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JACKSON TYPE ESTIMATES FOR PIECEWISE q-MONOTONE APPROXIMATION, $q \geq 3$, ARE NOT VALID

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Dedicated to Professor Amnon Jakimovski on the occasion of his 90th birthday

Abstract. We say that a function $f \in C[a,b]$ is q-monotone, $q \geq 3$, if $f \in C^{(q-2)}(a,b)$ and $f^{(q-2)}$ is a convex function on (a,b). Let $f \in C[-1,1]$ change its q-monotonicity $s \geq 1$ times, i.e., $f^{(q-2)}$ is alternatingly convex and concave $s+1$ times. We are interested in estimating the degree of approximation of f by polynomials which are co-q-monotone with it, namely, polynomials that change their q-monotonicity exactly at the points where f does. We prove, that no Jackson type estimates are valid, in general, for such approximation. This is in opposite to piecewise monotone ($q = 1$) and piecewise convex (q=2) approximations.

1. Introduction and the main results

We say that a function $f \in C[a,b]$ is q-monotone, $q \geq 2$, if $f \in C^{(q-2)}(a,b)$ and $f^{(q-2)}$ is a convex function on (a,b). Let $\mathbb{Y}_s[a,b] := \{Y_s[a,b]\}$ where $Y_s[a,b] = \{y_i\}_{i=0}^{s+1}$ such that $y_{s+1} := a < y_s < \cdots < y_1 < b =: y_0$, $s \geq 1$. We say that $f \in C[a,b]$ is piecewise q-monotone with respect to $Y_s[a,b]$ if it changes its q-monotonicity at the points $Y_s[a,b]$, that is, if $(-1)^{i-1} f^{(q-2)}$ is convex in (y_i, y_{i-1}), $1 \leq i \leq s+1$, and we denote by $\Delta^{(q)}(Y_s[a,b])$ the collection of all such piecewise q-monotone functions. If $q = 1$, then we use the same notation to denote piecewise monotone functions $f \in C[a,b]$, such that $(-1)^{i-1} f$ is nondecreasing in (y_i, y_{i-1}), $1 \leq i \leq s+1$. We also need the notation $W^r[a,b]$ for the Sobolev class of functions $f \in AC^{r-1}[a,b]$, such that $\|f^{(r)}\|_{[a,b]} \leq 1$. Note that we use the norm $\|g\|_{[a,b]} := \operatorname{esssup}_{x\in[a,b]} |g(x)|$. Of course, if $g \in C[a,b]$, then $\|g\|_{[a,b]} = \max_{x\in[a,b]} |g(x)|$. When $[a,b] = [-1,1]$, we denote $\mathbb{Y}_s := \mathbb{Y}_s[-1,1]$, $Y_s := Y_s[-1,1]$ (so that $\Delta^{(q)}(Y_s) = \Delta^{(q)}(Y_s[-1,1])$), $W^r := W^r[-1,1]$ and $\|\cdot\| := \|\cdot\|_{[-1,1]}$.

Let $\mathcal{P}_n$, be the space of algebraic polynomials of degree $\leq n$ and, for $g \in C[-1,1]$, let

$$E_n(g) := \inf_{P_n\in\mathcal{P}_n} \|g - P_n\|,$$

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denote the error of the best approximation of the function g. If $g \in \Delta^{(q)}(Y_s)$, then we denote by

$$E_n^{(q)}(g, Y_s) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^{(q)}(Y_s)} \|g - P_n\|,$$

the error of the best co-q-monotone approximation of the function g.

It is well known that if $f \in \Delta^{(q)}(Y_s) \cap W^r$, where $q = 1$ or $q = 2$, $Y_s \in \mathbb{Y}_s$, $s \geq 1$ and $r \geq 1$, then

$$E_n^{(q)}(f, Y_s) = O(1/n^r), \quad n \to \infty, \tag{1.1}$$

(see, e.g., [3] for details and references). It turns out, and proving this is the main purpose of this article, that for $q \geq 3$ the relationship (1.1) is, in general, invalid for any $r, s \in \mathbb{N}$ and every $Y_s \in \mathbb{Y}_s$.

Our main result is

Theorem 1.1. *For each $q \geq 3$, $r \in \mathbb{N}$, $s \in \mathbb{N}$ and any $Y_s \in \mathbb{Y}_s$, there exists a function $f \in \Delta^{(q)}(Y_s) \cap W^r$, such that*

$$\limsup_{n\to\infty} n^r E_n^{(q)}(f, Y_s) = \infty.$$

In fact, we will also prove the following less general but more precise statements.

Theorem 1.2. *Let $q \geq 3$, $s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. There is a function $f \in \Delta^{(q)}(Y_s) \cap W^{q-2}$, such that*

$$E_n^{(q)}(f, Y_s) \geq C(q, Y_s), \quad n \in \mathbb{N}, \tag{1.2}$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

Remark 1.3. Note that this, in particular, implies that for $r = q - 2$ even a Weierstrass type theorem is invalid. In fact, we will show that in Theorem 1.2, we may take the function $f(x) = \frac{1}{(q-2)!}|x - y_1|(x - y_1)^{q-3}$. Actually, we note that $\frac{1}{2}f \in \Delta^{(q)}(Y_s) \cap W^j$ for all $j = 1, \ldots, q-2$. Therefore we have an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let $q \geq 3$, $j \leq q-2, s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. There is a function $f \in \Delta^{(q)}(Y_s) \cap W^j$, such that*

$$E_n^{(q)}(f, Y_s) \geq C(q, Y_s), \quad n \in \mathbb{N},$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

Theorem 1.5. *Let $q \geq 3$, $s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. There is a function $f \in \Delta^{(q)}(Y_s) \cap W^{q-1}$, such that*

$$nE_n^{(q)}(f, Y_s) \geq C(q, Y_s), \quad n \in \mathbb{N}, \tag{1.3}$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

Our final result is

Theorem 1.6. *Let $r \geq 3$, $s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. For each sequence $\{\varepsilon_n\}_n^\infty$ of positive numbers, tending to infinity, there is a function $f \in \Delta^{(3)}(Y_s) \cap W^r$, such that*

$$\limsup_{n\to\infty} \varepsilon_n n^{r-1} E_n^{(3)}(f, Y_s) = \infty. \tag{1.4}$$

For the sake of completeness, we formulate a well-known result (see, e.g., [2], [1] and [6]),

Theorem 1.7. *Let $q \ge 4$, $r \ge q$, $s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. For each sequence $\{\varepsilon_n\}_n^\infty$ of positive numbers, tending to infinity, there is a function $f \in \Delta^{(q)}(Y_s) \cap W^r$, such that*

$$\limsup_{n\to\infty} \varepsilon_n n^{r-q+3} E_n^{(q)}(f, Y_s) = \infty.$$

Thus, Theorems 1.2, 1.5, 1.6 and Corollary 1.4, close the gap between the positive results for $q \le 2$ and the known negative ones for $q \ge 4$.

We prove Theorem 1.2 in Section 2 and Theorem 1.5 in Section 3. Then in Section 4 we prove Theorem 1.6. In the proofs we apply ideas from [1] and [2].

In the sequel, constants c and c_i depend only on q, r and s, while constants C may depend also on other parameters. Constants may differ from one another even if they appear in the same line.

Also, in the sequel, we will use the notation $l(\cdot; g; \alpha, \beta)$, for the linear function interpolating g at the points α and β, and

$$Y_1^* := \{1, 0, -1\}. \tag{1.5}$$

2. Proof of Theorem 1.2

Lemma 2.1. *Let $g \in C[-1,1]$ be such that g is convex on $(0,1)$ and concave on $(-1,0)$. Then there is an interval $[a,b] \subset [-1,1]$ of length $\frac{1}{4}$, such that $0 \notin [a,b]$ and*

$$\min_{x\in[a,b]} |\operatorname{sgn} x - g(x)| \ge \frac{1}{2}.$$

Proof. Denote by $x^* > 0$ and $x_* < 0$ two arbitrary points, such that $g(x^*) = 1$ and $g(x_*) = -1$, respectively. If x^* does not exist, we put $x^* = 1$, and if x_* does not exist, we put $x_* = -1$. Now we consider two cases.

Case 1. Suppose $g(0) \le 0$. Then, let $l(x) := l(x; g; 0, x^*)$. If $x^* \le \frac{1}{2}$, then the convexity of g implies that $g(x) \ge l(x), \quad x \in [x^*, 1]$. Hence, it is readily seen that

$$g(x) - \operatorname{sgn} x = g(x) - 1 \ge l(x) - 1 \ge \frac{1}{2}, \quad x \in [3/4, 1].$$

Otherwise, $x^* > \frac{1}{2}$, and the convexity of g implies $g(x) \le l(x), \quad x \in [0, x^*]$. Hence, it follows that

$$\operatorname{sgn} x - g(x) = 1 - g(x) \ge 1 - l(x) \ge \frac{1}{2}, \quad x \in (0, x^*/2].$$

Case 2. Otherwise, $g(0) > 0$, so let $l(x) := l(x; g; x_*, 0)$. If $x_* \ge -\frac{1}{2}$, then the concavity of g implies $g(x) \le l(x), \quad x \in [-1, x_*]$. Hence, it readily follows that

$$\operatorname{sgn} x - g(x) = -1 - g(x) \ge -1 - l(x) \ge \frac{1}{2}, \quad x \in [-1, -3/4].$$

Otherwise, $x_* < -\frac{1}{2}$, and the concavity of g implies $g(x) \ge l(x), \quad x \in [x_*, 0]$. Hence, we immediately have

$$g(x) - \operatorname{sgn} x = g(x) + 1 \ge l(x) + 1 \ge \frac{1}{2}, \quad x \in [x_*/2, 0).$$

This completes the proof. $\square$

Remark 2.2. Lemma 2.1 remains valid if $g \in C(-1,1)$, with the interval $[a,b]$ of length a little smaller, say, $1/5$. To see this, all that is needed is to realize that if $g(x) \neq 1$, $x \in (0,1)$, we may define, e.g., $x^* := 3/4$ and, similarly, if $g(x) \neq -1$, $x \in (-1,0)$, we may take, e.g., $x_* := -3/4$.

Fix $q \geq 3$ and denote

$$F_q(x) := \frac{|x|x^{q-3}}{(q-2)!}.$$

Clearly,

$$F_q \in \Delta^{(q)}(Y_1^*) \cap W^r, \quad r = 1, \dots, q-2,$$

and

$$F_q^{(q-2)}(x) = \operatorname{sgn} x, \quad x \neq 0.$$

We need the following lemma.

Lemma 2.3 ([4, Lemma 2])**.** *If* $G \in C^j[-1,1]$*, then*

$$2^{j-1} j! \|G\| \geq \min_{x \in [-1,1]} |G^{(j)}(x)|.$$

Applying this to $F \in C^j[a,b]$, by taking $G(x) := F\big(\frac{b-a}{2}x + \frac{a+b}{2}\big)$, we obtain

$$\frac{2^{2j-1} j!}{(b-a)^j} \|F\|_{[a,b]} \geq \min_{x \in [a,b]} |F^{(j)}(x)|. \tag{2.1}$$

We now have,

Lemma 2.4. *For each function* $g \in \Delta^{(q)}(Y_1^*) \cap C^{q-2}[-1,1]$, $q \geq 3$, *where* Y_1^* *was defined in* (1.5)*, we have*

$$\|F_q - g\| \geq c_0. \tag{2.2}$$

Proof. Since $g^{(q-2)} \in C[-1,1]$, $g^{(q-2)}$ is convex on $(0,1)$ and concave on $(-1,0)$, and since $F_q^{(q-2)}(x) = \operatorname{sgn} x$, $x \neq 0$, we can apply Lemma 2.1, and conclude that there exists an interval $[a,b] \subset [-1,1]$ of length $\frac{1}{4}$, such that

$$\min_{x \in [a,b]} |F_q^{(q-2)}(x) - g^{(q-2)}(x)| \geq 1/2,$$

and $0 \notin [a,b]$. The latter implies that $F_q - g \in C^{(q-2)}[a,b]$. Hence, it follows by (2.1) that

$$\|F_q - g\|_{[a,b]} \geq \frac{16^{2-q}}{(q-2)!},$$

and (2.2) follows with $c_0 = \frac{16^{2-q}}{(q-2)!}$. □

We are ready to prove Theorem 1.2.

Proof. Given $Y_s \in \mathbb{Y}_s$, $s \geq 1$, set

$$b := \begin{cases} \min\{1 - y_1, y_1 - y_2\}, & \text{if} \quad s > 1, \\ 1 - |y_1|, & \text{if} \quad s = 1. \end{cases} \tag{2.3}$$

Let

$$f(x) := F_q(x - y_1), \quad x \in [-1,1].$$

Clearly, $f \in \Delta^{(q)}(Y_s) \cap W^{q-2}$. We will prove that it yields (1.2).

To this end, let $P_n \in \mathcal{P}_n \cap \Delta^{(q)}(Y_s)$, be arbitrary. Define,

$$F(u) := b^{-q+2} f(bu + y_1)) \quad u \in [-1, 1],$$

and

$$Q_n(u) := b^{-q+2} P_n(bu + y_1)) \quad u \in [-1, 1].$$

Evidently, $F = F_q$, and

$$Q_n \in \Delta^{(q)}(Y_1^*).$$

Hence, by virtue of Lemma 2.4, we obtain

$$\|F - Q_n\| \geq c_0,$$

which implies

$$\|f - P_n\|_{[y_1-b, y_1+b]} \geq b^{q-2} c_0,$$

and (1.2) follows. □

3. Proof of Theorem 1.5

Lemma 3.1. *Given $q \geq 3$.*
(a) *Let $f \in C^{q-2}[0, 2b]$ have a convex $(q-2)$nd derivative $f^{(q-2)}$ on $[0, 2b]$. If there is a point $\tilde{x} \in (0, b]$, such that $f^{(q-2)}(0) \leq |f^{(q-2)}(\tilde{x})|$, then*

$$b^{q-2} \|f^{(q-2)}\|_{[0,b]} \leq c_1 \|f\|_{[0,2b]}. \tag{3.1}$$

(b) *Let $f \in C^{q-2}[-2b, 0]$ have a concave $(q-2)$nd derivative $f^{(q-2)}$ on $[-2b, 0]$. If there is a point $\tilde{x} \in [-b, 0)$, such that $f^{(q-2)}(0) \geq -|f^{(q-2)}(\tilde{x})|$, then*

$$b^{q-2} \|f^{(q-2)}\|_{[-b,0]} \leq c_1 \|f\|_{[-2b,0]}. \tag{3.2}$$

Proof. By symmetry it is enough to prove (a).

Let

$$M := \|f^{(q-2)}\|_{[0,b]}$$

If $M = 0$ there is nothing to prove, so assume $M > 0$. First assume that there is a point $x_* \in (0, b]$, such that $f^{(q-2)}(x_*) = M$. Put $l(x) := l(x; f^{(q-2)}; 0, x_*)$, and the convexity of $f^{(q-2)}$ implies that

$$f^{(q-2)}(x) \geq l(x) \geq M, \quad x \in [x_*, 2b]. \tag{3.3}$$

In particular, (3.3) holds for $x \in [b, 2b]$. Hence, we obtain, by virtue of (2.1),

$$\|f\|_{[0,2b]} \geq \|f\|_{[b,2b]} \geq \frac{2^{5-2q}}{(q-2)!} b^{q-2} M =: c_2 b^{q-2} M. \tag{3.4}$$

Otherwise, there is a point $x_* \in [0, b]$, such that $f^{(q-2)}(x_*) = -M$. Then, if $f^{(q-2)}(x) \leq 0$ for all $x \in [x_*, 2b]$, we put $l(x) := l(x; f^{(q-2)}; x_*, 2b)$. Since

$$f^{(q-2)}(x) \leq l(x), \quad x \in [x_*, 2b],$$

we have

$$|f^{(q-2)}(x)| \geq |l(x)| \geq \frac{M}{2}, \quad x \in [x_*, x_* + b/2],$$

Hence, (2.1) implies

$$\|f\|_{[0,2b]} \geq \|f\|_{[x_*,x_*+b/2]} \geq c_2 \left(\frac{b}{2}\right)^{q-2} \frac{M}{2}. \tag{3.5}$$

Finally, if $f^{(q-2)}(2b) > 0$ (recall $f^{(q-2)}(x_*) < 0$), then denote by $x^* \in (x_*, 2b)$ the (unique) point, such that $f^{(q-2)}(x^*) = 0$, and put $l(x) := l(x; f^{(q-2)}; x_*, x^*)$. If $x^* \geq \frac{3}{2}b$, then

$$f^{(q-2)}(x) \leq l(x), \quad x \in [x_*, x^*].$$

Hence,

$$|f^{(q-2)}(x)| \geq |l(x)| \geq \frac{M}{2}, \quad x \in [x_*, x_* + b/4],$$

and (2.1) implies

$$\|f\|_{[0,2b]} \geq \|f\|_{[x_*,x_*+b/4]} \geq c_2 \left(\frac{b}{4}\right)^{q-2} \frac{M}{2}. \tag{3.6}$$

Otherwise $x^* \in (x_*, \frac{3}{2}b)$. Then

$$f^{(q-2)}(x) \geq l(x), \quad x \in [x^*, 2b].$$

Hence,

$$|f^{(q-2)}(x)| \geq |l(x)| \geq l\left(\frac{3b}{4}\right) \geq \frac{1}{6}M, \quad x \in [3b/4, 2b],$$

and (2.1) implies

$$\|f\|_{[0,2b]} \geq \|f\|_{[3b/4,2b]} \geq c_2 \left(\frac{b}{4}\right)^{q-2} \frac{M}{6}. \tag{3.7}$$

Combining (3.4) through (3.7) yields (3.1) with an appropriate c_1 and completes the proof of (a). □

Remark 3.2. Clearly, (3.1) is guaranteed if $f^{(q-2)}(0) \leq 0$, and (3.2) is valid if $f^{(q-2)}(0) \geq 0$.

Lemma 3.3. *For the function* $F_{q+1}(x) = \frac{|x|x^{q-2}}{(q-1)!}$, *there exists* $c_3 > 0$ *such that,*

$$nE_n^{(q)}(F_{q+1}, Y_1^*) \geq c_3, \quad n \in \mathbb{N}. \tag{3.8}$$

Proof. Take $b = 1/2$. Let $P_n \in \mathcal{P}_n \cap \Delta^{(q)}(Y_1^*)$, so that $P_n^{(q-2)}$ is convex on $[0,1]$ and concave on $[-1,0]$. Note, that $F_{q+1}^{(q-2)}(x) = |x|$, and the difference $P_n^{(q-2)} - F_{q+1}^{(q-2)}$ is convex on $[0,1]$ and concave on $[-1,0]$. By S. N. Bernstein's result from 1914 (see, e.g., [5]), there exists $c_4 > 0$ such that,

$$M := \|P_n^{(q-2)} - F_{q+1}^{(q-2)}\|_{[-1/2,1/2]} \geq \frac{c_4}{n}.$$

If $P_n^{(q-2)}(0) \leq 0$, then by Lemma 3.1(a) we have

$$b^{q-2}\frac{c_4}{n} \leq c_1\|P_n - F_{q+1}\|_{[0,1]}.$$

Hence,

$$n\|P_n - F_{q+1}\| \geq \frac{c_4}{2^{q-2}c_1} =: c_3. \tag{3.9}$$

On the other hand, if $P_n^{(q-2)}(0) > 0$, then (3.9) follows, similarly, from of Lemma 3.1(b). This completes the proof of (3.8). □

Remark 3.4. Note that $\|F_{q+1}\| = \frac{1}{(q-1)!}$, and that (3.9), with $n = 1$ and $P_n = 0$, yields, $\|F_{q+1}\| \geq c_3$, so that $c_3 \leq \frac{1}{2}$.

We are ready to prove Theorem 1.5, following the ideas of the proof of Theorem 1.2.

Proof. Given $Y_s \in \mathbb{Y}_s$, $s \geq 1$, define

$$f(x) := F_{q+1}(x - y_1).$$

Then $f \in \Delta^{(q)}(Y_s) \cap W^{q-1}$ and we will prove that it yields (1.3).

Indeed, take any $P_n \in \mathcal{P}_n \cap \Delta^{(q)}(Y_s)$, and define for b from (2.3),

$$F(u) := b^{-q+1} f(bu + y_1) \quad u \in [-1, 1],$$

and

$$Q_n(u) := b^{-q+1} P_n(bu + y_1) \quad u \in [-1, 1].$$

Evidently, $F = F_{q+1}$, and

$$Q_n \in \Delta^{(q)}(Y_1^*).$$

By Lemma 3.3, we conclude that

$$n\|F - Q_n\| \geq c_3,$$

so that

$$n\|f - P_n\|_{[y_1-b, y_1+b]} \geq b^{q-1} c_3.$$

Hence (1.3) follows. □

4. Proof of Theorem 1.6

Denote by S a function having the properties:
(i) $S \in C^\infty(\mathbb{R})$,
(ii) $S - \frac{1}{2}$ is a monotone odd function,
and
(iii)

$$S(x) = \begin{cases} 1, & \text{if } \; x \geq 1, \\ 0, & \text{if } \; x \leq -1. \end{cases}$$

Put $s_0 := 1$,

$$s_j := \|S^{(j)}\|, \qquad j \in \mathbb{N}.$$

Note that (ii) implies,

$$\int_{-1}^{1} S(x)dx = 1$$

and, for the function $S_\lambda(x) := S(\frac{x}{\lambda})$, $\lambda > 0$, we have

$$\|S_\lambda^{(j)}\|_{[-\lambda,\lambda]} = \lambda^{-j} s_j, \qquad j \in \mathbb{N}. \tag{4.1}$$

Denote

$$f_n''(x) := S_{\lambda_n}(x - 2\lambda_n), \quad \lambda_n := \frac{c_3}{8n},$$

and

$$f_n(x) = \int_0^x (x - t) f_n''(t) dt.$$

Lemma 4.1. *We have*

$$f_n \in \Delta^{(3)}(Y_1^*), \tag{4.2}$$

$$n E_n^{(3)}(f_n, Y_1^*) \ge c_5, \tag{4.3}$$

$$\|f_n^{(j)}\| = \left(\frac{8n}{c_3}\right)^{j-2} s_{j-2} =: (c_6 n)^{j-2} s_{j-2}, \quad j \ge 2, \tag{4.4}$$

$$\|f_n'\| < 1, \tag{4.5}$$

and

$$\|f_n\| < 1. \tag{4.6}$$

Proof. Except for (4.3), all other statements are readily seen by straightforward computations, so we will prove (4.3). To this end, denote

$$g(x) := \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x \le 0. \end{cases}$$

and

$$G(x) := \int_0^x (x - t) g(t) dt = \begin{cases} \frac{1}{2} x^2, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x \le 0. \end{cases}.$$

Then,

$$\|f_n - G\| \le \int_0^{3\lambda_n} (1 - f_n''(t)) dt = 2\lambda_n.$$

Since $G(x) = \frac{1}{4} x|x| + \frac{1}{4} x^2 = \frac{1}{2} F_4(x) + \frac{1}{4} x^2$, it follows by Lemma 3.3 that $n E_n^{(3)}(G, Y_1^*) \ge \frac{c_3}{2}$, $n \ge 2$. Therefore

$$n E_n^{(3)}(f_n, Y_1^*) \ge n E_n^{(3)}(G, Y_1^*) - n\|f_n - G\| \ge \frac{c_3}{2} - 2n\lambda_n = \frac{c_3}{4} =: c_5.$$

□

Corollary 4.2. *For each $r \in \mathbb{N}$ and $n \in \mathbb{N}$ there is a function $f = f_n \in \Delta^{(3)}(Y_1^*) \cap W^r$, such that*

$$n^{r-1} E_n^{(3)}(f, Y_1^*) \ge c(r).$$

Now we fix $r \ge 3$, and for each $n \in \mathbb{N}$ and $b \in (0, 1)$, we denote

$$\lambda_{n,b} := b\lambda_n,$$

and

$$f_{n,b}(x) := A f_n\left(\frac{x}{b}\right),$$

where

$$A = b^r \|f_n^{(r)}\|^{-1} = \frac{c_6^{2-r} b^r}{s_{r-2} n^{r-2}}.$$

Lemma 4.1 readily implies

Lemma 4.3. *We have*

$$f_{n,b}(x) = 0, \quad x \le \lambda_{n,b}, \tag{4.7}$$

$$f_{n,b}^{(3)}(x) = 0, \quad x \ge 3\lambda_{n,b}, \tag{4.8}$$

$$f_{n,b} \in \Delta^{(3)}(Y_1^*), \tag{4.9}$$

for each polynomial $P_n \in \mathcal{P}_n \cap \Delta^{(3)}(Y_1^*)$ *the estimate*

$$\|f_{n,b} - P_n\|_{[-b,b]} \ge \frac{Ac_5}{n} =: \frac{c_7 b^r}{n^{r-1}}, \tag{4.10}$$

$$\|f_{n,b}^{(j)}\| = \frac{(c_6 n)^{j-2}}{b^j} A s_{j-2} = \left(\frac{b}{c_6 n}\right)^{r-j} \frac{s_{j-2}}{s_{r-2}}, \quad j \ge 2, \tag{4.11}$$

in particular,

$$\|f_{n,b}^{(r)}\| = 1, \tag{4.12}$$

and for each $\lambda > 0$*,*

$$\begin{aligned} \|f_{n,b}^{(j)}\|_{[-\lambda,\lambda]} \le \lambda^{2-j} \|f_{n,b}''\| &= \frac{c_6^{2-r} b^{r-2}}{s_{r-2} n^{r-2}} \lambda^{2-j} \\ &=: \frac{c_8 b^{r-2}}{n^{r-2}} \lambda^{2-j}, \quad j = 0, 1. \end{aligned} \tag{4.13}$$

Lemma 4.4. *Let* $r \ge 3$*. For each sequence* $\{\epsilon_n\}_n^\infty$ *of positive numbers, tending to infinity, there is a function* $f_* \in \Delta^{(3)}(Y_1^*) \cap W^r$*, such that*

$$\limsup_{n\to\infty} \epsilon_n n^{r-1} E_n^{(3)}(f_*, Y_1^*) = \infty. \tag{4.14}$$

Proof. First we define by induction a sequence $\{n_k\}_{k=1}^\infty$ of positive integers. Set $n_1 = 2$, $b_1 = 1$, and assume that n_{k-1} has already been chosen. Then we put

$$b_k := \lambda_{n_{k-1}, b_{k-1}}$$

and take $n_k > k^2 n_{k-1}$, so big, that for all $n \ge n_k$ the inequality

$$\epsilon_{n_k} b_k^r > k,$$

holds. Now we will prove that a desired function may be defined in the form of a convergent series

$$f_*(x) = \sum_{k=1}^\infty f_{n_k, b_k}(x), \quad x \in [-1, 1].$$

Indeed, by virtue of (4.11) and (4.13), we have, for $0 \le j \le r-1$,

$$\sum_{k=1}^\infty \|f_{n_k,b_k}^{(j)}\| \le c_* \sum_{k=1}^\infty \frac{1}{n_k} \le c_* \sum_{k=1}^\infty \frac{1}{k^2} < 2c_* < \infty.$$

Hence

$$f_*^{(j)}(x) = \sum_{k=1}^{\infty} f_{n_k,b_k}^{(j)}(x), \quad x \in [-1,1], \quad 1 \le j \le r-1, \tag{4.15}$$

and

$$f_* \in C^{r-1}[-1,1].$$

If $r > 3$, then for any $x_0 > 0$ there is a neighbourhood, O_{x_0} of x_0, such that all terms in the summation in (4.15) except perhaps one, say, $f_{n_k,b_k}^{(r-1)}(x)$, vanish for all $x \in O_{x_0}$. Hence, we may differentiate pointwise, so that $f_*^{(r)}$ is continuous in $[-1,1] \setminus \{0\}$, and by (4.12),

$$\|f_*^{(r)}\| = 1.$$

Thus, we conclude that

$$f_* \in W^r.$$

Similarly, if $r = 3$, then for any $x_0 > 0$ there is a neighbourhood, O_{x_0} of x_0, such that all terms in the summation in (4.15) except perhaps one, say, $f_{n_k,b_k}^{(2)}(x)$, are constants and add up to a finite constant, for all $x \in O_{x_0}$. Hence the sum in (4.15) equals $f_{n_k,b_k}^{(2)}(x)$ plus a constant. Therefore the same arguments yield

$$f_* \in W^3.$$

Finally, for all $r \ge 3$, it follows from the above discussion that

$$f_*^{(3)}(x) \ge 0, \quad x \in (0,1],$$

so that

$$f_* \in \Delta^{(3)}(Y_1^*).$$

We will show that (4.14) holds for this function. To this end, we fix $k \ge 1$ and take a polynomial $P_{n_k} \in \mathcal{P}_{n_k} \cap \Delta^{(3)}(Y_1^*)$. Then

$$\begin{aligned}
\|f_* - P_{n_k}\| \ge & \|f_* - P_{n_k}\|_{[-b_k,b_k]} = \Big\| \sum_{m=k}^{\infty} f_{n_m,b_m} - P_{n_k} \Big\|_{[-b_k,b_k]} \\
= & \Big\| (f_{n_k,b_k} - P_{n_k}) + \sum_{m=k+1}^{\infty} f_{n_m,b_m} \Big\|_{[-b_k,b_k]} \\
\ge & \|f_{n_k,b_k} - P_{n_k}\|_{[-b_k,b_k]} - \Big\| \sum_{m=k+1}^{\infty} f_{n_m,b_m} \Big\|_{[-b_k,b_k]} \\
\ge & c_7 \frac{b_k^r}{n_k^{r-1}} - \Big\| \sum_{m=k+1}^{\infty} f_{n_m,b_m} \Big\|_{[-b_k,b_k]}.
\end{aligned}$$

Now

$$b_m = b_{m-1}\lambda_{n_{m-1}} = \frac{c_3}{8} \frac{b_{m-1}}{n_{m-1}} =: c_9 \frac{b_{m-1}}{n_{m-1}},$$

so that

$$\Big\| \sum_{m=k+1}^{\infty} f_{n_m,b_m} \Big\|_{[-b_k,b_k]} = \sum_{m=k+1}^{\infty} \|f_{n_m,b_m}\|_{[-b_k,b_k]} \le c_8 b_k^2 \sum_{m=k+1}^{\infty} \frac{b_m^{r-2}}{n_m^{r-2}}$$

$$
\begin{aligned}
&= c_8 c_9^{r-2} b_k^2 \sum_{m=k+1}^{\infty} \frac{b_{m-1}^{r-2}}{(n_{m-1} n_m)^{r-2}} \\
&\le c_8 c_9^{r-2} \frac{b_k^r}{n_k^{r-1}} \sum_{m=k+1}^{\infty} \frac{1}{m^2 n_m^{r-3}} \\
&\le \frac{c_8 c_9^{r-2}}{k} \frac{b_k^r}{n_k^{r-1}} \le \frac{c_7}{2} \frac{b_k^r}{n_k^{r-1}},
\end{aligned}
$$

for all $k \ge k_0 := 2c_8 c_9^{r-2}/c_7$. Hence, for all $k \ge k_0$ we have

$$
\epsilon_{n_k} n_k^{r-1} E_{n_k}^{(3)}(f_*, Y_1^*) \ge \frac{c_7}{2} k \to \infty, \quad k \to \infty,
$$

and (4.14) is proved. □

Remark 4.5. Note that, for $j \ge 3$, $f_*^{(j)}(x) \equiv 0$, $x \notin [0, 1/2]$.

We are ready to prove Theorem 1.6.

Proof. Given $Y_s \in \mathbb{Y}_s$, $s \ge 1$, define for b from (2.3). Put

$$
f(x) := b^r f_* \left(\frac{x - y_1}{b} \right).
$$

Then $f \in \Delta^{(q)}(Y_s) \cap W^r$ and we will prove that it yields (1.4).

Indeed, note that

$$
f_*(u) = b^{-r} f(bu + y_1).
$$

Take any $P_n \in \mathcal{P}_n \cap \Delta^{(q)}(Y_s)$, and define

$$
Q_n(u) := b^{-r} P_n(bu + y_1).
$$

Then $Q_n \in \Delta^{(q)}(Y_1^*)$. Hence,

$$
\begin{aligned}
\|f - P_n\| &\ge \|f - P_n\|_{[y_1 - b, y_1 + b]} \\
&= b^r \|f_* - Q_n\| \\
&\ge b^r E_n^{(3)}(f_*, Y_1^*),
\end{aligned}
$$

so we conclude that

$$
E_n^{(3)}(f, Y_s) \ge b^r E_n^{(3)}(f_*, Y_1^*).
$$

By virtue of Lemma 4.4, (1.4) follows, and the proof is complete. □

References

[1] A. V. Bondarenko and A. V. Primak, *Negative results in shape-preserving higher-order approximations* (in Russian), Mat. Zametki **76** (2004), 812–823; transl. Math. Notes **76** (2004), 758–769.

[2] V. N. Konovalov and D. Leviatan, *Shape preserving widths of Sobolev-type classes of s-monotone functions on a finite interval*, Israel J. Math. **133** (2003), 239–268.

[3] K. A. Kopotun, D. Leviatan, A. Prymak and I. A. Shevchuk, *Uniform and pointwise shape preserving approximation by algebraic polynomials*, Surv. Approx. Theory **6** (2011), 24–74.

[4] D. Leviatan and I. A. Shevchuk, *Counter examples in convex and higher order constrained approximation*, East J. Approx. **1** (1995), 391–398.

[5] R. S. Varga and A. J. Carpenter, *On the Bernstein conjecture in approximation theory*, Constr. Approx. **1** (1985), 333–348.

[6] L. P. Yushchenko, *The negative results in q-coconvex approximation* (Ukrainian, with English summary), Visn., Mat. Mekh., Kyiv. Univ. Im. Tarasa Shevchenka **22** (2009), 8–10.

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