Pure and Applied Functional Analysis
Volume 1, Number 1, 2016, 97–122

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

EXISTENCE OF PULSES FOR A MONOTONE REACTION-DIFFUSION SYSTEM

MARTINE MARION AND VITALY VOLPERT

Abstract. We consider a monotone reaction-diffusion system of the form

$$w_1'' - w_1 + f_1(w_2) = 0, \;\; w_2'' - w_2 + f_2(w_1) = 0,$$

and address the question of the existence of pulses, that is of positive decaying at infinity solutions. We prove that pulses exist if and only if the wave speed of the associated travelling-wave problem is positive. The proofs are based on the Leray-Schauder method which uses topological degree for elliptic problems in unbounded domains and a priori estimates of solutions in weighted spaces.

1. Introduction

This work is concerned with the existence of pulses for systems of reaction-diffusion equations. Let us begin with the example of the scalar equation

$$w'' + f(w) = 0,$$

where the function f satisfies the conditions : $f(0) = f(1) = 0$, $f'(0) < 0$, there exists a single zero μ of this function in the interval $(0, 1)$ and $\int_0^1 f(s)ds > 0$. It can be verified that under these assumptions there exists a positive solution w defined on $\mathbb{R}$ and decaying at infinity. We call it the pulse solution.

Existence of pulses for the scalar equation can be easily proved by the phase space analysis or by an explicit calculation (see section 4.2 below). These approaches are not applicable for systems of equations. Existence of pulses has been proved for some reaction-diffusion systems: the Gray-Scott model [1], [3], [10], the Gierer-Meinhardt model [4], [11] and a three component system [2].

In this work we consider reaction-diffusion systems:

$$Dw'' + F(w) = 0, \tag{1.1}$$

Here $w = (w_1, w_2)$, $F = (F_1, F_2)$, D is a diagonal matrix with positive diagonal elements. Also F is a sufficiently smooth vector-function which satisfies the following condition

$$\frac{\partial F_i}{\partial w_j}(w) > 0, \quad i, j = 1, 2, \;\; i \neq j. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35K57, 47F05.
Key words and phrases. Existence of pulses, monotone systems, Leray-Schauder method.

Therefore this is a monotone system for which the maximum principle is applicable. Furthermore we suppose that the origin is a zero of F and the corresponding linearized matrix has all eigenvalues in the left-half plane. We consider system (1.1) on the real axis and look for an even positive solution vanishing at infinity:

$$w(x) > 0, \quad w(x) = w(-x), \;\; x \in \mathbb{R}, \quad w(\pm\infty) = 0.$$

Similarly to the scalar case we will call such solutions pulses. Here and everywhere below inequalities for vectors mean that each component of the vectors satisfies this inequality.

Instead of the problem on the whole axis, we can consider system (1.1) on the half-axis $\mathbb{R}_+$ with the boundary condition

$$w'(0) = 0. \tag{1.3}$$

We will look for decreasing solutions:

$$w'(x) < 0 \;\text{ for }\; x > 0. \tag{1.4}$$

The existence of pulses will be investigated for a particular case of system (1.1) which consists of two equations of a special form:

$$\begin{cases} w_1'' - w_1 + f_1(w_2) = 0, \\ \\ w_2'' - w_2 + f_2(w_1) = 0, \end{cases} \tag{1.5}$$

considered on the half-axis $x \geq 0$ with the boundary condition

$$w_1'(0) = w_2'(0) = 0. \tag{1.6}$$

As mentioned above, we look for positive decreasing solutions of this problem decaying at infinity:

$$w(x) > 0 \;\text{ and }\; w'(x) < 0 \;\text{ for }\; x > 0, \quad w(\infty) = 0. \tag{1.7}$$

The functions $f_i : \mathbb{R} \to \mathbb{R}$ are supposed to be sufficiently smooth. For simplicity, we can consider infinitely differentiable functions. They satisfy the following conditions:

$$f_i(0) = 0, \;\; f_i(1) = 1, \;\; f_i'(s) > 0 \;\text{ for }\; s \geq 0, \;\; i = 1, 2. \tag{1.8}$$

Hence the points $w^+ = (0, 0)$ and $w^- = (1, 1)$ are zeros of $F = (F_1, F_2)$ where we set $F_1(w) = -w_1 + f_1(w_2)$, $F_2(w) = -w_2 + f_2(w_1)$. We assume the existence of a unique additional zero

$$\text{The vector function } F \text{ has three zeros } w^+, w^-, \bar{w} = (\bar{w}_1, \bar{w}_2) \text{ in } \mathbb{R}^2_+; \\ \text{and } 0 < \bar{w}_i < 1. \tag{1.9}$$

Here $\mathbb{R}^2_+$ is the quarter plane $w_1 \geq 0, w_2 \geq 0$. We finally assume that w^+ and w^- (resp. $\bar{w}$) are stable (resp. unstable) stationary points of the system

$$\frac{du_1}{dt} = -u_1 + f_1(u_2), \quad \frac{du_2}{dt} = -u_2 + f_2(u_1),$$

that is the eigenvalues of the matrices of the linearized equations

$$F'(w) = \begin{pmatrix} -1 & f_1'(w_2) \\ f_2'(w_1) & -1 \end{pmatrix}$$

have negative real parts (resp. an eigenvalue with positive real part). Since off-diagonal elements of these matrices are positive, the eigenvalues are real and simple. Therefore we require that the principal eigenvalue is negative (resp. positive). It is straightforward that these assumptions amount to

$$f_1'(0)f_2'(0) < 1, \quad f_1'(1)f_2'(1) < 1, \quad f_1'(\bar{w}_2)f_2'(\bar{w}_1) > 1. \tag{1.10}$$

Under the assumptions (1.8)-(1.10), it is well known that there exists a travelling-wave solution of the reaction-diffusion system

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + F(v). \tag{1.11}$$

This solution reads $v(x,t) = u(x-ct)$ where u satisfies the system of equations

$$u'' + cu' + F(u) = 0 \tag{1.12}$$

and has the limits at infinity

$$u(\pm\infty) = w^{\pm}. \tag{1.13}$$

Moreover, it is a monotonically decreasing vector-function (component-wise). The wave is unique up to translation in space. This means that such solution exists for a unique value of c, and for the given value of c it is unique up to translation in space. We will use the results on existence, stability and the speed of propagation of travelling waves for monotone systems of equations [5], [6].

We can now formulate the main result of this work.

Theorem 1.1. *Under assumptions (1.8)-(1.10), problem (1.5)-(1.7) has a solution if and only if the value of the speed c in problem (1.12)-(1.13) is positive.*

Let us first comment on the particular case $f_1 \equiv f_2$. Then $w_1 \equiv w_2$ may provide a solution of system (1.5). For such a solution, the problem reduces to the scalar equation

$$w_1'' + f(w_1) = 0,$$

with $f(s) = -s + f_i(s)$. Under the assumptions of Theorem 1.1, it can be explicitly verified that a solution of this equation with conditions

$$w_1'(0) = 0, \;\; w_1'(x) < 0 \;\text{ for }\; x > 0, \quad w_1(\infty) = 0,$$

exists if and only if

$$\int_0^1 f(s)ds > 0. \tag{1.14}$$

On the other hand, for the uniquely defined travelling-wave solution, we have $u_1 = u_2$ and

$$u_1'' + cu_1' + f(u_1) = 0, \quad u_1(-\infty) = 1, \quad u_1(\infty) = 0.$$

It is well known that the speed c has the sign of the integral in (1.14), so that it is positive if and only if the condition (1.14) is satisfied. For this particular case these properties yield the assertion of Theorem 1.1.

When we consider systems of equations, the positiveness of the integral can not be used anymore to conclude that the pulse exists and/or the speed of the wave is positive. But the relation between the sign of the wave speed and the existence of pulses remains true.

Let us briefly describe the methods of proof of Theorem 1.1 that are much more involved than in the scalar case. The existence of solutions for $c > 0$ is derived thanks to the Leray-Schauder method. In section 2 we show separation of monotone solutions and we obtain a priori estimates of such solutions. Section 3 is devoted to some spectral properties. Next in section 4, we construct homotopy to some model problem such that $f_1 \equiv f_2$ and c remains positive (section 4.1). This system has a solution for which $w_1 = w_2$ (section 4.2). Its index equals -1 and the degree is different from zero. Since the value of the degree is preserved, then it is also different from zero for the original problem. Hence this proves the existence of a solution (section 4.3).

Non-existence of solutions for $c \leq 0$ can be proved by the comparison theorem using the wave solution (section 4.4).

We conclude the paper with an example of system for which we can show that the speed of the wave is positive so that Theorem 1.1 provides the existence of a pulse (section 5).

2. Properties of pulses

The results presented in this section are applicable for reaction-diffusion systems

$$Dw'' + F^\tau(w) = 0 \tag{2.1}$$

which are assumed to depend on some parameter $\tau \in [0,1]$. Here $w = (w_1, w_2)$ and D is a square diagonal matrix with positive diagonal elements d_i, $i = 1,2$. We aim to investigate solutions of such systems defined on the half-axis $x \geq 0$ and such that:

$$w_i(x) > 0 \ \text{ for } \ x \in \mathbb{R}_+, \quad w_i'(0) = 0, \quad w_i(\infty) = 0, \quad i = 1,2. \tag{2.2}$$

We will suppose for simplicity that the function $F^\tau(w) = (F_1^\tau(w), F_2^\tau(w))$ is infinitely differentiable with respect to both variables $w \in \mathbb{R}^2_+$ and $\tau \in [0,1]$. The system is assumed to be monotone :

$$\frac{\partial F_i^\tau}{\partial w_j}(w) > 0, \quad i,j = 1,2, \ \ i \neq j, \ \ w \in \mathbb{R}^2_+, \ \ \tau \in [0,1]. \tag{2.3}$$

Also we set $w^+ = (0,0)$ and we suppose that

$$F^\tau(w^+) = 0, \quad F^{\tau\prime}(w^+) \text{ has all eigenvalues in the left} - \text{half plane.} \tag{2.4}$$

For the functional setting let us introduce the Hölder space $C^{k+\alpha}(\mathbb{R}_+)$ consisting of vector-functions of class C^k, which are continuous and bounded on the half-axis $\mathbb{R}_+$ together with their derivatives of order k, and such that the derivatives of order k satisfy the Hölder condition with the exponent $\alpha \in (0,1)$. The norm in this space is the usual Hölder norm. Set

$$E^1 = \{w \in C^{2+\alpha}(\mathbb{R}_+), \ w_i'(0) = 0, \ i = 1,2\}, \quad E^2 = C^\alpha(\mathbb{R}_+).$$

Next we introduce the weighted spaces E^1_μ and E^2_μ where $\mu(x) = \sqrt{1+x^2}$ which are equipped with the norms:

$$\|w\|_{E^i_\mu} = \|w\mu\|_{E^i}, \quad i = 1, 2.$$

In view of (2.1), let us consider the operator

$$A^\tau(w) = Dw'' + F^\tau(w), \tag{2.5}$$

acting from E^1_μ into E^2_μ. Then the linearized operator about any function in E^1_μ satisfies the Fredholm property and has the zero index. The nonlinear operator is proper on closed bounded sets. This means that the inverse image of a compact set is compact in any closed bounded set in E^1_μ. Finally, the topological degree can be defined for this operator. All these properties can be found in [8], [9].

2.1. **Separation of monotone solutions.** We first suppose that all monotone solutions of (2.1)-(2.2) are uniformly bounded in the space E^1_μ (in section 2.2 below we will investigate how to derive such bounds).

We aim to derive a result of separation between the solutions of (2.1)-(2.2) that are monotonically decreasing for all their components that will be denoted by $w^M(x)$ and the solutions of (2.1)-(2.2) which do not satisfy this condition that will be denoted by $w^N(x)$. We will call the latter ones non-monotone solutions.

Theorem 2.1. *Under assumptions (2.3)-(2.4), we also suppose that all monotone solutions of (2.1)-(2.2) are uniformly bounded in the space E^1_μ. Then there exists a constant $r > 0$ such that for any monotone solution w^M and any non-monotone solution w^N of (2.1)-(2.2) and (for all $\tau \in [0,1]$) the following estimate holds:*

$$\|w^M - w^N\|_{E^1_\mu} \geq r.$$

Proof. Let us suppose that the assertion of the theorem does not hold true. Then there is a sequence of monotone solutions $(w^{M,k})_k$ and a sequence of non-monotone solutions $(w^{N,k})_k$ (possibly for various values of τ) such that the norm of their difference tends to 0 as $k \to \infty$.

Since all monotone solutions are uniformly bounded and the operator is proper, the set $\{w^{M,k},\ w^{N,k},\ k \in \mathbb{N}\}$ is relatively compact in E^1_μ. In particular the sequence of monotone solutions possesses some convergent subsequence still denoted by $w^{M,k}$. Let us denote the limiting function by $\widehat{w}(x)$. It is a solution of problem (2.1) for some $\tau = \tau_0$ and satisfies $\widehat{w}(x) \geq 0$ and $\widehat{w}'(x) \leq 0$ for $x \geq 0$, $\widehat{w}'(0) = 0$.

Let us first show that

$$\widehat{w}(0) \neq (0,0).$$

Arguing by contradiction suppose that $w^{M,k}(0)$ converges to w^+ as $k \to \infty$. The assumptions (2.3)-(2.4) provide the existence of some neighborhood of the origin in $\mathbb{R}^2_+$ such that for all τ sufficiently close to τ_0 and for each point $w \neq w^+$ in this neighborhood at least one component of $F^\tau(w)$ is negative. Indeed in view of (2.4) the matrix $F^{\tau_0\prime}(w^+)$ possesses some positive eigenvector p corresponding to the principal eigenvalue. Hence for sufficiently small ϵ, $F^{\tau_0}(\epsilon p) < 0$ so that $F^\tau(\epsilon p) < 0$ for all τ sufficiently close to τ_0. Consequently thanks to the assumption (2.3) in the neighborhood $w^+ \leq w \leq \epsilon p$, $w \neq w^+$ at least one component of $F^\tau(w)$ is negative. Coming back to the sequence $w^{M,k}(0)$ which converges to w^+, for sufficiently large

k, $w^{M,k}(0)$ enters this neighborhood and at least one component of $F^{\tau_k}(w^{M,k}(0))$ is negative. Hence for the corresponding index i by equations (2.1) the second derivative ${w_i^{M,k}}''(0)$ is positive, and the function $w_i^{M,k}(x)$ cannot be decreasing.

Next let us verify that $\widehat{w}'(x) < 0$ for all $x > 0$ (component-wise). Indeed, suppose that $\widehat{w}_i'(x_0) = 0$ for some i and for some $x_0 > 0$. Assuming for example $i = 1$ we differentiate the first equation of system (2.1) for $\tau = \tau_0$. Setting $v = -\widehat{w}_1'$ it provides:

$$-d_1 v'' - a(x) v = b(x),$$

where

$$a(x) = \frac{\partial F_1^{\tau_0}}{\partial w_1}(\widehat{w}) \ , \quad b(x) = -\frac{\partial F_1^{\tau_0}}{\partial w_2}(\widehat{w}) \ \widehat{w}_2'(x).$$

Since $b(x) \geq 0$, $v(x) \geq 0$ and $v(x_0) = 0$, the positiveness theorem yields $v(x) \equiv 0$ hence $\widehat{w}_1(x) \equiv 0$. Then $\widehat{w}_2$ satisfies

$$\widehat{w}_2'' - \widehat{w}_2 = 0, \quad \widehat{w}_2 \geq 0, \quad \widehat{w}_2' \leq 0, \quad \widehat{w}_2'(0) = 0,$$

so that $\widehat{w}_2 \equiv 0$. We obtain a contradiction with $\widehat{w}(0) \neq 0$. Hence $\widehat{w}'(x) < 0$ for all $x > 0$. In particular this yields

$$\widehat{w}(0) > 0. \tag{2.6}$$

Consider now a sequence of non-monotone solutions $w^{N,k}$ which converges to the monotone solution $\widehat{w}$ as $k \to \infty$. Without loss of generality we can suppose that the first components of the solutions are not monotone. Then there are values $x_k > 0$ such that ${w_1^{N,k}}'(x_k) = 0$ and up to some subsequence we have either $x_k \to x_* > 0$ or $x_k \to \infty$ or $x_k \to 0$ as $k \to \infty$.

If $x_k \to x_*$ for some $x_* > 0$, then $\widehat{w}_1'(x_*) = 0$ and we obtain a contradiction with the monotonicity of this function.

We claim that for sufficiently large $y > 0$ and for sufficiently large k,

$${w^{N,k}}' < 0 \ \text{ on } \ [y, \infty[. \tag{2.7}$$

Therefore the convergence $x_k \to \infty$ cannot hold.

Indeed considering again some eigenvector $p > 0$ corresponding to the principal eigenvalue λ_0 of ${F^{\tau_0}}'(w^+)$, clearly ${F^{\tau_0}}'(w^+)p = \lambda_0 p < 0$. Consequently there exists $\delta > 0$ and k_0 such that for $k \geq k_0$ and $|w| \leq \delta$ (where $|.|$ denotes the euclidian norm in $\mathbb{R}^2$) we have ${F^{\tau_k}}'(w)p < 0$. Next due to the exponential decay of the solutions and the convergence of $w^{N,k}$ to the monotone function $\widehat{w}$ we can select $y > 0$ and $k_1 \geq k_0$ such that:

$${w^{N,k}}'(y) < 0, \quad |\widehat{w}(x)| \leq \delta \ \text{ for } \ x \geq y, \quad |w^{N,k}(x)| \leq \delta \ \text{ for } \ x \geq y \ \text{ and } \ k \geq k_1.$$

Now suppose that (2.7) is not satisfied and a function $v^k(x) = -{w^{N,k}}'(x)$ is not positive for some $x > y$ and $k \geq k_1$. By differentiating system (2.1) v^k satisfies:

$$D {v^k}'' + {F^{\tau_k}}'(w^{N,k}) v^k = 0. \tag{2.8}$$

Since $v^k(y) > 0$ and $v^k(+\infty) = 0$ (due to the exponential decay of v^k), we can choose some $\alpha > 0$ such that $u^k(x) \equiv v^k(x) + \alpha p \geq 0$ for all $x \geq y$, and $u^k(x_1) = 0$

(for at least one of the components of this vector) for some $x_1 > y$. Taking into account system (2.8), we see that

$$Du^{k''} + F^{\tau_k \prime}(w^{N,k})u^k + q(x) = 0,$$

where $q(x) = -\alpha F^{\tau_k \prime}(w^{N,k})p > 0$ on $[y, +\infty)$. We obtain a contradiction in signs in the equation for the component of the vector-function u^k, which has a minimum at $x = x_1$ (by using also (2.3)).

It remains to study the case $x_k \to 0$. Let us verify that

$$F^{\tau_0}(\widehat{w}(0)) > 0. \tag{2.9}$$

Obviously, the inequality $F^{\tau_0}(\widehat{w}(0)) \geq 0$ holds because otherwise, if at least one of the components of this vector is negative, then the corresponding component of the vector $\widehat{w}''(0)$ is positive. Since $\widehat{w}'(0) = 0$, this would contradict the assumption that the function $\widehat{w}$ is decreasing. Thus, we need to verify that the components of the vector $F^{\tau_0}(\widehat{w}(0))$ can not equal zero. Suppose that this is not true, and for example the first component vanishes. Set $v(x) = -\widehat{w}_1'(x)$ and differentiate the first equation in (2.1). This gives

$$d_1 v'' + \frac{\partial F_1^{\tau_0}}{\partial w_1}(\widehat{w}(x))\, v + b(x) = 0,$$

where

$$b(x) = -\frac{\partial F_1^{\tau_0}}{\partial w_2}(\widehat{w}(x))\, \widehat{w}_2'(x) \geq 0.$$

Since $v(0) = 0$ and $v'(0) = 0$, then we obtain a contradiction with the Hopf lemma.

Thus, we proved that all components of the vector $F^{\tau_0}(\widehat{w}(0))$ are positive. Since the functions $w^{N,k}$ converge to $\widehat{w}$, then for all k sufficiently large $F^{\tau_k}(w^{N,k}(0)) > 0$. Therefore $w^{N,k}(x)'' < 0$ in some interval $0 < x < \delta$ independent of k. Hence $w^{N,k}(x)' < 0$ in this interval and the convergence $x_k \to 0$ cannot hold. This contradiction completes the proof of the lemma. □

Remark. The arguments in the proof of Theorem 2.1 yield the existence of some constant $\eta > 0$ such that for any monotone solution w^M and all $\tau \in [0, 1]$:

$$w_1^M(0) > \eta, \quad w_2^M(0) > \eta. \tag{2.10}$$

Indeed otherwise there exists a sequence of monotone solutions $w^{M,k}$ converging to some $\widehat{w}$ in E_μ^1 and at least one component of $\widehat{w}(0)$ vanishes. In view of (2.6) this is impossible.

This lower bound yields that the monotone solutions are also separated from the trivial solution $w \equiv 0$.

2.2. **Estimates of monotone solutions.** Here setting $w^- = (1, 1)$, we assume furthermore that

$$F^\tau(w^-) = 0, \;\; F^{\tau\prime}(w^-) \text{ has all eigenvalues in the left} - \text{half plane.} \tag{2.11}$$

and

$$\begin{gathered} F^\tau \text{has a unique zero } \bar{w}^\tau \in (w^+, w^-), \\ \text{the principal eigenvalue of } F^{\tau\prime}(\bar{w}^\tau) \text{ is positive} \end{gathered} \tag{2.12}$$

where (w^+, w^-) denotes the set of w such that $0 < w_i < 1$, $i = 1, 2$. Note that due to (2.4) F^τ only vanishes at the two points w^+ and w^- on the boundary of the unit square $0 < w_i < 1$.

Consider monotone solutions $w(x)$ of problem (2.1)-(2.2) that furthermore satisfy the inequality $w^+ < w(x) < w^-$ for all $x \geq 0$ (see Lemma 2.3 below). From this L^∞ estimate and from the assumption that the function F^τ is sufficiently smooth with all derivatives uniformly bounded it follows that the Hölder norm in $C^{2+\alpha}(\mathbb{R}_+)$ of the solutions is also uniformly bounded (there exists a bound independent of such a solution and of τ). However, this is not sufficient to conclude that the norm in the weighted space $E_1 = C_\mu^{2+\alpha}(\mathbb{R}_+)$ is uniformly bounded. It is clear from the following example. Let $u(x)$ be a positive function exponentially decaying at infinity. Consider the sequence of functions $u_k(x) = u(a_k x)$ where a_k is a sequence of positive numbers converging to zero. Then this sequence is uniformly bounded in the Hölder norm, each function $u_k(x)$ is bounded in the weighted norm, but this sequence is not uniformly bounded in the weighted norm. This is a typical situation for problems in unbounded domains. We need to impose some additional condition in order to get uniform estimates in the weighted norm.

This condition will be related to the travelling wave problem

$$Du'' + cu' + F^\tau(u) = 0. \tag{2.13}$$

In view of the assumptions (2.11)-(2.12) together with (2.3)-(2.4) there exists a unique value of c denoted by c^τ such that the system (2.13) has a monotonically decreasing solution with the limits at infinity:

$$u(\pm\infty) = w^\pm, \tag{2.14}$$

(see [5], [6]).

The following result provides the estimates for w in the weighted spaces under an appropriate assumption on c^τ.

Theorem 2.2. *Let assumptions (2.3)-(2.4), (2.11)-(2.12) hold. Moreover suppose that $c^\tau > 0$ for all $\tau \in [0,1]$. Then there exists some constant $R > 0$ such that for all $\tau \in [0,1]$ and for any arbitrary monotone solution w of (2.1)-(2.2) with $w^+ < w < w^-$, the following estimate holds:*

$$\|w\|_{E_\mu^1} \leq R.$$

Proof. Since such solutions are uniformly bounded in the Hölder norm without weight, it is sufficient to prove that the norm $\sup_x |w(x)\mu(x)|$ is uniformly bounded. Let us recall that solutions decay exponentially at infinity. So this norm is bounded for each solution. Suppose that solutions are not uniformly bounded in the weighted norm. Then there is a sequence of monotone solutions w^k (with $w^+ < w^k < w^-$) of problem (2.1)-(2.2) for possibly different values of τ for which

$$\sup_{x \geq 0} |w^k(x)\mu(x)| \to \infty \ \text{ as } k \ \to \infty.$$

Let $\epsilon > 0$ be sufficiently small so that the exponential decay of the solutions gives the estimate

$$|w^k(x)\mu(x)| \leq M,$$

for those values of x such that $|w^k(x)| \le \epsilon$, with some constant $M > 0$ independent of k. Choosing $\epsilon < \eta$ given by (2.10) we can select x_k so that $|w^k(x_k)| = \epsilon$ and

$$|w^k(x)\mu(x - x_k)| \le M \text{ for } x \ge x_k. \tag{2.15}$$

If the values x_k are uniformly bounded, then the values $|w^k(x)\mu(x)|$ are uniformly bounded for $0 \le x \le x_k$ since $w^k(x) \le w^+$. Together with (2.15), this provides the required estimate for all $x \ge 0$. Suppose that $x_k \to \infty$. Consider the sequence of functions $u^k(x) = w^k(x + x_k)$. We can choose a subsequence converging to some limiting function $u^0(x)$ in $C^2_{loc}(\mathbb{R})$. Then u^0 is a monotone function defined on the whole axis and satisfies the equation

$$Du^{0\prime\prime} + F^{\tau_0}(u^0) = 0$$

for some $\tau_0 \in [0, 1]$. The bound (2.15) yields that $u^0(\infty) = w^+$ while $u^0(-\infty) = u^-$, where u^- is such that $F^{\tau_0}(u^-) = 0$ and $u^- \ne w^+$ since $|u^0(0)| = \epsilon$. Hence we obtain a solution of system (2.13) with $c = 0$. Let us show that this is not possible. Indeed, if u^- is the unstable zero of F^{τ_0} in (w^+, w^-), then a solution of system (2.13) exists only for negative c [5], [6]. If $u^- = w^-$ (stable point), then by virtue of the assumption of Theorem 2.2, $c > 0$. Hence the function u^0 can not exist, and the sequence x_k is bounded. This completes the proof of Theorem 2.2. □

There remains to check conditions that guarantee that the monotone solutions take values in (w^+, w^-). Even though more general results can be obtained we focus on the particular system that we will consider hereafter that is

$$\begin{cases} d_1 w_1'' - w_1 + f_1^\tau(w_2) = 0, \\ \\ d_2 w_2'' - w_2 + f_2^\tau(w_1) = 0, \end{cases} \tag{2.16}$$

where $d_i > 0$, $i = 1, 2$. We will assume that the functions $f_i^\tau : \mathbb{R} \to \mathbb{R}$ are infinitely differentiable and satisfy conditions similar to (1.8)-(1.10) namely:

$$f_i^\tau(0) = 0,\ f_i^\tau(1) = 1,\ f_i^{\tau\prime}(s) > 0 \text{ for } s \ge 0,\ i = 1, 2, \tag{2.17}$$

(2.18)
The system $w_1 = f_1^\tau(w_2)$, $w_2 = f_2^\tau(w_1)$ has three solutions $w^+, w^-, \bar{w}^\tau$ in $\mathbb{R}^2_+$, furthermore $0 < \bar{w}_i^\tau < 1$,

$$f_1^{\tau\prime}(0) f_2^{\tau\prime}(0) < 1, \quad f_1^{\tau\prime}(1) f_2^{\tau\prime}(1) < 1, \quad f_1^{\tau\prime}(\bar{w}_2^\tau) f_2^{\tau\prime}(\bar{w}_1^\tau) > 1. \tag{2.19}$$

Lemma 2.3. *Under assumptions (2.17)-(2.19), assume that problem (2.16) has a solution $w(x)$ defined for $x \ge 0$ and such that*

$$w'(0) = 0, \quad w(x) > 0 \text{ and } w'(x) < 0 \text{ for } x > 0, \quad w(\infty) = 0. \tag{2.20}$$

Then for $x \ge 0$ and $i = 1, 2$ we have $w_i(x) \le 1$.

Proof. Let us omit the index τ. As usual we set $w^- = (1, 1)$ and $F(w) = (F_1(w), F_2(w))$ with $F_1(w) = -w_1 + f_1(w_2)$, $F_2(w) = -w_2 + f_2(w_1)$. Thanks to the assumptions (2.18)-(2.19), we have $f_2(w_1) < f_1^{-1}(w_1)$ for $w_1 > 1$ where f_1^{-1} denotes the inverse function of f_1. Consequently it is easy to check that for each point $w \in \mathbb{R}^2_+$ that does not belong to the unit square, at least one of the functions $F_i(w)$ is negative.

Consider now some solution w satisfying (2.20). Both functions w_1 and w_2 reach their maximum value at $x = 0$. If $(w_1(0), w_2(0))$ does not belong to the unit square then at least one of the $F_i(w(0))$ is negative, hence the second derivative $w_i''(0)$ is positive and w_i can not be decreasing. This concludes the proof of Lemma 2.3. □

The above assumptions (2.17)-(2.19) yield the conditions (2.3)-(2.4) and (2.11)-(2.12) for $F^\tau(w) = (-w_1 + f_1^\tau(w_2), -w_2 + f_2^\tau(w_1))$. Hence under these assumptions Theorems 2.1 and 2.2 and Lemma 2.3 apply. Consequently if $c^\tau > 0$ for all $\tau \in [0, 1]$ the monotone solutions are uniformly bounded in E^1_μ and they are separated from the non-monotone ones. This will be a crucial tool in section 4 below where we will apply the Leray-Schauder method to obtain the existence of monotone solutions. To be able to compute the degree we will need some spectral properties that are investigated in the next section.

3. Spectral properties

We consider the system

$$d_1 w_1'' - w_1 + f_1(w_2) = 0, \quad d_2 w_2'' - w_2 + f_2(w_1) = 0, \tag{3.1}$$

where $d_1,\ d_2 > 0$ and $f_1,\ f_2$ satisfy (1.8)-(1.10). We suppose that there is a solution (w_1, w_2) satisfying (1.6)-(1.7). Next consider the eigenvalue problem

$$d_1 v_1'' - v_1 + a_1 v_2 = \lambda v_1, \quad d_2 v_2'' - v_2 + a_2 v_1 = \lambda v_2, \tag{3.2}$$

$$v_1'(0) = v_2'(0) = 0, \quad v_1(\infty) = v_2(\infty) = 0, \tag{3.3}$$

where

$$a_1(x) = f_1'(w_2(x)), \quad a_2(x) = f_2'(w_1(x)).$$

Here note that $a_i(x) > 0$ for all $x \geq 0$, $i = 1, 2$. The principal eigenvalue λ_0 of problem (3.2)-(3.3) is real, simple, positive and the corresponding eigenfunction is positive [7]. We aim to show that all other eigenvalues are negative.

Theorem 3.1. *Let assumptions (1.8)-(1.10) hold. Suppose that w is a solution of (3.1) satisfying (1.6)-(1.7). Then the only real non-negative eigenvalue of problem (3.2)-(3.3) is the principal eigenvalue λ_0.*

Proof. Suppose that problem (3.2)-(3.3) has a real eigenvalue $\lambda \geq 0$, $\lambda \neq \lambda_0$, with some corresponding eigenfunction $v = (v_1, v_2)$. The system (3.2) also reads:

$$d_1 v_1'' - (1 + \lambda) v_1 = -a_1 v_2, \quad d_2 v_2'' - (1 + \lambda) v_2 = -a_2 v_1. \tag{3.4}$$

Here, since $\lambda \geq 0$, the operators in the left-hand sides of the above equations are invertible and this guarantees that $v_1 \not\equiv 0$ and $v_2 \not\equiv 0$.

Next we claim that if one of the functions v_i, $i = 1, 2$, is positive, then the other one is also positive. Indeed, suppose for example that $v_2(x) > 0$ for all $x > 0$. If v_1 has negative values then it has a point of minimum because it converges to zero at infinity. Since $a_1 v_2 > 0$ on $(0, +\infty)$, we obtain a contradiction in signs in the first equation in (3.4) at the point of minimum (which can be in particular $x = 0$).

Hence $v_1(x) \geq 0$. It can be easily verified that this inequality is strict (due to $v_1 \not\equiv 0$ and $v_1'(0) = 0$).

Therefore since a positive eigenfunction only corresponds to the principal eigenvalue, both functions v_1 and v_2 must have variable signs.

Let us check that there exists a value $x_1 \geq 0$ such that

$$v_1(x_1) \leq 0,\ v_2(x_1) \leq 0 \text{ and } v_i(x) > 0 \text{ for some } x > x_1 \text{ and } i. \tag{3.5}$$

Indeed, consider the values $v_1(0)$ and $v_2(0)$. If both of them are non-positive or non-negative then we set $x_1 = 0$ (and eventually consider the opposite of the eigenfunction). If these values have opposite signs with for example $v_1(0) < 0 < v_2(0)$, we set x_1 equal to the minimum from the two (possibly equal) values x where the functions v_1 and v_2 have their first zero. Then either $v_2(x_1) = 0$ and $v_1(x_1) \leq 0$, and v_1 will have positive values for some larger x since it has variable sign; or $v_1(x_1) = 0$ and $v_2(x_1) \geq 0$, this case is similar to the previous one by considering the opposite of the eigenfunction. Thus, in all cases, (3.5) holds true.

We aim to show that (3.5) contradicts the existence of some positive vector-function $v^0 = (v_1^0, v_2^0)$ which satisfies the equation

$$d_1 v_1^{0\prime\prime} - v_1^0 + a_1 v_2^0 = 0, \quad d_2 v_2^{0\prime\prime} - v_2^0 + a_2 v_1^0 = 0, \tag{3.6}$$

and the boundary condition

$$v_1^0(0) = v_2^0(0) = 0. \tag{3.7}$$

Such a solution exists since $-w'$ satisfies the above conditions.

Let us begin with the following assertion. Suppose that for some N we have the inequality

$$v_i^0(N) > v_i(N) \text{ for } i = 1, 2. \tag{3.8}$$

We claim that, if N is large enough (to be determined below), the following inequality is satisfied:

$$v_i^0(x) > v_i(x) \text{ for all } x \geq N \text{ and } i = 1, 2. \tag{3.9}$$

Indeed let us consider the functions $z_i = v_i^0 - v_i$, $i = 1, 2$, which satisfy the equations:

$$d_1 z_1'' - z_1 - \lambda z_1 + a_1 z_2 + \lambda v_1^0 = 0, \quad d_2 z_2'' - z_2 - \lambda z_2 + a_2 z_1 + \lambda v_2^0 = 0. \tag{3.10}$$

Suppose that these functions are not positive for all $x \geq N$. At least one of them has negative values. Since the matrix

$$F'(w^+) = \begin{pmatrix} -1 & f_1'(0) \\ f_2'(0) & -1 \end{pmatrix} = \begin{pmatrix} -1 & a_1(+\infty) \\ a_2(+\infty) & -1 \end{pmatrix}$$

is assumed to have negative principal eigenvalue, for some corresponding positive eigenvector p, we have the inequality $F'(w^+)p < 0$. Let us choose N sufficiently large so that the inequality

$$F'(w(x))p < 0 \text{ for all } x \geq N \tag{3.11}$$

holds true. Here $z_i(N) > 0$, $z_i(+\infty) = 0$ and at least one of these two functions has negative values for some $x > N$. Hence there exists a positive number τ such that the functions $u_i(x) = z_i(x) + \tau p_i$, where p_1, $p_2 > 0$ are the components of the vector p, satisfy the following conditions:

$$u_i(x) \geq 0 \text{ for } x \geq N \text{ and } i = 1, 2\ ; \quad \exists\, x_0 > N \text{ and } i \text{ such that } u_i(x_0) = 0. \tag{3.12}$$

On the other hand, from (3.10) it follows that the vector-function $u = (u_1, u_2)$ satisfies the system:

$$Du'' + F'(w)u - \lambda u + b = 0,$$

where $b(x) = -\tau F'(w(x))p + \tau\lambda p + \lambda v^0(x) > 0$. We obtain a contradiction in signs at the point x_0 where $u_i(x_0) = 0$. This contradiction proves that, under the condition (3.11), (3.9) follows from (3.8).

Let us now return to (3.5). We will compare the eigenfunction v with the function $v^0 = -w'$. First suppose that in (3.5) we have $x_1 > 0$. Since $v^0(x) > 0$ for $x > 0$, we can choose a positive number $\sigma > 0$ such that

$$\sigma v^0(x) > v(x) \text{ for } x_1 \leq x \leq N, \tag{3.13}$$

where $N > x_1$ is fixed and verifies (3.11). Then from (3.9) it follows that

$$\sigma v^0(x) > v(x) \text{ for } x \geq N. \tag{3.14}$$

Hence we have this inequality for all $x \geq x_1$. Denote by σ_0 the infinimum of all values σ for which (3.13) holds true. Since at least one of the components of the vector-function $v(x)$ has positive values for some $x > x_1$, then $\sigma_0 > 0$. Obviously, $\sigma_0 v^0(x) \geq v(x)$ for $x_1 \leq x \leq N$. Moreover there exists $x_2 \in (x_1, N]$ such that at this point the inequality is not strict. Indeed, if it is strict for all $x \in (x_1, N]$ and $v^0(x_1) > v(x_1)$, then we can take a value $\sigma < \sigma_0$ for which (3.13) holds true. Thus the function $\omega = \sigma_0 v^0 - v$ satisfies the following properties:

$$\omega(x) \geq 0 \text{ for } x \geq x_1\ ; \ \ \omega(x_1) > 0\ ; \quad \exists\, x_2 > x_1 \text{ and } i \text{ such that } \omega_i(x_2) = 0. \tag{3.15}$$

Moreover, ω satisfies the equation:

$$D\omega'' + F'(w)\omega - \lambda\omega + \lambda\sigma_0 v^0 = 0. \tag{3.16}$$

Since the last term in the left-hand side of this equation is non-negative and $\omega(x) \not\equiv 0$, we obtain a contradiction in signs at the point x_2 where (3.16) holds if $\lambda \neq 0$ and with the positiveness theorem if $\lambda = 0$.

We proved that (3.5) gives a contradiction if $x_1 > 0$. If $x_1 = 0$, the difference with the previous case is that $v^0(x_1)$ vanishes instead of being positive. Recalling (3.5), we have $v_1(0), v_2(0) \leq 0$. If both inequalities are strict, then we can proceed as above starting from (3.13). Suppose now that one of the values $v_i(0)$, $i = 1, 2$, (or both) equals 0. Then $\sigma v_i^0(0) = v_i(0)$. However $\sigma {v_i^0}'(0) > 0 = v_i'(0)$ (recall (2.9) in the proof of Theorem 2.1). Hence $\sigma v_i^0(\epsilon) > v_i(\epsilon)$ for ϵ sufficiently small and we can proceed as before by replacing x_1 by ϵ in (3.13) and after. $\square$

4. Monotone pulses and travelling waves for equal diffusion coefficients

We consider the problem

$$w'' + F(w) = 0, \tag{4.1}$$

where

$$F_1(w) = -w_1 + f_1(w_2), \quad F_2(w) = -w_2 + f_2(w_1), \tag{4.2}$$

and the functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ satisfy (1.8)-(1.10).

Under these assumptions, there exists a decreasing solution of the equations

$$u'' + cu' + F(u) = 0 \tag{4.3}$$

with the limits

$$u(-\infty) = w^-, \quad u(\infty) = w^+. \tag{4.4}$$

Such a solution exists for a unique value of c and is uniquely defined up to a translation in space, see [6].

The purpose of this section is to show that there exists a solution of (4.1) satisfying

$$w'(0) = 0, \;\; w(x) > 0 \;\text{ and }\; w'(x) < 0 \;\text{ for }\; x > 0, \quad w(\infty) = 0, \tag{4.5}$$

if and only if the wave speed c is positive.

We will first assume

$$c > 0 \tag{4.6}$$

and we will derive the existence of a monotone pulse thanks to the Leray-Schauder method. In sections 4.1 and 4.2, we construct a continuous deformation (homotopy) of our problem to a model problem for which the value of the topological degree is different from zero. Then in section 4.3, we use a priori estimates of solutions obtained above to conclude that the degree is preserved and, consequently, there is a solution of problem (4.1) satisfying (4.5).

Finally in section 4.4 we will prove the non existence of monotone pulses if $c \leq 0$.

We will use some properties of travelling waves that are valid for more general monotone systems and that are recalled in the following theorem. Namely, the speed of the wave increases if we increase the nonlinearity. The proof can be found in [6].

Theorem 4.1. *Consider the two problems*

$$u'' + cu' + F^k(u) = 0, \quad u(\pm\infty) = w^\pm, \;\; k = 1, 2, \tag{4.7}$$

where $F^k = (F_1^k, F_2^k)$ are assumed to satisfy (2.3)-(2.4), (2.11) and furthermore

$$F_i^1(u) \geq F_i^2(u) \;\text{ for }\; w_+ \leq u \leq w_- \;\text{ and }\; i = 1, 2.$$

If the two problems (4.7) possess monotone solutions with the values of the speed $c = c_k$, $k = 1, 2$, then we have $c_1 \geq c_2$.

4.1. **Homotopy.** We aim to construct a continuous deformation

$$w'' + F^\tau(w) = 0, \quad \tau \in [0,1], \tag{4.8}$$

with functions F^τ taking the form

$$F_1^\tau(w) = -w_1 + f_1^\tau(w_2), \quad F_2^\tau(w) = -w_2 + f_2^\tau(w_1), \tag{4.9}$$

in such a way that in its starting point ($\tau = 0$) we have the initial system (4.1) and in its final point ($\tau = 1$) the functions f_i^1 are equal to each other, that is:

$$f_i^0(s) = f_i(s) \text{ for } i = 1,2, \quad f_1^1(s) = f_2^1(s) = g(s), \quad s \geq 0. \tag{4.10}$$

We will require the functions $f_i^\tau(s)$ to be sufficiently smooth with respect to s and τ and to satisfy the assumptions (2.17)-(2.19). Furthermore considering the travelling wave problem:

$$u'' + c^\tau u' + F^\tau(u) = 0, \quad u(\pm\infty) = w_\pm,$$

we will require that its uniquely defined speed satisfies

$$c^\tau > 0 \text{ for } \tau \in [0,1], \tag{4.11}$$

(recall that we assume that this is true for the initial problem corresponding to $\tau = 0$). As noted at the end of section 2.2, these properties will provide estimates of the monotone pulses in the space E_μ^1 that are independent of $\tau \in [0,1]$. Furthermore the final problem ($\tau = 1$) will take the form

$$w'' + G(w) = 0 \text{ with } G(w) = (-w_1 + g(w_2), -w_2 + g(w_1)).$$

We will look for its solution $w = (w_1, w_2)$ for which $w_1 \equiv w_2$ so that this system will reduce to a scalar equation.

Let us list the properties of g that will be needed. Firstly we will ask for the properties (2.17)-(2.19) to be satisfied for $\tau = 1$; they read:

(4.12)

The equation $g(s) = s$ has three zeros in $\mathbb{R}_+ : 0,\ 1$ and μ ; furthermore $0 < \mu < 1$,

$$g'(0) < 1, \ \ g'(\mu) > 1, \ \ g'(1) < 1, \ \ g'(s) > 0 \text{ for } \ s \geq 0. \tag{4.13}$$

Furtermore we will require that

$$\int_0^1 (g(s) - s)ds > 0. \tag{4.14}$$

As already mentioned in the introduction this last condition will both guarantee the positivity of the speed for the corresponding travelling wave problem and the existence of a monotone pulse for the final problem.

The construction of the homotopy consists of several steps.

Preliminary step. In view of the assumption (1.10) at least one of the values $f_i'(0)$ is less than one and we can assume that $f_1'(0) \le f_2'(0)$ and $f_1'(0) < 1$.

We claim that we can choose smooth functions h_1 and h_2 from $\mathbb{R}_+$ into $\mathbb{R}$ such that the following conditions are satisfied. Firstly the functions h_i satisfy the analog of (1.8)-(1.10), more precisely setting $H(w) = (-w_1 + h_1(w_2), -w_2 + h_2(w_1))$, we require that:

$$\begin{array}{c}\text{the equation } H(w) = 0 \text{ has three solutions in } \mathbb{R}_+^2 : \\ w^+,\ w^-,\ \hat{w} \text{ with } 0 < \hat{w}_i < 1,\end{array} \tag{4.15}$$

$$\begin{array}{c}h_1'(0)h_2'(0) < 1,\ \ h_1'(1) < 1,\ \ h_2'(1) < 1,\ \ h_1'(\hat{w}_2)h_2'(\hat{w}_1) > 1, \\ h_2'(s) > 0 \text{ for } s \ge 0.\end{array} \tag{4.16}$$

Next we want the h_i to be bounded from below by the f_i :

$$f_i(s) < h_i(s) \text{ for } 0 < s < 1 \text{ and } i = 1, 2. \tag{4.17}$$

Moreover we will need that

$$h_1(s) \le h_2(s) \text{ for } 0 \le s \le 1, \tag{4.18}$$

and

$$h_1 \text{ satisfies the properties } (4.12) - (4.14) \tag{4.19}$$

(stated above for g).

Suppose that such functions are constructed. Then the homotopy will proceed as follows. We will first reduce the functions f_i to the functions h_i. The properties (4.15)-(4.17) will guarantee that the wave speed is positive for the travelling wave problem associated to h_i (thanks to the monotony property in Theorem 4.1). Next, the second step in the homotopy will consist of reducing h_2 to h_1 thanks to (4.18). The final scalar problem for the monotone pulse will involve the function h_1 which satisfies (4.19). As already mentioned this will provide the existence of a solution for this problem.

We first construct some piecewise linear continuous functions satisfying the conditions (4.15)-(4.19) (except for the derivatives at the points where they do not exist). Let α_1 and α_2 be chosen so that

$$f_1'(0) < \alpha_1 < 1,\ \ f_2'(0) < \alpha_2,\ \ \alpha_1 < \alpha_2,\ \ \alpha_1\alpha_2 < 1.$$

Also let

$$M_2 > M_1 > 1, \quad M_i > \max_{s\in[0,1]} f_i'(s) \text{ for } i = 1, 2,$$

$$0 < m_2 < m_1 < 1, \quad m_i < \min_{s\in[0,1]} f_i'(s) \text{ for } i = 1, 2.$$

For $i = 1, 2$ we set:

$$h_i(s) = \begin{cases} \alpha_i s \text{ for } 0 \le s < \delta, \\ M_i s + \alpha_i\delta - M_i\delta \text{ for } \delta \le s \le \gamma_i, \\ m_i s + 1 - m_i \text{ for } \gamma_i \le s \le 1, \end{cases} \tag{4.20}$$

where γ_i is determined by expressing the continuity of h_i at this point. Tedious calculations provide that by choosing δ, m_1 and m_2 sufficiently small and M_1 and M_2 sufficiently large all required above conditions can be satisfied. In particular we have $\gamma_2 < \gamma_1 < 1$, $h_1 < h_2$ on $(0,1)$, and at the point $\hat{w}$ such that (4.15) holds ${h_1}'(\hat{w}_2) = M_1$ so that the product ${h_1}'(\hat{w}_2){h_2}'(\hat{w}_1)$ can be made larger than one.

To conclude it is straightforward to slightly modify the functions h_i in the neighborhoods of the points δ and γ_i to make them smooth on $[0,1]$. Also they can be easily extended to $\mathbb{R}_+$ so that the equation $H(w) = 0$ has no additional zero outside the unit square.

First step ($\tau \in [0,1/2]$). As already mentioned the first step of the homotopy consists of reducing the functions f_i to the functions h_i.

We first keep f_1 *fixed* and aim to construct some homotopy f_2^τ, $\tau \in [0,1/4]$, of the function f_2 to the function h_2 so that (2.17)-(2.19) are satisfied for (f_1, f_2^τ) and furthermore

$$f_2(s) \leq f_2^\tau(s) \text{ for } 0 \leq s \leq 1 \text{ and } \tau \in [0,1/4]. \tag{4.21}$$

This last condition will provide that

$$c^\tau \geq c > 0 \text{ for } \tau \in [0,1/4],$$

for the speed c^τ of the travelling wave corresponding to $F^\tau(w) = (-w_1 + f_1(w_2), -w_2 + f_2^\tau(w_1))$.

It is convenient to rewrite the conditions (2.18)-(2.19) in terms of the inverse function of f_1 that we denote by θ. For $\tau \in [0,1/4]$, we want the function $\varphi^\tau = f_2^\tau - \theta$ to vanish at 0 and 1 and to possess some unique additional zero s^τ in $(0,1)$; furthermore we need that ${\varphi^\tau}'(0) < 0$, ${\varphi^\tau}'(1) < 0$ and ${\varphi^\tau}'(s^\tau) > 0$.

Thanks to assumptions (1.8)-(1.10) these properties hold true for the starting point $f_2^0 = f_2$ $(\tau = 0)$. Let $\underline{s} \in (0,1)$ denote the unique zero of $f_2 - \theta$ in $(0,1)$. For $\tau = 1/4$, the final point $f_2^{1/4} = h_2$ is a perturbation of the piecewise linear function (4.20). By eventually increasing $1/\delta$ and M_2, the function $h_2 - \theta$ satisfies the required properties. Let $\bar{s} > \underline{s}$ denote its unique zero in $(0,1)$.

For $a > 0$ sufficiently large (to be specified later), we introduce the function

$$f_2^\tau(s) = \begin{cases} f_2(s), & 0 \leq s \leq 1 - 4\tau, \\ a(s - 1 + 4\tau) + f_2(1 - 4\tau), & 1 - 4\tau \leq s \leq \gamma^\tau, \\ h_2(s), & \gamma^\tau \leq s \leq 1, \end{cases} \tag{4.22}$$

where γ^τ is determined by expressing the continuity of f_2^τ at this point.

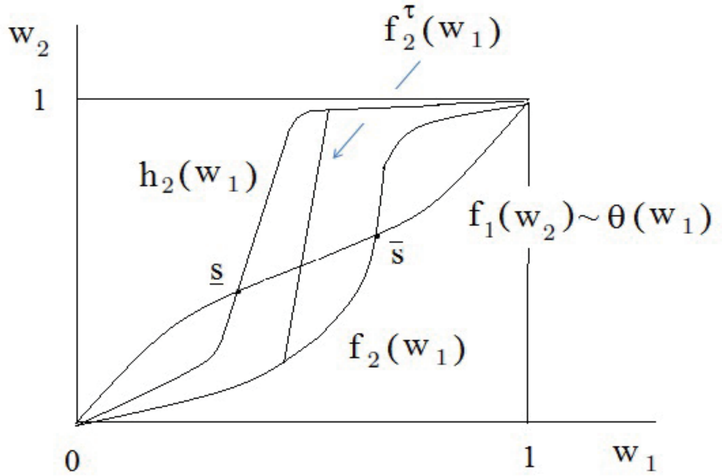

Figure 1. Schematic representation of the first step of homotopy.

Let $\epsilon > 0$ be given such that $\epsilon < \underline{s}$ and $\bar{s} + \epsilon < 1$. The above function f_2^τ is introduced for $\tau \in [\tau_0, \tau_1] \subset (0, \frac{1}{4})$ with $1 - 4\tau_0 = \bar{s}$ and $1 - 4\tau_1 = \underline{s} - \epsilon$. Then it is easy to see that for sufficiently large a we can define f_2^τ such that $\gamma^\tau \le 1 - 4\tau + \epsilon < 1$.

Next, let us investigate the zeros of $\varphi^\tau = f_2^\tau - \theta$ on $(0, 1)$. If $1 - 4\tau = \bar{s}$ (that is $\tau = \tau_0$), the unique zero of φ^{τ_0} is $\bar{s}$. If $\underline{s} \le 1 - 4\tau < \bar{s}$ the zeros of φ^τ necessarily belong to $(1 - 4\tau, \gamma^\tau)$ and on this interval the derivative ${\varphi^\tau}'(s) = a - \theta'(s)$ is positive if $a > \max_{s \in [0,1]} \theta'(s)$. If $\underline{s} - \epsilon \le 1 - 4\tau \le \underline{s}$ the zeros either belong to $(1 - 4\tau, \gamma^\tau)$ or take the value $\underline{s}$. In all cases, we conclude that φ^τ possesses a unique zero $s^\tau \in (0, 1)$ and ${\varphi^\tau}'(s^\tau) > 0$ (when this derivative exists).

To complete the definition of the homotopy, we start by reducing f_2 to $f_2^{\tau_0}$ on $[0, \tau_0]$ thanks to a linear homotopy

$$f_2^\tau(s) = (1 - \frac{\tau}{\tau_0}) f_2(s) + \frac{\tau}{\tau_0} f_2^{\tau_0}(s) \quad \text{for} \quad 0 \le \tau \le \tau_0.$$

Then on $[\tau_0, \tau_1]$, we go from $f_2^{\tau_0}$ to $f_2^{\tau_1}$ as described above. We conclude by reducing $f_2^{\tau_1}$ to h_2 on $[\tau_1, 1/4]$:

$$f_2^\tau(s) = \frac{1 - 4\tau}{1 - 4\tau_1} f_2^{\tau_1}(s) + \frac{4\tau - 4\tau_1}{1 - 4\tau_1} h_2(s) \quad \text{for} \quad \tau_1 \le \tau \le \frac{1}{4}.$$

For $\tau \notin [\tau_0, \tau_1]$, the unique zero of the corresponding $\varphi^\tau = f_2^\tau - \theta$ is either $\bar{s}$ or $\underline{s}$, the condition on the derivative is satisfied.

Up to now we have been considering piecewise C^1 functions and we conclude by slightly modifying them in the neighborhoods of the points where they are not differentiable. Also we extend them to $\mathbb{R}_+$ so that the equation $F^\tau(w) = 0$ has no additional zero outside the unit square.

For $\tau \in [1/4, 1/2]$, we next reduce the function f_1 to the function h_1. We proceed as above. The homotopy takes the form

$$F^\tau(w) = (-w_1 + f_1^\tau(w_2), -w_2 + h_2(w_1)).$$

Denoting by ξ the inverse function to h_2 we want the function $f_1^\tau - \xi$ to possess a unique zero in $(0,1)$ with a positive derivative at that point. We construct f_1^τ, $\tau \in [1/4, 1/2]$, as we did above for f_2^τ, $\tau \in [0, 1/4]$.

Second step ($\tau \in [1/2, 1]$). We now aim to reduce the function h_2 to h_1. The homotopy takes the form:

$$f_1^\tau(s) = h_1(s), \quad f_2^\tau(s) = 2(1-\tau)h_2(s) + (2\tau - 1)h_1(s), \quad \tau \in [1/2, 1].$$

Recalling that the h_i are slight perturbations of the functions in (4.20), the properties (2.17)-(2.19) for F^τ are easily checked (eventually by increasing the constant M_1). Also we claim that

$$c^\tau > 0 \text{ for } \tau \in [1/2, 1]. \tag{4.23}$$

Indeed note that (4.18) yields $f_2^\tau \geq h_1$ on $[0,1]$. Therefore $c^\tau \geq \hat{c}$ where $\hat{c}$ is the speed of the travelling wave for the problem

$$u'' + \hat{c}\, u' + G(u) = 0, \qquad G(u) = (-u_1 + g(u_2), -u_2 + g(u_1))$$

with $g = h_1$. For this uniquely defined wave, we have $u_1 = u_2$ and the sign of the speed is the same as the sign of the integral in (4.14). Hence $c^\tau \geq \hat{c} > 0$.

4.2. **Model problem.** As a result of the homotopy above, we obtained the model system

$$w_1'' - w_1 + g(w_2) = 0, \quad w_2'' - w_2 + g(w_1) = 0, \tag{4.24}$$

where $g = h_1$ satisfies (4.12)-(4.14).

Let us look for its solution for which $w_1 = w_2$ so that the system amounts to the scalar equation

$$w_1'' - w_1 + g(w_1) = 0. \tag{4.25}$$

The following lemma yields the existence of a monotone pulse for this problem.

Lemma 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f(0) = f(1) = 0$, $f'(0) < 0$, there exists a single zero μ of this function in $(0,1)$ and $\int_0^1 f(s)ds > 0$. Then the scalar equation*

$$w'' + f(w) = 0,$$

possesses a unique solution defined for $x \geq 0$ such that $w'(0) = 0$, $w' < 0$ on $(0,\infty)$ and $w(\infty) = 0$.

Proof. Let us rewrite the equation as a system of two first-order equations:

$$\begin{cases} w' = v, \\ v' = -f(w). \end{cases}$$

The equilibrium point $(0,0)$ is a saddle and we can consider the solution on the stable manifold such that $w'(x) < 0$ for large x. For this solution

$$\frac{w'(x)^2}{2} = -\int_0^{w(x)} f(s)ds$$

and Lemma 4.2 follows readily. $\square$

Lemma 4.2 with $f(s) = g(s) - s$ provides a solution w_1 of (4.25). Then, setting $w_2 = w_1$, the system (4.24) has a positive monotone solution with $w_i'(0) = 0$ and $w_i(\infty) = 0$, $i = 1, 2$. We do not know whether this solution is unique but we do not need its uniqueness. We now use Theorem 3.1 about spectral properties of the operator linearized at monotone solutions. We can introduce the number

$$\gamma^* = \sum_{k=1}^{K} (-1)^{\nu_k},$$

where K is the number of monotone solutions, ν_k the number of positive eigenvalues of the operator linearized about these solutions together with their multiplicity. Since these operators have a single real non-negative eigenvalue and it is simple, then $\gamma^* = -K \neq 0$. This is the value of the topological degree for the model problem. We will use separation of monotone solutions and their a priori estimates in order to show that this value is preserved for all τ. This will allow us to conclude the proof of the existence of solutions in the next section.

4.3. **Existence of solutions.** We can now prove the existence of monotone pulses for problem (4.1) under the assumption $c > 0$.

In section 4.1 we constructed the homotopy F^τ satisfying (4.10)-(4.11) and (2.17)-(2.19). Let us now consider the spaces introduced in section 2 and the operator A^τ given by (2.5).

It satisfies the conditions imposed to obtain a priori estimates of monotone solutions (Theorem 2.2 and Lemma 2.3). Denote by $B \subset E^1_\mu$ a ball which contains all monotone solutions. Since the operator A^τ is proper on closed bounded sets with respect to both variables w and τ, then the set of monotone solutions of the equation $A^\tau(w) = 0$ is compact. Since they are separated from non-monotone solutions, then we can construct a domain $D \subset B \subset E^1_\mu$ such that all monotone solutions (for all $\tau \in [0, 1]$) are located inside D and there are no non-monotone solutions in the closure $\bar{D}$. Indeed it is sufficient to take a union of small balls of the radius r (Theorem 2.1) around each monotone solution.

Let us note that only strictly monotone solutions belong to this domain. In particular, the trivial solution $w \equiv 0$ does not belong to it since, as indicated at the end of the section 2.1, the monotone solutions are separated from the trivial solution. This is an important remark because the index of the trivial solution equals one. If it belongs to the domain D, then the sum of the indices, which equals the value of the degree, can be zero. If it was the case, then we could not affirm the existence of solutions for the original problem.

We can define the topological degree $\gamma(A^\tau, D)$. We have

$$\gamma(A^0, D) = \gamma(A^1, D) = \gamma^* \neq 0.$$

Hence the degree is different from zero for the original problem, and the equation $A^0(w) = 0$ has a solution in D. This concludes the proof of the existence result in Theorem 1.1.

4.4. **Non-existence of solutions.** Here we suppose that the speed c in problem (4.3)-(4.4) is negative. We aim to show that problem (4.1) can not possess any solution w satisfying (4.5).

Let us begin with the case $c < 0$. Suppose that such a solution exists. Let us extend it to $\mathbb{R}$ by setting $w(x) = w(-x)$. Consider the parabolic equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + F(v) \tag{4.26}$$

on the whole axis (for x) with some initial condition v_0 such that $v_0(-\infty) = w^-, v_0(\infty) = w^+$ and $v_0(x) > w(x)$ for all $x \in \mathbb{R}$. The solution v of this Cauchy problem will satisfy the inequality

$$v(x,t) > w(x), \quad \forall\, x \in \mathbb{R},\ t > 0. \tag{4.27}$$

On the other hand, this solution converges to the wave $u(x - ct - h)$ where h is an appropriate real number [6]:

$$\sup_{x \in \mathbb{R}} |v(x,t) - u(x - ct - h)| \to 0, \quad t \to \infty.$$

Since $c < 0$, then $v(0,t) \to w^+$ as $t \to \infty$. We obtain a contradiction with inequality (4.27).

Consider now the case $c = 0$. Suppose again that (4.1) possesses some positive solution w satisfying (4.5) that we extend to $\mathbb{R}$ as above. Let u denote a travelling wave with speed 0, so that

$$u'' + F(u) = 0, \quad u(-\infty) = w^-, \quad u(\infty) = w^+.$$

Introduce $u^h(x) = u(x-h)$. We can choose h sufficiently large such that $u^h(x) > w(x)$ for all $x \in \mathbb{R}$. Indeed, first let us select $N > 0$ sufficiently large so that from the inequality $u^h(N) > w(N)$ it follows that $u^h(x) > w(x)$ for all $x \geq N$ (the existence of such N can be derived by arguments close to the ones used to infer (3.9) from (3.8) by using the Taylor formulas). Then since $w(x) < w^-$ we can find h such that $u^h > w$ on $[0, N]$. Consequently since w is even, $u^h(x) > w(x)$ for all $x \leq 0$.

Let h_0 be the infimum of all h for which $u^h > w$ on $\mathbb{R}$. Then

$$u^{h_0}(x) \geq w(x) \text{ for } x \in \mathbb{R},$$

and there exists some point x_0 where this inequality is not strict. This contradicts the positiveness theorem which is valid for monotone systems. Thus we have proved the non-existence part of Theorem 1.1.

5. Sign of the wave speed

As already mentioned, for the scalar case the wave speed c has the sign of the integral of the nonlinearity. This is no more true for systems. For monotone systems, the wave speed admits a minimax representation [6]. It allows one to estimate the speed if a good test function is chosen.

We will use here another approach. In section 5.1, we will introduce a system with discontinuous nonlinear terms for which we show the existence of travelling waves and explicitly determine the sign of the corresponding speed. Then in section 5.2 we will consider a regularized system obtained by smoothing the nonlinear terms. We will show the convergence of the corresponding wave speed to the wave speed of the system with discontinuous nonlinearities. This result will allow us to make conclusion about the sign of the wave speed for the regularized system and to apply the results on the existence of pulses to this system.

5.1. **A system with discontinuous nonlinearities.** We consider the system

$$\left\{ \begin{array}{l} u_1'' + cu_1' - u_1 + f_1(u_2) = 0, \\ \\ u_2'' + cu_2' - u_2 + f_2(u_1) = 0, \end{array} \right. \tag{5.1}$$

where

$$f_1(s) = \left\{ \begin{array}{l} 0 \text{ for } s < s_2 \\ 1 \text{ for } s \geq s_2 \end{array} \right. , \quad f_2(s) = \left\{ \begin{array}{l} 0 \text{ for } s < s_1 \\ 1 \text{ for } s \geq s_1 \end{array} \right.$$

for some $0 < s_2, s_1 < 1$. Without loss of generality we will suppose that $s_2 \leq s_1$. We look for a monotonically decreasing solution of this system with the limits

$$u_i(-\infty) = 1, \quad u_i(\infty) = 0, \quad i = 1, 2. \tag{5.2}$$

Since the solution is invariant with respect to translation in space, we can suppose that $u_1(0) = s_1$. Let L be such that $u_2(L) = s_2$. Then we can write equations (5.1) as follows:

$$u_1'' + cu_1' - u_1 + 1 = 0 \text{ for } x < L, \quad u_1'' + cu_1' - u_1 = 0 \text{ for } x > L, \tag{5.3}$$

$$u_2'' + cu_2' - u_2 + 1 = 0 \text{ for } x < 0, \quad u_2'' + cu_2' - u_2 = 0 \text{ for } x > 0. \tag{5.4}$$

The solutions of (5.3)-(5.4) can be easily computed. Consider

$$\lambda_1 = -\frac{c}{2} - \sqrt{\frac{c^2}{4} + 1} < 0\,, \quad \lambda_2 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1} > 0\,. \tag{5.5}$$

Setting $u_1(L) = u_1^*$ and $u_2(0) = u_2^*$, and expressing the limits at infinity, we find readily that

$$u_1(x) = 1 - (1 - u_1^*)e^{\lambda_2(x-L)} \text{ for } x < L, \quad u_1(x) = u_1^* e^{\lambda_1(x-L)} \text{ for } x > L,$$
$$u_2(x) = 1 - (1 - u_2^*)e^{\lambda_2 x} \text{ for } x < 0, \quad u_2(x) = u_2^* e^{\lambda_1 x} \text{ for } x > 0.$$

We now need to distinguish the cases $L \geq 0$ and $L < 0$. First suppose that $L \geq 0$. Then the conditions $u_1(0) = s_1$ and $u_2(L) = s_2$ take the form:

$$1 - (1 - u_1^*)e^{-\lambda_2 L} = s_1\,, \quad u_2^* e^{\lambda_1 L} = s_2, \tag{5.6}$$

while the continuity of the derivatives for u_1 at L and for u_2 at 0 reads:

$$\lambda_1 u_1^* = -\lambda_2(1 - u_1^*), \quad \lambda_1 u_2^* = -\lambda_2(1 - u_2^*). \tag{5.7}$$

The conditions (5.6)-(5.7) provide four equations for the four unknowns u_1^*, u_2^*, L and c. We can eliminate the unknowns u_1^* and u_2^* thanks to the equations (5.7):

$$u_1^* = u_2^* = \frac{\lambda_2}{\lambda_2 - \lambda_1},$$

so that it suffices to solve the following system of two equations for the unknowns c and L:

$$\frac{-\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 L} = 1 - s_1, \quad \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 L} = s_2. \tag{5.8}$$

Recall that λ_1 and λ_2 are given by (5.5).

Lemma 5.1. *For any $0 < s_2 \le s_1 < 1$, $i = 1, 2$, the system (5.8) has a unique solution (c, L); moreover $L \ge 0$. Furthermore $c > 0$ if $s_1 + s_2 < 1$, $c = 0$ if $s_1 + s_2 = 1$ and $c < 0$ if $s_1 + s_2 > 1$.*

Proof. Let us express L from the first equation in (5.8) and denote the corresponding function $L_1(c)$, and also express it from the second equation with the corresponding function $L_2(c)$. We obtain:

$$L_1(c) = -\frac{1}{\lambda_2}\log\frac{(1-s_1)(\lambda_2-\lambda_1)}{-\lambda_1}\ , \quad L_2(c) = \frac{1}{\lambda_1}\log\frac{s_2(\lambda_2-\lambda_1)}{\lambda_2}\ .$$

It can be easily verified that $L_1(c) \to \infty$ and $L_2(c) \to 0$ as $c \to \infty$. On the other hand, $L_1(c) \to 0$ and $L_2(c) \to \infty$ as $c \to -\infty$. Therefore the equation $L_1(c) = L_2(c)$ has at least a solution.

Next, we have $L_1(0) = -\log(2(1-s_1))$ and $L_2(0) = -\log(2s_2)$. If $L_1(0) < L_2(0)$, then there is a solution $c > 0$. In the case of equality, $c = 0$ is a solution, and if the inequality is opposite, then there is a solution $c < 0$.

In order to prove the uniqueness of the solution, we now consider L_1 and L_2 as functions of the variable λ_2 and we will use for them the same notations. Since $\lambda_1 = -1/\lambda_2$, we have:

$$L_1(\lambda_2) = -\frac{1}{\lambda_2}\log((1-s_1)(\lambda_2^2+1)), \quad L_2(\lambda_2) = -\lambda_2\log(s_2(1+1/\lambda_2^2)). \tag{5.9}$$

Then introducing $y = \lambda_2^2$, the equation $L_1(\lambda_2) = L_2(\lambda_2)$ amounts to

$$\Phi(y) \equiv y(\log(1+1/y) + \log s_2) - \log(1+y) - \log(1-s_1) = 0, \quad y > 0.$$

We note that $\Phi(0) = -\log(1-s_1) > 0$ and $\Phi(y) < 0$ for sufficiently large $y > 0$. Furthermore since $\Phi''(y) = \frac{y-1}{y(1+y)^2}$, we have $\Phi''(y) < 0$ for $y < 1$ and $\Phi''(y) > 0$ for $y > 1$. It is straightforward to conclude that the equation $\Phi(y) = 0$ has a unique solution $y_0 > 0$. Hence $\lambda_2 = \sqrt{y_0}$ and $\lambda_1 = -1/\lambda_2$ are uniquely defined and this provides c thanks to (5.5). Finally by (5.9):

$$L = -\frac{1}{\lambda_2}\log((1-s_1)(y_0+1)). \tag{5.10}$$

There remains to investigate the sign of L. Clearly in view of (5.10) $L \ge 0$ amounts to $y_0 < \frac{s_1}{1-s_1}$. Checking the sign of $\Phi(\frac{s_1}{1-s_1})$ we find readily that $L \ge 0$ if and only if $s_2 \le s_1$. □

Lemma 5.1 provides the existence of a travelling-wave with $L \ge 0$ in (5.3)-(5.4). Let us check that solution with $L < 0$ does not exist. Indeed if $L < 0$ the condition (5.6) is replaced by

$$u_1^* e^{-\lambda_1 L} = s_1\ , \quad 1 - (1-u_2^*)e^{\lambda_2 L} = s_2, \tag{5.11}$$

while (5.7) is unchanged. Here (5.11) is similar to (5.6) by replacing L by $-L$ and interverting the index 1 and 2. Hence Lemma 5.1 allows us to conclude that for $0 < s_2 \le s_1 < 1$ solution of (5.11)-(5.7) with $L < 0$ does not exist.

In conclusion we proved that Problem (5.1)-(5.2) possesses a monotone solution for a unique value of the speed which is given by Lemma 5.1 and that hereafter we denote by c^0. Furthermore c^0 is positive if and only if $s_1 + s_2 < 1$.

Let us give some interpretation of this last condition. It also reads $S_+ > S_-$, where $S_+ = (1 - s_1)(1 - s_2)$ and $S_- = s_1 s_2$. Coming back to the functions $F_1(w_1, w_2) = -w_1 + f_1(w_2)$ and $F_2(w_1, w_2) = -w_2 + f_2(w_1)$, then S_+ is the area of the domain (in the unit square) where both functions F_1 and F_2 are positive while S_- is the area of the domain where both functions are negative.

5.2. **Passage to the limit for a regularized system.** Consider the regularized system

$$\left\{ \begin{array}{l} u_1'' + cu_1' - u_1 + f_1^\rho(u_2) = 0, \\ \\ u_2'' + cu_2' - u_2 + f_2^\rho(u_1) = 0, \end{array} \right. \tag{5.12}$$

where f_1^ρ and f_2^ρ are infinitely differentiable functions with non-negative first derivatives and

$$f_1^\rho(s) = \left\{ \begin{array}{l} 0 \text{ for } s \le s_2 - \rho \\ 1 \text{ for } s \ge s_2 + \rho \end{array} \right. , \quad f_2^\rho(s) = \left\{ \begin{array}{l} 0 \text{ for } s \le s_1 - \rho \\ 1 \text{ for } s \ge s_1 + \rho \end{array} \right.$$

for some $0 < s_1,\ s_2 < 1$. Here $0 < \rho \le \rho_0$ where $\rho_0 > 0$ is given and satisfies $s_i - 2\rho_0 > 0$, $s_i + 2\rho_0 < 1$, $i = 1, 2$. We also suppose that the system

$$u_1 = f_1^\rho(u_2), \ \ u_2 = f_2^\rho(u_1)$$

possesses only one additional zero $\bar{u}^\rho = (\bar{u}_1^\rho, \bar{u}_2^\rho)$ with $0 < \bar{u}_i^\rho < 1$ and this zero is unstable: $f_1^{\rho\prime}(\bar{u}_2^\rho) f_2^{\rho\prime}(\bar{u}_1^\rho) > 1$. Functions f_i^ρ satisfying these conditions can be easily constructed.

System (5.12) has a unique monotone solution (up to translation in space) $u^\rho(x) = (u_1^\rho(x), u_2^\rho(x))$ with the limits

$$u_i^\rho(-\infty) = 1, \quad u_i^\rho(\infty) = 0, \quad i = 1, 2 \tag{5.13}$$

with the speed $c = c^\rho$. We will show that c^ρ converges to the speed c^0 of the discontinuous system (5.1) as $\rho \to 0$ together with the convergence of the corresponding waves.

Denote by $C_b^l(\mathbb{R})$ the class of vector functions of class l on $\mathbb{R}$ which are bounded together with all their derivatives of order $\le l$. The following result is standard.

Lemma 5.2. *The monotone solution u^ρ of (5.12)-(5.13) is bounded in $C_b^2(\mathbb{R})$ independently of $0 < \rho \le \rho_0$.*

Proof. See Lemma 2.1 in [6, p.158] □

We now aim to derive a priori bounds for c^ρ independent of $\rho \in (0, \rho_0]$.

Lemma 5.3. *There exist two constants $0 < c_* < c^*$ such that for all $0 < \rho \le \rho_0$, we have the bounds $c_* \le c^\rho \le c^*$.*

Proof. Let h_i and H_i be infinitely differentiable monotone functions such that

$$h_1(s) = \left\{ \begin{array}{l} 0 \text{ for } s \le s_2 + \rho_0 \\ 1 \text{ for } s \ge s_2 + 2\rho_0 \end{array} \right. , \quad h_2(s) = \left\{ \begin{array}{l} 0 \text{ for } s \le s_1 + \rho_0 \\ 1 \text{ for } s \ge s_1 + 2\rho_0 \end{array} \right.$$

and

$$H_1(s) = \begin{cases} 0 \text{ for } s \leq s_2 - 2\rho_0 \\ 1 \text{ for } s \geq s_2 - \rho_0 \end{cases}, \quad H_2(s) = \begin{cases} 0 \text{ for } s \leq s_1 - 2\rho_0 \\ 1 \text{ for } s \geq s_1 - \rho_0 \end{cases}.$$

We suppose that the system of equations $u_1 = h_1(u_2)$, $u_2 = h_2(u_1)$ has a unique solution for $0 < u_1, u_2 < 1$ and this solution is unstable. A similar condition is imposed on the functions H_i. Such functions can be easily constructed.

Then for $0 < \rho \leq \rho_0$

$$h_i(s) \leq f_i^\rho(s) \leq H_i(s) \text{ for } 0 \leq s \leq 1 \text{ and } i = 1, 2.$$

Consider the systems

$$\begin{cases} u_1'' + cu_1' - u_1 + h_1(u_2) = 0, \\ \\ u_2'' + cu_2' - u_2 + h_2(u_1) = 0, \end{cases} \tag{5.14}$$

and

$$\begin{cases} u_1'' + cu_1' - u_1 + H_1(u_2) = 0, \\ \\ u_2'' + cu_2' - u_2 + H_2(u_1) = 0. \end{cases} \tag{5.15}$$

They have unique solutions with the limits $u_i(-\infty) = 1$, $u_i(+\infty) = 0$, $i = 1, 2$. Denote by c_* the corresponding value of c for system (5.14) and by c^* for system (5.15). By virtue of Theorem 4.1, we have the estimates $c_* \leq c^\rho \leq c^*$. □

We can now prove the theorem about the convergence of the travelling waves. Since the waves are unique up to some translation in space we need to fix the value of one of the unknowns at some point. For example let us set

$$u_1^\rho(0) = \frac{s_1}{2} \text{ for } 0 < \rho \leq \rho_0. \tag{5.16}$$

Theorem 5.4. *As $\rho \to 0$, c^ρ converges to the speed c^0 for Problem (5.1). Moreover under the additional condition (5.16), the corresponding solution (u_1^ρ, u_2^ρ) converges in the topology of $C_b^1(\mathbb{R})$ to the unique solution (u_1^0, u_2^0) of problem (5.1) satisfying $u_1^0(0) = \frac{s_1}{2}$.*

Proof. Thanks to the previous lemmae the speed c^ρ is bounded independently of $0 < \rho \leq \rho_0$ while u^ρ is bounded in $C_b^2(\mathbb{R})$. Classical compactness arguments yield the existence of a sequence ρ_k with $\lim_{k\to\infty} \rho_k = 0$ such that $c^k = c^{\rho_k}$ converges to some limiting value $\hat{c}$ and $u^k = u^{\rho_k}$ converges to some $\hat{u}$ in $C_{loc}^1(\mathbb{R})$ as $k \to \infty$. We also set $(f_1^k, f_2^k) = (f_1^{\rho_k}, f_2^{\rho_k})$.

Let us denote by λ_i^k, $i = 1, 2$, the values in (5.5) corresponding to $c = c^k$ and by $\hat{\lambda}_i$, $i = 1, 2$, the values for $c = \hat{c}$. Since the sequence c^k is bounded it is easily seen that there existe $\delta > 0$ such that

$$\forall k, \;\; \lambda_1^k \leq -\delta < 0. \tag{5.17}$$

The monotony of u_1^k together with the condition (5.16) yield $u_1^k(x) < \frac{s_1}{2}$ for $x \geq 0$. Hence $f_2^k(u_1^k) = 0$ for k large enough so that thanks to the equation for u_2^k and $0 \leq u_2^k \leq 1$:

$$u_2^k(x) = u_2^k(0) e^{\lambda_1^k x} \text{ for } x \geq 0 \text{ and } k \geq k_0. \tag{5.18}$$

Hence for $k \geq k_0$, (5.17) and (5.18) provide $u_2^k(x) \leq e^{-\delta x}$ for $x \geq 0$, so that $u_2^k(x) < \frac{s_2}{2}$ for $x \geq x_1$ where x_1 is independent of k. Then $f_1^k(u_2^k) = 0$ for k large enough and the equation for u_1^k yields

$$u_1^k(x) = u_1^k(x_1)e^{\lambda_1^k(x-x_1)} \text{ for } x \geq x_1 \geq 0 \text{ and } k \geq k_1 \geq k_0. \tag{5.19}$$

By taking the limit $k \to \infty$ in (5.18) and (5.19) we obtain

$$\hat{u}_2(x) = \hat{u}_2(0)e^{\hat{\lambda}_1 x} \text{ for } x \geq 0,\ \ \hat{u}_1(x) = \hat{u}_1(x_1)e^{\hat{\lambda}_1(x-x_1)} \text{ for } x \geq x_1 \geq 0. \tag{5.20}$$

Hence $\hat{u}_1(\infty) = \hat{u}_2(\infty) = 0$ and the function $\hat{u}_1$ is not constant since $\hat{u}_1(0) = \frac{s_1}{2}$.

Next let us check that for some $x_0 < 0$ we have $\hat{u}_1(x_0) = s_1$. Indeed otherwise $\hat{u}_1 < s_1$ on $\mathbb{R}$. Consider $a \in \mathbb{R}^-$. The uniform convergence of the sequence u_1^k on bounded sets together with the monotony guarantee that for k large enough $\hat{u}_1^k(x) \leq \hat{u}_1^k(a) < s_1$ for $x \geq a$. As above we conclude that $\hat{u}_2(x) = \hat{u}_2(0)e^{\hat{\lambda}_1 x}$ on (a, ∞) hence on $\mathbb{R}$ since a is arbitrary. Since $\hat{u}_2$ is bounded we necessarily have $\hat{u}_2 \equiv 0$. Then $\hat{u}_2 < s_2$ on $\mathbb{R}$ which thanks to similar arguments yields $\hat{u}_1 \equiv 0$, hence the contradiction.

Further properties of solutions are given in the following lemmas. The proof of the theorem is completed after them.

Lemma 5.5.

(i) *There existe a unique $x_0 \in \mathbb{R}$ such that $\hat{u}_1(x_0) = s_1$. Furthermore $x_0 < 0$, $\hat{u}_1 > s_1$ on $(-\infty, x_0)$ and $\hat{u}_1 < s_1$ on (x_0, ∞).*

(ii) *We have $\hat{u}_2(x) = \hat{u}_2(x_0)e^{\hat{\lambda}_1(x-x_0)}$ for $x \geq x_0$ and $\hat{u}_2(x) = 1-(1-\hat{u}_2(x_0))e^{\hat{\lambda}_2(x-x_0)}$ for $x \leq x_0$.*

Proof. (i) The function $\hat{u}_1$ is (possibly not strictly) decaying with $\hat{u}_1(\infty) = 0$, $\hat{u}_1(0) = \frac{s_1}{2}$. Furthermore we proved that $\hat{u}_1(x_0) = s_1$ for some $x_0 < 0$. Let us argue by contradiction to derive the uniqueness of x_0. If not denote by x_0 the largest x such that $\hat{u}_1(x) = s_1$. Then we have $x_0 < 0$, $\hat{u}_1 \equiv s_1$ on $[b, x_0]$ for some $b < x_0$ and $\hat{u}_1 < s_1$ on (x_0, ∞).

Let us show that $\hat{u}_1$ can not be differentiable at x_0 which will provide the desired contradiction. Since $\hat{u}_1$ is constant on $[b, x_0]$ the derivative should be equal to 0. Next since $\hat{u}_1 < s_1$ on (x_0, ∞), we obtain that $\hat{u}_2(x) = \hat{u}_2(x_0)e^{\hat{\lambda}_1(x-x_0)}$ for $x \geq x_0$. Then there are two possibilities. Either $\hat{u}_2 < s_2$ on (x_0, ∞). Then for $x \geq x_0$, $\hat{u}_1(x) = \hat{u}_1(x_0)e^{\hat{\lambda}_1(x-x_0)}$ and $\hat{u}_1'(x_0) = s_1\lambda_1 < 0$. Or there exists some $x_1 > x_0$ such that $\hat{u}_2(x_1) = s_2$. Since $\hat{u}_2(x) = \hat{u}_2(x_0)e^{\hat{\lambda}_1(x-x_0)}$ is strictly monotone on $[x_0, \infty)$ we have $\hat{u}_2 > s_2$ on $(-\infty, x_1)$ so that $\hat{u}_1(x) = 1 + Be^{\hat{\lambda}_2 x}$ on $(-\infty, x_1)$. Here $\hat{u}_1(x_0) = s_1 < 1$ hence $B < 0$ and $\hat{u}_1'(x_0) = \hat{\lambda}_2 Be^{\hat{\lambda}_2 x_0} < 0$. To conclude, the derivative of $\hat{u}_1$ at x_0 should both vanish and be strictly negative, which is impossible. This shows (i) in Lemma 5.5. Then (ii) follows readily. □

Using analogous argument it is possible to derive the following lemma. Its proof is left to the reader.

Lemma 5.6.

(i) *There exists a unique* $y_0 \in \mathbb{R}$ *such that* $\hat{u}_2(y_0) = s_2$. *Furthermore* $\hat{u}_2 > s_2$ *on* $(-\infty, y_0)$ *and* $\hat{u}_2 < s_2$ *on* (y_0, ∞).

(ii) *We have* $\hat{u}_1(x) = \hat{u}_1(y_0)e^{\hat{\lambda}_1(x-y_0)}$ *for* $x \geq y_0$ *and* $\hat{u}_2(x) = 1-(1-\hat{u}_2(y_0))e^{\hat{\lambda}_2(x-y_0)}$ *for* $y \leq x_0$.

In view of Lemmae 5.5 and 5.6 we conclude that $(\hat{u}_1, \hat{u}_2)$ is a solution of system (5.1)-(5.2) for $c = \hat{c}$ and $\hat{u}_1(0) = \frac{s_1}{2}$. Hence $\hat{c} = c^0$ and $\hat{u} = u^0$. Also since the solution of the limiting system is unique, then the convergence holds true for the whole family (c^ρ, u^ρ) as $\rho \to 0$. It remains to note that the local convergence combined with the behavior at infinity imply the uniform convergence on the whole axis. This completes the proof of Theorem 5.4. □

References

[1] A. Doelman, R. A. Gardner and T. J. Kaper, *Stability analysis of singular patterns in the 1-D Gray-Scott model*, Physica D **122** (1998), 1–36.

[2] A. Doelman, P. van Heijster and T. J. Kaper, *Pulse dynamics in a three-component system: existence analysis*, J. Dyn. Diff. Equat. **21** (2009), 73–115.

[3] J. K. Hale, L.A. Peletier and W. C. Troy, *Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocatalysis*, SIAM J. Appl. Math. **61** (2000), 102–130.

[4] Y. Nec and M. J. Ward, *The stability and slow dynamics of two-spike patterns for a class of reaction-diffusion system*, Math. Model. Nat. Phenom. **8** (2013), 206–232.

[5] A. I. Volpert and V. A. Volpert, *Applications of the rotation theory of vector fields to the study of wave solutions of parabolic equations*, Trans. Moscow Math. Soc. **52** (1990), 59–108.

[6] A. Volpert, V. Volpert and V. Volpert, *Travelling Wave Solutions of Parabolic Systems*, Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.

[7] V. Volpert and A. Volpert, *Spectrum of elliptic operators and stability of travelling waves*, Asymptotic Analysis **23** (2000), 111–134.

[8] V. Volpert and A. Volpert, *Properness and topological degree for general elliptic operators*, Abstract and Applied Analysis (2003), 129–182.

[9] V. Volpert, *Elliptic Partial Differential Equations. Volume 1*, Fredholm theory of elliptic problems in unbounded domains, Birkhäuser, 2011.

[10] J. Wei and M. Winter, *Stability of spiky solutions in a reaction-diffusion system with four morphogens on the real line*, SIAM J. Math. Anal. **42** (2010), 2818–2841.

[11] J. Wei and M. Winter, *Existence, classification and stability analysis of multiple-peaked solutions for the Gierer-Meinhardt system in* $\mathbb{R}^1$, Methods Appl. Anal. **14** (2007), 119–163.

Manuscript received 19 October 2015
revised 10 January 2016

M. Marion
Institut Camille Jordan, UMR 5585 CNRS, Ecole Centrale de Lyon, 69134 Ecully, France
E-mail address: martine.marion@ec-lyon.fr

V. Volpert
Institut Camille Jordan, UMR 5585 CNRS, University Lyon 1, 69622 Villeurbanne, France
E-mail address: volpert@math.univ-lyon1.fr