 0$ *otherwise.*

To prove *Claim B*, denote by $s(w)$ the sign of w and select from $I_{r,1}$ a set of k real numbers $u_{i_1}, \dots, u_{i_k}$ such that $u_{i_1} \cdots u_{i_k} = w$.

Define

$$u^{(\mathbf{s})} := \sum_{j=1}^{k} \mathbf{s}_j |u_{i_j}| \mathbf{e}_{i_j} \quad \text{for every } \mathbf{s} := (s_1, \dots, s_k) \in S_k(s(w)).$$

By construction one has $u^{(\mathbf{s})} \in [-r, r]^m = U_r$ and

$$u^{(\mathbf{s})}_{i_1} \cdots u^{(\mathbf{s})}_{i_k} = w.$$

By *Claim A*, for every $h < k$ and every increasing finite subsequence $i_1 \leq i_{j_1} < \cdots < i_{j_h} \leq i_k$ of $i_1, \dots, i_k$, one has

$$\sum_{\mathbf{s} \in S_k(s(w))} u^{(\mathbf{s})}_{i_{j_1}} \cdots u^{(\mathbf{s})}_{i_{j_h}} = |u_{i_{j_h}}| \cdots |u_{i_{j_h}}| \sum_{(s_1,\dots,s_k) \in S_k(s)} s_{j_h} \cdots s_{j_h} = 0.$$

Notice that 2^{k-1} is the cardinality of $S_k(s(w))$. Hence by the definition of near-control-affine system it easily follows that

$$\sum_{\mathbf{s} \in S_k(s(w))} \frac{1}{2^{k-1}} f(x, u^{(\mathbf{s})}) = f_0(x) +$$

$$\sum_{h=1}^{k} \frac{1}{2^{k-1}} \left(\sum_{i_1 \leq i_{j_1} < \cdots < i_{j_h} \leq i_k} \left(\sum_{\mathbf{s} \in S_k(s(w))} u^{(\mathbf{s})}_{i_{j_1}} \cdots u^{(\mathbf{s})}_{i_{j_h}} \right) f_{\mathbf{e}_{i_{j_1}} + \cdots + \mathbf{e}_{i_{j_h}}}(x) \right)$$

$$= f_0(x) + \frac{1}{2^{k-1}} \left(\sum_{i_1 < \cdots < i_k} \left(\sum_{\mathbf{s} \in S_k(s(w))} u^{(\mathbf{s})}_{i_1} \cdots u^{(\mathbf{s})}_{i_k} \right) f_{\mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k}}(x) \right)$$

$$= f_0(x) + w \; f_{\alpha_1, \dots, \alpha_m}(x),$$

which concludes the proof of *Claim B*.

To end the proof of Lemma 4.4 in case $K = (1, \dots, 1)$, it suffices to remark that for every $k = 1, \dots, d$, by the definition of $\bar{r}$ given in (4.2)

$$[-\bar{r}, \bar{r}] \subseteq M[-r^k, r^k].$$

Therefore *Claim B* implies that for every

$$w = (w_{\mathbf{e}_1}, \dots, w_{\mathbf{e}_d}, w_{\mathbf{e}_1+\mathbf{e}_2}, w_{\mathbf{e}_1+\mathbf{e}_3}, \dots, w_{\mathbf{e}_1+\cdots+\mathbf{e}_d}) \in [-\bar{r}, \bar{r}]^M = \bar{U}_r$$

$$f_{\mathit{aff}}(x, w) = \sum_{k=1}^{d} \sum_{i_1 < \cdots < i_k} \frac{1}{M} (f_0(x) + M w_{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}} f_{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}}(x)) \in co\; f(x, U_r).$$

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