Pure and Applied Functional Analysis
Volume 1, Number 2, 2016, 159–184

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

NECESSARY CONDITIONS OF OPTIMALITY OF OUTPUT FEEDBACK CONTROL LAW FOR INFINITE DIMENSIONAL UNCERTAIN DYNAMIC SYSTEMS

N. U. AHMED

Abstract. In this paper we consider a class of partially observed dynamic systems on infinite dimensional Banach spaces subject to dynamic and measurement uncertainty. The problem is to find an output feedback control law, an operator valued function, that minimizes the maximum risk. Recently a result on existence of optimal feedback law was proved by the author. Inspired by the existence result, here in this paper, we consider the question of characterization of the optimal feedback operator. We develop the necessary conditions of optimality which an optimal feedback law must satisfy. A conceptual algorithm for computation of the optimal operator valued function is also presented.

1. Introduction

Over the last fifty years control theory has made impressive progress both in the theory of optimal Control and its applications in diverse areas such as Physics, Engineering, Economics, Space sciences, Biological Sciences, Ecology and Environment and Industry and Social planing etc. Control theory for finite as well as infinite dimensional systems have been extensively developed as indicated by the encyclopedic presentations in Cesari [8], Fattorini [9] and Ahmed and Teo [7]. The results presented in these books are based on the concept of open loop controls. However, there are many problems in engineering and physical sciences where open loop controls are not practically feasible and/or not desirable. One must construct feedback control based on available information leading to the notion of fully observed or partially observed feedback controls. In case the full state information is available, one can construct an optimal feedback control law based on HJB equation arising from the application of the principle of dynamic programming due to Bellman. This leads to nonlinear Partial Differential equations for the value function on finite or infinite dimensional spaces (a formidable problem); see [2] and the references therein. In case full state information is not available this theory fails. One must use feedback control based only on available data from sensors which

2010 *Mathematics Subject Classification.* 49J24, 49J27, 49K24, 49K35, 93E20, 93B52, 35R60, 35R70, 34G25, 34H05, 47A62.

Key words and phrases. Partially observed, uncertain systems, operator valued functions, feedback operators, existence of optimal operators in the presence of uncertainty, necessary conditions of optimality, conceptual computational algorithm.

is generally noisy. So far in the literature, this problem remains largely unsolved. Here in this paper, we develop a direct approach and consider feedback control for infinite dimensional systems subject to noisy sensor data and uncertain system model. We determine the feedback control law based on only available information. This involves optimization of nonlinear functionals on the space of bounded linear Operators. The feedback control law must be chosen from a prescribed class of operator valued functions satisfying certain constraints so as to optimize the performance of the system. The question of optimization on the space of bounded linear operators also arises in the study of inverse (or equivalently identification) problems [3]. In such problems, it is assumed that the state is fully or partially observable without any measurement uncertainty. The problems considered here are also different from those of optimal controls of differential inclusions [5] where the controls are strongly measurable vector valued functions of time. In this paper, one may view the controls as operator valued functions operating on the space of available noisy information and delivering control forces. In a recent paper [1], we proved existence of optimal feedback control laws, in the presence of both system as well as sensor uncertainty, minimizing the maximum loss or equivalently maximizing minimum payoff. The emphasis there was the question of existence. Here, we consider the problem of characterization and construction of optimal feedback control laws given the existence. This is done by developing the necessary conditions of optimality characterizing the optimal feedback control operator. The results presented here substantially generalize our previous results on similar topics for finite dimensional systems [7], where numerical results were also presented. This paper also generalizes our recent results on similar topic for infinite dimensional uncertain systems [4].

The rest of the paper is organized as follows. In section 2, we present some typical notations. In section 3, we present the mathematical model describing the system and formulate the problem considered in the paper. The basic assumptions used are given in section 4 followed by a result on existence and regularity of solutions of the feedback system. In section 5, we present a result of Mayoral [14] characterizing compact subsets of the Banach space of compact linear operators. This result is used for proof of existence of an optimal feedback operator. Further, we collect together some relevant results from [1] on continuous dependence of solutions on feedback operators and the operators representing perturbation of the semigroup (generator) and the process representing measurement noise. Also a result on the existence of optimal feedback operator is included. In section 6, necessary conditions for extremality of the system and sensor uncertainty are presented leading to a pair of forward-backward evolution inclusions (FBEI). Existence of solutions for the FBEI is presented leading to the set of extremal solutions. In section 7, based on the results of section 6, we present the necessary conditions of optimality. We conclude the paper after presenting in section 8 a conceptual algorithm whereby one can numerically determine the optimal feedback operator using the necessary conditions.

2. Some notations

Let $\{X, Y, U\}$ denote a triple of real Banach spaces representing the state space, the output(measurement) space and the control space respectively. Let $I = [0, T]$ denote any closed bounded interval. For any separable reflexive Banach space Z, we let $L_1(I, Z)$ denote the space of Bochner integrable functions with values in Z, and its dual by $L_\infty(I, Z^*)$. Let Z_1, Z_2 be any pair of real Banach spaces and $\mathcal{L}(Z_1, Z_2)$ the Banach space of bounded linear operators from Z_1 to Z_2. Let $B_1(Z)$ denote the closed unit ball in any Banach space Z. An operator $S \in \mathcal{L}(Z_1, Z_2)$ is said to be compact if $S(B_1(Z_1))$ is a relatively compact subset of Z_2. Let $B_\infty(I, \mathcal{L}(Z_1, Z_2))$ denote the space of operator valued functions which are measurable in the uniform operator topology and uniformly bounded on the interval I in the sense that

$$\sup\{\| T(t) \|_{\mathcal{L}(Z_1,Z_2)}, t \in I\} < \infty$$

for $T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$. Suppose this is furnished with the topology of strong convergence (convergence in the strong operator topology) uniformly on I in the sense that, given $T_n, T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$, $T_n \to T$ in this topology if and only if for every $z \in Z_1$,

$$\sup\{|T_n(t)z - T(t)z|_{Z_2}, t \in I\} \to 0$$

as $n \to \infty$. Let $\mathcal{K}(Z_1, Z_2))$ denote the class of compact linear operators from Z_1 to Z_2. It is well known that this is a closed linear subspace of $\mathcal{L}(Z_1, Z_2)$ in the uniform operator topology and hence a Banach space. Let Γ be a closed bounded (possibly convex) subset of $\mathcal{K}(Z_1, Z_2)$. We are interested in the set $B_\infty(I, \Gamma) \subset B_\infty(I, \mathcal{K}(Z_1, Z_2))$ endowed with the relative topology of convergence in the strong operator topology of the space $\mathcal{L}(Z_1, Z_2)$ point wise in $t \in I$.

In the sequel we need the tensor product of Banach spaces. Let V, W be a pair real Banach spaces and denote the algebraic tensor product of V and W by $V \otimes W$. An element $z \in V \otimes W$ has the representation $z \equiv \sum_{i=1}^n v_i \otimes w_i$ for $v_i \in V$ and $w_i \in W$ and finite $n \in N$. The largest cross norm, also known as the projective norm, (denoted by π) is given by

$$|z|_\pi \equiv \inf\left\{\sum_{i=1}^n |v_i|_V |w_i|_W : \; z = \sum_{i=1}^n v_i \otimes w_i\right\}$$

where the infimum is taken over all such representations of $z \in V \otimes W$. With respect to this norm topology $V \otimes W$ is a normed space denoted by $V \otimes_\pi W$. Completion of $V \otimes_\pi W$ with respect to this norm topology is a Banach space denoted by $V \hat{\otimes}_\pi W$. An element $z \in V \hat{\otimes}_\pi W$ has the Grothendieck representation $z = \sum_{i=1}^\infty v_i \otimes w_i$. For any $C \in \mathcal{L}(V, W^*)$ one can introduce the pairing

$$\langle C, z \rangle = \sum (C v_i, w_i)_{W^*,W}.$$

Clearly, $|\langle C, z \rangle| \leq \| C \|_{\mathcal{L}(V,W^*)} |z|_\pi$. Thus every element of $\mathcal{L}(V, W^*)$ induces a continuous linear functional on $V \hat{\otimes}_\pi W$ and hence the embedding $\mathcal{L}(V, W^*) \hookrightarrow (V \hat{\otimes}_\pi W)^*$ is continuous (in the locally convex topology of uniform convergence on compacts). In fact, it follows from a well known result reported in (Hájek and

Smith [12], Proposition 1.1) that the equality holds, that is, $\mathcal{L}(V, W^*) = (V\hat{\otimes}_\pi W)^*$. Clearly then, the cross norm π of any $u \in V\hat{\otimes}_\pi W$ can be evaluated by

$$|u|_\pi = \sup\{|\langle C, u\rangle| : C \in B_1(\mathcal{L}(V, W^*))\}$$

where $B_1(\mathcal{L}(V, W^*))$ is the closed unit ball in $\mathcal{L}(V, W^*)$. It follows from this result that if X is a reflexive Banach space then $(X\hat{\otimes}_\pi X^*)^* = \mathcal{L}(X)$. We use this result in section 6. For many other interesting results on tensor product spaces the interested reader is referred to the excellent paper of Hájek and Smith [12].

3. System with uncertainties and problem formulation

Let X, Y, U be real Banach spaces, with X denoting the state space, Y denoting the output space, and U the space where controls take their values from. The complete system is governed by the following system of equations:

$$\dot{x} = Ax + R(t)x + F(x) + B(t)u, \text{ in } X, \tag{3.1}$$

$$y = L(t)x + \xi \text{ in } Y, \tag{3.2}$$

$$u = K(t)y \text{ in } U, \tag{3.3}$$

where the first equation describes the dynamics of the system in the state space X giving the state $x(t)$ at any time $t \geq 0$, the second equation describes the (measurement) output process that observes the status of the system in a noisy environment characterized by the random process ξ and delivers the output $y(t), t \geq 0$, with values from the Banach space Y. The operator valued process R perturbing the semigroup generator is also random or uncertain and takes values from the Banach space $\mathcal{L}(X)$ of bounded linear operators in X. This represents the uncertainty in the dynamics, in the sense that the exact value of R at any given time is not known, but it is known that it takes values from a bounded set in $\mathcal{L}(X)$, for example, the closed unit ball around the origin $B_1(\mathcal{L}(X))$. We denote this class of operator valued functions by $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X)))$. In order to regulate the system (3.1), the third equation provides the control based on the noisy data y through the operator valued function K. In general the operator A is an unbounded linear operator with domain and range in X. The operator F is a nonlinear map in X, the operator valued function B takes values from $\mathcal{L}(U, X)$, the operator L, representing the sensor (or measurement system), takes values from $\mathcal{L}(X, Y)$ and the output feedback control operator K is an operator valued function taking values from the space $\mathcal{K}(Y, U)$. Let $\mathcal{F}_{ad}$, whose precise characterization is given later, denote the class of admissible feedback operator valued functions $\{K(t), t \geq 0\}$ with values in $\mathcal{K}(Y, U)$. The process $\xi(t), t \geq 0$, represents the uncertainty in the measurement data and takes values from the Banach space Y. For most practical situations, it is reasonable to assume that the disturbance process is bounded. And so, without any loss of generality, we may assume that the process ξ is strongly measurable taking values from the closed unit ball $B_1(Y)$ centered at the origin. We denote this class of disturbance processes by $\mathcal{D}$.

The performance of the system over the time horizon $I \equiv [0,T]$ is measured by the following functional (called cost functional)

$$J(K,R,\xi) \equiv \int_I \ell(t,x(t))dt + \Phi(x(T)) \tag{3.4}$$

where $\ell : I \times X \longrightarrow [0,\infty]$ and $\Phi : X \longrightarrow [0,\infty]$. The cost functional depends on the choice of the control law K in the presence of dynamic uncertainty $R \in \mathcal{V}$ and imperfect measurement induced by $\xi \in \mathcal{D}$. Our objective is to find an operator valued function $K \in \mathcal{F}_{ad}$ that minimizes the maximum possible cost. In other words, we want a feedback law that minimizes the maximum risk posed by system and measurement uncertainties. This problem can be formulated as min-max problem:

$$\inf_{K\in\mathcal{F}_{ad}} \sup_{(R,\xi)\in\mathcal{V}\times\mathcal{D}} J(K,R,\xi).$$

Given this pessimistic view, an element $K_o \in \mathcal{F}_{ad}$ is said to be optimal if and only if

$$J_o(K_o) \equiv \sup_{(R,\xi)\in\mathcal{V}\times\mathcal{D}} J(K_o,R,\xi) \leq \sup_{(R,\xi)\in\mathcal{V}\times\mathcal{D}} J(K,R,\xi) \equiv J_o(K),\ \forall\ K \in \mathcal{F}_{ad}. \tag{3.5}$$

4. Basic assumptions and preliminaries

To consider the problem as stated above, we introduce the following basic assumptions:

(A0): The Banach spaces $\{X,Y\}$ are reflexive and U is any real Banach space.

(A1): The operator A is the infinitesimal generator of a C_0-semigroup of operators $S(t), t \geq 0$, on X.

(A2): The vector field $F : X \longrightarrow X$ is uniformly Lipschitz with Lipschitz constant $C_1 > 0$.

(A3): Both B and L are measurable in the uniform operator topology, with $B \in L_1(I,\mathcal{L}(U,X))$ and $L \in B_\infty(I,\mathcal{L}(X,Y))$.

(A4): Let $\Gamma \subset \mathcal{K}(Y,U)$ be a nonempty closed bounded convex set and denote the admissible feedback control laws by

$$\mathcal{F}_{ad} \equiv \{K \in B_\infty(I,\mathcal{K}(Y,U)) : K(t) \in \Gamma\ \forall\ t \in I\}.$$

(A5): The process R perturbing the semigroup is any uniformly measurable operator valued function defined on I and taking values from the closed unit ball $B_1(\mathcal{L}(X))$. This is denoted by $\mathcal{V} \equiv B_\infty(I,B_1(\mathcal{L}(X)))$.

(A6): The disturbance (noise) process $\xi : I \longrightarrow Y$, is any measurable function taking values from the closed unit ball $B_1(Y)$ of the B-space Y. We denote this family by $\mathcal{D} \equiv B_\infty(I,B_1(Y))$. This represents the uncertainty in the measurement data.

Some comments on the uncertainties in dynamics $\mathcal{V}$ and measurement $\mathcal{D}$ are in order. We do not assume any probabilistic structure for these process except that they are bounded measurable process and hence locally square integrable.

(A7): The integrand $\ell : I \times X \longrightarrow (-\infty, \infty]$ is measurable in the first variable and continuous in the second argument and there exists a $p \in [1, \infty)$ such that

$$|\ell(t,x)| \leq g(t) + c_1 \parallel x \parallel_X^p, x \in X, t \geq 0$$

with $0 \leq g \in L_1(I)$ and $c_1 \geq 0$. The function Φ is also continuous on X and there exist constants $c_2, c_3 \geq 0$ such that

$$|\Phi(x)| \leq c_2 + c_3 \parallel x \parallel_X^p$$

for the same p.

Substituting the equations (3.1) and (3.2) into (3.1) we obtain the following uncertain feedback system

$$\dot{x} = Ax + Rx + F(x) + BKLx + BK\xi, x_0 \in X \text{ (fixed) }, K \in \mathcal{F}_{ad}, \tag{4.1}$$

subject to the (unstructured) disturbances $\{R, \xi\} \in \mathcal{V} \times \mathcal{D}$. Before we conclude this section we present the following standard result on the existence and regularity of solutions of the feedback system. This is used later in the paper.

Lemma 4.1. *Consider the uncertain feedback system given by (4.1) over any finite time horizon $I \equiv [0,T]$, and suppose the assumptions* **(A1)**-**(A6)** *hold. Then, for every initial state $x(0) = x_0 \in X$, and any feedback law $K \in \mathcal{F}_{ad}$ and any element from the set of uncertainty, $(R, \xi) \in \mathcal{V} \times \mathcal{D}$, the system (4.1) has a unique mild solution $x \in C(I, X)$. Further, the solution set*

$$\mathcal{X} \equiv \left\{ x(K, R, \xi)(\cdot) \in C(I, X) : K \in \mathcal{F}_{ad}, R \in \mathcal{V}, \xi \in \mathcal{D} \right\}$$

is a bounded subset of $C(I, X)$.

Proof. See [1, Lemma 4.1]. □

5. Existence of optimal feedback operator

For proof of existence of optimal feedback operator we need continuity of solutions with respect to the operator and vector valued processes $\{K, R, \xi\}$. Since continuity is crucially dependent on the topology of both the domain and the target spaces, it is necessary to specify the admissible topologies. Let Z be any bounded subset of a topological space and let $B_\infty(I, Z)$ denote the class of Borel measurable functions defined on I and taking values from Z. It was shown in [1] that for the target space $C(I, X)$, the sup-norm topology is natural and for the domain space,

$$\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D} \equiv B_\infty(I, \Gamma) \times B_\infty(I, B_1(\mathcal{L}(X))) \times B_\infty(I, B_1(Y)),$$

which is a subset of the B-space $B_\infty(I, \mathcal{K}(Y, U)) \times B_\infty(I, \mathcal{L}(X)) \times B_\infty(I, Y)$, the Tychonoff product topology is the most appropriate one. For the class of feedback operator valued functions let $\Gamma \subset \mathcal{K}(Y, U)$ be a closed bounded convex set and $B_\infty(I, \Gamma)$ denote the class of strongly measurable operator valued functions defined on I and taking values from Γ endowed with the topology of convergence in the strong operator topology point wise in $t \in I$. In particular, we need the set Γ to satisfy certain compactness property. The following result due to Mayoral [12] characterizes relatively compact subsets of $\mathcal{K}(Y, U)$.

Proposition 5.1 (Mayoral [14, Theorem 1, p79]). *If the B-space Y does not contain a copy of ℓ_1, a set $\Gamma \subset \mathcal{K}(Y,U)$ is relatively compact iff (i): Γ is uniformly completely continuous (ucc) and (ii): for every $y \in Y$, the y-section, $\Gamma(y) \equiv \{L(y), L \in \Gamma\}$, is relatively compact in U.*

(H1) (Admissible Feedback Operators $\mathcal{F}_{ad}$): By assumption, Y is a reflexive Banach space, so it does not contain a copy of ℓ_1. So Mayoral's result holds. We assume that $\Gamma \subset \mathcal{K}(Y,U)$ satisfies the above characterization for relative compactness and further that it is closed so that it is compact and convex. Then we consider the Tychonoff product topology τ_T on the function space $B_\infty(I,\Gamma) \equiv \mathcal{F}_{ad}$ which turns this into a compact Hausdorff (topological) space.

(H2) (System Uncertainty Set $\mathcal{V}$): Next, we consider the set $\mathcal{V}$ representing uncertainty in the system model. Since X is a reflexive Banach space, it is well known that the closed unit ball $B_1(\mathcal{L}(X))$ is compact with respect to the weak operator topology τ_{wo}. Using this fact we may now equip $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X))$ with the Tychonoff product topology and denote this by τ_{Two}. With respect to this topology $\mathcal{V}$ is a compact Hausdorff space.

(H3) (Measurement Uncertainty Set $\mathcal{D}$) Next we consider the set $\mathcal{D} \equiv B_\infty(I, B_1(Y))$ with $B_1(Y)$ denoting the closed unit ball (centered at the origin) representing the measurement uncertainty. Reflexivity of Y implies that $B_1(Y)$ is weakly compact. The set $\mathcal{D}$ is endowed with the Tychonoff product topology τ_{Tw}. With respect to this topology $\mathcal{D}$ is a compact Hausdorff space.

Remark 5.2. The assumptions that the uncertainties are given by the closed unit balls $B_1(\mathcal{L}(X))$ and $B_1(Y)$ do not impose any practical limitation. In fact, one can choose the closed balls $B_r(\mathcal{L}(X), R_0(t))$ and $B_\theta(Y, \xi_0(t))$ of radius $r \geq 0$ and $\theta \geq 0$ respectively and $t \in I$, where $R_0 \in B_\infty(I, \mathcal{L}(X))$ and $\xi_0 \in B_\infty(I,Y))$ are bounded measurable functions.

Now we are prepared to consider the question of continuity. We need the continuity of the solution with respect to the operators and processes on which it depends. In particular we have the following result.

Theorem 5.3. *Consider the feedback system (4.1) and suppose the assumptions (A0)-(A6) and (H1)-(H3) hold and that the operator A is the infinitesimal generator of a compact C_0-semigroup $S(t), t > 0$. Then the map $(K,R,\xi) \longrightarrow x(K,R,\xi)$ is jointly continuous from $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ to $C(I,X)$ with respect to their respective topologies.*

Proof. See [1, Theorem 5.2]. □

From the above result one obtains the following continuity result of the functional $J(K,R,\xi)$.

Corollary 5.4. *Suppose the assumptions of Theorem 5.3 hold and the functions ℓ and Φ satisfy the assumption (A7). Then, the functional $(K,R,\xi) \longrightarrow J(K,R,\xi)$ is jointly continuous on $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ with respect to the product topology $\tau_T \times \tau_{Two} \times \tau_w$.*

Proof. See [1, Corollary 5.3]. □

Proof of existence of an optimal operator valued function $K_o \in \mathcal{F}_{ad}$ that solves the min-max problem requires the notions of upper and lower semi-continuity of multi functions as follows.

Definition 5.5. Let Z_1, Z_2 be any pair of topological spaces. A multi function $G : Z_1 \longrightarrow 2^{Z_2} \setminus \emptyset$ is upper semi-continuous if for every closed set $C \subset Z_2$, the preimage $G^{-1}(C) \equiv \{x \in Z_1 : G(x) \cap C \neq \emptyset\}$ is closed. And it is lower semi-continuous if for every open set $D \subset Z_2$ the preimage $G^{-1}(D) \equiv \{x \in Z_1 : G(x) \cap D \neq \emptyset\}$ is open.

For details on multi-functions see the Handbook by Hu and Papageorgiou [12]. The following existence result was proved in [1, Theorem 6.1].

Theorem 5.6. *Consider the feedback system (4.1). Suppose the assumptions of Theorem 5.3 and Corollary 5.4 hold. Then there exists an optimal feedback operator valued function $K_o \in \mathcal{F}_{ad}$ such that*

$$J_o(K_o) \leq J_o(K) \ \forall \ K \in \mathcal{F}_{ad}$$

where

$$J_o(K) \equiv \sup\{J(K, R, \xi), (R, \xi) \in \mathcal{V} \times \mathcal{D}\}.$$

Proof. [1, Theorem 6.1]. □

Remark 5.7. Since J is jointly continuous on $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ and the set $\mathcal{V} \times \mathcal{D}$ is compact, for each $K \in \mathcal{F}_{ad}$ the set

$$\Pi(K) \equiv \{(R, \xi) \in \mathcal{V} \times \mathcal{D} : J(K, R, \xi) = J_o(K)\}$$

is nonempty. In view of the Theorem 5.6, there exists a $K_o \in \mathcal{F}_{ad}$ such that $J_o(K_o) \leq J_o(K)$ for all $K \in \mathcal{F}_{ad}$. Since $J_o(K) \equiv J(K, \Pi(K))$ for any $K \in \mathcal{F}_{ad}$, we have

$$J(K_o, R, \xi) \leq J(K_o, \Pi(K_o)) \leq J(K, \Pi(K)) \ \forall \ K \in \mathcal{F}_{ad} \text{ and } \forall \ (R, \xi) \in \mathcal{V} \times \mathcal{D}.$$

Clearly, the right side inequality says that the optimal feedback operator minimizes the maximum risk (maximum potential cost), while the left side inequality tells that the cost in all other situations will never exceed the pessimistic (conservative) cost. This is precisely what is desired in the presence of uncertainty (in the system model and measurement (sensor)).

6. Necessary conditions for extremality

In order to solve the problem (3.5), it is clear that we must solve first the extremality problem,

$$J_o(K) \equiv \sup\{J(K, R, \xi) : (R, \xi) \in \mathcal{V} \times \mathcal{D}\}, \tag{6.1}$$

for arbitrary $K \in \mathcal{F}_{ad}$. Since J is jointly continuous in all the variables, and $\mathcal{V} \times \mathcal{D}$ is compact in the Tychonoff product topology $\tau_{Two} \times \tau_w$, the supremum in (6.1) is attained. Thus we can characterize them and then proceed to determine the optimal feedback law. Denote the corresponding set of extremals by

$$\mathcal{E}_K \equiv \{(R, \xi) \in \mathcal{V} \times \mathcal{D} : J_o(K) = J(K, R, \xi)\} \subset \mathcal{V} \times \mathcal{D}.$$

Clearly, for each $K \in \mathcal{F}_{ad}$, the set $\mathcal{E}_K \neq \emptyset$. In the following theorem we characterize the set $\mathcal{E}_K$.

Theorem 6.1. *Consider the feedback system (4.1). Suppose the assumptions of Theorem 5.6 hold. In addition, suppose that F is once continuously Fréchet differentiable with the F-derivative being continuous and uniformly bounded in operator norm with values in $\mathcal{L}(X)$; and ℓ and Φ are once continuously Gâteaux differentiable along any path $x \in C(I,X)$ with the derivatives $\ell_x \in L_1(I,X^*)$ and $\Phi_x \in X^*$. Then, in order that the pair $(\hat{R},\hat{\xi}) \in \mathcal{E}_K$, it is necessary that there exists a pair $(x,\psi) \in C(I,X) \times C(I,X^*)$ satisfying the following evolution equations:*

$$\dot{x} = Ax + \hat{R}x + F(x) + BKLx + BK\hat{\xi}, x(0) = x_0 \tag{6.2}$$

$$-\dot{\psi} = A^*\psi + \hat{R}^*\psi + F_x^*(x)\psi + (BKL)^*\psi + \ell_x(t,x), \psi(T) = \Phi_x(x(T)), \tag{6.3}$$

and the following inequality (extremality condition):

$$\begin{aligned}\int_0^T \big(\hat{R}x + BK\hat{\xi}, \psi\big)_{X,X^*} dt \\ \geq \int_0^T \big(Rx + BK\xi, \psi\big)_{X,X^*} dt,\ \forall\ (R,\xi) \in \mathcal{V} \times \mathcal{D}.\end{aligned} \tag{6.4}$$

Proof. Let $K \in \mathcal{F}_{ad}$ be any fixed element and let $(\hat{R},\hat{\xi}) \in \mathcal{E}_K$. Then by definition $J(K,\hat{R},\hat{\xi}) \geq J(K,R,\xi)$ for all $(R,\xi) \in \mathcal{V}\times\mathcal{D}$. For any $\varepsilon > 0$, define $R^\varepsilon \equiv \hat{R}+\varepsilon(R-\hat{R})$ and $\xi^\varepsilon \equiv \hat{\xi} + \varepsilon(\xi - \hat{\xi})$ for any pair $(R,\xi) \in \mathcal{V} \times \mathcal{D}$. Since the set $\mathcal{V} \times \mathcal{D}$ is closed and convex, it is clear that $(R^\varepsilon, \xi^\varepsilon) \in \mathcal{V} \times \mathcal{D}$. Thus $J(K,R^\varepsilon,\xi^\varepsilon)$ is well defined and

$$J(K,\hat{R},\hat{\xi}) \geq J(K,R^\varepsilon,\xi^\varepsilon)\ \forall\ \varepsilon \in [0,1]\ \forall\ (R,\xi) \in \mathcal{V} \times \mathcal{D}. \tag{6.5}$$

Let $x \in C(I,X)$ denote the mild solution of the evolution equation (4.1) corresponding to the triple $(K,\hat{R},\hat{\xi})$ and $x^\varepsilon \in C(I,X)$ the mild solution corresponding to the triple $(K,R^\varepsilon,\xi^\varepsilon)$. It is easy to verify that $x^\varepsilon \longrightarrow x$ strongly in $C(I,X)$ as $\varepsilon \to 0$, and further, the limit $\lim_{\varepsilon\downarrow 0}(1/\varepsilon)(x^\varepsilon - x)$ exists and it is given by $\lim_{\varepsilon\downarrow 0}(1/\varepsilon)(x^\varepsilon - x) \overset{s}{\longrightarrow} z$ in $C(I,X)$ with $z \in C(I,X)$ being the mild solution of the following evolution equation

$$\dot{z} = Az + \hat{R}z + F_x(x(t))z + BKLz + (R-\hat{R})x + BK(\xi - \hat{\xi}), z(0) = 0. \tag{6.6}$$

Since the Fréchet derivative of F is continuous and uniformly bounded and the assumptions (A1)-(A5) hold, it follows from Banach fixed point theorem that the following integral equation

$$\begin{aligned} z(t) = & \int_0^t S(t-r)\hat{R}(r)z(r)dr + \int_0^t S(t-r)F_x(x(r))z(r)dr \\ & + \int_0^t S(t-r)(BKL)(r)z(r)dr + \int_0^t S(t-r)(R(r)-\hat{R}(r))x(r)dr \\ & + \int_0^t S(t-r)(BK)(r)(\xi(r)-\hat{\xi}(r))dr \end{aligned} \tag{6.7}$$

has a unique solution $z \in C(I,X)$. Using (3.4) and (6.5) for computing the Gâteaux differential of J at the point $(K,\hat{R},\hat{\xi}) \in \mathcal{F}_{ad}\times\mathcal{V}\times\mathcal{D}$ in the direction $(K,R-\hat{R},\xi-\hat{\xi})$, we find that

$$dJ(K, \hat{R}, \hat{\xi}; R - \hat{R}, \xi - \hat{\xi}) = \int_0^T (\ell_x(t, x(t)), z(t))_{X^*,X} dt + (\Phi_x(x(T)), z(T))_{X^*,X} \leq 0 \tag{6.8}$$

for all $(R, \xi) \in \mathcal{V} \times \mathcal{D}$ where z is the solution of the integral equation (6.7), equivalently, the mild solution of the variational equation (6.6). Define the functional

$$\eta(z) \equiv \int_0^T (\ell_x(t, x(t)), z(t))_{X^*,X} dt + (\Phi_x(x(T)), z(T))_{X^*,X}. \tag{6.9}$$

Since $z \in C(I, X) \subset L_\infty(I, X)$ and by assumption on ℓ and Φ, $\ell_x(\cdot, x(\cdot)) \in L_1(I, X^*)$ and $\Phi_x(x(T)) \in X^*$, it is clear that the map $z \longrightarrow \eta(z)$ is a continuous (bounded) linear functional on $C(I, X)$. On the other hand, under the assumptions (A3)-(A6), it follows from the integral equation (6.7) that the map

$$(R - \hat{R})x + BK(\xi - \hat{\xi}) \longrightarrow z$$

is a continuous linear map from $L_1(I, X)$ to $C(I, X)$. Thus the composition map

$$h \equiv (R - \hat{R})x + BK(\xi - \hat{\xi}) \longrightarrow z \longrightarrow \eta(z)$$

is a continuous linear functional on $L_1(I, X)$. Since X is assumed to be reflexive, the dual of $L_1(I, X)$ is $L_\infty(I, X^*)$. Thus by Riesz representation theorem there exists a $\psi \in L_\infty(I, X^*)$ such that the functional η is also given by

$$\begin{aligned} \eta(z) \equiv \tilde{\eta}(h) &= \int_0^T < h(s), \psi(s) >_{X,X^*} ds \\ &= \int_0^T ((R - \hat{R})(s)x(s) + (BK)(s)(\xi - \hat{\xi})(s), \psi(s))_{X,X^*} ds. \end{aligned} \tag{6.10}$$

Hence, it follows from the inequality (6.8) that

$$\begin{aligned} &\tilde{\eta}(h) \\ &= \int_0^T ((R - \hat{R})(s)x(s) + (BK)(s)(\xi - \hat{\xi})(s), \psi(s))_{X,X^*} ds \leq 0 \end{aligned} \tag{6.11}$$

for all $(R, \xi) \in \mathcal{V} \times \mathcal{D}$. Clearly, from this inequality we arrive at the inequality (6.4) and hence the extremality of the pair $(\hat{R}, \hat{\xi}) \in \mathcal{V} \times \mathcal{D}$ with $x \in C(I, X)$ being the mild solution of equation (6.2). Now we prove that $\psi \in L_\infty(I, X^*)$ is the mild solution of equation (6.2). Using the variational equation (6.6) into the identity (6.11) and integrating by parts, we have

$$\begin{aligned} \tilde{\eta}(h) &= (z(T), \psi(T))_{X,X^*} \\ &- \int_0^T \langle z(t), \dot{\psi} + A^*\psi + \hat{R}^*(t)\psi + F_x^*(x(t))\psi + (BKL)^*\psi \rangle_{X,X^*} dt. \end{aligned} \tag{6.12}$$

Since we are interested in the mild solutions (not strong solution), this is rigorously justified by use of Yosida approximation of A and then taking limits. Now setting

$\psi(T) \equiv \Phi_x(x(T))$ and $\dot{\psi} + A^*\psi + F_x^*(x(t))\psi + (BKL)^*\psi = -\ell_x(t, x(t))$ in the above equation we find that

$$\tilde{\eta}(h) = (z(T), \Phi_x(x(T)))_{X,X^*} + \int_0^T \langle z(t), \ell_x(t, x(t))\rangle_{X,X^*} dt.$$

This is precisely the functional as defined by the expression (6.9). Thus we conclude that ψ is the mild solution of the backward (or adjoint) evolution equation

$$-\dot{\psi} = A^*\psi + \hat{R}^*\psi + F_x^*(x(t))\psi + (BKL)^*\psi + \ell_x(t, x(t)), \psi(T) = \Phi_x(x(T)).$$

Again using Banach fixed point theorem one can show that this equation has a unique solution in $C(I, X^*) \subset L_\infty(I, X^*)$. So it is more regular than what was predicted by representation theorem. This completes the proof. □

In the sequel we need the following result. This result may be known in the literature though the author is not unaware of any such result explicitly stated.

Proposition 6.2. *Let X be a separable reflexive Banach space having separable dual X^*. Consider the space of bounded linear operators $\mathcal{L}(X)$ on X equipped with the weak operator topology τ_{wo} denoted by $\mathcal{L}_{wo}(X)$. This topology is metrizable with respect to which it becomes a complete separable metric space and hence a Polish space.*

Proof. Since X is a reflexive Banach space, the space $\mathcal{L}(X)$ equipped with the weak operator topology τ_{wo}, denoted by $\mathcal{L}_{wo}(X)$, is a locally convex sequentially complete Hausdorff topological vector space. As both X and its dual X^* are separable, there exist a countable set $\{x_n\} \subset X$ dense in X and a countable set $\{x_n^*\} \subset X^*$ which is dense in X^*. We use this family to construct a family of seminorms $\{\rho_n\}$ as follows. For any $T \in \mathcal{L}(X)$ define $\rho_n(T) \equiv |x_n^*(Tx_n)|$. It is easy to verify that $\{\rho_n\}$ is a family of seminorms. Equipped with this family of seminorms, $\mathcal{L}_{wo}(X)$ turns into a Fréchet space. This space is metrizable with a translation invariant metric as follows,

$$d(T, S) \equiv \sum_{n=1}^{\infty} (1/2^n) \frac{\rho_n(T - S)}{1 + \rho_n(T - S)}.$$

Equipped with this metric topology, $\mathcal{L}_{wo}(X)$ turns into a metric space. In other words, under the given assumptions, the original topology is compatible with the metric topology. Since the weak operator topology is sequentially complete, the space $(\mathcal{L}_{wo}(X), d)$ is a complete metric space. We verify that it is also separable. Let Q_0 denote the set of nonnegative rational numbers. For each $n \in N$ and $r \in Q_0$, define the set $\Gamma_n(r) \equiv \{T \in \mathcal{L}(X) : \rho_n(T) < r\}$. Clearly, the family of sets $\{\Gamma_n(r), r \in Q_0, n \in N\}$ forms a countable base for the topology consisting of convex neighborhoods of the origin. Let Γ be any open set in $\mathcal{L}_{wo}(X)$. By translation, if necessary, we may consider this set around the origin. Then there exists a pair $x^* \in X^*$ and $x \in X$ and $r \in Q_0$ such that $\Gamma = \{T \in \mathcal{L}_{wo}(X) : |x^*(Tx)| < r\}$. Since $\{x_i\}$ and $\{x_i^*\}$ are dense in X and X^* respectively, it is clear that $\Gamma \subset \cup_{n=1}^{\infty} \Gamma_n(r)$. Thus this metric topology satisfies the axiom of second countability. Hence we conclude that $(\mathcal{L}_{wo}(X), d)$ is a complete separable metric space and hence a Polish space. □

Note A: Since X is a reflexive Banach space, the closed unit ball $B_1(\mathcal{L}(X))$ is compact in the weak operator topology. Thus under the assumptions of the Proposition 6.2, the closed unit ball $B_1(\mathcal{L}(X))$ of $\mathcal{L}(X)$ is a compact metric space.

Note B: Since separability of X implies $w*$-separability of X^*, the assumption that X^* is separable can be omitted altogether.

Now we return to our control problem. We assume throughout the rest of the paper that X is reflexive and separable having separable dual. It is clear from the necessary conditions (Theorem 6.1), in particular the inequality (6.4), that the extremal pair $(\hat{R}, \hat{\xi}) \in \mathcal{V} \times \mathcal{D}$ must maximize the functional,

$$\varrho(R, \xi) \equiv \int_0^T (Rx + BK\xi, \psi)_{X,X^*} dt, \tag{6.13}$$

over the set $\mathcal{V} \times \mathcal{D}$ with (x, ψ) being the corresponding mild solution of the evolution equations (6.2)-(6.2). Using the tensor product notation we can rewrite this functional in the form

$$\varrho(R, \xi) \equiv \int_0^T \{\langle R, x \otimes \psi \rangle_{\mathcal{L}(X), X\hat{\otimes}_\pi X^*} + (\xi, (BK)^*\psi)_{Y,Y^*}\} dt, \tag{6.14}$$

where the first bracket denotes the duality pairing between the projective tensor product space $X\hat{\otimes}_\pi X^*$ and its dual $\mathcal{L}(X)$ as discussed in section 2, see also Hájek & Smith [11, Proposition 1.1]. The second duality pairing is the standard pairing between the Banach space Y and its dual Y^*. We must choose $(R, \xi) \in \mathcal{V} \times \mathcal{D}$ that maximizes this functional. Since $\mathcal{V} \times \mathcal{D}$ is compact with respect to the product topology $\tau_{Two} \times \tau_{Tw}$ and the functional ϱ is also continuous in this topology, there exists $(\hat{R}, \hat{\xi}) \in \mathcal{V} \times \mathcal{D}$ at which it attains its maximum. Let $J_1 : X\hat{\otimes}_\pi X^* \longrightarrow 2^{B_1(\mathcal{L}(X))} \setminus \emptyset$ denote the normalized duality map with values given by

$$J_1(z) \equiv \{R \in B_1(\mathcal{L}(X)) : \langle R, z \rangle_{\mathcal{L}(X), X\hat{\otimes}_\pi X^*} = \| z \|_\pi\} \tag{6.15}$$

for any $z \in X\hat{\otimes}_\pi X^*$. By virtue of Hahn-Banach theorem, this is a nonempty set. More precisely, this is a multi valued map, convex, demi-continuous (strong to weak) and τ_{wo} closed. Thus $t \longrightarrow J_1(z(t)) \equiv J_1(x(t) \otimes \psi(t))$ is a measurable multi function with values in $B_1(\mathcal{L}(X))$ equipped with the relative weak operator topology. It follows from Proposition 2, that this is a compact Polish space. Therefore, by virtue of either Kuratowski-Ryll Nardzewski or Yankov-Von Neumann-Aumann measurable selection theorem [12, Theorem 2.1, p154]; Theorem 2.14, p158], it has measurable selections. Let $J_2 : Y^* \longrightarrow 2^{B_1(Y)} \setminus \emptyset$ denote the normalized duality map with values

$$J_2(y^*) \equiv \{y \in B_1(Y) : \langle y^*, y \rangle_{Y^*,Y} = \| y^* \|_{Y^*}\}. \tag{6.16}$$

Since Y is reflexive, again by Hahn-Banach theorem this is a nonempty set valued map, demi-continuous, convex, τ_w closed. Therefore, $t \longrightarrow J_2((BK)^*(t)\psi(t))$ is a measurable multifunction having measurable selections. Thus we conclude that there exist measurable selections $\{\hat{R}, \hat{\xi}\}$ such that $\hat{R}(t) \in J_1(x(t) \otimes \psi(t)), \hat{\xi}(t) \in J_2((BK)^*(t)\psi(t)), t \in I$, at which the functional (6.14) attains its maximum. Using the above multi functions we arrive at the following result:

Corollary 6.3. *Under the assumptions of Theorem 6.1 and Proposition 6.2, for each $K \in \mathcal{F}_{ad}$, the solution of the extremality problem is given by the set of mild solutions of the following forward-backward evolution inclusions,*

$$\dot{x} \in Ax + J_1(x(t) \otimes \psi(t))x + F(x) + BKLx + BKJ_2((BK)^*(t)\psi(t)), \tag{6.17}$$

$$-\dot{\psi} \in A^*\psi + J_1^*(x(t) \otimes \psi(t))\psi + F_x^*(x)\psi + (BKL)^*\psi + \ell_x(t, x(t)), \tag{6.18}$$

satisfying the initial boundary conditions $x(0) = x_0, \psi(T) = \Phi_x(x(T))$ where, for any $z \in X\hat{\otimes}_\pi X^*$, $J_1^*(z) \equiv \{R^* : R \in J_1(z)\} \subset B_1(\mathcal{L}(X^*))$.

In view of Corollary 6.3, the original optimal feedback control problem subject to dynamic uncertainty and imperfect measurement reduces to the following optimal output feedback control problem of the forward-backward system of evolution inclusions (6.17)-(6.18) with the cost functional

$$J_o(K) \equiv \int_0^T \ell(t, x(t))dt + \Phi(x(T)), \tag{6.19}$$

where x is the state component of the mild solution $(x, \psi) \in C(I, X) \times C(I, X^*)$ of the system (6.17)-(6.18) (if one exists). For convenience of reference we call this problem P_e. Before we can proceed further we must prove that the system (6.17)-(6.18) has a nonempty set of (mild) solutions.

Theorem 6.4. *Consider the system (6.17)-(6.18) and suppose the assumptions (A0)-(A7) including those of Theorem 6.1 and Proposition 6.2 hold. Then the system of evolution inclusions (6.17)-(6.18) with initial boundary conditions as stated above has a nonempty set of mild solutions $(x, \psi) \in C(I, X) \times C(I, X^*)$.*

Proof. Consider the system of forward-backward evolution inclusions (6.17)-(6.18) and let $K \in \mathcal{F}_{ad}$ fixed. Take any pair $(R, \xi) \in \mathcal{V} \times \mathcal{D}$ and consider the associated evolution equations

$$\dot{x} = Ax + Rx + F(x) + BKLx + BK\xi, x(0) = x_0 \tag{6.20}$$

$$-\dot{\psi} = A^*\psi + R^*\psi + F_x^*(x)\psi + (BKL)^*\psi + \ell_x(t, x), \psi(T) = \Phi_x(x(T)). \tag{6.21}$$

The mild solution (if one exists) of these equations is given by the solution of the following system of integral equations

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(x(r))dr \\ &\quad + \int_0^t S(t-r)(BKL)(r)x(r)dr + \int_0^t S(t-r)(BK)(r)\xi(r)dr, t \in I, \end{aligned} \tag{6.22}$$

$$\begin{aligned} \psi(t) &= S^*(T-t)\Phi_x(x(T)) + \int_t^T S^*(r-t)R^*(r)\psi(r)dr \\ &\quad + \int_t^T S^*(r-t)F_x^*(x(r))\psi(r)dr + \int_t^T S^*(r-t)(BKL)^*(r)\psi(r)dr \\ &\qquad + \int_t^T S(r-t)\ell_x(r, x(r))dr, t \in I. \end{aligned} \tag{6.23}$$

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the integral operators defining the expressions on the right-hand side of the equations (6.22) and (6.23) respectively. Using these notations, the equations (6.22)-(6.23) can be compactly described by the following functional equations

$$x = \mathcal{G}_1(x), \ \psi = \mathcal{G}_2(x, \psi) \tag{6.24}$$

on the Banach space $C(I, X) \times C(I, X^*)$. Under the given assumptions, it is easy to verify that $\mathcal{G}_1 : C(I, X) \longrightarrow C(I, X)$ and $\mathcal{G}_2 : C(I, X) \times C(I, X^*) \longrightarrow C(I, X^*)$ are bounded and continuous (nonlinear) maps. Define the product map $\mathcal{G} \equiv \mathcal{G}_1 \times \mathcal{G}_2$ giving $(x, \psi) = \mathcal{G}(x, \psi)$. By partitioning the interval $I \equiv [0, T]$ into a finite number of subintervals of suitable length, $I = \cup_i^n I_i, i = 1, 2, \cdots n, n < \infty$, and respecting the boundary conditions, one can prove, as in Ahmed [1, Lemma 4.1], that the restriction of the map $\mathcal{G}$ to each of the Banach spaces $\mathcal{Z}_i \equiv C(I_i, X) \times C(I_i, X^*), i = 1, 2 \cdots n$, is a contraction and thereby guaranteeing, by Banach fixed point theorem, a unique fixed point $(x_i, \psi_i) \in C(I_i, X) \times C(I_i, X^*)$ for each $i \in [1, 2 \cdots n]$. By concatenation of the sequence $\{(x_i, \psi_i)\}$ one then obtains a unique fixed point (x, ψ) of the map $\mathcal{G}$ in the Banach space $C(I, X) \times C(I, X^*)$ proving existence of a unique solution of the system of integral equation (6.22)-(6.23) corresponding to the pair $(R, \xi) \in \mathcal{V} \times \mathcal{D}$ for a fixed $K \in \mathcal{F}_{ad}$. Further, one can verify that the solution (x, ψ) is sequentially continuous with respect to the variable $(R, \xi) \in \mathcal{V} \times \mathcal{D}$ in the sense as stated below. There exist two bounded continuous maps $G_1 : \mathcal{V} \times \mathcal{D} \longrightarrow C(I, X)$ and $G_2 : \mathcal{V} \times \mathcal{D} \longrightarrow C(I, X^*)$ such that $x = G_1(R, \xi), \psi = G_2(R, \xi)$ and that, as $(R^n, \xi^n) \longrightarrow (R^o, \xi^o)$ in the product topology $\tau_{Two} \times \tau_{Tw}$, the corresponding solutions $(x^n, \psi^n) = (G_1(R^n, \xi^n), G_2(R^n, \xi^n)) \xrightarrow{s} (G_1(R^o, \xi^o), G_2(R^o, \xi^o)) = (x^o, \psi^o)$ in $C(I, X) \times C(I, X^*)$. It follows from the forward-backward evolution inclusions (6.17)-(6.18) that if the following inclusions hold

$$R(t) \in J_1(x(t) \otimes \psi(t)) \equiv J_1(G_1(R, \xi)(t) \otimes G_2(R, \xi)(t)), t \in I,$$
$$\xi(t) \in J_2((BK)^*(t) G_2(R, \xi)(t)), t \in I,$$

then the pair (x, ψ) is a (mild) solution of these evolution inclusions and conversely. Define the multivalued maps

$$\mathcal{J}_1(R, \xi) \equiv \{J_1(G_1(R, \xi)(t) \otimes G_2(R, \xi)(t)), t \in I\},$$
$$\mathcal{J}_2(R, \xi) \equiv \{J_2((BK)^*(t) G_2(R, \xi)(t)), t \in I\}$$

and note that $\mathcal{J}_1 : \mathcal{V} \times \mathcal{D} \longrightarrow 2^{\mathcal{V}} \setminus \emptyset$ and $\mathcal{J}_2 : \mathcal{V} \times \mathcal{D} \longrightarrow 2^{\mathcal{D}} \setminus \emptyset$, and their cartesian product $\mathcal{J} \equiv \mathcal{J}_1 \times \mathcal{J}_2 : \mathcal{V} \times \mathcal{D} \longrightarrow 2^{\mathcal{V} \times \mathcal{D}} \setminus \emptyset$. Thus the question of existence of a (mild) solution of the forward-backward evolution system (6.17)-(6.18) is equivalent to the question of existence of a fixed point of the multivalued map $\mathcal{J} : \mathcal{V} \times \mathcal{D} \longrightarrow 2^{\mathcal{V} \times \mathcal{D}} \setminus \emptyset$ in the topological space $\mathcal{V} \times \mathcal{D}$. Since the weak operator topology on $\mathcal{L}(X)$ and the weak topology on Y (equivalently weak star topology because Y is reflexive) are locally convex topologies, the Tychonoff product topology on $\mathcal{V} \times \mathcal{D}$ is also a locally convex topology. Thus $\mathcal{V} \times \mathcal{D}$ is a nonempty closed convex and compact subset of a locally convex topological space. Since the duality map J_1 is continuous with respect to the topology induced by the cross norm π on $X \otimes_\pi X^*$ and weak operator topology on $B_1(\mathcal{L}(X))$, and the solution maps G_1 and G_2 are continuous with respect to the $\tau_{Two} \times \tau_{Tw}$ topology, the map $\mathcal{J}_1$ is continuous with respect to this topology (so both upper and lower semi-continuous in this topology). Clearly, the set $\mathcal{J}_1(R,\xi)$ is nonempty and, since the duality map J_1 is convex and closed valued, $\mathcal{J}_1(R,\xi)$ is also convex and closed valued. Considering the map $\mathcal{J}_2$, we recall that the duality map J_2 is demi continuous, convex and τ_w closed. Thus the map $\mathcal{J}_2$ is also continuous with respect to the $\tau_{Two} \times \tau_{Tw}$ topology and closed convex valued. It is well known that a multivalued map is continuous if and only if it is both upper and lower semicontinuous. Hence we conclude that the map $\mathcal{J}$ is upper semi-continuous from $\mathcal{V}\times\mathcal{D}$ to $2^{\mathcal{V}\times\mathcal{D}}\setminus\emptyset$ and closed convex valued. The set $\mathcal{V}\times\mathcal{D}$ is a nonempty compact convex subset of a locally convex topological space. Further, we have just seen that, for each $(R,\xi)\in\mathcal{V}\times\mathcal{D}$, the set $\mathcal{J}(R,\xi)\subset\mathcal{V}\times\mathcal{D}$ is closed and convex. Thus, it follows from the generalized Schauder fixed point theorem for multi valued maps, due to Kakutani (Zeidler [15, Theorem 9.B, p452]), that $\mathcal{J}$ has a nonempty set of fixed points. Hence, we conclude that for any given $K\in\mathcal{F}_{ad}$, the system of forward-backward evolution inclusions (6.17)-(6.18) has a nonempty set of mild solutions. This completes the proof. □

Now we consider the optimization problem P_e. Note that, in the preceding theorem, $K\in\mathcal{F}_{ad}$ was fixed and so the multivalued map $\mathcal{J}$ is in fact dependent on K. So for correct notation, let us denote this map by $\mathcal{J}_K$ and the associated set of fixed points by $Fix(\mathcal{J}_K)$. So for each $K\in\mathcal{F}_{ad}$, it follows from the above theorem that the set $Fix(\mathcal{J}_K)\neq\emptyset$ and hence the evolution inclusions (6.17)-(6.18) has a nonempty set of (mild) solutions given by

(6.25)

$$\mathcal{S}_K \equiv \Big\{(x,\psi)\in C(I,X)\times C(I,X^*) : x = G_1(R,\xi),$$
$$\psi = G_2(R,\xi),\ \text{for } (R,\xi)\in Fix(\mathcal{J}_K)\Big\}.$$

This is the set of extremal solutions corresponding to $K\in\mathcal{F}_{ad}$ and it is a closed and bounded subset of $C(I,X)\times C(I,X^*)$.

7. Necessary conditions of optimality

Now we are prepared to consider the optimization problem P_e. Our objective is to find a $K^o\in\mathcal{F}_{ad}$ that minimizes the functional (6.19) subject to the dynamic constraints imposed by the evolution inclusions (6.17)-(6.18). According to theorem 5.5 this optimization problem has a solution. Here we present necessary conditions characterizing the optimality. For this we first construct the so called variational equations around the potentially optimal $K^o\in\mathcal{F}_{ad}$.

Lemma 7.1. *Consider the system (6.17)-(6.18) and suppose the assumptions of Theorem 6.4 hold and further the nonlinear operator F is twice continuously Fréchet differentiable with the second F-derivative uniformly bounded in $\mathcal{L}(X,\mathcal{L}(X))$, and Φ,ℓ are also twice Gâteaux differentiable with the second G-derivatives being uniformly bounded in $\mathcal{L}(X,X^*)$. Let $K^o\in\mathcal{F}_{ad}$ be optimal and K any other element of $\mathcal{F}_{ad}$. Then the corresponding variational equations are given by the following system* *of evolution equations on the product space $X \times X^*$:*

$$\begin{aligned} (7.1) \qquad \dot{z} &= Az + R^o z + Q_1 z + Q_2\varphi + F_x(x^o)z + (BK^oL)z + BK^oQ_5\varphi \\ &\quad + B(K - K^o)(Lx^o + \xi^o), \end{aligned}$$

$$\begin{aligned} (7.2) \qquad -\dot{\varphi} &= A^*\varphi + (R^o)^*\varphi + Q_3 z + Q_4\varphi + F_x^*(x^o)\varphi + Q_6 z \\ &\quad + (BK^oL)^*\varphi + \ell_{xx}(t, x^o)z + (B(K - K^o)L)^*\psi^o, \end{aligned}$$

with the boundary conditions $z(0) = 0, \varphi(T) = \Phi_{xx}(x^o(T))z(T)$, where the operators $\{Q_i, i = 1, 2, \cdots, 6\}$ (dependent on (x^o, ψ^o)) are identified in the body of the proof. Further, the system of variational equations (7.1)-(7.2) has a unique mild solution $(z, \varphi) \in C(I, X) \times C(I, X^*)$.

Proof. Let $K^o \in \mathcal{F}_{ad}$ denote the optimal operator minimizing the functional (6.19) and $K \in \mathcal{F}_{ad}$ any other element. For any $\varepsilon \in [0, 1]$, define $K^\varepsilon \equiv K^o + \varepsilon(K - K^o)$. Since $\mathcal{F}_{ad}$ is closed and convex, it is clear that $K^\varepsilon \in \mathcal{F}_{ad}$ and $J_o(K^o) \le J_o(K^\varepsilon)$ for all $\varepsilon \in [0, 1]$. Thus the Gâteaux differential of J at K^o in the direction $K - K^o$, denoted by $dJ(K^o; K - K^o)$, satisfies the inequality $dJ(K^o; K - K^o) \ge 0$, for all $K \in \mathcal{F}_{ad}$. Let $(x^o, \psi^o) \in \mathcal{S}_{K^o}$ and $(x^\varepsilon, \psi^\varepsilon) \in \mathcal{S}_{K^\varepsilon}$. Suppressing the time variable (for convenience of notation), it follows from Theorem 6.4 that there exist measurable selections

$$R^\varepsilon \in J_1(x^\varepsilon \otimes \psi^\varepsilon), R^o \in J_1(x^o \otimes \psi^o), \xi^\varepsilon \in J_2((BK^\varepsilon)^*\psi^\varepsilon), \xi^o \in J_2((BK^o)^*\psi^o)$$

such that the pairs $(x^\varepsilon, \psi^\varepsilon)$ and (x^o, ψ^o) are the mild solutions of the following pairs of evolution equations with the boundary conditions as indicated:

$$\begin{aligned} (7.3) \qquad \dot{x}^\varepsilon &= Ax^\varepsilon + R^\varepsilon x^\varepsilon + F(x^\varepsilon) + BK^\varepsilon L x^\varepsilon + BK^\varepsilon \xi^\varepsilon, \\ x^\varepsilon(0) &= x_0, \\ (7.4) \qquad -\dot{\psi}^\varepsilon &= A^*\psi^\varepsilon + (R^\varepsilon)^*\psi^\varepsilon + F_x^*(x^\varepsilon)\psi^\varepsilon + (BK^\varepsilon L)^*\psi^\varepsilon + \ell_x(t, x^\varepsilon), \\ \psi^\varepsilon(T) &= \Phi_x(x^\varepsilon(T)). \\ (7.5) \qquad \dot{x}^o &= Ax^o + R^o x^o + F(x^o) + BK^o L x^o + BK^o \xi^o, \\ x^o(0) &= x_0 \\ (7.6) \qquad -\dot{\psi}^o &= A^*\psi^o + (R^o)^*\psi^o + F_x^*(x^o)\psi^o + (BK^o L)^*\psi^o + \ell_x(t, x^o), \\ \psi^o(T) &= \Phi_x(x^o(T)). \end{aligned}$$

By Theorem 5.2, $x^\varepsilon \xrightarrow{s} x^o$ in $C(I, X)$, $\psi^\varepsilon \xrightarrow{s} \psi^o$ in $C(I, X^*)$ as $\varepsilon \to 0$. Let $(z, \varphi) \in C(I, X) \times C(I, X^*)$ denote the following limits

$$\lim_{\varepsilon \downarrow 0}(1/\varepsilon)(x^\varepsilon - x^o) \equiv z \in C(I, X), \text{ and } \lim_{\varepsilon \downarrow 0}(1/\varepsilon)(\psi^\varepsilon - \psi^o) \equiv \varphi \in C(I, X^*).$$

By straightforward variation using the above equations, one can easily verify that the pair (z, φ) is the mild solution of the following pair of evolution equations on the product space $X \times X^*$:

$$\begin{aligned} (7.7) \qquad \dot{z} &= Az + R^o z + Q_1(x^o, \psi^o)z + Q_2(x^o, \psi^o)\varphi + F_x(x^o)z + BK^o Lz \\ &\quad + BK^o\eta(\psi^o; \varphi) + B(K - K^o)(Lx^o + \xi^o), \\ z(0) &= 0, \end{aligned}$$

$$
\begin{aligned}
-\dot{\varphi} &= A^*\varphi + (R^o)^*\varphi + Q_3(x^o,\psi^o)z + Q_4(x^o,\psi^o)\varphi \\
&\quad + F_x^*(x^o)\varphi + F_{xx}^*(x^o;z)\psi^o + (BK^oL)^*\varphi \\
&\quad + \ell_{xx}(t,x^o)z + (B(K-K^o)L)^*\psi^o, \\
\varphi(T) &= \Phi_{xx}(x^o(T))z(T),
\end{aligned} \tag{7.8}
$$

where, it follows from the inclusion relations and compactness of the set $\mathcal{V}$ in the τ_{Two} topology that the operators $\{R^o, Q_1, Q_2\}$, evaluated at the point $x^o \otimes \psi^o \in X \otimes_\pi X^*$ along the direction $(z \otimes \varphi)$, are given by the following weak limit in X

$$
w - \lim_{\varepsilon \downarrow 0}(1/\varepsilon)(R^\varepsilon x^\varepsilon - R^o x^o) \equiv Q_1(x^o,\psi^o)z + Q_2(x^o,\psi^o)\varphi + R^o z \tag{7.9}
$$

with $Q_1 \in \mathcal{L}(X), Q_2 \in \mathcal{L}(X^*, X), R^o \in \mathcal{L}(X)$. Similarly, considering the adjoint counterpart, we have

$$
w^* - \lim_{\varepsilon \downarrow 0}\big((R^\varepsilon)^*\psi^\varepsilon - (R^o)^*\psi^o\big) = Q_3(x^o,\psi^o)z + Q_4(x^o,\psi^o)\varphi + (R^o)^*\varphi \tag{7.10}
$$

with $Q_3 \in \mathcal{L}(X, X^*), Q_4 \in \mathcal{L}(X^*)$ and $(R^o)^* \in \mathcal{L}(X^*)$. Since the pair $(x^o, \psi^o) \in C(I, X \times X^*)$ corresponds to the optimal K^o and so fixed, for simplicity of notation we shall continue to omit these arguments. Similarly, the variable η evaluated at ψ^o in the direction φ, is given by the weak limit

$$
w - \lim_{\varepsilon \downarrow 0}(1/\varepsilon)(\xi^\varepsilon - \xi^o) \equiv \eta(\psi^o;\varphi)
$$

in Y and it is also linear in φ. Thus there exists an operator $Q_5(\psi^o) \in \mathcal{L}(X^*, Y)$ (parameterized by ψ^o) such that $\eta(\psi^o;\varphi) = Q_5(\psi^o)\varphi$. Thus equation (7.7) can be written as

$$
\begin{aligned}
\dot{z} &= Az + R^o z + Q_1 z + Q_2\varphi + F_x(x^o)z + BK^oLz + BK^oQ_5\varphi \\
&\quad + B(K-K^o)(Lx^o + \xi^o), \\
z(0) &= 0.
\end{aligned} \tag{7.11}
$$

Similarly, considering the term $F_{xx}^*(x^o;z)\psi^o$ in equation (7.8), because of linearity in both z and ψ^o, it follows from our assumption on F that there exists an operator $Q_6(x^o,\psi^o) \in \mathcal{L}(X, X^*)$, parameterized by (x^o,ψ^o), such that $F_{xx}^*(x^o;z)\psi^o = Q_6(x^o,\psi^o)z$. Thus equation (7.8) can be written as

$$
\begin{aligned}
-\dot{\varphi} &= A^*\varphi + (R^o)^*\varphi + Q_3 z + Q_4\varphi + F_x^*(x^o)\varphi + Q_6 z + (BK^oL)^*\varphi \\
&\quad + \ell_{xx}(t,x^o)z + (B(K-K^o)L)^*\psi^o, \\
\varphi(T) &= \Phi_{xx}(x^o(T))z(T).
\end{aligned} \tag{7.12}
$$

This completes the proof of the first part of the Lemma. The last part asserting existence and uniqueness of solution follows as a Corollary of Theorem 6.4. Also direct proof is similar to the one given in the next theorem. □

Now we are prepared to prove the necessary conditions of optimality.

Theorem 7.2. *Consider the system of evolution inclusions (6.17)-(6.18) with the objective functional (6.19) to be minimized on the set of admissible operator valued functions $\mathcal{F}_{ad}$. Suppose the assumptions of Lemma 7.1 hold. Then, for an element $K^o \in \mathcal{F}_{ad}$ to be optimal, it is necessary that there exist (multipliers) $(\varphi_1, \varphi_2) \in$*

$C(I,X^*)\times C(I,X)$ which are the mild solutions of the following system of evolution equations,

$$\begin{aligned}
-\dot{\varphi}_1 &= A^*\varphi_1 + (R^o)^*\varphi_1 + Q_1^*\varphi_1 + F_x^*(x^o)\varphi_1 + (BK^oL)^*\varphi_1 \\
&\quad +(Q_3+Q_6)^*\varphi_2 + \ell_{xx}^*(t,x^o)\varphi_2 + \ell_x(t,x^o) \qquad (7.13)\\
\dot{\varphi}_2 &= (A\varphi_2 + R^o\varphi_2 + Q_4^*\varphi_2 + F_x(x^o)\varphi_2 + (BK^oL)\varphi_2 \\
&\quad +(Q_2+BK^oQ_5)^*\varphi_1, \qquad (7.14)
\end{aligned}$$

subject to the two point boundary conditions,

$$\varphi_1(T) - \Phi_{xx}^*(x^o(T))\varphi_2(T) = \Phi_x(x^o(T)),\ \ \varphi_2(0)=0,$$

satisfying the following inequality,

$$\int_0^T \big\{(B(K-K^o)(Lx^o+\xi^o),\varphi_1)_{X,X^*} + ((B(K-K^o)L)^*\psi^o,\varphi_2)_{X^*,X}\big\}dt \geq 0, \qquad (7.15)$$

for all $K\in\mathcal{F}_{ad}$.

Proof. Let $K^o\in\mathcal{F}_{ad}$ be optimal and $K\in\mathcal{F}_{ad}$ any other element and, for any $\varepsilon\in[0,1]$, let $K^\varepsilon \equiv K^o+\varepsilon(K-K^o)$. By convexity of the set $\mathcal{F}_{ad}$, $K^\varepsilon\in\mathcal{F}_{ad}$ for all $\varepsilon\in[0,1]$. Clearly, the Gâteaux differential of J_o at K^o in the direction $K-K^o$, denoted by $dJ_o(K^o,K-K^o)$, satisfies the inequality, $dJ_o(K^o;K-K^o)\geq 0$ for all $K\in\mathcal{F}_{ad}$. Let $(x^o,\psi^o)\in\mathcal{S}_{K^o}$ and $(x^\varepsilon,\psi^\varepsilon)\in\mathcal{S}_{K^\varepsilon}$. Using the definition of J_o, it follows from standard variation that

$$\begin{aligned}
dJ_o(K^o;K-K^o) &= \int_0^T \langle \ell_x(t,x^o(t)),z(t)\rangle_{X^*,X}dt \\
&\quad +\langle\Phi_x(x^o(T)),z(T)\rangle_{X^*,X} \ \geq\ 0 \qquad (7.16)
\end{aligned}$$

for all $K\in\mathcal{F}_{ad}$ where, it follows from Lemma 7.1 that, z is the first component of the solution (z,φ) of the variational evolution equations (7.1)-(7.2). For convenience of reference, let us denote the functional appearing in (7.16) by

$$L(z)\equiv\int_0^T \langle \ell_x(t,x^o(t)),z(t)\rangle_{X^*,X}dt + \langle\Phi_x(x^o(T)),z(T)\rangle_{X^*,X}. \qquad (7.17)$$

Note that even though this functional appears like the functional (6.9), it corresponds to the extremals $\mathcal{S}_{K^o},\mathcal{S}_{K^\varepsilon}$ unlike (6.9). Here the variational equations are (7.1) and (7.2) whereas for (6.9) the variational equation is given by (6.6). Since, by our assumption, $\ell_x(\cdot,x^o(\cdot))\in L_1(I,X^*)$ and $\Phi_x(x^o(T))\in X^*$ and, by Lemma 7.1, $z\in C(I,X)$, it follows from (7.17) that $z\longrightarrow L(z)$ is a continuous linear functional on $C(I,X)$. By our assumption (A3), $B\in L_1(I,\mathcal{L}(U,X))$ and the operator valued functions K and L are bounded on I with values in $\mathcal{L}(Y,U)$ and $\mathcal{L}(X,Y)$ respectively, $\xi^o\in B_\infty(I,B_1(Y))$ and, by Theorem 6.1, we have $x^o\in C(I,X)$ and $\psi^o\in C(I,X^*)$. Thus $B(K-K^o)(Lx^o+\xi^o)\in L_1(I,X)$ and $(B(K-K^o)L)^*\psi^o\in L_1(I,X^*)$. In view of the variational equations (7.1)-(7.2) or (7.11)-(7.12), we note that they are linear and driven by the elements

$B(K-K^o)(Lx^o+\xi^o) \in L_1(I,X)$ and $(B(K-K^o)L)^*\psi^o \in L_1(I,X^*)$ respectively, and hence the map

$$\begin{pmatrix} B(K-K^o)(Lx^o+\xi^o) \\ \\ (B(K-K^o)L)^*\psi^o \end{pmatrix} \longrightarrow \begin{pmatrix} z \\ \\ \varphi \end{pmatrix}$$

is a bounded linear operator from $L_1(I,X) \times L_1(I,X^*)$ to $C(I,X) \times C(I,X^*)$. Clearly, the projection map $\begin{pmatrix} z \\ \varphi \end{pmatrix} \longrightarrow z$ is continuous linear from $C(I,X)\times C(I,X^*)$ to $C(I,X)$. Hence the composition map

$$\begin{pmatrix} B(K-K^o)(Lx^o+\xi^o) \\ \\ (B(K-K^o)L)^*\psi^o \end{pmatrix} \longrightarrow \begin{pmatrix} z \\ \\ \varphi \end{pmatrix} \longrightarrow z \longrightarrow L(z)$$

is a continuous linear functional on $L_1(I,X)\times L_1(I,X^*)$. Since, by our assumption, X is reflexive, the topological dual of $L_1(I,X)\times L_1(I,X^*)$ is given by $L_\infty(I,X^*)\times L_\infty(I,X)$. Thus there exist $\varphi_1 \in L_\infty(I,X^*)$ and $\varphi_2 \in L_\infty(I,X)$ such that

$$\begin{aligned} (7.18)\quad L(z) = \int_0^T \Big\{ &\langle B(K-K^o)(Lx^o+\xi^o), \varphi_1\rangle_{X,X^*} \\ &+ \langle (B(K-K^o)L)^*\psi^o, \varphi_2\rangle_{X^*,X} \Big\} dt \\ \equiv \Lambda(B(K-K^o)&(Lx^o+\xi^o), (B(K-K^o)L)^*\psi^o) \;\ge\; 0 \;\forall\; K \in \mathcal{F}_{ad}. \end{aligned}$$

The last inequality follows from (7.16) and (7.17). This proves the inequality (7.15). It remains to verify that the pair (φ_1,φ_2) is given by the mild solution of the pair of forward-backward evolution equations (7.13)-(7.14) with the initial-boundary conditions as stated. Now using the variational equations (7.1)-(7.2) of Lemma 7.1 in the righthand side of the following expression

$$\begin{aligned} (7.19)\quad &\Lambda(B(K-K^o)(Lx^o+\xi^o), (B(K-K^o)L)^*\psi^o) \\ &= \int_0^T \Big\{ \langle B(K-K^o)(Lx^o+\xi^o), \varphi_1\rangle_{X,X^*} \\ &\qquad\qquad + \langle (B(K-K^o)L)^*\psi^o, \varphi_2\rangle_{X^*,X} \Big\} dt \end{aligned}$$

and using integration by parts and the boundary conditions from (7.11)-(7.12), $z(0)=0$, $\varphi(T)=\Phi_{xx}(x^o(T))z(T)$, we arrive at the following expression

$$\begin{aligned} (7.20)\qquad \Lambda \;=\; &\langle z(T), \varphi_1(T)\rangle_{X,X^*} \\ &-\int_0^T \big\{ \langle z, \dot{\varphi}_1 + (A+R^o+Q_1+F_x(x^o))^*\varphi_1 \\ &\qquad + (BK^oL)^*\varphi_1 + \ell_{xx}^*\varphi_2 + (Q_3+Q_6)^*\varphi_2\rangle_{X,X^*} \big\} dt \\ &-\langle \Phi_{xx}(x^o(T))z(T), \varphi_2(T)\rangle_{X^*,X} + \langle \varphi(0), \varphi_2(0)\rangle_{X^*,X} \\ &-\int_0^T \big\{ \langle \varphi, -\dot{\varphi}_2 + (A+R^o+Q_4^*+F_x(x^o)+BK^oL)\varphi_2 \end{aligned}$$

$$+(Q_2 + BK^o Q_5)^* \varphi_1 \rangle_{X^*,X} dt \}.$$

Now setting

$$(7.21) \quad \varphi_1(T) - \Phi_{xx}^*(x^o(T))\varphi_2(T) = \Phi_x(x^o(T)), \ \varphi_2(0) = 0,$$

$$(7.22) \quad \dot{\varphi}_1 + (A + R^o + Q_1 + F_x(x^o) + (BK^o L))^* \varphi_1 + (Q_3 + Q_6)^* \varphi_2 + \ell_{xx}^*(t, x^o)\varphi_2 = -\ell_x(t, x^o),$$

$$(7.23) \quad -\dot{\varphi}_2 + (A + R^o + Q_4^* + F_x(x^o) + BK^o L)\varphi_2 + (Q_2 + BK^o Q_5)^* \varphi_1 = 0,$$

on the righthand side of the expression (7.20), we obtain

$$(7.24) \qquad \Lambda = \langle z(T), \Phi_x(x^o(T)) \rangle_{X,X^*} + \int_0^T \langle z(t), \ell_x(t, x^o(t)) \rangle_{X,X^*} dt.$$

This shows that the identities (7.21)-(7.23) yield the same functional $L(z)$ as defined by the expression (7.17). Thus we conclude that the multipliers $\{\varphi_1, \varphi_2\}$, whose existence was guaranteed by the representation of the dual of the Banach space $L_1(I, X) \times L_1(I, X^*)$ by the Banach space $L_\infty(I, X^*) \times L_\infty(I, X)$, satisfy in the mild sense the evolution equations (7.22)-(7.23) subject to the boundary conditions (7.21). Collecting all the above facts we arrive at the following initial-boundary value problem on the Banach space $\mathcal{Z} \equiv X^* \times X$:

$$\begin{aligned} (7.25) \qquad -\dot{\varphi}_1 &= A^* \varphi_1 + (R^o)^* \varphi_1 + Q_1^* \varphi_1 + F_x^*(x^o)\varphi_1 + (BK^o L)^* \varphi_1 \\ &\quad + (Q_3 + Q_6)^* \varphi_2 + \ell_{xx}^* \varphi_2 + \ell_x(t, x^o) \\ (7.26) \qquad \dot{\varphi}_2 &= A\varphi_2 + R^o \varphi_2 + Q_4^* \varphi_2 + F_x(x^o)\varphi_2 \\ &\quad + (BK^o L)\varphi_2 + (Q_2 + BK^o Q_5)^* \varphi_1 \end{aligned}$$

subject to the two point boundary conditions:

$$(7.27) \qquad \varphi_1(T) - \Phi_{xx}^*(x^o(T))\varphi_2(T) = \Phi_x(x^o(T)), \ \varphi_2(0) = 0.$$

Thus we have obtained the necessary conditions as stated in Theorem 7.2. To complete the proof, we must show that the evolution equations (7.13)-(7.14) with the two point boundary conditions as stated have mild solutions $(\varphi_1, \varphi_2) \in C(I, X^*) \times C(I, X)$. We present a brief outline of the proof. Consider the following associated integral equations on the Banach space $C(I, \mathcal{Z}) \equiv C(I, X^*) \times C(I, X)$ (with the usual supnorm topology):

$$\begin{aligned} (7.28) \quad \varphi_1(t) &= H_1(\varphi_1, \varphi_2)(t) \equiv S^*(T - t)[\Phi_x(x^o(T)) + \Phi_{xx}^*(x^o(T))\varphi_2(T)] \\ &\quad + \int_t^T S^*(r - t)[(R^o)^* + Q_1^* + F_x^*(x^o) + (BK^o L)^*](r)\varphi_1(r) dr \\ &\quad + \int_t^T S^*(r - t)(Q_3 + Q_6 + \ell_{xx})^*(r)\varphi_2(r) dr \\ &\quad + \int_t^T S^*(r - t)\ell_x(r, x^o(r)) dr, \\ (7.29) \quad \varphi_2(t) &= H_2(\varphi_1, \varphi_2)(t) \\ &\equiv \int_0^t S(t - r)[R^o + Q_4^* + F_x(x^o) + BK^o L](r)\varphi_2(r) dr \end{aligned}$$

$$+\int_0^t S(t-r)(Q_2+BK^oQ_5)^*(r)\varphi_1(r)dr, t\in I.$$

For convenience of presentation we introduce the following operator valued functions $\{E_i, i=1,2,3,4\}$ taking values in the spaces indicated:

$$E_1(t)\equiv[(R^o)^*+Q_1^*+F_x^*(x^o)+(BK^oL)^*](t)\in\mathcal{L}(X^*), t\in I, \tag{7.30}$$

$$E_2(t)\equiv(Q_3+Q_6+\ell_{xx})^*(t)\in\mathcal{L}(X,X^*), t\in I, \tag{7.31}$$

$$E_3(t)\equiv[R^o+Q_4^*+F_x(x^o)+BK^oL](t)\in\mathcal{L}(X), t\in I, \tag{7.32}$$

$$E_4(t)\equiv(Q_2+BK^oQ_5)^*(t)\in\mathcal{L}(X^*,X), t\in I\equiv[0,T]. \tag{7.33}$$

Under our assumptions, all the operators, except B, appearing in the above expressions, are uniformly norm bounded on I and $\| B(\cdot)\|_{\mathcal{L}(U,X)}\in L_1^+(I)$. Thus the norms

$$\| E_1(\cdot)\|_{\mathcal{L}(X^*)}\equiv h_1(\cdot)\in L_1^+(I), \| E_2(\cdot)\|_{\mathcal{L}(X,X^*)}\equiv h_2(\cdot)\in L_1^+(I),$$
$$\| E_3(\cdot)\|_{\mathcal{L}(X)}\equiv h_3(\cdot)\in L_1^+(I), \| E_4(\cdot)\|_{\mathcal{L}(X^*,X)}\equiv h_4(\cdot)\in L_1^+(I).$$

Recall that, by assumption $\ell_x\in L_1(I,X^*)$ along the path x^o, and there exists $M\geq 1$ such that the semigroup S has the bound $\sup_{t\in I}\| S(t)\|_{\mathcal{L}(X)}\leq M$. Using these facts and Gronwall inequality it is easy to verify that the solutions of the above integral equations (if they exist) are bounded, that is, there exists an apriori bound $b>0$ such that

$$\| \varPhi\|_{C(I,\mathcal{Z})}\equiv\| (\varphi_1,\varphi_2)\|_{C(I,\mathcal{Z})}\equiv\| \varphi_1\|_{C(I,X^*)}+\| \varphi_2\|_{C(I,X)}\leq b<\infty.$$

Define the operator $\mathbf{H}: C(I,\mathcal{Z})\longrightarrow C(I,\mathcal{Z})$ as follows:

$$\mathbf{H}(\varPhi)=(\mathbf{H}_1(\varPhi),\mathbf{H}_2(\varPhi))\equiv(H_1(\varphi_1,\varphi_2),H_2(\varphi_1,\varphi_2))$$

with values $\mathbf{H}(\varPhi)(t)\in\mathcal{Z}, t\in I$. We show that the operator $\mathbf{H}$ has a unique fixed point in $C(I,\mathcal{Z})$. Choose any subinterval $I_i\equiv[T_{i-1},T_1]\subset I$ and consider the restriction of the operator $\mathbf{H}$ on $C(I_i,\mathcal{Z})$ and note that, for any $(\Phi,\Psi)\in C(I_i,\mathcal{Z})$, we have

$$\begin{aligned}\sup_{t\in I_i}|\mathbf{H}(\varPhi)(t)-\mathbf{H}(\varPsi)(t)|_{\mathcal{Z}} &\leq |\Phi_{xx}(x^o(T_i))[\varphi_2(T_i)-\psi_2(T_i)]|_{X^*}\\ &+M\int_{T_{i-1}}^{T_i} h_1(r)|\varphi_1(r)-\psi_1(r)|_{X^*}dr+M\int_{T_{i-1}}^{T_i} h_2(r)|\varphi_2(r)-\psi_2(r)|_X dr\\ &M\int_{T_{i-1}}^{T_i} h_3(r)|\varphi_2(r)-\psi_2(r)|_X dr+M\int_{T_{i-1}}^{T_i} h_4(r)|\varphi_1(r)-\psi_1(r)|_{X^*}dr.\end{aligned} \tag{7.34}$$

Considering the first term on the righthand side of the above expression, note that by our hypothesis, $\| \Phi_{xx}\|_{\mathcal{L}(X,X^*)}$ is uniformly bounded, say, by C_0. Then, using equation (7.29) one can easily verify that

$$\begin{aligned}|\Phi_{xx}(x^o(T_i))[\varphi_2(T_i)-\psi_2(T_i)]|_{X^*} &\leq C_0|\varphi_2(T_i)-\psi_2(T_i)|_X\\ &\leq C_0M\int_{I_i} h_3(r)|\varphi_2(r)-\psi_2(r)|_X dr\\ &+C_0M\int_{I_i} h_4(r)|\varphi_1(r)-\psi_1(r)|_{X^*}dr.\end{aligned} \tag{7.35}$$

Substituting (7.35) into (7.34) we obtain

$$\sup_{t\in I_i}|\mathbf{H}(\Phi)(t)-\mathbf{H}(\Psi)(t)|_{\mathcal{Z}} \leq \int_{T_{i-1}}^{T_i} M[h_1(r)+(1+C_0)h_4(r)]|\varphi_1(r)-\psi_1(r)|_{X^*}dr$$
$$+\int_{T_{i-1}}^{T_i} M[h_2(r)+(1+C_0)h_3(r)]|\varphi_2(r)-\psi_2(r)|_X dr. \tag{7.36}$$

Let $\mathcal{B}(I)$ denote the class of Borel subsets of the set I. Define the set function α by

$$\alpha(\sigma)\equiv\int_\sigma M\{h_1(t)+(1+C_0)h_4(t)+h_2(t)+(1+C_0)h_3(t)\}dt \tag{7.37}$$

for $\sigma\in\mathcal{B}(I)$. This is a nonnegative set function and since $\{h_1,h_2,h_3,h_4\}\in L_1^+(I)$ it has bounded variation on I and it is absolutely continuous with respect to the Lebesgue measure. Upon using this set function, it follows from the expression (7.36) that

$$\sup_{t\in I_i}|\mathbf{H}(\Phi)(t)-\mathbf{H}(\Psi)(t)|\leq\alpha(I_i)\parallel\Phi-\Psi\parallel_{C(I_i,\mathcal{Z})}. \tag{7.38}$$

Partition the interval $I\equiv[0,T]$ into a finite number of subintervals, $I=\cup_{i=1}^n I_i$, such that, $I_1=[0,T_1]$, $\{I_i=[T_{i-1},T_i], i=2,3,\cdots,n-1\}$ and $I_n=[T_{n-1},T]$, and that $\alpha(I_i)<1$ for all $i=1,2,\cdots,n$. Since I is a finite interval and α is absolutely continuous with respect to Lebesgue measure, there exists an integer $n\in N$ (finite) for which the above partition is feasible. Thus it follows from (7.38) that the restriction of the operator $\mathbf{H}$ to each of the Banach spaces $C(I_i,\mathcal{Z})$ is a contraction and hence by Banach fixed point theorem it has a unique fixed point on each of these spaces $C(I_i,\mathcal{Z})$. By piecing together these solutions we conclude that the system of integral equations (7.28)-(7.29) has a unique solution $\Phi\in C(I,\mathcal{Z})$. This proves that the two point boundary value problem (7.13)-(7.14) has a unique mild solution. This completes the proof. □

Remark 7.3a In the absence of uncertainty (both dynamic and measurement) we recover the standard minimum principle. Indeed, by setting $R\equiv 0,\xi\equiv 0$ in the necessary conditions of optimality given by Theorem 7.2 we find that $Q_1=0,Q_2=0,Q_3=0,Q_4=0,Q_5=0$. Thus the equations (7.13)-(7.14) reduce to

$$-\dot{\varphi}_1 = A^*\varphi_1+F_x^*(x^o)\varphi_1+(BK^oL)^*\varphi_1+(\ell_{xx}+Q_6)^*\varphi_2+\ell_x(t,x^o) \tag{7.39}$$
$$\dot{\varphi}_2 = A\varphi_2+F_x(x^o)\varphi_2+(BK^oL)\varphi_2 \tag{7.40}$$

with the boundary conditions $\varphi_1(T)-\Phi_{xx}^*(x^o(T))\varphi_2(T)=\Phi_x(x^o(T)),\varphi_2(0)=0$. Note that equation (7.40) is homogeneous with initial condition $\varphi_2(0)=0$. Thus this equation has the trivial (mild) solution $\varphi_2(t)\equiv 0$. Hence equation (7.39) reduces to

$$-\dot{\varphi}_1=A^*\varphi_1+F_x^*(x^o)\varphi_1+(BK^oL)^*\varphi_1+\ell_x(t,x^o) \tag{7.41}$$

with the boundary condition $\varphi_1(T)=\Phi_x(x^o(T))$. The inequality (7.15) reduces to

$$\int_0^T\{(B(K-K^o)(Lx^o),\varphi_1)_{X,X^*}\}dt\geq 0\ \forall\ K\in\mathcal{F}_{ad}. \tag{7.42}$$

where x^o is the mild solution of the state equation (4.1) with the uncertainty removed

$$\dot{x}^o = Ax^o + F(x^o) + (BK^oL)x^o, x^o(0) = x_0. \tag{7.43}$$

The evolution equations (7.41) and (7.43) along with the boundary conditions and the inequality (7.42) provide the necessary conditions of optimality in the absence of uncertainty.

Remark 7.3b Similarly, in the absence of only dynamic uncertainty ($R(t) \equiv 0$), the necessary conditions (7.13)-(7.14) simplify with $R^o = 0, Q_1 = 0, Q_2 = 0, Q_3 = 0, Q_4 = 0$. The inequality (7.15) remains unchanged. The system of inclusions (6.17)-(6.18) are modified by removing the duality map J_1.

Remark 7.4 It is interesting to mention that in the absence of uncertainty, the assumptions on the admissible feedback control laws $\mathcal{F}_{ad} \equiv B_\infty(I, \Gamma)$ can be relaxed. One can take a compact convex subset Γ of the locally convex topological space $(\mathcal{L}(Y,U), \tau_{so})$ ($\mathcal{L}(Y,U)$ endowed with the strong operator topology).

8. A conceptual algorithm

Here we present a conceptual algorithm for computation of optimal feedback operator. Throughout this section we assume that $\{X, Y, U\}$ are all reflexive Banach spaces. Using the necessary conditions of optimality as stated in Theorem 7.2, we construct a sequence of operator valued functions $\{K^n\} \in \mathcal{F}_{ad}$ that converges to a point in $\mathcal{F}_{ad}$ at which J_o attains it's local minimum. Before we proceed with the algorithm, let us note that the necessary inequality (7.15) can be written in the form of duality products using the projective tensor product space as introduced in section 2. Since $\mathcal{K}(Y,U)$ is a Banach space, it follows from Hahn-Banach theorem that it has a nontrivial continuous dual denoted by $\mathcal{K}^*(Y,U)$. Using the tensor product, the reader can easily verify that the inequality (7.15) can be expressed in the following form:

$$\begin{aligned} dJ_o(K^o, K - K^o) &= \int_0^T \{\langle K - K^o, (Lx^o + \xi^o) \otimes (B^*\varphi_1) + (L\varphi_2) \otimes (B^*\psi^o)\rangle\} dt \\ &\equiv \int_0^T \langle K - K^o, Z^o\rangle_{K(Y,U),K^*(Y,U)} dt \geq 0, \ \forall\ K \in \mathcal{F}_{ad}, \end{aligned} \tag{8.1}$$

where $\mathcal{K}^*(Y,U)$ denotes the (topological) dual (see, Feder & Saphar, [10] for characterization of the dual) of the space $\mathcal{K}(Y,U)$. Since Y is a reflexive Banach space it has RNP (Radon Nikodym Property) and hence the result of Feder and Saphar [10, Theorem 1, p40] holds. Note that Z^o given by $Z^o \equiv (Lx^o + \xi^o) \otimes (B^*\varphi_1) + (L\varphi_2) \otimes (B^*\psi^o) \in Y \hat{\otimes}_\pi U^* \subset \mathcal{K}^*(Y,U)$ is a sum of two elementary tensors. Since U is also a reflexive Banach space, the duality pairing (8.1) is equivalent to the pairing between the Banach space $Y \hat{\otimes}_\pi U^*$ and its dual $(Y \hat{\otimes}_\pi U^*)^* = \mathcal{L}(Y,U)$. Now we describe the algorithm:

Step1: Suppose at the n-th stage of iteration we have $K^n \in \mathcal{F}_{ad}$.

Step2: Using this K^n, solve the corresponding two point boundary value problem (given by differential inclusions) (6.17)-(6.18) following Theorem 6.4 giving

$\{x^n, \psi^n\} \in \mathcal{S}_{K^n} \subset C(I, X) \times C(I, X^*)$ corresponding to any pair $(R^n, \xi^n) \in Fix(\mathcal{J}_{K^n}) \subset \mathcal{V} \times \mathcal{D}$.

Step 3: Use $\{x^n, \psi^n\}$ to compute the operators $\{Q_i, i = 1, 2, \cdots, 6\}$ as introduced in Lemma 7.1 by replacing the pair $\{x^o, \psi^o\}$ in their arguments by the pair $\{x^n, \psi^n\}$ and solve the system (7.13)-(7.14) with the two point boundary conditions as stated in Theorem 7.2 giving the pair $\{\varphi_1^n, \varphi_2^n\}$.

Step 4: Use the optimality condition (7.15) or equivalently (8.1) (corresponding to the $n-th$ stage) and verify if the following inequality holds

$$
\begin{aligned}
(8.2) \qquad dJ_o(K^n, K - K^n) &= \int_0^T \langle K - K^n, Z^n \rangle_{\mathcal{K}(Y,U),\mathcal{K}^*(Y,U)} dt \\
&\equiv \langle\langle K - K^n, DJ_o(K^n) \rangle\rangle_{(Y\hat{\otimes}_\pi U^*)^*, Y\hat{\otimes}_\pi U^*} \ \geq 0,
\end{aligned}
$$

$\forall\ K \in \mathcal{F}_{ad}$ where the tensor Z^n is given by

$$Z^n(t) \equiv (Lx^n + \xi^n)(t) \otimes (B^*\varphi_1^n)(t) + (L\varphi_2^n)(t) \otimes (B^*\psi^n)(t), t \in I,$$

and $DJ_o(K^n) = \{Z^n(t), t \in I\}$. Note that $Z^n(t) \in Y\hat{\otimes}_\pi U^* \subset \mathcal{K}^*(Y, U)$ for $t \in I$. If the inequality (8.2) holds, the algorithm is complete and the optimal operator is given by K^n. If it fails, go to step 5.

Step 5: Define the normalized duality map

$$\Delta : (Y\hat{\otimes}_\pi U^*) \longrightarrow 2^{(Y\hat{\otimes}_\pi U^*)^*} \setminus 0$$

by

$$\Delta(z) \equiv \{S \in (Y\hat{\otimes}_\pi U^*)^* :< S, z >= \parallel z \parallel_\pi\}$$

for $z \in Y\hat{\otimes}_\pi U^*$ and choose K^{n+1} given by

$$(8.3) \qquad K^{n+1}(t) \in K^n(t) - \varepsilon\Delta(DJ_o(K^n))(t) = K^n(t) - \varepsilon\Delta(Z^n)(t), t \in I,$$

for $\varepsilon > 0$, sufficiently small, so that $K^{n+1} \in \mathcal{F}_{ad}$. In general $t \longrightarrow \Delta(Z^n(t))$ is a uniformly measurable multifunction with closed convex values $\Delta(Z^n(t)) \subset \mathcal{L}(Y, U)$. But since, for each $t \in I$, $Z^n(t)$ is the sum of a finite number (two) of elementary tensors there exists a uniformly measurable operator valued function Υ with values $\Upsilon(t) \in \mathcal{K}(Y, U) \cap (Y \otimes_\pi U^*)^*$ of norm 1 such that $\langle \Upsilon(t), Z^n(t) \rangle = \parallel Z^n(t) \parallel_\pi$. For example, a finite rank operator valued function Υ of the form $\Upsilon(t) = V_1(t) \otimes W_1^*(t) + V_2(t) \otimes W_2^*(t) \subset \mathcal{K}(Y, U) \subset (Y\hat{\otimes}_\pi U^*)^*$, satisfies the required properties for suitable choice of $V_1(t), V_2(t) \in B_1(U)$ and $W_1^*(t), W_2^*(t) \in B_1(Y^*)$. As U is also a reflexive Banach space, existence of such elements follows from Hahn-Banach separation theorem. Now returning to the algorithm and computing the cost functional at K^{n+1} we have

$$
\begin{aligned}
(8.4) \qquad J_o(K^{n+1}) &= J_o(K^n) + dJ_o(K^n; K^{n+1} - K^n) + o(\parallel K^{n+1} - K^n \parallel) \\
&= J_o(K^n) + \langle\langle K^{n+1} - K^n, DJ_o(K^n) \rangle\rangle + o(\cdot) \\
&= J_o(K^n) - \varepsilon\langle\langle \Delta(Z^n), Z^n \rangle\rangle + o(\varepsilon) \\
&= J_o(K^n) - \varepsilon \parallel Z^n \parallel_\pi + o(\varepsilon).
\end{aligned}
$$

This inequality implies that, for sufficiently small $\varepsilon > 0$, $\{J_o(K^n)\}$ is a (possibly monotone) decreasing sequence of real numbers. It follows from Corollary 5.4 and compactness of $\mathcal{F}_{ad}$ in the τ_T topology that $\sup\{|J_o(K)|, K \in \mathcal{F}_{ad}\} < \infty$. Thus there exists a real number m such that, as $n \to \infty$, $J_o(K^n) \longrightarrow m$ where m is (possibly) a local minimum.

Remark 8.1 Since, generally, the space $\mathcal{K}(Y,U)$ is not reflexive Kalton[Corollary 2, p268]12 the duality map $\Gamma : \mathcal{K}^*(Y,U) \longrightarrow B_1(\mathcal{K}(Y,U))$ may not be well defined in the sense that for any $Z \in \mathcal{K}^*(Y,U)$ the following set

$$\Gamma(Z) \equiv \{T \in B_1(\mathcal{K}(Y,U)) : \langle T, Z\rangle_{\mathcal{K}(Y,U),\mathcal{K}^*(Y,U)} = \parallel Z \parallel_*\}$$

may be empty. In our particular case (see the expression (8.2)), Z^n is given by a sum of two elementary tensors. We can take advantage of this and replace the $\mathcal{K}(Y,U) - \mathcal{K}^*(Y,U)$ pairing by $(Y\hat{\otimes}_\pi U^*)^* - (Y\hat{\otimes}_\pi U^*)$ pairing.

Open Problem It would be interesting to extend the above results to stochastic evolution equations. In particular, equation (3.1) is replaced by the following stochastic evolution equation

$$dx = Axdt + R(t)xdt + F(x)dt + B(t)u(t)dt + G(x)dW, x(0) = x_0, t \in I, \tag{8.5}$$

where W is an H-cylindrical Brownian motion (or Wiener process) on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ and $G : X \longrightarrow \mathcal{L}(H,Y)$. If the stochastic component appear additively, that is G is independent of the state x, the optimal feedback operator valued function K^o given by the deterministic analysis remain unchanged. For multiplicative noise substantial modification of the results given here is required.

References

[1] N. U. Ahmed, *Infinite dimensional uncertain dynamic systems on Banach spaces and their optimal output feedback control*, Differential Inclusions, Control and optimization (DICO) **35** (2015), 65–87.

[2] N. U. Ahmed, *Optimal control of infinite dimensional stochastic systems via generalized Solutions of HJB equations*, Discussiones Mathematicae Differential Inclusions, Control and Optimization, DICO **21** (2001), 97–126.

[3] N. U. Ahmed, *Optimization and Identification Systems Governed by Evolution Equations on Banach Space*, Pitman research Notes in Mathematics series, 184, Longman Scientific and Technical, U.K; and Co-published with John-Wiely and Sons, Inc. New York, 1988.

[4] N. U. Ahmed, *Optimal Output Feedback control law for a class of uncertain infinite dimensional dynamic systems*, JADEA **7** (2016), 11–29.

[5] N. U. Ahmed and X. Xiang, *Differential inclusions on Banach spaces and their optimal control*, Nonlinear Funct. Anal.& Appl. **8** (2003),461–488.

[6] N. U. Ahmed and S. Miah, *Optimal feedback control law for a class of partially observed dynamic systems: A min-max problem*, Dynamic Systems and Applications **20** (2011), 149–168.

[7] N. U. Ahmed and K. L. Teo, *Optimal Control of Distributed Parameter Systems*, Elsevier North Holland, Inc., New York, Oxford, 1981.

[8] L. Cesari, *Optimization Theory and Applications*, Springer-Verlag, 1983.

[9] H. O.Fattorini, *Infinite Dimensional optimization and Control Theory, Encyclopedia of mathematics and its applications*, vol. 62, Cambridge University Press, Cambridge, 1999.

[10] M. Feder and P. Saphar,*Spaces of compact operators and their dual spaces*, Israel Journal of Mathematics, **21** (1975).

[11] P. Hájek and R.J. Smith, *On Some Duality Relations in the Theory of Tensor Products*, Institute of Mathematics, Academy of Sciences of the Czech Republic, Preprint no. 28-2010,(old series no. 233) Praha 2010.

[12] S. Hu and N.S.Papageorgiou, *Handbook of Multivalued Analysis, (Theory)*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1997.

[13] N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

[14] F. Mayoral, *Compact sets of compact operators in absence of* ℓ_1, Proc. Am. Math. Soc. **129** (2001), 79–82.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol 1*, Fixed Point Theorems, Springer-Verlag, New York, 1986.

Manuscript received 28 October 2015
revised 22 March 2016

N. U. Ahmed
University of Ottawa,Ottawa, Canada
E-mail address: ahmed@site.uottawa.ca