Pure and Applied Functional Analysis
Volume 1, Number 4, 2016, 475–503

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

LEVITIN-POLYAK WELL-POSEDNESS OF NONCOMPACT GENERALIZED MIXED VARIATIONAL INEQUALITIES IN REFLEXIVE BANACH SPACES

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Abstract. We introduce the concept of Levitin-Polyak well-posedness for a noncompact generalized mixed variational inequality in a real reflexive Banach space X, and establish some characterizations of its Levitin-Polyak well-posedness. We prove that the Levitin-Polyak well-posedness of a noncompact generalized mixed variational inequality is equivalent both to the Levitin-Polyak well-posedness of a corresponding inclusion problem and to the Levitin-Polyak well-posedness of a corresponding fixed point problem. Some conditions under which a noncompact generalized mixed variational inequality in X is Levitin-Polyak well-posed are given. It is worth pointing out that there is no compactness assumption in our results.

1. Introduction

In 1966, Tykhonov [24] initially introduced and considered the well-posedness of a minimization problem, which consists of the existence and uniqueness of minimizers, and the convergence of every minimizing sequence to the unique minimizer. In many practical situations, there are more than one minimizer for a minimization problem. In a natural way, the concept of Tykhonov well-posedness in the generalized sense was introduced, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence to a minimizer. Because it plays an important role in the study of optimization problems, various concepts of well-posedness have been introduced and studied widely for minimization problems in past decades. For details, the readers refer to [28–30] and the references therein.

The Tykhonov well-posedness of a constrained minimization problem requires that every minimizing sequence should lie in the constraint set. It is well known that in many practical situations, the minimizing sequence produced by a numerical optimization method usually fails to be feasible but gets closer and closer to the constraint set. Such a sequence is called a generalized minimizing sequence

2010 *Mathematics Subject Classification.* 49J40, 49K40, 90C31.

Key words and phrases. Noncompact generalized mixed variational inequality, Inclusion problem, fixed point problem, Levitin-Polyak well-posedness.

*The first author was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

†The third author was partially supported by the Grant MOST 103-2923-E-039-001-MY3.

for constrained minimization problems. Taking into account this case, Levitin and Polyak [17] strengthened the concept of Tykhonov well-posedness by requiring the existence and uniqueness of minimizers, and the convergence of every generalized minimizing sequence to the unique minimizer, which is called Levitin and Polyak (for short, LP) well-posedness. In terms of the recent literature on the research of LP well-posedness for variational inequalities, most researchers mainly focus on the introduction of various kinds of LP well-posedness for different variational inequalities, the establishment of metric characterizations for LP well-posed variational inequalities, the necessary and sufficient conditions of LP well-posedness for variational inequalities, and the links of LP well-posedness between variational inequalities and their related problems, for instance, minimization problems, fixed pointed problems, inclusion problems, etc. There have been a large number of results involving Tykhonov well-posedness, LP well-posedness and their generalizations for minimization problems. For details, we refer the readers to [1,3–6,9–12,15–20,24,27–30].

In 2008, Fang, Huang and Yao [10] considered and studied the well-posedness of a mixed variational inequality in a real Hilbert space H, which includes as a special case the classical variational inequality, and derived some results for the well-posedness of such a mixed variational inequality, the corresponding inclusion problem and the corresponding fixed-point problem. Subsequently, Ceng and Yao [6] extended the concept of well-posedness to a generalized mixed variational inequality in H, which includes as a special case the mixed variational inequality, and gave some characterizations of its well-posedness. Under suitable conditions, the authors [6] proved that the well-posedness of the generalized mixed variational inequality is equivalent both to the well-posedness of the corresponding inclusion problem and to the corresponding fixed-point problem, and derived some conditions under which the generalized mixed variational inequality is well-posed. Recently, some authors made the further extension and development on the concept of well-posedness; see e.g., [3–5,12,27] and the references therein.

Furthermore, Hu and Huang [11] considered the LP well-posedness of a general variational inequality in $\mathbf{R}^n$. They derived some characterizations of the LP well-posedness by considering the size of LP approximating solution sets of general variational inequalities. They also proved that the LP well-posedness of a general variational inequality is closely related to the LP well-posedness of a minimization problem and a fixed point problem. Finally, they proved that under suitable conditions, the LP well-posedness of a general variational inequality is equivalent to the uniqueness and existence of its solutions.

Let X be a real reflexive Banach space. In 2012, Li and Xia [20] extended the notion of LP well-posedness to a generalized mixed variational inequality in X, and gave some characterizations of its LP well-posedness. Under suitable conditions, they proved that the LP well-posedness of a generalized mixed variational inequality is closely related to the LP well-posedness of a corresponding inclusion problem and a corresponding fixed point problem, and derived some conditions under which a generalized mixed variational inequality is LP well-posed. However, it is worth emphasizing that in their results, there is the compactness requirement for set-valued mapping $F : X \to 2^{X^*}$ in the generalized mixed variational inequality, that is, $F : X \to 2^{X^*}$ is compact-valued. Meantime, we note that there is no result for the LP well-posedness of a noncompact generalized mixed variational inequality. Therefore, it is worth studying implementable results for the LP well-posedness of a noncompact generalized mixed variational inequality.

Motivated and inspired by the research work going on this field, we extend the notion of LP well-posedness to a noncompact generalized mixed variational inequality in a real reflexive Banach space X, and give some characterizations of its LP well-posedness. Under suitable conditions, we prove that the LP well-posedness of a noncompact generalized mixed variational inequality is equivalent both to the LP well-posedness of a corresponding inclusion problem and to the LP well-posedness of a corresponding fixed point problem. Finally, we also derive some conditions under which a noncompact generalized mixed variational inequality is LP well-posed. It is worth pointing out that there is no compactness assumption in our results. Our results improve, extend and develop the early and recent ones announced by some others, e.g., Ceng and Yao [6] and Li and Xia [20]. For recent related results, we refer readers [7, 23, 25] and the references therein.

2. Preliminaries

Let X be a real reflexive Banach space with its dual X^* and K be a nonempty, closed and convex subset of X. Let $F : X \to 2^{X^*}$ be a set-valued mapping, and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Denote by domϕ the efficient domain of ϕ, i.e.,

$$\mathrm{dom}\phi = \{x \in X : \phi(x) < +\infty\}.$$

In this paper, we always assume that $\mathrm{dom}\phi \cap K \neq \emptyset$. Consider the following generalized mixed variational inequality associated with (F, ϕ, K):

$$\begin{aligned} &\mathrm{GMVI}(F, \phi, K): \text{ find } x \in K \text{ such that for some } u \in F(x),\\ &\langle u, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K. \end{aligned}$$

It is easy to see that the GMVI(F, ϕ, K) is equivalent to the following inclusion problem associated with $F + \partial(\phi + \delta_K)$:

$$\mathrm{IP}(F + \partial(\phi + \delta_K), K): \text{ find } x \in K \text{ such that } 0 \in F(x) + \partial(\phi + \delta_K)(x),$$

where δ_K denotes the indicator function associated with K (i.e., $\delta_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise) and $\partial(\phi + \delta_K)(x)$ denotes the subdifferential of the convex function $\phi + \delta_k$ at x.

If F is the subdifferential of a finite-valued convex continuous function f defined on X, then the problem GMVI(F, ϕ, K) reduces to the following nondifferentiable convex optimization problem:

$$\min_{x\in X}\{f(x) + \varphi(x)\}.$$

Furthermore, the usual GMVI(F, ϕ, K) formulation admits various modifications and extensions which also can be in principle applied to economic equilibrium problems. For example(see [14] and the references therein), we consider a market structure with perfect competition. The model deals in n commodities. Then, given a price vector $p \in \mathbb{R}^n_+$, we can define the value $E(p)$ of the excess demand mapping $E : \mathbb{R}^n_+ \to 2^{\mathbb{R}^n}$, which is multiv-alued in general. Traditionally, a vector $p^* \in \mathbb{R}^n$

is said to be an equilibrium price vector if it solves the following complementarity problem:

$$p^* \geq 0, \quad \exists q^* \in E(p^*) : q^* \leq 0, \quad \langle p^*, q^* \rangle = 0.$$

or equivalently, the following variational inequality: find $p^* \geq 0$ such that

$$\exists q^* \in E(p^*) : \langle -q^*, p - p^* \rangle \geq 0, \quad \forall p \geq 0.$$

We now specialize our model from this very general one. First, we suppose that each price of a commodity which is involved in the market structure has a lower positive bound and may have an upper bound. It follows that the feasible prices are assumed to be contained in the box constrained set

$$K = \prod_{i=1}^{n} K_i, \quad K_i = \{t \in \mathbb{R} : 0 < \tau_i \leq t \leq \varsigma_i \leq +\infty\}, \quad i = 1, 2, \ldots, n.$$

Next, as usual, the excess demand mapping is represented as follows:

$$E(p) = D(p) - S(p)$$

where D and S are the demand and supply mappings, respectively. We suppose that the demand mapping is single-valued and set $G = -D$. Then, the problem of finding an equilibrium price can be formulated as follows: find $p^* \in K$ such that

$$\exists s^* \in S(p^*), \quad \langle G(p^*), p - p^* \rangle + \langle s^*, p - p^* \rangle \geq 0, \quad \forall p \in K.$$

Under some suitable conditions, this problem can be reduced to GMVI(F, ϕ, K).

Now, we give some useful propositions and definitions.

Proposition 2.1. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a nonempty set-valued mapping, and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then the following conclusions are equivalent:*

(i) *x solves GMVI(F, ϕ, K);*
(ii) *x solves* IP$(F + \partial(\phi + \delta_K), K)$.

Definition 2.2. A nonempty set-valued mapping $F : X \to 2^{X^*}$ is said to be monotone, if for all $x, y \in X, u \in F(x)$ and $v \in F(y)$

$$\langle u - v, x - y \rangle \geq 0.$$

Definition 2.3. Let X and Y be two topological spaces, and $F : X \to 2^Y$ be a set-valued mapping.

(i) F is said to upper semicontinuous (u.s.c.) at $x \in X$ if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(y) \subset V \ \forall y \in U$. If F is u.s.c. at each point of X, we say that F is u.s.c. on X.
(ii) F is said to be closed (resp. open) if the set $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed (resp. open) in $X \times Y$.

Definition 2.4 (see also [22], [21])**.** Let X, Y be two Banach spaces, and $F : X \to 2^Y$ be a nonempty set-valued mapping. Then F is said to be locally bounded if for each $x \in X$, there exists a neighborhood of x and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \leq \ell$.

Let A, B be nonempty subsets of a normed vector space $(X, \|\cdot\|)$. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A,B) = \max\{e(A,B), e(B,A)\},$$

where $e(A,B) = \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B}\|a-b\|$. Note that [19] if A and B are compact subsets in X, then for each $a \in A$ there exists $b \in B$ such that

$$\|a-b\| \le \mathcal{H}(A,B).$$

Definition 2.5 (see [6])**.** A nonempty set-valued mapping $F : X \to 2^{X^*}$ is said to be

(i) $\mathcal{H}$-hemicontinuous, if for any $x, y \in X$, the function $t \mapsto \mathcal{H}(F(x+t(y-x)), F(x))$ from $[0,1]$ into $\mathbf{R}^+ = [0,+\infty)$ is continuous at 0^+ where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X^*)$.

(ii) $\mathcal{H}$-uniformly continuous, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x-y\| < \delta$, one has $\mathcal{H}(F(x), F(y)) < \epsilon$ where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X^*)$.

Definition 2.6 (see [13])**.** Let A be a nonempty subset of X. The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i,\ \mathrm{diam} A_i < \epsilon,\ i = 1,2,...,n\},$$

where diam means the diameter of a set.

Motivated and inspired by Lemma 2.2 in [6], we present the following proposition.

Proposition 2.7. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper and convex functional. Then for a given $x \in K$, the following statements are equivalent:*

(i) *there exists $u \in F(x)$ such that $\langle u, x-y\rangle + \phi(x) - \phi(y) \le 0,\ \forall y \in K$;*

(ii) $\langle v, x-y\rangle + \phi(x) - \phi(y) \le 0,\ \forall y \in K, v \in F(y)$.

Proof. Suppose that for some $u \in F(x)$,

$$\langle u, x-y\rangle + \phi(x) - \phi(y) \le 0,\ \forall y \in K.$$

Since F is monotone, one has

$$\langle u - v, x-y\rangle \ge 0, \quad \forall y \in K, v \in F(y),$$

and hence

$$\langle v, x-y\rangle + \phi(x) - \phi(y) \le \langle u, x-y\rangle + \phi(x) - \phi(y) \le 0, \quad \forall y \in K, v \in F(y).$$

Consequently,

$$\langle v, x-y\rangle + \phi(x) - \phi(y) \le 0, \quad \forall y \in K, v \in F(y).$$

Conversely, suppose that the last inequality is valid. Given any $y \in K$ we define $y_t = ty + (1-t)x$ for all $t \in (0,1)$. Since K is a nonempty, closed and convex subset, we have $y_t \in K$ for all $t \in (0,1)$. Replacing y by y_t in the left-hand side of the last inequality, one derives for each $v_t \in F(y_t)$,

$$\begin{aligned}0 &\geq \langle v_t, x - y_t\rangle + \phi(x) - \phi(y_t)\\ &= \langle v_t, x - (ty + (1-t)x)\rangle + \phi(x) - \phi(ty + (1-t)x)\\ &\geq \langle v_t, t(x-y)\rangle + \phi(x) - t\phi(y) - (1-t)\phi(x)\\ &= t[\langle v_t, x - y\rangle + \phi(x) - \phi(y)],\end{aligned}$$

which hence implies that

$$\langle v_t, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall v_t \in F(y_t), t \in (0,1).$$

Since F is locally bounded, there exists a neighborhood of x and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \leq \ell$. Since $y_t \to x$ as $t \to 0$, for t sufficiently small $\|v_t\| \leq \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $v_t \to u \in Y$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $v_t \to u \in Y$ in the weak topology of X^*. Since F has weakly closed graph, $y_t \to x$ and $v_t \to u \in Y$ weakly, we have $u \in F(x)$ and

$$\langle u, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K.$$

This completes the proof. □

3. Levitin-Polyak Well-Posedness of GMVI(F, ϕ, K)

In this section, we extend the concepts of Levitin-Polyak well-posedness to the noncompact generalized mixed variational inequality and establish its metric characterizations. In the sequel, we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha : X \to [0, +\infty)$ be a given continuous functional with $\alpha(tz) = t^p\alpha(z), \forall t \geq 0$ and $\forall z \in X$, where $p > 1$, and let X, K, F, ϕ be defined as in the previous section.

Definition 3.1. A sequence $\{x_n\} \subset X$ is called a LP α-approximating sequence for GMVI(F, ϕ, K), if there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$ for all $n \in \mathbf{N}$, and there exists $u_n \in F(x_n)$ such that

$$x_n \in \mathrm{dom}\phi, \quad \langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \leq \alpha(x_n - y) + \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}.$$

If $\alpha_1(z) \geq \alpha_2(z) \geq 0\ \forall z \in X$, then every LP α_2-approximating sequence is LP α_1-approximating. When $\alpha(z) = 0\ \forall z \in X$, we say that $\{x_n\}$ is a LP approximating sequence for GMVI(F, ϕ, K).

Definition 3.2. We say that GMVI(F, ϕ, K) is strongly (resp. weakly) LP α-well-posed if GMVI(F, ϕ, K) has a unique solution and every LP α-approximating sequence converges strongly (resp. weakly) to the unique solution. In the sequel, strong (resp. weak) LP 0-well-posedness is always known as strong (resp. weak) LP well-posedness. If $\alpha_1(z) \geq \alpha_2(z) \geq 0\ \forall z \in X$, then strong (resp. weak) LP α_1-well-posedness implies strong (resp. weak) LP α_2-well-posedness.

Definition 3.3. We say that GMVI(F,ϕ,K) is strongly (resp. weakly) LP α-well-posed in the generalized sense if GMVI(F,ϕ,K) has nonempty solution set S and every LP α-approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of S. In the sequel, strong (resp. weak) LP 0-well-posedness in the generalized sense is always known as strong (resp. weak) LP well-posedness in the generalized sense. If $\alpha_1(z) \geq \alpha_2(z) \geq 0\ \forall z \in X$, then strong (resp. weak) LP α_1-well-posedness in the generalized sense implies strong (resp. weak) LP α_2-well-posedness in the generalized sense.

Remark 3.4. It is easy to see that the above Definitions 3.1, 3.2 and 3.3 extend the Definitions 3.1, 3.2 and 3.3 in [20], respectively. When X is a real Hilbert space, $K = X$ and $w_n \equiv 0$, Definitions 3.2 and 3.3 in [20] reduce to the Definitions 3.2 and 3.3 in [10], respectively. When $X = \mathbf{R}^n$, $\alpha \equiv 0$, $\phi = \delta_K$ and F is single-valued, Definitions 3.2 and 3.3 reduce to the Definitions 3.3 and 3.4 of [11], respectively.

To derive the metric characterizations of LP α-well-posedness, we consider the following LP α-approximating solution set of GMVI(F,ϕ,K):

$$\begin{aligned}\Omega_\alpha(\epsilon) = \{&x \in \mathrm{dom}\phi : d(x,K) \leq \epsilon, \text{and there exists } u \in F(x) \text{ such that} \\ &\forall y \in K,\ \langle u, x-y\rangle + \phi(x) - \phi(y) \leq \alpha(x-y) + \epsilon\}, \quad \forall \epsilon \geq 0.\end{aligned}$$

Theorem 3.5. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then* GMVI(F,ϕ,K) *is strongly LP α-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \text{ and } \mathrm{diam}(\Omega_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.1}$$

Proof. Suppose that GMVI(F,ϕ,K) is strongly LP α-well-posed and $x^* \in K$ is the unique solution of GMVI(F,ϕ,K). It is obvious that $x^* \in \Omega_\alpha(\epsilon)$. If $\mathrm{diam}(\Omega_\alpha(\epsilon)) \not\to 0$ as $\epsilon \to 0$, then there exist constant $l > 0$ and sequences $\{\epsilon_n\} \subset \mathbf{R}^+$ with $\epsilon_n \to 0$, and $\{x_n^{(1)}\}, \{x_n^{(2)}\}$ with $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$ such that

$$\|x_n^{(1)} - x_n^{(2)}\| > l, \quad \forall n \in \mathbf{N}. \tag{3.2}$$

Since $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$, for $x_n^{(1)}$ we have

$$d(x_n^{(1)}, K) \leq \epsilon_n < \epsilon_n + \frac{1}{n},$$

and there exists $u_n \in F(x_n^{(1)})$ such that

$$\langle u_n, x_n^{(1)} - y\rangle + \phi(x_n^{(1)}) - \phi(y) \leq \alpha(x_n^{(1)} - y) + \epsilon_n, \quad \forall y \in K.$$

Since K is closed and convex, then there exists $\bar{x}_n^{(1)} \in K$ such that $\|x_n^{(1)} - \bar{x}_n^{(1)}\| < \epsilon_n + \frac{1}{n}$. Putting $w_n = \bar{x}_n^{(1)} - x_n^{(1)}$, we have $w_n + x_n^{(1)} = \bar{x}_n^{(1)} \in K$ and $\|w_n\| = \|x_n^{(1)} - \bar{x}_n^{(1)}\| \to 0$. This implies that $w_n \to 0$. Thus, $\{x_n^{(1)}\}$ is a LP approximating sequence for GMVI(F,ϕ,K). By the similar argument, we obtain that $\{x_n^{(2)}\}$ is a LP approximating sequence for GMVI(F,ϕ,K). So they have to converge strongly to the unique solution of GMVI(F,ϕ,K), a contradiction to (3.2).

Conversely, suppose that the conclusion (3.1) holds. Let $\{x_n\} \subset X$ be a LP α-approximating sequence for GMVI(F, ϕ, K). Then there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exist $0 < \epsilon'_n \to 0$ and $u_n \in F(x_n)$ such that

$$x_n \in \mathrm{dom}\phi, \quad \langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \alpha(x_n - y) + \epsilon'_n, \ \forall y \in K, n \in \mathbf{N}. \tag{3.3}$$

Since $x_n + w_n \in K$, then there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. It is easy to see that $d(x_n, K) \le \|x_n - \bar{x}_n\| = \|w_n\| \to 0$. Set $\epsilon_n = \max\{\epsilon'_n, \|w_n\|\}$, it follows that $x_n \in \Omega_\alpha(\epsilon_n)$. From (3.1), we deduce that $\{x_n\}$ is a Cauchy sequence and so it converges strongly to a point $\bar{x} \in K$. Since F is monotone, ϕ is lower semicontinuous and α is continuous, it follows from (3.3) that for any $y \in K, v \in F(y)$,

$$\begin{aligned} \langle v, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) &\le \liminf_{n\to\infty}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \\ &\le \liminf_{n\to\infty}\{\alpha(x_n - y) + \epsilon'_n\} \\ &= \alpha(\bar{x} - y). \end{aligned} \tag{3.4}$$

For any $y \in K$, put $y_t = \bar{x} + t(y - \bar{x})$ for all $t \in (0, 1)$. Since K is a nonempty, closed and convex subset, we have $y_t \in K$ for all $t \in (0, 1)$. Then

$$\langle v_t, \bar{x} - y_t\rangle + \phi(\bar{x}) - \phi(y_t) \le \alpha(\bar{x} - y_t), \quad \forall v_t \in F(y_t).$$

Since ϕ is convex, from the properties of α we get

$$t[\langle v_t, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y)] \le t^p \alpha(\bar{x} - y), \quad \forall v_t \in F(y_t), y \in K;$$

that is,

$$\langle v_t, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \le t^{p-1} \alpha(\bar{x} - y), \quad \forall v_t \in F(y_t), y \in K, \tag{3.5}$$

where $p > 1$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \le \ell$. Since $y_t \to \bar{x}$ as $t \to 0$, for t sufficiently small $\|v_t\| \le \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $v_t \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $v_t \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $y_t \to \bar{x}$ and $v_t \to u \in X^*$ weakly, we have $u \in F(\bar{x})$ and

$$\langle u, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \le 0, \quad \forall y \in K$$

(due to (3.5)). Therefore, $\bar{x}$ solves GMVI(F, ϕ, K).

To complete the proof, we need only to prove that GMVI(F, ϕ, K) has a unique solution. Assume by contradiction that GMVI(F, ϕ, K) has two distinct solutions x_1 and x_2 in K. Then it is easy to see that $x_1, x_2 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \le \mathrm{diam}(\Omega_\alpha(\epsilon)) \to 0,$$

a contradiction to (3.1). The proof is complete. □

Theorem 3.6. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a nonempty, weakly closed and locally bounded set-valued mapping, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then* GMVI(F, ϕ, K) *is strongly LP α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \ne \emptyset, \quad \forall \epsilon > 0 \text{ and } \mu(\Omega_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.6}$$

Proof. Suppose that GMVI(F,ϕ,K) is strongly LP α-well-posed in the generalized sense. Let S be the solution set of GMVI(F,ϕ,K). Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S. Then $\{x_n\}$ is a LP α-approximating sequence for GMVI(F,ϕ,K). Since GMVI(F,ϕ,K) is strongly LP α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. It is obvious that $\emptyset \neq S \subset \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$. Now we show that

$$\mu(\Omega_\alpha(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0.$$

Observe that for every $\epsilon > 0$,

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S).$$

Taking into account the compactness of S, we get

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) + \mu(S) = 2e(\Omega_\alpha(\epsilon), S).$$

To prove (3.6), it is sufficient to show that

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Indeed, if $e(\Omega_\alpha(\epsilon), S) \not\to 0$ as $\epsilon \to 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset \mathbf{R}^+$ with $\epsilon_n \to 0$, and $x_n \in \Omega_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0,l), \quad \forall n \in \mathbf{N}, \tag{3.7}$$

where $B(0,l)$ is the closed ball centered at 0 with radius l. By the definition of $\Omega_\alpha(\epsilon_n)$, we know $d(x_n, K) \leq \epsilon_n < \epsilon_n + \frac{1}{n}$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \leq \alpha(x_n - y) + \epsilon_n, \quad \forall y \in K.$$

Thus, there exists $\bar{x}_n \in K$ such that $\|\bar{x}_n - x_n\| < \epsilon_n + \frac{1}{n}$. Let $w_n = \bar{x}_n - x_n$, then we have $w_n + x_n \in K$ with $w_n \to 0$. So $\{x_n\}$ is a LP α-approximating sequence for GMVI(F,ϕ,K). Since GMVI(F,ϕ,K) is strongly LP α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to some point of S. This contradicts (3.7) and so

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Conversely, assume that (3.6) holds. We first show that $\Omega_\alpha(\epsilon)$ is closed for all $\epsilon > 0$. Let $\{x_n\} \subset \Omega_\alpha(\epsilon)$ with $x_n \to x$. Then there exists $u_n \in F(x_n)$ such that $d(x_n, K) \leq \epsilon$ and

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \leq \alpha(x_n - y) + \epsilon, \quad \forall y \in K, n \in \mathbf{N}. \tag{3.8}$$

Since F is locally bounded, there exists a neighborhood of x and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \leq \ell$. Since $x_n \to x$ as $n \to \infty$, for n sufficiently large $\|u_n\| \leq \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_n \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_n \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_n \to x$ and $u_n \to u \in X^*$ weakly, we have $u \in F(x)$ and

$$\langle u, x - y\rangle + \phi(x) - \phi(y) \leq \alpha(x - y) + \epsilon, \quad \forall y \in K$$

(due to (3.8)). It is easy to see $d(x, K) \leq \epsilon$. This shows that $x \in \Omega_\alpha(\epsilon)$ and so $\Omega_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Observe that

$$S = \bigcap_{\epsilon > 0} \Omega_\alpha(\epsilon).$$

Since $\mu(\Omega_\alpha(\epsilon)) \to 0$, the theorem in page 412 of [13] can be applied and one concludes that S is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \to 0.$$

Let $\{\widehat{x}_n\} \subset X$ be a LP α-approximating sequence for GMVI(F, ϕ, K). Then there exists $w_n \in X$ with $w_n \to 0$ such that $\widehat{x}_n + w_n \in K$, and there exist $\widehat{u}_n \in F(\widehat{x}_n)$ and $0 < \epsilon'_n \to 0$ such that

$$\langle \widehat{u}_n, \widehat{x}_n - y\rangle + \phi(\widehat{x}_n) - \phi(y) \leq \alpha(\widehat{x}_n - y) + \epsilon'_n, \quad \forall y \in K, n \in \mathbf{N}.$$

Since $\widehat{x}_n + w_n \in K$, then there exists $\bar{x}_n \in K$ such that $\widehat{x}_n + w_n = \bar{x}_n$. It follows that

$$d(\widehat{x}_n, K) \leq \|\widehat{x}_n - \bar{x}_n\| = \|w_n\| \to 0.$$

Set $\epsilon_n = \max\{\|w_n\|, \epsilon'_n\}$, we get $\widehat{x}_n \in \Omega_\alpha(\epsilon_n)$. From (3.6) and the definition of $\Omega_\alpha(\epsilon_n)$, we have

$$d(\widehat{x}_n, S) \leq e(\Omega_\alpha(\epsilon_n), S) \to 0.$$

Since S is compact, there exists $p_n \in S$ such that

$$\|p_n - \widehat{x}_n\| = d(\widehat{x}_n, S) \to 0.$$

Again from the compactness of S, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which converges strongly to $\bar{p} \in S$. Hence the corresponding subsequence $\{\widehat{x}_{n_k}\}$ of $\{\widehat{x}_n\}$ converges strongly to $\bar{p} \in S$. Therefore, GMVI(F, ϕ, K) is strongly LP α-well-posed in the generalized sense. The proof is complete. □

Remark 3.7. Theorems 3.5 and 3.6 improve, extend and develop Theorems 3.1 and 3.2 in [20] to a great extent because we drop the compactness, $\mathcal{H}$-hemicontinuity and upper semicontinuity of F.

4. Links with Levitin-Polyak Well-Posedness of Inclusion Problems

In this section, we shall show that the Levitin-Polyak well-posedness of a noncompact generalized mixed variational inequality is closely related to the Levitin-Polyak well-posedness of an inclusion problem. Let $A : X \to 2^{X^*}$ be a set-valued mapping. The inclusion problem associated with (A, K) is defined by

$$\text{IP}(A, K): \text{ find } x \in K \text{ such that } 0 \in A(x).$$

Definition 4.1 (see [20]). A sequence $\{x_n\} \subset X$ is called a LP approximating sequence for IP(A, K) if there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$ and $d(0, A(x_n)) \to 0$ as $n \to \infty$, or equivalently, there exists $y_n \in A(x_n)$ such that $\|y_n\| \to 0$ as $n \to \infty$.

Definition 4.2 (see [20])**.** We say that IP(A, K) is strongly (resp. weakly) LP well-posed if it has a unique solution and every LP approximating sequence converges strongly (resp. weakly) to the unique solution of IP(A, K). IP(A, K) is said to be strongly (resp. weakly) LP well-posed in the generalized sense if the solution set S of IP(A, K) is nonempty and every LP approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S.

Remark 4.3. When X is a Hilbert space, $K = X$ and $w_n \equiv 0$, Definitions 4.1 and 4.2 reduce to the Definitions 4.1 and 4.2 in [10], respectively.

Theorem 4.4. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. If* GMVI(F, ϕ, K) *is strongly (resp. weakly) LP well-posed, then* IP$(F + \partial(\phi + \delta_K), K)$ *is strongly (resp. weakly) LP well-posed.*

Proof. Let x^* be the unique solution of GMVI(F, ϕ, K). Then, by Proposition 2.1 we know that x^* is also the unique solution of IP$(F + \partial(\phi + \delta_K), K)$. Let $\{x_n\}$ be a LP approximating sequence for IP$(F + \partial(\phi + \delta_K), K)$. Then, there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exists $y_n \in F(x_n) + \partial(\phi + \delta_K)(x_n)$ such that $\|y_n\| \to 0$ as $n \to \infty$. It is easy to see that $\{x_n\} \subset K$ and there exists $u_n \in F(x_n)$ such that

$$\phi(y) - \phi(x_n) \geq \langle y_n - u_n, y - x_n \rangle, \quad \forall y \in K. \tag{4.1}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0, 1]$ and $z_n \rightharpoonup z (\neq x^*)$. Since $x^*, x_n \in K$, this together with $t_n \in (0, 1]$ yields that $z_n \in K$ and so $z \in K$. For any $y \in K, v \in F(y)$,

$$\begin{aligned} \langle v, z - y \rangle &= \langle v, z - z_n \rangle + \langle v, z_n - x^* \rangle + \langle v, x^* - y \rangle \\ &= \langle v, z - z_n \rangle + t_n \langle v, x_n - x^* \rangle + \langle v, x^* - y \rangle \\ &= \langle v, z - z_n \rangle + t_n \langle v, x_n - y \rangle + (1 - t_n) \langle v, x^* - y \rangle. \end{aligned} \tag{4.2}$$

Since x^* is the unique solution of GMVI(F, ϕ, K), there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^* - y \rangle + \phi(x^*) - \phi(y) \leq 0, \quad \forall y \in K. \tag{4.3}$$

Since F is monotone, we have

$$\langle v, x^* - y \rangle \leq \langle u^*, x^* - y \rangle, \ \langle v, x_n - y \rangle \leq \langle u_n, x_n - y \rangle. \tag{4.4}$$

Note that ϕ is convex. Hence it follows from (4.1)-(4.4) that

$$\begin{aligned}\langle v, z-y\rangle &\leq \langle v, z-z_n\rangle + t_n\phi(y) - t_n\phi(x_n)\\ &\quad + t_n\langle y_n, x_n - y\rangle + (1-t_n)(\phi(y)-\phi(x^*))\\ &= \langle v, z-z_n\rangle + \phi(y) - [t_n\phi(x_n)+(1-t_n)\phi(x^*)] + \frac{\langle y_n, x_n-y\rangle}{\|x_n-x^*\|}\\ &\leq \langle v, z-z_n\rangle + \phi(y) - \phi(z_n) + \frac{\langle y_n, x_n-y\rangle}{\|x_n-x^*\|}, \quad \forall y \in K.\end{aligned} \tag{4.5}$$

Since $\|y_n\| \to 0$, it follows from (4.5) that for any $y \in K, v \in F(y)$,

$$\langle v, z-y\rangle \leq \liminf_{n\to\infty}\left\{\langle v, z-z_n\rangle + \phi(y) - \phi(z_n) + \frac{\langle y_n, x_n-y\rangle}{\|x_n-x^*\|}\right\} \leq \phi(y)-\phi(z).$$

This together with Proposition 2.7 yields that z solves GMVI(F,ϕ,K), a contradiction. Thus, $\{x_n\}$ is bounded.

Suppose that GMVI(F,ϕ,K) is strongly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \leq \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \leq \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_{n_k} \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_{n_k} \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_{n_k} \to \bar{x}$ and $u_{n_k} \to u \in X^*$ weakly, we have $u \in F(\bar{x})$ and

$$\langle u, \bar{x}-y\rangle + \phi(\bar{x}) - \phi(y) \leq 0, \quad \forall y \in K$$

(due to (4.1)). Therefore, $\bar{x}$ solves GMVI(F,ϕ,K). Since GMVI(F,ϕ,K) has a unique solution x^*, we get $\bar{x} = x^*$. This means that $x_n \to x^*$ as $n \to \infty$. Therefore, IP$(F+\partial(\phi+\delta_K),K)$ is strongly LP well-posed.

Suppose that GMVI(F,ϕ,K) is weakly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. It follows from (4.1) that

$$\phi(y) - \phi(x_{n_k}) \geq \langle y_{n_k} - u_{n_k}, y - x_{n_k}\rangle, \quad \forall y \in K.$$

Since F is monotone, ϕ is convex and lower semicontinuous, and $\|y_n\| \to 0$, we have

$$\begin{aligned}\langle v, \bar{x}-y\rangle + \phi(\bar{x}) - \phi(y) &\leq \liminf_{k\to\infty}\{\langle v, x_{n_k}-y\rangle + \phi(x_{n_k}) - \phi(y)\}\\ &\leq \liminf_{k\to\infty}\{\langle u_{n_k}, x_{n_k}-y\rangle + \phi(x_{n_k}) - \phi(y)\}\\ &\leq \liminf_{k\to\infty}\langle y_{n_k}, x_{n_k}-y\rangle = 0, \quad \forall y \in K, v \in F(y).\end{aligned} \tag{4.6}$$

This together with Proposition 2.7, implies that $\bar{x}$ solves GMVI(F,ϕ,K). Since GMVI(F,ϕ,K) has a unique solution x^*, we get $\bar{x} = x^*$. Thus $x_n \rightharpoonup x^*$ and so IP$(F+\partial(\phi+\delta_K),K)$ is weakly LP well-posed. The proof is complete. □

Theorem 4.5. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and differentiable functional. If*

IP$(F+\partial(\phi+\delta_K),K)$ *is strongly (resp. weakly) LP well-posed, then* GMVI(F,ϕ,K) *is strongly (resp. weakly) LP well-posed.*

Proof. Let x^* be the unique solution of IP$(F+\partial(\phi+\delta_K),K)$. Then, by Proposition 2.1 we know that x^* is also the unique solution of GMVI(F,ϕ,K). Let $\{x_n\}$ be a LP approximating sequence for GMVI(F,ϕ,K). Then there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ satisfying

$$\phi(x_n) \le \phi(y) + \langle u_n, y - x_n\rangle + \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}. \tag{4.7}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0,1]$ and $z_n \rightharpoonup z(\neq x^*)$. Since $x^*, x_n \in K$, this together with $t_n \in (0,1]$ yields that $z_n \in K$ and so $z \in K$. For any $y \in K, v \in F(y)$,

$$\begin{aligned} \langle v, z-y\rangle &= \langle v, z-z_n\rangle + \langle v, z_n - x^*\rangle + \langle v, x^*-y\rangle \\ &= \langle v, z-z_n\rangle + t_n\langle v, x_n - x^*\rangle + \langle v, x^*-y\rangle \\ &= \langle v, z-z_n\rangle + t_n\langle v, x_n - y\rangle + (1-t_n)\langle v, x^*-y\rangle. \end{aligned} \tag{4.8}$$

Since x^* is the unique solution of GMVI(F,ϕ,K), there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^*-y\rangle + \phi(x^*) - \phi(y) \le 0, \quad \forall y \in K. \tag{4.9}$$

Since F is monotone, we have

$$\langle v, x^*-y\rangle \le \langle u^*, x^*-y\rangle, \ \langle v, x_n - y\rangle \le \langle u_n, x_n - y\rangle. \tag{4.10}$$

Note that ϕ is convex. Hence it follows from (4.7)-(4.10) that

$$\begin{aligned} \langle v, z-y\rangle &\le \langle v, z-z_n\rangle + t_n\phi(y) - t_n\phi(x_n) \\ &\quad + t_n\epsilon_n + (1-t_n)(\phi(y)-\phi(x^*)) \\ &= \langle v, z-z_n\rangle + \phi(y) - [t_n\phi(x_n) + (1-t_n)\phi(x^*)] + \frac{\epsilon_n}{\|x_n - x^*\|} \\ &\le \langle v, z-z_n\rangle + \phi(y) - \phi(z_n) + \frac{\epsilon_n}{\|x_n-x^*\|}, \quad \forall y \in K. \end{aligned} \tag{4.11}$$

Since $\epsilon_n \to 0$, it follows from (4.11) that for any $y \in K, v \in F(y)$,

$$\langle v, z-y\rangle \le \liminf_{n\to\infty}\{\langle v, z-z_n\rangle + \phi(y) - \phi(z_n) + \frac{\epsilon_n}{\|x_n-x^*\|}\} \le \phi(y) - \phi(z).$$

This together with Proposition 2.7 yields that z solves GMVI(F,ϕ,K), a contradiction. Thus, $\{x_n\}$ is bounded.

Since $x_n + w_n \in K$, there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. Define $\tilde{\phi}_n : X \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y) + \langle u_n, y - x_n\rangle, \quad \forall y \in K, n \in \mathbf{N}.$$

Since ϕ is proper, convex and differentiable, we know that $\tilde{\phi}_n$ is proper, convex and differentiable for all $n \in \mathbf{N}$. It follows from Proposition 2.2.6 of [2] that $\tilde{\phi}_n$ is

Lipschitz continuous. Since $\|w_n\| = \|\bar{x}_n - x_n\| \to 0$, then there exists $0 < \delta_n \to 0$ such that

$$\tilde{\phi}_n(\bar{x}_n) - \tilde{\phi}_n(x_n) \le \delta_n. \tag{4.12}$$

By (4.7) and (4.12), we have

$$\tilde{\phi}_n(\bar{x}_n) \le \tilde{\phi}_n(y) + \delta_n + \epsilon_n, \quad \forall y \in K.$$

By Ekeland Theorem [8], there exists $\widehat{x}_n \in K$ such that

$$\|\widehat{x}_n - \bar{x}_n\| \le \sqrt{\delta_n + \epsilon_n},$$

and

$$\tilde{\phi}_n(\widehat{x}_n) \le \tilde{\phi}_n(y) + \sqrt{\delta_n + \epsilon_n}\|y - \widehat{x}_n\|, \quad \forall y \in K.$$

Thus, $\widehat{x}_n$ minimizes the function $\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n}\|\cdot - \widehat{x}_n\|$. It follows that $0^* \in \partial(\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n}\|\cdot - \widehat{x}_n\|)(\widehat{x}_n)$. That is,

$$0^* \in \partial\tilde{\phi}_n(\widehat{x}_n) + \sqrt{\delta_n + \epsilon_n} B_{X^*}.$$

So there exists

$$x_n^* \in \partial\tilde{\phi}_n(\widehat{x}_n) = \partial\phi(\widehat{x}_n) + u_n \tag{4.13}$$

such that

$$\|x_n^*\| \le \sqrt{\delta_n + \epsilon_n}.$$

Since $\|x_n - \bar{x}_n\| \to 0$ and $\|\widehat{x}_n - \bar{x}_n\| \to 0$, this implies that $\|\widehat{x}_n - x_n\| \le \|x_n - \bar{x}_n\| + \|\widehat{x}_n - \bar{x}_n\| \to 0$. From (4.13) and $0^* \in \partial\delta_k(\widehat{x}_n)$, we have

$$x_n^* - u_n \in \partial\phi(\widehat{x}_n) + \partial\delta_k(\widehat{x}_n) = \partial(\phi + \delta_k)(\widehat{x}_n),$$

which hence leads to

$$\langle u_n - x_n^*, \widehat{x}_n - y\rangle + \phi(\widehat{x}_n) - \phi(y) \le 0, \quad \forall y \in K. \tag{4.14}$$

Suppose that IP$(F + \partial(\phi + \delta_K), K)$ is strongly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \le \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \le \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_{n_k} \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_{n_k} \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_{n_k} \to \bar{x}$ and $u_{n_k} \to u \in X^*$ weakly, we have $u \in F(\bar{x})$. Note that $\|\widehat{x}_n - x_n\| \to 0$ as $n \to \infty$. Thus, we have $\widehat{x}_{n_k} \to \bar{x}$ as $k \to \infty$. Since $u_{n_k} \to u \in X^*$ weakly and $x_n^* \to 0$ as $n \to \infty$, we obtain that $u_{n_k} - x_{n_k}^* \to u$ weakly. Consequently, from (4.14) and the differentiability of ϕ we deduce that

$$\langle u, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \le 0, \quad \forall y \in K.$$

Therefore, $\bar{x}$ solves GMVI(F, ϕ, K). Since GMVI(F, ϕ, K) has a unique solution x^*, we get $\bar{x} = x^*$. This means that $x_n \to x^*$ as $n \to \infty$. Therefore, GMVI(F, ϕ, K) is strongly LP well-posed.

Suppose that IP$(F + \partial(\phi + \delta_K), K)$ is weakly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. It follows from (4.7) that

$$\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y) \le \epsilon_{n_k}, \quad \forall y \in K.$$

Since F is monotone, ϕ is convex and lower semicontinuous and $\epsilon_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \langle v, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) &\le \liminf_{k\to\infty}\{\langle v, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\epsilon_{n_k} = 0, \quad \forall y \in K, v \in F(y). \end{aligned} \tag{4.15}$$

This together with Proposition 2.7, implies that $\bar{x}$ solves GMVI(F, ϕ, K). Since GMVI(F, ϕ, K) has a unique solution x^*, we get $\bar{x} = x^*$. Thus $x_n \rightharpoonup x^*$ and so GMVI(F, ϕ, K) is weakly LP well-posed. The proof is complete. □

Theorem 4.6. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a nonempty set-valued mapping, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. If* GMVI(F, ϕ, K) *is strongly (resp. weakly) LP α-well-posed in the generalized sense with $\alpha(z) = \frac{1}{2}\|z\|^2$ $\forall z \in X$, then* IP$(F + \partial(\phi + \delta_K), K)$ *is strongly (resp. weakly) LP well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be a LP approximating sequence for IP$(F + \partial(\phi + \delta_K), K)$. Then there exists $y_n \in F(x_n) + \partial(\phi + \delta_K)(x_n)$ such that $\|y_n\| \to 0$. It is easy to see that $\{x_n\} \subset K$ and there exists $u_n \in F(x_n)$ such that

$$\phi(y) - \phi(x_n) \ge \langle y_n - u_n, y - x_n\rangle, \quad \forall y \in K.$$

Thus,

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \langle y_n, x_n - y\rangle \le \frac{1}{2}\|x_n - y\|^2 + \frac{1}{2}\|y_n\|^2, \quad \forall y \in K, n \in \mathbf{N}.$$

This together with $\|y_n\| \to 0$ implies that $\{x_n\}$ is a LP α-approximating sequence for GMVI(F, ϕ, K) with $\alpha(z) = \frac{1}{2}\|z\|^2$ $\forall z \in X$. Since GMVI(F, ϕ, K) is strongly (resp. weakly) LP α-well-posed in the generalized sense with $\alpha(z) = \frac{1}{2}\|z\|^2$ $\forall z \in X$, $\{x_n\}$ has a subsequence that converges strongly (resp. weakly) to some solution x^* of GMVI(F, ϕ, K). By Proposition 2.1, x^* is also a solution of IP$(F+\partial(\phi+\delta_K), K)$ and so IP$(F + \partial(\phi + \delta_K), K)$ is strongly (resp. weakly) LP well-posed in the generalized sense. The proof is complete. □

Theorem 4.7. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and differentiable functional. If* IP$(F + \partial(\phi+\delta_K), K)$ *is strongly (resp. weakly) LP well-posed in the generalized sense, then* GMVI(F, ϕ, K) *is strongly (resp. weakly) LP well-posed in the generalized sense.*

Proof. The conclusion follows from similar arguments to those in the proof of Theorem 4.5. □

Remark 4.8. Theorems 4.4, 4.5, 4.6 and 4.7 improve, extend and develop Theorems 4.1-4.4 in [20] to a great extent because we drop the compactness, $\mathcal{H}$-hemicontinuity, and $\mathcal{H}$-uniform continuity of F. In addition, if X is a Hilbert space, $K = X$ and F is a single valued mapping, Theorems 4.1-4.4 in [20] reduce to Theorems 4.1-4.4 of [10], respectively.

5. Links with Levitin-Polyak Well-Posedness of Fixed Point Problems

In this section, we shall investigate the relations between the Levitin-Polyak well-posedness of noncompact generalized mixed variational inequalities and the Levitin-Polyak well-posedness of the corresponding fixed point problems. Let $T : X \to 2^{X^*}$ be a set-valued mapping. The fixed point problem associated with (T, K) is defined by

$$\mathrm{FP}(T, K): \text{ find } x \in K \text{ such that } x \in T(x).$$

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere. A Banach space X is said to be (i) strictly convex if for any $x, y \in U,\ x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1$; (ii) smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. The modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in U,\ \|x - y\| \geq \epsilon\},$$

and the modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U,\ \|y\| \leq \tau\}.$$

In this section, we suppose that $q > 1$ and $s > 1$ are fixed numbers.

Definition 5.1. A Banach space X is said to be

(i) q-uniformly convex if there exists a constant $c > 0$ such that $\delta_X(\epsilon) \geq c\epsilon^q$ for all $\epsilon \in (0, 2)$;

(ii) q-uniformly smooth if there exists a constant $k > 0$ such that $\rho_X(\tau) \leq k\tau^q$.

The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle j_q(x), x\rangle = \|x\|^q,\ \|j_q(x)\| = \|x\|^{q-1}\}.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is well known that J_q has the following properties: (a) J_q is bounded; (b) if X is smooth, then J_q is single-valued; (c) if X is strictly convex, then J_q is one-to-one and strictly monotone.

Lemma 5.2 (see [26]). *Let X be a q-uniformly smooth Banach space. Then there exists a constant $L_q > 0$ such that*

$$\|J_q(x) - J_q(y)\| \leq L_q\|x - y\|^{q-1}, \quad \forall x, y \in X.$$

Lemma 5.3 (see [26]). *Let X be a q-uniformly convex Banach space. Then there exists a constant $K_q > 0$ such that*

$$\langle J_q(x) - J_q(y), x - y\rangle \geq K_q\|x - y\|^q, \quad \forall x, y \in X.$$

Lemma 5.4 (see [9])**.** *Let X be a q-uniformly convex Banach space and $M : X \to 2^{X^*}$ be a maximal monotone operator. Then for every $\lambda > 0$, $(J_q + \lambda M)^{-1}$ is well-defined and single-valued.*

We denote $\Pi_\lambda^\phi = (J_q + \lambda\partial\phi)^{-1}$. By the definition of Π_λ^ϕ and Lemma 5.4, it is easy to prove the following proposition.

Proposition 5.5. *Let X be a q-uniformly convex Banach space, and K be a nonempty, closed and convex subset of X, and $F : X \to 2^{X^*}$ be a nonempty set-valued mapping. Let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then the following conclusions are equivalent:*

(i) *x solves* GMVI(F, ϕ, K)*;*

(ii) *x solves the fixed-point problem*

$$\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K): \text{ find } x \in K \text{ such that } x \in \Pi_\lambda^{\phi+\delta_k}(J_q(x) - \lambda F(x)).$$

Definition 5.6. A sequence $\{x_n\} \subset X$ is called a LP approximating sequence for $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K)$ if there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ such that $y_n = \Pi_\lambda^{\phi+\delta_k}(J_q(x_n) - \lambda u_n)$, $\|x_n - y_n\| \to 0$ as $n \to \infty$, and $\lim_{n\to\infty}\{\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)\} = 0$.

Definition 5.7. We say that $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K)$ is strongly (resp. weakly) LP well-posed if it has a unique solution and every LP approximating sequence for $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K)$ converges strongly (resp. weakly) to the unique solution. $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K)$ is said to be strongly (resp. weakly) LP well-posed in the generalized sense if the solution set S of $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_k}(J_q - \lambda F), K)$ is nonempty and every LP approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S.

Theorem 5.8. *Let X be a s-uniformly convex and q-uniformly smooth Banach space and K be a nonempty, closed and convex subset of X. Let $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. If* GMVI(F, ϕ, K) *is strongly (resp. weakly) LP well-posed, then* $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ *is strongly (resp. weakly) LP well-posed, where $\lambda > 0$ is a constant.*

Proof. Let x^* be the unique solution of GMVI(F, ϕ, K). Then, by Proposition 5.5 we know that x^* is also the unique solution of $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Let $\{x_n\}$ be a LP approximating sequence for $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Then, there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ such that $y_n = \Pi_\lambda^{\phi+\delta_k}(J_q(x_n) - \lambda u_n)$, $\|x_n - y_n\| \to 0$ as $n \to \infty$, and $\lim_{n\to\infty}\{\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)\} = 0$. By the definition of $\Pi_\lambda^{\phi+\delta_K}$, we get

$$\frac{J_q(x_n) - J_q(y_n)}{\lambda} - u_n \in \partial(\phi + \delta_K)(y_n).$$

It is easy to see that $\{y_n\} \subset K$ and

$$\phi(y) - \phi(y_n) \geq \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda} - u_n, y - y_n \Big\rangle, \quad \forall y \in K, n \in \mathbf{N}. \tag{5.1}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0,1]$ and $z_n \rightharpoonup z(\neq x^*)$. Since $x^*, x_n \in K$, this together with $t_n \in (0,1]$ yields that $z_n \in K$ and so $z \in K$. For any $y \in K, v \in F(y)$,

$$\begin{aligned} \langle v, z - y\rangle &= \langle v, z - z_n\rangle + \langle v, z_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n - y\rangle + (1 - t_n)\langle v, x^* - y\rangle. \end{aligned} \tag{5.2}$$

Since x^* is the unique solution of GMVI(F, ϕ, K), there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^* - y\rangle + \phi(x^*) - \phi(y) \le 0, \quad \forall y \in K. \tag{5.3}$$

Since F is monotone, we have

$$\langle v, x^* - y\rangle \le \langle u^*, x^* - y\rangle,\ \langle v, x_n - y\rangle \le \langle u_n, x_n - y\rangle. \tag{5.4}$$

Note that ϕ is convex. Hence it follows from (5.1)-(5.4) that

$$\begin{aligned} \langle v, z - y\rangle &\le \langle v, z - z_n\rangle + t_n\langle u_n, y_n - y + x_n - y_n\rangle + (1 - t_n)(\phi(y) - \phi(x^*)) \\ &\le \langle v, z - z_n\rangle + t_n\Big[\langle u_n, x_n - y_n\rangle + \phi(y) - \phi(y_n) \\ &\quad + \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, y_n - y\Big\rangle\Big] + (1 - t_n)(\phi(y) - \phi(x^*)) \\ &= \langle v, z - z_n\rangle + t_n\Big[\langle u_n, x_n - y_n\rangle + \phi(y) - \phi(x_n) + \phi(x_n) - \phi(y_n) \\ &\quad + \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, y_n - y\Big\rangle\Big] + (1 - t_n)(\phi(y) - \phi(x^*)) \\ &= \langle v, z - z_n\rangle + \phi(y) - [t_n\phi(x_n) + (1 - t_n)\phi(x^*)] \\ &\quad + t_n[\langle u_n, x_n - y_n\rangle \\ &\quad + \phi(x_n) - \phi(y_n)] + \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, t_n(y_n - y)\Big\rangle \\ &\le \langle v, z - z_n\rangle + \phi(y) - \phi(z_n) + t_n[\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)] \\ &\quad + \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, t_n(y_n - y)\Big\rangle, \quad \forall y \in K. \end{aligned} \tag{5.5}$$

According to Lemma 5.2 and $\|x_n - y_n\| \to 0$, we have

$$\|J_q(x_n) - J_q(y_n)\| \le L_q\|x_n - y_n\|^{q-1} \to 0$$

and

$$t_n(y_n - y) = \frac{x_n - x^*}{\|x_n - x^*\|} + t_n(y_n - x_n + x^* - y).$$

Since $\lim_{n\to\infty}\{\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)\} = 0$, it follows from (5.5) that for any $y \in K, v \in F(y)$,

$$\langle v, z - y\rangle \le \limsup_{n\to\infty}\Big\{\langle v, z - z_n\rangle + \phi(y) - \phi(z_n) + t_n[\langle u_n, x_n - y_n\rangle$$

$$+ \phi(x_n) - \phi(y_n)] + \frac{1}{\lambda}\langle J_q(x_n) - J_q(y_n), t_n(y_n - y)\rangle\Big\}$$
$$\le \phi(y) - \phi(z).$$

This together with Proposition 2.7 yields that z solves GMVI(F, ϕ, K), a contradiction. Thus, $\{x_n\}$ is bounded.

Suppose that GMVI(F, ϕ, K) is strongly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \le \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \le \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_{n_k} \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_{n_k} \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_{n_k} \to \bar{x}$ and $u_{n_k} \to u \in X^*$ weakly, we have $u \in F(\bar{x})$. It follows from (5.1) that

$$\langle u_{n_k}, y_{n_k} - y\rangle + \phi(y_{n_k}) - \phi(y) \le \Big\langle \frac{J_q(x_{n_k}) - J_q(y_{n_k})}{\lambda}, y_{n_k} - y\Big\rangle, \quad \forall y \in K. \tag{5.6}$$

Since ϕ is lower semicontinuous, $x_{n_k} \to \bar{x}$, $\|x_n - y_n\| \to 0$ and $u_{n_k} \rightharpoonup u$, we have

$$\langle u, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \le 0, \quad \forall y \in K.$$

Therefore, $\bar{x}$ solves GMVI(F, ϕ, K). Since GMVI(F, ϕ, K) has a unique solution x^*, we get $\bar{x} = x^*$. This means that $x_n \to x^*$ as $n \to \infty$. Therefore, FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is strongly LP well-posed.

Suppose that GMVI(F, ϕ, K) is weakly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is monotone, ϕ is convex and lower semicontinuous, and $\|x_n - y_n\| \to 0$, from (5.6) we have

$$\begin{aligned}
\langle v, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) &\le \liminf_{k\to\infty}\{\langle v, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\}\\
&\le \liminf_{k\to\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\}\\
&\le \limsup_{k\to\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\}\\
&= \limsup_{k\to\infty}\{\langle u_{n_k}, y_{n_k} - y\rangle + \phi(y_{n_k}) - \phi(y)\\
&\quad + \langle u_{n_k}, x_{n_k} - y_{n_k}\rangle + \phi(x_{n_k}) - \phi(y_{n_k})\}\\
&\le \limsup_{k\to\infty}\Big\{\Big\langle \frac{J_q(x_{n_k}) - J_q(y_{n_k})}{\lambda}, y_{n_k} - y\rangle\\
&\quad + \langle u_{n_k}, x_{n_k} - y_{n_k}\rangle + \phi(x_{n_k}) - \phi(y_{n_k})\Big\}\\
&\le 0, \quad \forall y \in K, v \in F(y).
\end{aligned} \tag{5.7}$$

This together with Proposition 2.7, implies that $\bar{x}$ solves GMVI(F, ϕ, K). Since GMVI(F, ϕ, K) has a unique solution x^*, we get $\bar{x} = x^*$. Thus $x_n \rightharpoonup x^*$ and so FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is weakly LP well-posed. The proof is complete. □

Theorem 5.9. *Let X be a s-uniformly convex and q-uniformly smooth Banach space and K be a nonempty, closed and convex subset of X. Let $F : X \to 2^{X^*}$ be a* *weakly closed set-valued mapping which is locally bounded and monotone, and let* $\phi : X \to \mathbf{R} \cup \{+\infty\}$ *be a proper, convex and differentiable functional. If* $\mathrm{FP}(\varPi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ *is strongly (resp. weakly) LP well-posed, then* $\mathrm{GMVI}(F, \phi, K)$ *is strongly (resp. weakly) LP well-posed, where* $\lambda > 0$ *is a constant.*

Proof. Let x^* be the unique solution of $\mathrm{FP}(\varPi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Then, by Proposition 5.5 we know that x^* is also the unique solution of $\mathrm{GMVI}(F, \phi, K)$. Let $\{x_n\}$ be a LP approximating sequence for $\mathrm{GMVI}(F, \phi, K)$. Then there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ satisfying

$$\phi(x_n) \leq \phi(y) + \langle u_n, y - x_n \rangle + \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}. \tag{5.8}$$

We claim that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0, 1]$ and $z_n \rightharpoonup z (\neq x^*)$. Since $x^*, x_n \in K$, this together with $t_n \in (0, 1]$ yields that $z_n \in K$ and so $z \in K$. For any $y \in K, v \in F(y)$,

$$\begin{aligned} \langle v, z - y \rangle &= \langle v, z - z_n \rangle + \langle v, z_n - x^* \rangle + \langle v, x^* - y \rangle \\ &= \langle v, z - z_n \rangle + t_n \langle v, x_n - x^* \rangle + \langle v, x^* - y \rangle \\ &= \langle v, z - z_n \rangle + t_n \langle v, x_n - y \rangle + (1 - t_n) \langle v, x^* - y \rangle. \end{aligned} \tag{5.9}$$

Since x^* is the unique solution of $\mathrm{GMVI}(F, \phi, K)$, there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^* - y \rangle + \phi(x^*) - \phi(y) \leq 0, \quad \forall y \in K. \tag{5.10}$$

Since F is monotone, we have

$$\langle v, x^* - y \rangle \leq \langle u^*, x^* - y \rangle, \ \langle v, x_n - y \rangle \leq \langle u_n, x_n - y \rangle. \tag{5.11}$$

Note that ϕ is convex. Hence it follows from (5.8)-(5.11) that

$$\begin{aligned} \langle v, z - y \rangle &\leq \langle v, z - z_n \rangle \\ &\quad + t_n \phi(y) - t_n \phi(x_n) + t_n \epsilon_n + (1 - t_n)(\phi(y) - \phi(x^*)) \\ &= \langle v, z - z_n \rangle + \phi(y) - [t_n \phi(x_n) + (1 - t_n)\phi(x^*)] + \frac{\epsilon_n}{\|x_n - x^*\|} \\ &\leq \langle v, z - z_n \rangle + \phi(y) - \phi(z_n) + \frac{\epsilon_n}{\|x_n - x^*\|}, \quad \forall y \in K. \end{aligned} \tag{5.12}$$

Since $\epsilon_n \to 0$, it follows from (5.12) that for any $y \in K, v \in F(y)$,

$$\langle v, z - y \rangle \leq \liminf_{n \to \infty} \{\langle v, z - z_n \rangle + \phi(y) - \phi(z_n) + \frac{\epsilon_n}{\|x_n - x^*\|}\} \leq \phi(y) - \phi(z).$$

This together with Proposition 2.7 yields that z solves $\mathrm{GMVI}(F, \phi, K)$, a contradiction. Thus, $\{x_n\}$ is bounded.

Since $x_n + w_n \in K$, there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. Define $\tilde{\phi}_n : X \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y) + \langle u_n, y - x_n \rangle, \quad \forall y \in K, n \in \mathbf{N}.$$

Since ϕ is proper, convex and differentiable, we know that $\tilde{\phi}_n$ is proper, convex and differentiable for all $n \in \mathbf{N}$. It follows from Proposition 2.2.6 of [2] that $\tilde{\phi}_n$ is Lipschitz continuous. Since $\|w_n\| = \|\bar{x}_n - x_n\| \to 0$, then there exists $0 < \delta_n \to 0$ such that

$$\tilde{\phi}_n(\bar{x}_n) - \tilde{\phi}_n(x_n) \leq \delta_n. \tag{5.13}$$

It follows from (5.8) and (5.13) that

$$\tilde{\phi}_n(\bar{x}_n) \leq \tilde{\phi}_n(y) + \delta_n + \epsilon_n, \quad \forall y \in K.$$

By Ekeland Theorem [8], there exists $\widehat{x}_n \in K$ such that

$$\|\widehat{x}_n - \bar{x}_n\| \leq \sqrt{\delta_n + \epsilon_n},$$

and

$$\tilde{\phi}_n(\widehat{x}_n) \leq \tilde{\phi}_n(y) + \sqrt{\delta_n + \epsilon_n}\|y - \widehat{x}_n\|, \quad \forall y \in K.$$

Thus, $\widehat{x}_n$ minimizes the function $\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n}\| \cdot - \widehat{x}_n\|$. It follows that $0^* \in \partial(\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n}\| \cdot - \widehat{x}_n\|)(\widehat{x}_n)$. That is,

$$0^* \in \partial\tilde{\phi}_n(\widehat{x}_n) + \sqrt{\delta_n + \epsilon_n} B_{X^*}.$$

So there exists

$$x_n^* \in \partial\tilde{\phi}_n(\widehat{x}_n) = \partial\phi(\widehat{x}_n) + u_n \tag{5.14}$$

such that

$$\|x_n^*\| \leq \sqrt{\delta_n + \epsilon_n}.$$

Since $\|x_n - \bar{x}_n\| \to 0$ and $\|\widehat{x}_n - \bar{x}_n\| \to 0$, we have $\|\widehat{x}_n - x_n\| \leq \|x_n - \bar{x}_n\| + \|\widehat{x}_n - \bar{x}_n\| \to 0$. From (5.14) and $0^* \in \partial\delta_k(\widehat{x}_n)$, we have

$$x_n^* - u_n \in \partial\phi(\widehat{x}_n) + \partial\delta_k(\widehat{x}_n) = \partial(\phi + \delta_k)(\widehat{x}_n),$$

which hence leads to

$$\widehat{x}_n = \Pi_\lambda^{\phi+\delta_K}(J_q(\widehat{x}_n) + \lambda x_n^* - \lambda u_n). \tag{5.15}$$

Suppose that $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is strongly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \leq \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \leq \ell$. Furthermore, according to Lemma 5.2 and $\|\widehat{x}_n - x_n\| \to 0$, we have

$$\|J_q(\widehat{x}_n) - J_q(x_n)\| \leq L_q\|\widehat{x}_n - x_n\|^{q-1} \to 0.$$

Meantime, from (5.15) we observe that

$$\begin{aligned}
\|x_{n_k} - \Pi_\lambda^{\phi+\delta_K}(J_q(x_{n_k}) - \lambda u_{n_k})\| &\leq \|x_{n_k} - \widehat{x}_{n_k}\| + \|\widehat{x}_{n_k} - \Pi_\lambda^{\phi+\delta_K}(J_q(x_{n_k}) - \lambda u_{n_k})\| \\
&= \|x_{n_k} - \widehat{x}_{n_k}\| + \|\Pi_\lambda^{\phi+\delta_K}(J_q(\widehat{x}_{n_k}) + \lambda x_{n_k}^* \\
&\quad - \lambda u_{n_k}) - \Pi_\lambda^{\phi+\delta_K}(J_q(x_{n_k}) - \lambda u_{n_k})\| \\
&\leq \|x_{n_k} - \widehat{x}_{n_k}\| + \|J_q(\widehat{x}_{n_k}) - J_q(x_{n_k}) + \lambda x_{n_k}^*\| \\
&\leq \|x_{n_k} - \widehat{x}_{n_k}\| + \|J_q(\widehat{x}_{n_k}) - J_q(x_{n_k})\| + \lambda\|x_{n_k}^*\| \\
&\quad \to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

and

$$\limsup_{k\to\infty}\{\langle u_{n_k}, x_{n_k}-y_{n_k}\rangle + \phi(x_{n_k}) - \phi(y_{n_k})\} \le 0$$

where $y_{n_k} = \Pi_\lambda^{\phi+\delta_K}(J_q(x_{n_k}) - \lambda u_{n_k})$ for all $k \in \mathbf{N}$. So, it follows that $\{x_{n_k}\}$ is a LP approximating sequence for $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Since x^* is the unique solution of $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ and $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is strongly LP well-posed, we deduce that $x_{n_k} \to x^*$ as $k \to \infty$. This means that $x_n \to x^*$ as $n \to \infty$. Therefore, $\mathrm{GMVI}(F, \phi, K)$ is strongly LP well-posed.

Suppose that $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is weakly LP well-posed. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. It follows from (5.8) that

$$\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y) \le \epsilon_{n_k}, \quad \forall y \in K.$$

Since F is monotone, ϕ is convex and lower semicontinuous and $\epsilon_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \langle v, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) &\le \liminf_{k\to\infty}\{\langle v, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\epsilon_{n_k} = 0, \quad \forall y \in K, v \in F(y). \end{aligned} \tag{5.16}$$

This together with Proposition 2.7, implies that $\bar{x}$ solves $\mathrm{GMVI}(F, \phi, K)$. Since $\mathrm{GMVI}(F, \phi, K)$ has a unique solution x^*, we get $\bar{x} = x^*$. Thus $x_n \rightharpoonup x^*$ and so $\mathrm{GMVI}(F, \phi, K)$ is weakly LP well-posed. The proof is complete. □

Remark 5.10. The $\mathcal{H}$-uniform continuity and compactness of F in Theorems 5.1-5.2 of [20] are replaced by the local boundedness and weak closedness of F in our Theorems 5.8 and 5.9. If $X = \mathbf{R^n}$, $\phi = \delta_k$ and F is a single valued mapping, Theorem 5.1-5.2 in [20] reduce to Theorem 5.3 of Hu and Fang [11].

Theorem 5.11. *Let X be a s-uniformly convex and q-uniformly smooth Banach space and K be a nonempty, closed and convex subset of X. Let $F : X \to 2^{X^*}$ be a nonempty set-valued mapping. Let $\phi : X \to \mathbf{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous functional. If* $\mathrm{GMVI}(F, \phi, K)$ *is strongly (resp. weakly) LP α-well-posed in the generalized sense with $\alpha(z) = \frac{1}{\lambda}\|z\|^2$ $\forall z \in X$, then* $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ *is strongly (resp. weakly) LP well-posed in the generalized sense, where $\lambda > 0$ is a constant.*

Proof. Let $\{x_n\}$ be a LP approximating sequence for $\mathrm{FP}(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Then there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ such that $y_n = \Pi_\lambda^{\phi+\delta_k}(J_q(x_n) - \lambda u_n)$, $\|x_n - y_n\| \to 0$ as $n \to \infty$, and $\lim_{n\to\infty}\{\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)\} = 0$. By the definition of $\Pi_\lambda^{\phi+\delta_K}$, we get

$$\frac{J_q(x_n) - J_q(y_n)}{\lambda} - u_n \in \partial(\phi + \delta_K)(y_n).$$

It is easy to see that $\{y_n\} \subset K$ and

$$\phi(y) - \phi(y_n) \ge \langle \frac{J_q(x_n) - J_q(y_n)}{\lambda} - u_n, y - y_n\rangle, \quad \forall y \in K, n \in \mathbf{N},$$

which hence implies that

$$\begin{aligned}
\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) &= \langle u_n, y_n - y\rangle + \phi(y_n) - \phi(y) \\
&\quad + \langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n) \\
&\le \Big\langle \frac{J_q(x_n) - J_q(y_n)}{\lambda}, y_n - y \Big\rangle + \langle u_n, x_n - y_n\rangle \\
&\quad + \phi(x_n) - \phi(y_n) \\
&\le \frac{1}{2\lambda}(\|J_q(x_n) - J_q(y_n)\|^2 + \|y_n - y\|^2) \\
&\quad + |\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)| \\
&\le \frac{1}{2\lambda}(\|J_q(x_n) - J_q(y_n)\|^2 + 2\|x_n - y_n\|^2 + 2\|x_n - y\|^2) \\
&\quad + |\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)| \\
&= \frac{1}{\lambda}\|x_n - y\|^2 + \frac{1}{2\lambda}(\|J_q(x_n) - J_q(y_n)\|^2 + 2\|x_n - y_n\|^2) \\
&\quad + |\langle u_n, x_n - y_n\rangle + \phi(x_n) - \phi(y_n)|.
\end{aligned}$$

By Lemma 5.2, we get

$$\|J_q(x_n) - J_q(y_n)\| \le L_q\|x_n - y_n\|^{q-1} \to 0.$$

Thus, we know that $\{x_n\}$ is a LP α-approximating sequence for GMVI(F, ϕ, K) with $\alpha(z) = \frac{1}{\lambda}\|z\|^2\ \forall z \in X$. If GMVI$(F, \phi, K)$ is strongly LP α-well-posed in the generalized sense with $\alpha(z) = \frac{1}{\lambda}\|z\|^2\ \forall z \in X$, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$, where x^* is a solution of GMVI(F, ϕ, K). By Proposition 5.5, we get x^* is also a solution of FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Thus, FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is strongly LP well-posed in the generalized sense.

If GMVI(F, ϕ, K) is weakly LP α-well-posed in the generalized sense with $\alpha(z) = \frac{1}{\lambda}\|z\|^2\ \forall z \in X$, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$, where x^* is a solution of GMVI(F, ϕ, K). By Proposition 5.5, we get x^* is also a solution of FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$. Thus, FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ is weakly LP well-posed in the generalized sense. The proof is complete. □

Theorem 5.12. *Let X be a s-uniformly convex and q-uniformly smooth Banach space and K be a nonempty, closed and convex subset of X. Let $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and differentiable functional. If* FP$(\Pi_\lambda^{\phi+\delta_K}(J_q - \lambda F), K)$ *is strongly (resp. weakly) LP well-posed in the generalized sense, then* GMVI(F, ϕ, K) *is strongly (resp. weakly) LP well-posed in the generalized sense, where $\lambda > 0$ is a constant.*

Proof. The conclusion follows from the arguments similar to those in the proof of Theorem 5.9. □

Remark 5.13. Theorems 5.8, 5.9, 5.11 and 5.12 improve, extend and develop Theorems 5.1-5.4 in [20] to a great extent because we drop the compactness and $\mathcal{H}$-uniform continuity of F. In addition, if X is a Hilbert space, $K = X$ and F is a single valued mapping, Theorems 5.1-5.4 of [20] reduce to Theorems 5.1-5.4 of [10], respectively.

6. Conditions for Levitin-Polyak Well-Posedness

In this section, we shall derive some conditions under which a noncompact generalized mixed variational inequality in Banach spaces is Levitin-Polyak well-posed.

Theorem 6.1. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a weakly closed set-valued mapping which is locally bounded and monotone, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and uniformly continuous functional. Then the following conclusions are equivalent:*

(i) GMVI(F, ϕ, K) *has a unique solution;*
(ii) GMVI(F, ϕ, K) *is strongly LP well-posed;*
(iii) GMVI(F, ϕ, K) *is weakly LP well-posed.*

Proof. It is clear that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). Next, we show that and (i) $\Rightarrow$ (iii).
Indeed, suppose that GMVI(F, ϕ, K) has a unique solution.

(i) $\Rightarrow$ (ii). If GMVI(F, ϕ, K) is not strongly LP well-posed, then there exists a LP approximating sequence $\{x_n\}$ for GMVI(F, ϕ, K) such that $\{x_n\}$ does not converge strongly to x^*. Thus, there exists $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}. \tag{6.1}$$

Since $x_n + w_n \in K$, then there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. Thus,

$$d(x_n, K) \le \|x_n - \bar{x}_n\| = \|w_n\| \to 0 \quad \text{as } n \to \infty.$$

We claim that $\{x_n\}$ is bounded. As a matter of fact, if $\{x_n\}$ is unbounded, then $\{\bar{x}_n\}$ is an unbounded sequence in K. Without loss of generality we may assume that $\|\bar{x}_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|\bar{x}_n - x^*\|}, \quad z_n = x^* + t_n(\bar{x}_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0, 1]$ and $z_n \rightharpoonup z(\ne x^*)$. Then we have for each $y \in K, v \in F(y)$,

$$\begin{aligned}\langle v, z - y\rangle &= \langle v, z - z_n\rangle + \langle v, z_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, \bar{x}_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n + w_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n - y\rangle + (1 - t_n)\langle v, x^* - y\rangle + t_n\langle v, w_n\rangle.\end{aligned} \tag{6.2}$$

Since x^* is the unique solution of GMVI(F, ϕ, K), there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^* - y\rangle + \phi(x^*) - \phi(y) \le 0, \quad \forall y \in K. \tag{6.3}$$

Since F is monotone, we have

$$\langle v, x^* - y\rangle \le \langle u^*, x^* - y\rangle, \ \langle v, x_n - y\rangle \le \langle u_n, x_n - y\rangle. \tag{6.4}$$

It follows from (6.1)-(6.4) and the convexity of ϕ that for all $v \in F(y)$,

$$
\begin{aligned}
\langle v, z-y\rangle &\le \langle v, z-z_n\rangle + t_n\phi(y) - t_n\phi(x_n) + t_n\epsilon_n \\
&\quad + (1-t_n)(\phi(y)-\phi(x^*)) + t_n\langle v, w_n\rangle \\
&= \langle v, z-z_n\rangle + \phi(y) \\
&\quad - [t_n\phi(x_n) + (1-t_n)\phi(x^*)] + t_n\epsilon_n + t_n\langle v, w_n\rangle \\
&= \langle v, z-z_n\rangle + \phi(y) \qquad (6.5)\\
&\quad - [t_n\phi(\bar{x}_n) + (1-t_n)\phi(x^*) + t_n\phi(x_n) - t_n\phi(\bar{x}_n)] \\
&\quad + t_n\epsilon_n + t_n\langle v, w_n\rangle \\
&\le \langle v, z-z_n\rangle + \phi(y) - \phi(z_n) - t_n[\phi(x_n)-\phi(\bar{x}_n)] + t_n\epsilon_n \\
&\quad + t_n\langle v, w_n\rangle, \quad \forall y \in K.
\end{aligned}
$$

Since ϕ is uniformly continuous, we have

$$
\begin{aligned}
\langle v, z-y\rangle &\le \liminf_{n\to\infty}\{\langle v, z-z_n\rangle + \phi(y) - \phi(z_n) - t_n[\phi(x_n)-\phi(\bar{x}_n)] \\
&\quad + t_n\epsilon_n + t_n\langle v, w_n\rangle\} \\
&\le \phi(y) - \phi(z), \quad \forall y \in K.
\end{aligned}
$$

This together with Proposition 2.7 yields that z solves GMVI(F,ϕ,K), a contradiction. Thus, $\{x_n\}$ is bounded.

We claim that $x_n \to x^*$ as $n \to \infty$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \le \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \le \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_{n_k} \to u \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_{n_k} \to u \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_{n_k} \to \bar{x}$ and $u_{n_k} \to u \in X^*$ weakly, we have $u \in F(\bar{x})$ and

$$\langle u, \bar{x}-y\rangle + \phi(\bar{x}) - \phi(y) \le 0, \quad \forall y \in K$$

(due to (6.1)). This together with Proposition 2.7 yields that $\bar{x}$ solves GMVI(F,ϕ,K). Since GMVI(F,ϕ,K) has a unique solution x^*, we have $\bar{x} = x^*$. Thus $x_n \to x^*$, a contradiction. Therefore, GMVI(F,ϕ,K) is strongly LP well-posed.

(i) $\Rightarrow$ (iii). If GMVI(F,ϕ,K) is not weakly LP well-posed, then there exists a LP approximating sequence $\{x_n\}$ for GMVI(F,ϕ,K) such that $\{x_n\}$ does not converge weakly to x^*. Thus, there exists $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}. \tag{6.6}$$

Since $x_n + w_n \in K$, then there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. Thus,

$$d(x_n, K) \le \|x_n - \bar{x}_n\| = \|w_n\| \to 0 \quad \text{as } n \to \infty.$$

We claim that $\{x_n\}$ is bounded. As a matter of fact, repeating the same arguments as in the proof of (i) $\Rightarrow$ (ii), we can prove that $\{x_n\}$ is bounded.

We claim that $x_n \rightharpoonup x^*$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Clearly $\bar{x} \in K$. It follows from (6.6) that

$$\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y) \le \epsilon_{n_k}, \quad \forall y \in K.$$

Since F is monotone and ϕ is lower semicontinuous, we have

$$\begin{aligned} \langle v, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) &\le \liminf_{k\to\infty}\{\langle v, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\to\infty}\epsilon_{n_k} = 0, \quad \forall y \in K, v \in F(y). \end{aligned} \tag{6.7}$$

This together with Proposition 2.7, implies that $\bar{x}$ solves GMVI(F, ϕ, K). Since GMVI(F, ϕ, K) has a unique solution x^*, we have $\bar{x} = x^*$. Thus $x_n \rightharpoonup x^*$, a contradiction. So GMVI(F, ϕ, K) is weakly LP well-posed. The proof is complete. □

Remark 6.2. The $\mathcal{H}$-hemicontinuity and compactness of F in Theorem 6.1 of [20] is replaced by the local boundedness and weak closedness of F in our Theorem 6.1. If X ia a Hilbert space, $k = X$ and F is a single valued mapping, Theorem 6.1 in [20] reduces to Theorem 6.1 in [10].

Now, for any $\delta_0 \ge 0$, we denote $M(\delta_0) = \{x \in X : d_K(x) \le \delta_0\}$. In addition, we say that a bounded LP α-approximating sequence $\{x_n\}$ for GMVI(F, ϕ, K) has the approximation property (AP) if there exists a strongly convergent subsequence of $\{x_n\}$. Then we have the following result.

Theorem 6.3. *Let K be a nonempty, closed and convex subset of X, $F : X \to 2^{X^*}$ be a nonempty, weakly closed and locally bounded set-valued mapping, and let $\phi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. If there exists some δ_0 with $\delta_0 > 0$ such that $M(\delta_0)$ is bounded and every bounded LP α-approximating sequence $\{x_n\}$ for* GMVI(F, ϕ, K) *has the AP. Then* GMVI(F, ϕ, K) *is strongly LP α-well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be a LP α-approximating sequence for GMVI(F, ϕ, K). Then there exist $0 < \epsilon'_n \to 0$ and $w_n \in X$ with $w_n \to 0$ such that

$$x_n + w_n \in K,$$

and there exists $u_n \in F(x_n)$ satisfying

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \alpha(x_n - y) + \epsilon'_n, \quad \forall y \in K, n \in \mathbf{R}. \tag{6.8}$$

Since $x_n + w_n \in K$, then there exists $\bar{x}_n \in K$ such that $x_n + w_n = \bar{x}_n$. Thus,

$$d(x_n, K) \le \|x_n - \bar{x}_n\| = \|w_n\| \to 0.$$

Set $\epsilon_n = \max\{\epsilon'_n, \|w_n\|\}$, we can get $d(x_n, K) \le \epsilon_n$. Without loss of generality, we may assume that $\{x_n\} \subset M(\delta_0)$ for n sufficiently large. By the boundedness of $M(\delta_0)$, we know that $\{x_n\}$ is bounded. Since every bounded LP α-approximating sequence $\{x_n\}$ for GMVI(F, ϕ, K) has the AP, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$. It is easy to see $\bar{x} \in K$. Since F is locally bounded, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that, for each z in this neighborhood and $u \in F(z)$, we have $\|u\| \le \ell$. Since $x_{n_k} \to \bar{x}$ as $k \to \infty$, for k sufficiently large $\|u_{n_k}\| \leq \ell$; hence, from the reflexivity of X we may assume, without loss of generality, that $u_{n_k} \to \bar{u} \in X^*$ in the weak* topology of X^*. Note that the reflexivity of X implies that the weak topology of X^* coincides with the weak* topology of X^*. Thus, we know that $u_{n_k} \to \bar{u} \in X^*$ in the weak topology of X^*. Since F has weakly closed graph, $x_{n_k} \to \bar{x}$ and $u_{n_k} \to \bar{u} \in X^*$ weakly, we have $\bar{u} \in F(\bar{x})$. Since ϕ is proper, convex and lower semicontinuous, it follows from (6.8) that

$$\langle \bar{u}, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \leq \alpha(\bar{x} - y), \quad \forall y \in K. \tag{6.9}$$

For any $y \in K$, put $y_t = \bar{x} + t(y - \bar{x})$ for all $t \in (0, 1)$, it is easy to see $y_t \in K$. Now, utilizing (6.9), one has

$$\langle \bar{u}, \bar{x} - y_t\rangle + \phi(\bar{x}) - \phi(y_t) \leq \alpha(\bar{x} - y_t).$$

By the convexity of ϕ and the property of α, we deduce that for each $t \in (0, 1)$, one has

$$\langle \bar{u}, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \leq t^{p-1}\alpha(\bar{x} - y), \quad \forall y \in K,$$

where $p > 1$. Letting $t \to 0^+$ in the last inequality, we have

$$\langle \bar{u}, \bar{x} - y\rangle + \phi(\bar{x}) - \phi(y) \leq 0, \quad \forall y \in K.$$

This shows that $\bar{x}$ solves GMVI(F, ϕ, K). Thus, GMVI(F, ϕ, K) is strongly LP α-well-posed in the generalized sense. The proof is complete. □

Remark 6.4. Theorems 6.1 and 6.3 improve, extend and develop Theorems 6.1-6.2 in [20] to a great extent because we drop the compactness, $\mathcal{H}$-semicontinuity and upper semisemicontinuity of F, and the compactness of $M(\delta_0)$.

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Manuscript received December 16 2016
revised January 15 2016

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