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EXTENSIONS OF BROWDER'S DEMICLOSEDNESS PRINCIPLE AND REICH'S LEMMA AND THEIR APPLICATIONS

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Abstract. In this paper, we study some fundamental properties of nonexpansive mappings. Then, we obtain extensions of Browder's demiclosedness principle and Reich's lemma. Using these results, we prove extensions of Reich's weak convergence theorem and Khan and Suzuki's weak convergence theorem.

1. Introduction

The fixed point approximation theory started from the Banach contraction principle [2]. After Banach, many researchers have attempted to generalize his result. One important branch is headed for the study of nonexpansive mappings. In this direction, the following Browder's demiclosedness principle [4] is crucial.

Theorem 1.1 ([4]). *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self–mapping on C. Suppose $\{u_n\}$ is a sequence in C which converges weakly to some point $u \in C$ and satisfies $\lim_n \|Tu_n - u_n\| = 0$. Then, $u \in F(T)$, where $F(T)$ is the set of fixed points of T.*

The following lemma due to Reich [17] is also important when we prove weak convergence theorems for nonexpansive mappings. We note that Reich's proof is too short and the detail can be found in Takahashi and Kim [22]. In this connection, see also lines 9–11 on the page 548 of the paper by Reich [18].

Lemma 1.2 ([17]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a convex subset of E. Let $\{T_n\}$ be a sequence of nonexpansive self–mappings on C with $\cap_n F(T_n) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C defined by*

$$u_1 \in C \quad \text{and} \quad u_{n+1} = T_n u_n, \quad \forall n \in N.$$

Let $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ be subsequences of $\{u_n\}$. Suppose that $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ converge weakly to $v, w \in \cap_n F(T_n)$, respectively. Then $v = w$.

Let C be a subset of a Banach space E and let T be a mapping of C into E. Let $F(T)$ be the set of fixed points of T and let $A(T)$ be the set of attractive points

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of T, i.e., $A(T) = \{z \in E : \|Tx - z\| \le \|x - z\|,\ \forall x \in C\}$. Consider the following conditions:

(N_1) $F(T) \subset A(T)$.
(N_2) For some $s \in [0, \infty)$,

$$\|x - Ty\| \le s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there are a sequence $\{\alpha_n\}$ in $[0, 1]$ and a sequence $\{u_n\}$ in C such that $u_{n+1} = \alpha_n T u_n + (1 - \alpha_n)u_n$ for all $n \in N$. This procedure is called Krasnoselskii-Mann iteration [12, 14]. Under these setting, consider the following condition:

$$(N_3) \qquad \|Tu_{n+1} - Tu_n\| \le \alpha_n\|Tu_n - u_n\| = \|u_{n+1} - u_n\|, \quad \forall n \in N.$$

Obviously, a nonexpansive mapping satisfies all these conditions $(N_1) - (N_3)$. We note that a nonexpansive mapping satisfies (N_2) with $s = 1$ and (N_2) is stronger than (N_1). These conditions are fundamental pieces of the properties of nonexpansive mappings. Recently, some researchers study these pieces; for example, see Suzuki [20], Takahashi and Takeuchi [23], Kubota and Takeuchi [13] and Falset et.al. [7]. Suzuki [20] introduced a new class of mappings. A mapping T on C is said to satisfy Condition (C) if

$$\text{(C)} \qquad \tfrac{1}{2}\|x - Tx\| \le \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. We call such a mapping T a class (C) mapping in this paper. It is obvious that if T is nonexpansive then T is in Class (C). Suzuki [20] proved that if T is in Class (C) then T satisfies (N_2) with $s = 3$ and it also satisfies (N_3) as $\alpha_n = c \in [1/2, 1)$ for all $n \in N$. It is useful that if a mapping T satisfies (N_2) for some $s \in [0, \infty)$, then the following holds:

$$(N_2') \qquad \|y - Ty\| \le s\|x - Tx\| + 2\|x - y\|, \quad \forall x, y \in C.$$

Motivated by Suzuki [20], Falset et.al. [7] studied (N_2). In this direction, Khan and Suzuki [10] obtained some results and they proved the following theorem which is connected with Reich's weak convergence theorem [17].

Theorem 1.3 ([10]). *Let $c \in [1/2, 1)$. Let E be a uniformly convex Banach space whose dual E^* has the Kadec–Klee property. Let C be a bounded, closed and convex subset of E and let T be a self–mapping on C. Assume that T is in* Class (C). *Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = cTu_n + (1 - c)u_n, \quad \forall n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Theorem 1.4 ([17]). *Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_n \alpha_n(1-\alpha_n) = \infty$. Let E be a uniformly convex Banach space whose norm is Fréchet differentiable. Let C be a closed and convex subset of E and let T be a nonexpansive self–mapping on C with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = \alpha_n T u_n + (1 - \alpha_n)u_n, \quad \forall n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Let $\{T_n\}$ be a sequence of mappings from C into C and let I be the identity mapping on C. For non–negative integers n, k, we define:

$$V_0 = T_0 = I, \quad V_n^{n+k} = T_{n+k} \cdots T_n \ (V_n^n = T_n), \quad V_n = V_0^n = T_n \cdots T_1 T_0.$$

Obviously, $T_n \cdots T_1 T_0 = T_n \cdots T_1$ for all $n \in N$. Assume that each T_n is quasi–nonexpansive and $\cap_n F(T_n) \neq \emptyset$. Then, we define the condition (N_4) which is connected with Lemma 1.2.

(N_4) For $c \in [0,1]$ and $z \in \cap_n F(T_n)$,

$$\limsup_n \sup_k \Big(\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| - (1-c)\|V_n u_1 - z\| \Big) \leq 0.$$

We note that (N_4) holds if each T_i is nonexpansive and (N_4) is weaker than (N_4').

(N_4') There is $n_0 \in N$ such that, for any $n > n_0$ and $k \in N$,

$$\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| \leq (1-c)\|V_n u_1 - z\|,$$

where $c \in [0,1]$ and $z \in \cap_n F(T_n)$.

In this paper, we improve the conditions regarding the spaces or the mappings in Theorem 1.1 or Lemma 1.2. Then, studying conditions $(N_1) - (N_4)$, we obtain extensions of Browder's demiclosedness principle and Reich's lemma. Using these results, we prove extensions of Theorem 1.4 and Theorem 1.3.

2. Preliminaries

We denote by R the set of real numbers, by N the set of positive integers and by N_0 the set of nonnegative integers. For $i, j \in N_0$ satisfying $i \leq j$, $N(i,j)$ denotes the set $\{k \in N_0 : i \leq k \leq j\}$. We denote by E a real Banach space with norm $\|\cdot\|$ and by E^* its dual. For simplicity, we remove "real". For $x \in E$ and $y^* \in E^*$, we denote $y^*(x)$ by $\langle x, y^* \rangle$. S_E denotes the unit sphere and rB_E denotes the closed ball with radius $r > 0$ centered at $0 \in E$. In particular, B_E denotes the unit ball. That is,

$$S_E = \{x \in E : \|x\| = 1\}, \quad rB_E = \{x \in E : \|x\| \leq r\}, \quad B_E = \{x \in E : \|x\| \leq 1\}.$$

Let C be a nonempty subset of E and let T be a mapping of C into E. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. $A(T)$ denotes the set of attractive points of T, i.e.,

$$A(T) = \{x \in E : \|Ty - x\| \leq \|x - y\| \ for\ all\ y \in C\}.$$

The attractive fixed points set $A_F(T)$ is defined by $A_F(T) = A(T) \cap C$; see Takahashi and Takeuchi [23]. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Usually, T is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - v\| \leq \|x - v\|$ for all $x \in C$ and $v \in F(T)$. Then, the condition $A_F(T) = F(T) \subset A(T)$ always holds if T is quasi-nonexpansive.

The normalized duality mapping J of E into 2^{E^*} is defined by, for any $x \in E$,

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

We know the following basic properties of J: $J(x) \neq \emptyset$ for all $x \in E$ and

$$\|x\|^2 - \|y\|^2 \geq 2 \langle x - y, h \rangle, \quad \forall x, y \in E, \quad h \in J(y).$$

A Banach space E is called reflexive if the canonical embedding of E is E^{**}. In a reflexive Banach space, a bounded, closed and convex subset is weakly compact and a bounded sequence has a weakly convergent subsequence. Convexity is one of the most elementary property for a norm. A Banach space E is called strictly convex if

$$\|ax + (1-a)y\|^2 < a\|x\|^2 + (1-a)\|y\|^2$$

for all $a \in (0,1)$, $x, y \in E$ with $x \neq y$. The modulus δ of convexity of E is the function of $[0,2]$ into $[0,1]$ defined by

$$\delta(t) = \inf\{1 - \|\tfrac{x+y}{2}\| : x, y \in B_E,\ t \leq \|x-y\|\}, \quad \forall t \in [0,2].$$

A Banach space E is called uniformly convex if $\delta(t) > 0$ for $t \in (0,2]$. It is known that if E is uniformly convex then δ is strictly increasing with $\delta(0) = 0$. It is also known that a uniformly convex Banach space is strictly convex and reflexive.

The norm of a Banach space E is said to be Fréchet differentiable if for each $x \in S_E$, the limit $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ is attained uniformly for $y \in S_E$. A Banach space E is said to have the Opial property [15] if a sequence $\{x_n\}$ in E converging weakly to $u \in E$ satisfies the following condition:

$$\liminf_n \|x_n - u\| < \liminf_n \|x_n - v\|, \quad \forall v \in E \text{ with } v \neq u.$$

A Banach space E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ in E converges strongly to a point $x \in E$ under the conditions that $\{x_n\}$ converges weakly to x and $\{\|x_n\|\}$ converges to $\|x\|$. It is known that if a uniformly convex Banach space E has a Fréchet differentiable norm then E^* has the Kadec–Klee property. However, there is a uniformly convex Banach space E whose dual E^* has the Kadec–Klee property even if neither E has the Opial property nor E has Fréchet differentiable norm.

3. Lemmas

In this section, we start with the following lemma which is related to Schu [19].

Lemma 3.1. *Let E be a uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E. Assume that $a \in (0,1)$ and $\lim_n \|ax_n + (1-a)y_n\| = 1$. Suppose one of the followings holds:*

(1) $\{x_n\}, \{y_n\} \subset B_E$.
(2) $\lim_n \|x_n\| = \lim_n \|y_n\| = 1$.
(3) $\limsup_n \|x_n\| \leq 1$ and $\limsup_n \|y_n\| \leq 1$.

Then, $\lim_n \|x_n - y_n\| = 0$.

Proof. Let δ be the modulus of convexity of E. We assume (1) and consider the case $a = 1/2$. Assume that there exist $\varepsilon \in (0,2]$ and a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - y_{n_i}\| \geq \varepsilon$ for all $i \in N$. By the uniform convexity of E and definition of δ, we have $0 < \delta(\varepsilon) \leq 1 - \|\frac{x_{n_i}+y_{n_i}}{2}\|$ for all $i \in N$. However, $\lim_i \|\frac{x_{n_i}+y_{n_i}}{2}\| = 1$ implies $\delta(\varepsilon) = 0$. This contradicts to $0 < \delta(\varepsilon)$.

In the case (1), we show $\lim_n \|x_n - y_n\| = 0$. By $\{x_n\}, \{y_n\} \subset B_E$, it is obvious that $\limsup_n \|\frac{x_n+y_n}{2}\| \leq 1$. We consider the case that $a \in (0, 1/2]$. Then we have

$$\|ax_n + (1-a)y_n\| = \|2a\tfrac{x_n+y_n}{2} + (1-2a)y_n\| \leq 2a\|\tfrac{x_n+y_n}{2}\| + (1-2a)\|y_n\|$$

for all $n \in N$. Since $\{x_n\}, \{y_n\} \subset B_E$ and $\lim_n \|ax_n + (1-a)y_n\| = 1$, we have

$$1 = \liminf_n \|ax_n + (1-a)y_n\| \le 2a \liminf_n \|\tfrac{x_n+y_n}{2}\| + (1-2a) \times 1.$$

This implies that $1 \le \liminf_n \|\frac{x_n+y_n}{2}\|$ and hence $\lim_n \|\frac{x_n+y_n}{2}\| = 1$. Thus, we have $\lim_n \|x_n - y_n\| = 0$. In the case of $a \in (1/2, 1)$, the proof is similar by considering $1 - a \in (0, 1/2]$.

We assume (2). Set $r_n = \max\{\|x_n\|, \|y_n\|\}$ for all $n \in N$. Then, $\lim_n r_n = 1$. We may assume $r_n \neq 0$ for all $n \in N$. By the assumptions, we have that, for each $n \in N$,

$$\begin{aligned} &u_n = \tfrac{x_n}{r_n}, v_n = \tfrac{y_n}{r_n} \in B_E, \\ &\|au_n + (1-a)v_n\| = \left\| a\tfrac{x_n}{r_n} + (1-a)\tfrac{y_n}{r_n} \right\| = \tfrac{1}{r_n} \|ax_n + (1-a)y_n\|. \end{aligned}$$

By $\lim_n \|ax_n + (1-a)y_n\| = 1$ and $\lim_n r_n = 1$, we have $\lim_n \|au_n + (1-a)v_n\| = 1$. From the case (1), it follows that $\lim_n \|u_n - v_n\| = 0$. Thus, $\lim_n \|x_n - y_n\| = 0$.

We assume (3). That is, we assume $\limsup_n \|x_n\| \le 1$ and $\limsup_n \|y_n\| \le 1$. Then, by the case (2), if we show $\liminf_n \|x_n\| \ge 1$ and $\liminf_n \|y_n\| \ge 1$ then we have the result. It is obvious that

$$\|ax_n + (1-a)y_n\| - (1-a)\|y_n\| \le a\|x_n\|$$

for all $n \in N$. Then

$$\begin{aligned} a \liminf_n \|x_n\| &\ge \liminf_n (\|ax_n + (1-a)y_n\| - (1-a)\|y_n\|) \\ &= \lim_n \|ax_n + (1-a)y_n\| - (1-a) \limsup_n \|y_n\| \\ &\ge 1 - (1-a) = a. \end{aligned}$$

Thus, we have $\liminf_n \|x_n\| \ge 1$. In the same way, we have $\liminf_n \|y_n\| \ge 1$. □

Let $r \in (0, \infty)$. Let g_r be a strictly increasing function of $[0, 2r]$ into $[0, \infty)$ with $g_r(0) = 0$. We know that g_r is integrable in the sense of Riemann. Let G and $g_{\underline{r}}$ be functions defined by

$$G(t) = \int_0^t g_r(s)ds, \quad g_{\underline{r}}(t) = \tfrac{1}{2r} G(t) = \tfrac{1}{2r} \int_0^t g_r(s)ds, \quad \forall t \in [0, 2r].$$

Then, we can easily show that $g_{\underline{r}}$ is a strictly increasing continuous convex function of $[0, 2r]$ into $[0, \infty)$ with $g_{\underline{r}}(0) = 0$ and $g_{\underline{r}} \le g_r$.

The following lemma is essentially due to Zalinescu [25]; also see Xu [24]. However, their proofs are not so simple and not so easy to read. We can find an excellent proof of the lemma in Prus [16]. Our proof of this lemma is also elementary.

Lemma 3.2. *Let E be a Banach space. Then the followings are equivalent:*

1. *E is uniformly convex, that is, $\delta(t) > 0$ for $t \in (0, 2]$.*
2. *For $r > 0$, there exists a strictly increasing function g_r of $[0, 2r]$ into $[0, \infty)$ with $g_r(0) = 0$ such that, for all $x, y \in rB_E$ and $t \in [0, 2r]$ with $t \le \|x - y\|$,*

$$\|\tfrac{1}{2}(x+y)\|^2 \le \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \tfrac{1}{4} g_r(t).$$

(3) *For $r>0$, there exists a strictly increasing function f_r of $[0,2r]$ into $[0,\infty)$ with $f_r(0)=0$ such that, for all $x,y\in rB_E$ and $t\in[0,2r]$ with $t\le\|x-y\|$ and $a\in[0,1]$,*

$$\|ax+(1-a)y\|^2\le a\|x\|^2+(1-a)\|y\|^2-a(1-a)f_r(t).$$

We note the followings: Since $\|x-y\|\le\|x-y\|$ for $x,y\in rB_E$, it is obvious that we can replace $g_r(t)$ and $f_r(t)$ by $g_r(\|x-y\|)$ and $f_r(\|x-y\|)$, respectively. It is also obvious that we can replace g_r and f_r by $g_{\underline{r}}$ and $f_{\underline{r}}$, respectively.

Proof. We prove (1)⇒(2). Let $\delta(\cdot)$ be the modulus of convexity of E. We set $g_r(t)=\frac{1}{2^4}t^2\delta(\frac{t}{r})^2$ for all $t\in[0,2r]$ and prove that g_r satisfies conditions in (2). By the properties of δ, we have that $g_r(0)=0$ and g_r is strictly increasing. We show that, for all $t\in[0,2r]$ and $x,y\in rB_E$ with $t\le\|x-y\|$,

$$\|\tfrac12(x+y)\|^2\le\tfrac12\|x\|^2+\tfrac12\|y\|^2-\tfrac14 g_r(t).$$

Let $x,y\in rB_E$ with $\|x-y\|\ge t$ for some $t\in[0,2r]$. We consider the cases (I) and (II).

Case (I) $t=0$: By the convexity of $\|\cdot\|^2$, it is obvious that, for any $x,y\in rB_E$,

$$\tfrac12\|x\|^2+\tfrac12\|y\|^2-\|\tfrac12(x+y)\|^2\ge 0=\tfrac14 g_r(0).$$

Case (II) $t>0$: We set $s=\max\{\|x\|,\|y\|\}$ and $s'=\min\{\|x\|,\|y\|\}\ge0$. Without loss of generality, we can assume $\|y\|\le\|x\|=s$. We note that $r\ge s\ge t/2>0$. We set $u=x/s, v=y/s$ and $k=t/s$. Then $u,v\in B_E$. We can easily have

$$0<k=\tfrac{t}{s}\le\tfrac1s\|x-y\|=\|u-v\|\le\|u\|+\|v\|\le2,\quad k\in(0,2],$$
$$\|\tfrac12(x+y)\|\le\tfrac12\|x\|+\tfrac12\|y\|=\tfrac12(s+s'),\quad 0<\delta(\tfrac{t}{r})\le\delta(\tfrac{t}{s})=\delta(k).$$

We consider the following two cases:

$$\text{(i)}\quad s-s'>\tfrac12 s\delta(k),\quad \text{(ii)}\quad s-s'\le\tfrac12 s\delta(k).$$

Case (i): We can easily have the following relation:

$$\begin{aligned}&\tfrac12\|x\|^2+\tfrac12\|y\|^2-\|\tfrac12(x+y)\|^2\\&\quad\ge\tfrac12 s^2+\tfrac12 s'^2-(\,\tfrac12(s+s')\,)^2\\&\quad=\tfrac14(s-s')^2>\tfrac14\tfrac1{2^2}s^2\delta(k)^2\\&\quad\ge\tfrac14\tfrac1{2^2}(\tfrac12 t)^2\delta(\tfrac{t}{r})^2=\tfrac14\tfrac1{2^4}t^2\delta(\tfrac{t}{r})^2=\tfrac14 g_r(t).\end{aligned}$$

Case (ii): By the definition of δ, it is obvious that

$$0<\delta(k)\le1-\tfrac12\|u+v\|=1-\tfrac1{2s}\|x+y\|,\quad \tfrac12\|x+y\|\le s(1-\delta(k)).$$

Then, from $1-\delta(k)\in[0,1)$, it follows that

$$\|\tfrac12(x+y)\|^2\le s^2(1-\delta(k))^2\le s^2(1-\delta(k))\le s^2-s^2\delta(k),$$
$$\text{(a)}\qquad s^2\delta(k)\le s^2-\|\tfrac12(x+y)\|^2.$$

We note that

$$s^2-s'^2=(s-s')(s+s')\le2s(s-s'),\quad \tfrac14\tfrac1{2^2}s^2\delta(k)^2\ge\tfrac14 g_r(t).$$

Thus, by (ii) and (a), we have

$$
\begin{aligned}
&\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \|\tfrac{1}{2}(x+y)\|^2 \\
&\qquad = \|x\|^2 - \|\tfrac{1}{2}(x+y)\|^2 - \tfrac{1}{2}(\|x\|^2 - \|y\|^2) \\
&\qquad = s^2 - \|\tfrac{1}{2}(x+y)\|^2 - \tfrac{1}{2}(s^2 - s'^2) \\
&\qquad \geq s^2\delta(k) - s(s-s') \geq s^2\delta(k) - \tfrac{1}{2}s^2\delta(k) \\
&\qquad = \tfrac{1}{2}s^2\delta(k) > \tfrac{1}{4}\tfrac{1}{2^2}s^2\delta(k)^2 \geq \tfrac{1}{4}g_r(t).
\end{aligned}
$$

We prove (2)⇒(1). Set $r = 1$. Then there is g_1 satisfying conditions in (2). Let $t \in (0,2]$ and $x, y \in B_E$ with $t \leq \|x-y\|$. By $\|\frac{1}{2}(x+y)\| \leq 1$ and (2), we have

$$
\begin{aligned}
0 < \tfrac{1}{4}g_1(t) &\leq \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \|\tfrac{1}{2}(x+y)\|^2 \leq 1 - \|\tfrac{1}{2}(x+y)\|^2 \\
&= (1 - \|\tfrac{1}{2}(x+y)\|)(1 + \|\tfrac{1}{2}(x+y)\|) \leq 2(1 - \|\tfrac{1}{2}(x+y)\|).
\end{aligned}
$$

Then it follows that

$$
0 < \tfrac{1}{8}g_1(t) \leq \inf\{1 - \|\tfrac{1}{2}(x+y)\| : x, y \in B_E,\ t \leq \|x-y\|\} = \delta(t).
$$

Thus, E is uniformly convex.

We prove (2)⇒(3). We set $f_r(t) = \frac{1}{2}g_r(t)$ for all $t \in [0, 2r]$ and prove that f_r satisfies conditions in (3). Let $t \in [0, 2r]$ and $x, y \in rB_E$ with $t \leq \|x-y\|$. In the case $a \in (0, 1/2]$, by (2) and the convexity of $\|\cdot\|^2$, we have

$$
\begin{aligned}
\|ax + (1-a)y\|^2 &= \|2a(\tfrac{1}{2}(x+y)) + (1-2a)y\|^2 \\
&\leq 2a\|\tfrac{1}{2}(x+y)\|^2 + (1-2a)\|y\|^2 \\
&\leq 2a(\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \tfrac{1}{4}g_r(t)) + (1-2a)\|y\|^2 \\
&= a\|x\|^2 + a\|y\|^2 - \tfrac{1}{2}ag_r(t) + (1-2a)\|y\|^2 \\
&\leq a\|x\|^2 + (1-a)\|y\|^2 - \tfrac{1}{2}a(1-a)g_r(t) \\
&= a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)f_r(t).
\end{aligned}
$$

By the properties of g_r, it is obvious that f_r has the desired properties.

In the case $a \in (1/2, 1)$, it is obvious that $1 - a \in (0, 1/2]$. Then, the proof is similar. In the cases $a = 0$ or $a = 1$, the proof is trivial.

We prove (3)⇒(2). We set $g_r(t) = f_r(t)$ for $t \in [0, 2r]$ and $a = 1/2$. By the properties of f_r, it is obvious that g_r satisfies conditions in (2). □

We also have the following lemma.

Lemma 3.3. *Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Let C be a subset of a Banach space E and let T be a mapping of C into C. Suppose that a sequence $\{u_n\}$ in C satisfies*

$$
(1)\quad u_{n+1} = \alpha_n Tu_n + (1-\alpha_n)u_n, \qquad (2)\quad \|Tu_{n+1} - Tu_n\| \leq \alpha_n \|Tu_n - u_n\|
$$

for all $n \in N$. Then, the followings hold:

(a) *$\{\|Tu_n - u_n\|\}$ is non-increasing and converges.*
(b) *If $v \in A(T)$ then $\{\|u_n - v\|\}$ is non-increasing and converges.*
(c) *If either $\{u_n\}$ or $\{Tu_n\}$ is bounded and $\{\alpha_n\}$ is in $[0,b] \subset [0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\lim_n \|Tu_n - u_n\| = 0$.*
(d) *If E is uniformly convex, $A(T) \neq \emptyset$ and $\sum_n \alpha_n(1-\alpha_n) = \infty$, then $\lim_n \|Tu_n - u_n\| = 0$.*

Proof. We prove (a). By (1) and (2), it is obvious that, for any $n \in N$,

$$\begin{aligned}\|Tu_{n+1} - u_{n+1}\| &\le \|Tu_{n+1} - Tu_n\| + \|Tu_n - u_{n+1}\| \\ &\le \alpha_n\|Tu_n - u_n\| + (1-\alpha_n)\|Tu_n - u_n\| = \|Tu_n - u_n\|.\end{aligned}$$

Then, $\{\|Tu_n - u_n\|\}$ is non-increasing and $\lim_n \|Tu_n - u_n\|$ exists.

We prove (b). It is obvious that

$$\|u_{n+1} - v\| \le \alpha_n\|Tu_n - v\| + (1-\alpha_n)\|u_n - v\| \le \|u_n - v\|$$

for all $n \in N$. Then $\{\|u_n - v\|\}$ is non–increasing and converges.

The proof of (c) is in [13].

We prove (d). Set $r = \|u_1 - v\|$, where $v \in A(T)$ and $u_1 \neq v$. If $u_1 = v$, then $v \in F(T)$ and (d) holds. By (b), we have that $Tu_n - v,\ u_n - v \in rB_E$ and $\|Tu_n - u_n\| \le \|Tu_n - v\| + \|u_n - v\| \le 2r$ for all $n \in N$. Assume $\lim_n \|Tu_n - u_n\| > 0$. Then, there is $\varepsilon \in (0, 2r]$ such that $\|Tu_n - u_n\| \ge \varepsilon$ for sufficiently large $n \in N$. Since E is uniformly convex, by Lemma 3.2, there is a strictly increasing function f_r of $[0, 2r]$ into $[0, \infty)$ such that $f_r(0) = 0$ and

$$\begin{aligned}\alpha_n(1-\alpha_n)f_r(\varepsilon) &\le \alpha_n\|Tu_n - v\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &\qquad - \|\alpha_n(Tu_n - v) + (1-\alpha_n)(u_n - v)\|^2 \\ &\le \|u_n - v\|^2 - \|u_{n+1} - v\|^2\end{aligned}$$

for all $n \in N$. It follows that, for any $n \in N$,

$$f_r(\varepsilon)\textstyle\sum_{i=1}^n \alpha_i(1-\alpha_i) = \sum_{i=1}^n \alpha_i(1-\alpha_i)f_r(\varepsilon) \le \|u_1 - v\|^2 - \|u_{n+1} - v\|^2.$$

Since $\{\|u_n - v\|\}$ is bounded and $\sum_{i=1}^\infty \alpha_i(1-\alpha_i) = \infty$, this inequality implies $f_r(\varepsilon) = 0$. This contradicts to $f_r(\varepsilon) > 0$. Thus, $\lim_n \|Tu_n - u_n\| = 0$. □

We prove the following two lemmas. The first one was essentially proved in [20].

Lemma 3.4. *Let E be a Banach space which has the Opial property. Let C be a closed convex subset of E and let S be a self–mapping on C which satisfies (N_2). Let $\{x_n\}$ be a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. Assume $u \neq Su$. Since $\{x_n\}$ converges weakly to u, by the Opial property, we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. Since S satisfies condition (N_2) for some $s \in [0, \infty)$, the following holds:

$$\|x_n - Su\| \le s\|x_n - Sx_n\| + \|x_n - u\|, \quad \forall n \in N.$$

By $\lim_n \|Sx_n - x_n\| = 0$, this implies $\liminf_n \|x_n - Su\| \le \liminf_n \|x_n - u\|$. We have a contradiction. This completes the proof. □

Lemma 3.5. *Let E be a reflexive Banach space which has the Opial property. Let C be a subset of E. Let $\{u_n\}$ be a sequence in E such that $\{\|u_n - w\|\}$ converges for any $w \in C$. Suppose the weak limit of any weakly convergent subsequence of $\{u_n\}$ is in C. Then $\{u_n\}$ converges weakly to some $z \in C$.*

Proof. In our setting, the bounded sequence $\{u_n\}$ has a weakly convergent subsequence. Suppose $\{u_{n_i}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converge weakly to $u, v \in C$, respectively. Assume $u \neq v$. Let $w \in C$. Since $\{\|u_n - w\|\}$ converges, any subsequence of $\{\|u_n - w\|\}$ converges to the same real number. Then, by $u, v \in C$ and the Opial property, we have

$$\begin{aligned} \liminf_i \|u_{n_i} - u\| &< \liminf_i \|u_{n_i} - v\| = \liminf_j \|u_{n_j} - v\| \\ &< \liminf_j \|u_{n_j} - u\| = \liminf_i \|u_{n_i} - u\|. \end{aligned}$$

This is a contradiction. Thus we have $u = v$ and the result. □

4. Browder's demiclosedness principle

In this section, we obtain an extension of Browder's demiclosedness principle which was proved for nonexpansive mappings in uniformly convex Banach spaces. Before obtaining the result, we need the following lemma which is connected with the condition (N_2).

Lemma 4.1. *Let C be a bounded and convex subset of a uniformly convex Banach space E. Let T be a self–mapping on C satisfying (N_2) for $s \in [0, \infty)$, that is,*

$$\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in C$ satisfy $\|Tx - x\| < \delta$ and $\|Ty - y\| < \delta$ then, for any $c \in [0, 1]$,

$$\|T(cx + (1 - c)y) - (cx + (1 - c)y)\| < \varepsilon.$$

Proof. Assume that there are $\varepsilon_0 > 0$, a sequence $\{c_n\}$ in $[0, 1]$ and sequences $\{u_n\}, \{v_n\}$ in C such that

(i) $$\|Tu_n - u_n\| < \tfrac{1}{n}, \quad \|Tv_n - v_n\| < \tfrac{1}{n},$$

(ii) $$\|T(c_n u_n + (1 - c_n)v_n) - (c_n u_n + (1 - c_n)v_n)\| \geq \varepsilon_0$$

for all $n \in N$. We set $d_n = \|u_n - v_n\|$ for all $n \in N$. For simplicity, we set

$$A_n = \|T(c_n u_n + (1 - c_n)v_n) - (c_n u_n + (1 - c_n)v_n)\|$$

for all $n \in N$. Since C is bounded, $\{d_n\}$ is also bounded in $[0, \infty)$. We know that $\{c_n\} \subset [0, 1]$. Without loss of generality, we can assume that there are $d \in [0, \infty)$ and $b \in [0, 1]$ satisfying $\lim_n d_n = d$ and $\lim_n c_n = b$. It follows that $\lim_n c_n d_n = bd$ and $\lim_n (1 - c_n) d_n = (1 - b)d$. We note that (i) and (ii) are still satisfied. We know that (N_2') is derived from (N_2). Then, we have that, for $n \in N$,

(iii) $$\begin{aligned} A_n &\leq s\|Tu_n - u_n\| + 2\|u_n - (c_n u_n + (1 - c_n)v_n)\| \\ &\leq s\|Tu_n - u_n\| + 2(1 - c_n)\|u_n - v_n\| \leq \tfrac{s}{n} + 2(1 - c_n)d_n. \end{aligned}$$

In the same way, we have

(iv) $$A_n \leq s\|Tv_n - v_n\| + 2c_n\|u_n - v_n\| \leq \tfrac{s}{n} + 2c_n d_n.$$

In the case of $bd = 0$, by (iv), it is easy to see that

$$\limsup_n A_n \leq \lim_n (\tfrac{s}{n} + 2c_n d_n) = 0$$

and hence $\lim_n A_n = 0$. This contradicts to (ii). In the case of $b = 1$, by (iii), we also have $\lim_n A_n = 0$. Then, we have $b \in (0, 1)$ and $d > 0$. Without loss of generality, we can assume that there is $a \in (0,1)$ such that $d_n \geq a$ and $c_n, 1-c_n \in [a, 1-a]$ for all $n \in N$. We set

$$w_n = T(c_n u_n + (1-c_n)v_n),\ x_n = \frac{(u_n - w_n)}{(1-c_n)d_n},\ y_n = \frac{(w_n - v_n)}{c_n d_n}$$

for all $n \in N$. Then, from (N_2), we have:

$$\begin{aligned}\|x_n\| &= \tfrac{1}{(1-c_n)d_n}\|u_n - w_n\| \\ &\leq \tfrac{1}{(1-c_n)d_n}(s\|u_n - Tu_n\| + \|u_n - (c_n u_n + (1-c_n)v_n)\|) \\ &= \tfrac{1}{(1-c_n)d_n}(s\|Tu_n - u_n\| + (1-c_n)\|u_n - v_n\|) \\ &\leq 1 + \tfrac{1}{n}\tfrac{s}{(1-c_n)d_n} \leq 1 + \tfrac{s}{na^2}\end{aligned}$$

for all $n \in N$. In the same way, we have

$$\begin{aligned}\|y_n\| &= \tfrac{1}{c_n d_n}\|w_n - v_n\| \\ &= \tfrac{1}{c_n d_n}(s\|Tv_n - v_n\| + c_n\|u_n - v_n\|) \leq 1 + \tfrac{s}{na^2}\end{aligned}$$

for all $n \in N$. Since $\lim_n(1 + \frac{s}{na^2}) = 1$, we have that $\limsup_n \|x_n\| \leq 1$ and $\limsup_n \|y_n\| \leq 1$. On the other hand, we have

$$\|(1-c_n)x_n + c_n y_n\| = \tfrac{1}{d_n}\|u_n - v_n\| = 1$$

for all $n \in N$. Obviously, $\lim_n \|(1-c_n)x_n + c_n y_n\| = 1$.

It is easy to see that

$$\begin{aligned}&\Big| \|(1-c_n)x_n + c_n y_n\| - \|(1-b)x_n + by_n\| \Big| \\ &\qquad \leq \|(1-c_n)x_n + c_n y_n - (1-b)x_n - by_n\| \\ &\qquad \leq |c_n - b|(\|x_n\| + \|y_n\|)\end{aligned}$$

for all $n \in N$. Then, since $\{c_n\}$ converges to $b \in (0,1)$, we have

$$\lim_n \|(1-b)x_n + by_n\| = \lim_n \|(1-c_n)x_n + c_n y_n\| = 1.$$

By Lemma 3.1, we have $\lim_n \|x_n - y_n\| = 0$. It is also obvious that

$$\begin{aligned}\|x_n - y_n\| &= \tfrac{1}{c_n(1-c_n)}\|c_n(u_n - w_n) - (1-c_n)(w_n - v_n)\| \\ &= \tfrac{1}{c_n(1-c_n)}\|c_n u_n + (1-c_n)v_n - w_n\| \geq \tfrac{1}{(1-a)^2}A_n\end{aligned}$$

for all $n \in N$. Thus, we have $\lim_n A_n = 0$. This contradicts to (ii). □

The following is an extension of Browder's demiclosedness principle.

Theorem 4.2. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E. Let T be a self–mapping on C satisfying (N_2) for some $s \in [0,\infty)$, that is, $\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|$ for all $x, y \in C$. Suppose that $\{u_n\}$ is a sequence in C which converges weakly to u and $\lim_n \|Tu_n - u_n\| = 0$. Then, $u \in F(T)$.*

Proof. Let $\varepsilon > 0$ and set $\delta_1 = \varepsilon/(s+2)$. By Lemma 4.1, we can take a sequence $\{\delta_k\}$ in $(0,\infty)$ with $\delta_{k+1} < \delta_k$ such that for any $x, y \in C$ satisfying $\|Tx - x\| < \delta_{k+1}$ and $\|Ty - y\| < \delta_{k+1}$, and for any $c \in [0,1]$,

$$\|T(cx + (1-c)y) - (cx + (1-c)y)\| < \delta_k.$$

By $\lim_n \|Tu_n - u_n\| = 0$, we can take a sequence $\{n_k\}$ in N such that, for each k,

$$n_k < n_{k+1}, \qquad \|Tu_{n_k} - u_{n_k}\| < \delta_{k+1}.$$

Obviously, $\{u_{n_k}\}$ converges weakly to u. Let H_c^l be the convex hull of $\{u_{n_k}\}_{k=1}^l$ for each $l \in N$ and let H_c be the closed and convex hull of $\{u_{n_k}\}$. We know that H_c is a closed and convex subset of C. Then H_c is weakly closed. We have $u \in H_c$. Then, there exists the smallest $L \in N$ such that there is $v \in H_c^L$ satisfying $\|v - u\| < \delta_1$.

We show $\|v - Tv\| < \delta_1$. By $v \in H_c^L$, there is a sequence $\{a_k\}_{k=1}^L$ in $[0,1]$ with $\sum_{k=1}^L a_k = 1$ and $v = \sum_{k=1}^L a_k u_{n_k}$, where $a_L \neq 0$.

We set $b_L = a_L$ and

$$b_k = a_k + \cdots + a_L, \quad \forall k \in N(1, L-1).$$

This implies $b_k > 0$ for all $k \in N(1, L)$. It is obvious that $b_1 = 1$. We set $v_L = u_{n_L}$ and

$$v_k = \tfrac{1}{b_k} \textstyle\sum_{i=k}^L a_i u_{n_i}, \qquad \forall k \in N(1, L-1).$$

It is also obvious that $v_L = u_{n_L} = \frac{1}{a_L} a_L u_{n_L} = \frac{1}{b_L} a_L u_{n_L} = \frac{1}{b_L} \sum_{i=L}^L a_i u_{n_i}$.

By induction, we have $v_1 = \frac{1}{b_1} v = v$. It is obvious that $a_k/b_k, b_{k+1}/b_k \in [0,1]$ and

$$\tfrac{a_k}{b_k} + \tfrac{b_{k+1}}{b_k} = 1, \qquad \forall k \in N(1, L-1).$$

We know that $\|Tv_L - v_L\| = \|Tu_{n_L} - u_{n_L}\| < \delta_{L+1} < \delta_L$. We assume that $\|Tv_{k+1} - v_{k+1}\| < \delta_{k+1}$ for some $k \in N(1, L-1)$. Since $\|Tu_{n_k} - u_{n_k}\| < \delta_{k+1}$, by the definition of δ_{k+1}, we have

$$\|Tv_k - v_k\| = \|T(\tfrac{a_k}{b_k} u_{n_k} + \tfrac{b_{k+1}}{b_k} v_{k+1}) - (\tfrac{a_k}{b_k} u_{n_k} + \tfrac{b_{k+1}}{b_k} v_{k+1})\| < \delta_k.$$

By induction, we have $\|Tv_1 - v_1\| = \|Tv - v\| < \delta_1$.

Thus, $\|v - u\| < \delta_1$ and $\|Tv - v\| < \delta_1$. By (N_2'), we have

$$\|Tu - u\| \leq s\|Tv - v\| + 2\|v - u\| < (s+2)\delta_1 = \varepsilon.$$

Since ε is arbitrary, we have the desired result $u \in F(T)$. □

We note that Khan and Suzuki [10] proved similar result for mappings of Class (C). We prove the following lemma which is derived from Lemmas 3.3, 3.4 and Theorem 4.2.

Lemma 4.3. *Let E be a Banach space satisfying either of the followings:*

(e_1) *E is uniformly convex.*

(e_2) *E is reflexive and has the Opial property.*

Let C be a bounded, closed and convex subset of E and let T be a self–mapping on C satisfying (N_2). Let $\{\alpha_n\}$ be a sequence in $[0,b] \subset [0,1)$ satisfying $\sum_{n=1}^\infty \alpha_n = \infty$. Let $u_1 \in C$ and $\{u_n\}$ be the sequence defined by

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n) u_n, \quad \forall n \in N.$$

Suppose that (N_3) holds. Then, there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some $u \in F(T)$.

Proof. We know that $\{u_n\}$ is bounded. Since (N_3) holds, we have

$$\|Tu_{n+1} - Tu_n\| \le \alpha_n \|Tu_n - u_n\|, \quad \forall n \in N.$$

By Lemma 3.3 (c), we have $\lim_n \|Tu_n - u_n\| = 0$. Since E is reflexive, we have that C is weakly compact. Then, there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some $u \in C$. Recall that T satisfies (N_2). Thus, by Theorem 4.2 or Lemma 3.4, we have $u \in F(T)$. □

5. Reich's lemma

In this section, we obtain an extension of Reich's lemma [17] which was proved in a uniformly convex Banach space with a Fréchet differentiable norm. The following lemma was essentially proved by Falset et.al [6]. We give an elementary proof.

Lemma 5.1. *Let E be a reflexive Banach space such that E^* has the Kadec–Klee property. Let $\{u_n\}$ be a sequence in E. Let $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ be subsequences of $\{u_n\}$ which converge weakly to $v, w \in E$, respectively. Assume that, for each $k \in N$, there exists* $\lim_n \|\frac{1}{k}u_n + (1-\frac{1}{k})v - w\|$. *Then $v = w$.*

Proof. By the assumptions, for each $k \in N$, $\{\|\frac{1}{k}u_n + (1-\frac{1}{k})v - w\|\}$ converges. By setting $k = 1$, we have that $\{\|u_n - w\|\}$ converges. Then $\{u_n\}$ is bounded. We set $M = \sup_n \|u_n - v\|$. It is obvious that, for each $k \in N$, $\{\|\frac{1}{k}u_n + (1-\frac{1}{k})v - w\|^2\}$ also converges and that, for $n \in N$,

$$\|\tfrac{1}{k}u_n + (1-\tfrac{1}{k})v - w\| = \|(v-w) + \tfrac{1}{k}(u_n - v)\|.$$

Let J be the normalized duality mapping of E into 2^{E^*}. We note that $J(x) \neq \emptyset$ for $x \in E$. We need a basic and well known property of J as follows:

$$\|x\|^2 - \|y\|^2 \ge 2\langle x - y, h\rangle, \quad \forall x, y \in E, \quad h \in J(y).$$

We note that $h \in J(y)$ implies $\|h\| = \|y\|$. Since $\{\|\frac{1}{k}u_n + (1-\frac{1}{k})v - w\|^2\}$ converges, there is $n_k \in N$ such that

$$\tfrac{2}{k^2} \ge \|\tfrac{1}{k}u_n + (1-\tfrac{1}{k})v - w\|^2 - \|\tfrac{1}{k}u_m + (1-\tfrac{1}{k})v - w\|^2 \ge 2 \times \tfrac{1}{k}\langle u_n - u_m, h\rangle$$

for all $m, n > n_k$ and $h \in J(v - w + \frac{1}{k}(u_m - v))$. That is,

$$\tfrac{1}{k} \ge \langle u_n - u_m, h\rangle, \quad \forall m, n > n_k \text{ and } h \in J(v - w + \tfrac{1}{k}(u_m - v)).$$

Taking subsequences of $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$, we can assume that $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ have the following property:

$$\tfrac{1}{k} \ge \left\langle u_{i(k)} - u_{j(k)}, h\right\rangle \quad \text{for} \quad k \in N \text{ and } h \in J((v-w) + \tfrac{1}{k}(u_{j(k)} - v)).$$

Let $\{f_k\}$ be a sequence in E^* such that $f_k \in J((v-w) + \frac{1}{k}(u_{j(k)} - v))$ for all k. Since $\{\|(v-w) + \frac{1}{k}(u_{j(k)} - v)\|\}$ is bounded, so is $\{f_k\}$. Since E is reflexive, there exists a subsequence of $\{f_k\}$ which converges weakly to some $g \in E^*$. Taking subsequences again, we can assume that $\{f_k\}$ itself converges weakly to g.

Then, we can easily see that $1/k > \left\langle u_{i(k)} - u_{j(k)}, f_k\right\rangle$ for all $k \in N$. That is, we have

$$\limsup_k \left\langle u_{i(k)} - u_{j(k)}, f_k\right\rangle \le 0. \tag{1}$$

Let $f \in J(v-w)$. It is easy to see that

$$
\begin{aligned}
|\,\|f_k\| - \|f\|\,| &= |\,\|(v-w) + \tfrac{1}{k}(u_{j(k)} - v)\| - \|v-w\|\,| \\
&\le \|(v-w) + \tfrac{1}{k}(u_{j(k)} - v) - (v-w)\| = \tfrac{1}{k}\|u_{j(k)} - v\| \le \tfrac{1}{k}M
\end{aligned}
$$

for all $k \in N$. Then, $\lim_k \|f_k\| = \|f\|$. We know that $\|\cdot\|$ is weakly lower semi–continuous. Then, since $\{f_k\}$ converges weakly to g, we have

$$
\|g\| \le \liminf_k \|f_k\| = \lim_k \|f_k\| = \|f\|. \tag{2}
$$

Let $\delta > 0$ be arbitrary. Since $\lim_k \|f_k\| = \|f\|$, $f_k \in J((v-w) + \frac{1}{k}(u_{j(k)} - v))$ and $f \in J(v-w)$, we have, for sufficiently large k,

$$
\begin{aligned}
\|v-w\|^2 = \|f\|^2 &\le \|f_k\|^2 + \delta \\
&= \big\langle (v-w) + \tfrac{1}{k}(u_{j(k)} - v), f_k \big\rangle + \delta \\
&\le \langle v-w, f_k\rangle + \tfrac{1}{k}M\|f_k\| + \delta.
\end{aligned}
$$

We know that $\{f_k\}$ converges weakly to g. Then,

$$
\|v-w\|^2 \le \lim_k \big(\langle v-w, f_k\rangle + \tfrac{1}{k}M\|f_k\| \big) + \delta = \langle v-w, g\rangle + \delta.
$$

Since δ is arbitrary, by (2), we have

$$
\|v-w\|^2 \le \langle v-w, g\rangle \le \|v-w\|\,\|g\| \le \|v-w\|\,\|f\| = \|v-w\|^2.
$$

These imply that $\lim_k \|f_k\| = \|f\| = \|g\| = \|v-w\|$ and $g \in J(v-w)$. Since E^* has the Kadec–Klee property, it follows that $\{f_k\}$ converges strongly to $g \in J(v-w)$. Since $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ converge weakly to v and w, respectively and $f_k \to g$, by (1), we have

$$
\|v-w\|^2 = \langle v-w, g\rangle = \lim_k \big\langle u_{i(k)} - u_{j(k)}, f_k \big\rangle \le 0.
$$

Then, it follows that $\|v-w\|^2 = 0$ and hence $v = w$. □

Lemma 5.2. *Let C be a convex subset of a uniformly convex Banach space E. Let $\{T_n\}$ be a sequence of quasi–nonexpansive self–mappings on C with $\cap_n F(T_n) \ne \emptyset$. Let $u_1 \in C$. Suppose that (N_4) holds. That is, for any $c \in [0,1]$ and $z \in \cap_n F(T_n)$,*

$$
\limsup_n \sup_k \Big(\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| - (1-c)\|V_n u_1 - z\| \Big) \le 0.
$$

Then, either of the followings hold

(1) $\lim_n \|V_n u_1 - z\| = 0$.

(2) *For $c \in [0,1]$, $z \in \cap_n F(T_n)$ and $\varepsilon > 0$, there exists $n_0 \in N$ such that, for $n > n_0$ and $k \in N$,*

$$
\|cV_{n+k}u_1 + (1-c)z - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| < \varepsilon.
$$

Proof. Let $z \in \cap_n F(T_n)$. In the case of $c = 0$, we easily have $\|z - V_{n+1}^{n+k}(z)\| = 0$. In the case of $c = 1$, we have $\|V_{n+k}u_1 - V_{n+1}^{n+k}(V_n u_1)\| = 0$. In both cases, we obviously have (2). Then, we assume $c \in (0,1)$. Since each T_i is quasi–nonexpansive, it is obvious that $\{\|V_n u_1 - z\|\}$ is non–increasing and $\lim_n \|V_n u_1 - z\|$ exists.

We show that (2) holds if $\lim_n \|V_n u_1 - z\| \ne 0$. Assume that there are $\varepsilon_0 > 0$ and a sequence $\{k_n\}$ of positive integers such that

$$
\|cV_{n+k_n}u_1 + (1-c)z - V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z)\| \ge \varepsilon_0 \tag{i}
$$

for all $n \in N$. For simplicity, we set

$$x_n = V_{n+k_n} u_1 - V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z), \quad y_n = V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z) - z$$

for all $n \in N$. It follows that

$$\text{(ii)} \qquad cx_n - (1-c)y_n = cV_{n+k_n} u_1 + (1-c)z - V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z)$$

for all $n \in N$. Since each T_i is quasi–nonexpansive, we have

$$\text{(iii)} \qquad \|y_n\| = \|V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z) - z\| \le c\|V_n u_1 - z\|.$$

We set $d_n = \|V_n u_1 - z\|$ for all $n \in N$. By $\lim_n \|V_n u_1 - z\| \neq 0$, we know that $\{d_n\}$ converges to some $d \in (0, \infty)$, that is, $d_n \ge d > 0$ for all $n \in N$. It is obvious that $d_n \ge d_{n+k_n} \ge d > 0$ for all $n \in N$. Then, $\{d_{n+k_n}\}$ also converges to d. By $c, d_n > 0$, we set

$$x'_n = \tfrac{x_n}{(1-c)d_n} = \tfrac{V_{n+k_n} u_1 - V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z)}{(1-c)d_n}, \quad y'_n = \tfrac{y_n}{cd_n} = \tfrac{V_{n+1}^{n+k_n}(cV_n u_1 + (1-c)z) - z}{cd_n}$$

for all $n \in N$. By (iii), it is obvious that $\limsup_n \|y'_n\| \le 1$. Since $\{(1-c)d_n\}$ converges, by (N_4), it is easy to see that $\limsup_n \|x'_n\| \le 1$. It is also obvious that

$$\|(1-c)x'_n + cy'_n\| = \tfrac{1}{d_n}\|V_{n+k_n} u_1 - z\| = \tfrac{1}{d_n} d_{n+k_n}.$$

That is, $\lim_n \|(1-c)x'_n + cy'_n\| = 1$. By Lemma 3.1, we have $\lim_n \|x'_n - y'_n\| = 0$. We know that $\{d_n\}$ is non–increasing and $c(1-c)d_1 > 0$. For any $n \in N$, we have

$$\|x'_n - y'_n\| = \tfrac{1}{c(1-c)d_n}\|cx_n - (1-c)y_n\| \ge \tfrac{1}{c(1-c)d_1}\|cx_n - (1-c)y_n\|.$$

Thus, we have $\lim_n \|cx_n - (1-c)y_n\| = 0$. This contradicts to (i). □

Remark 1. In Lemma 5.2, the condition (2) holds if the condition (1) holds. Let $\varepsilon > 0$. In the case of $\lim_n \|V_n u_1 - z\| = 0$. it is easy to see that, for any $n, k \in N$,

$$\begin{aligned} &\|cV_{n+k} u_1 + (1-c)z - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| \\ &\qquad \le \|cV_{n+k} u_1 + (1-c)z - z\| + \|V_{n+1}^{n+k}(cV_n u_1 + (1-c)z) - z\| \\ &\qquad \le \|cV_{n+k} u_1 - cz\| + \|cV_n u_1 + (1-c)z - z\| \le 2c\|V_n u_1 - z\|. \end{aligned}$$

By $\lim_n \|V_n u_1 - z\| = 0$, there exists $n_0 \in N$ such that, for any $n > n_0$ and $k \in N$,

$$\|cV_{n+k} u_1 + (1-c)z - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| < \varepsilon.$$

The following is an extension of Reich's lemma [17].

Lemma 5.3. *Let E be a uniformly convex Banach space such that E^* has the Kadec–Klee property. Let C be a convex subset of E. Let $\{T_n\}$ be a sequence of quasi–nonexpansive self–mappings on C with $\cap_n F(T_n) \neq \emptyset$. Let $u_1 \in C$ and let $\{u_n\}$ be a sequence in C defined by $u_{n+1} = T_n u_n = V_n u_1$ for all $n \in N$. Let $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$ be subsequences of $\{u_n\}$ which converge weakly to $v, w \in \cap_n F(T_n)$, respectively. Suppose (N_4) holds. Then $v = w$.*

Proof. Let $\varepsilon > 0$. Since $v, w \in \cap_n F(T_n)$ and each T_i is quasi–nonexpansive, $\{\|V_n u_1 - v\|\}$ and $\{\|V_n u_1 - w\|\}$ are non–increasing. Then, it follows that $\{u_n\}$ is bounded. So $\{\|cu_n + (1-c)v - w\|\}$ is bounded for each $c \in [0, 1]$.

We fix $c \in [0,1]$ arbitrary. Since (N_4) holds for $v \in \cap_n F(T_n)$, Lemma 5.2 (1) or (2) holds. If $\lim_n \|V_n u_1 - v\| = 0$ then $\{u_n\}$ converges and $v = w$. We thus assume from now on that Lemma 5.2 (2) holds. For simplicity, we set, for any $n, k \in N$,

$$A_{n,k} = \left\| cV_{n+k}\, u_1 + (1-c)v - V_{n+1}^{n+k}(cV_n\, u_1 + (1-c)v) \right\|.$$

Then, there exists $n_0 \in N$ such that $A_{n,k} < \varepsilon$ for all $n > n_0$ and $k \in N$. Since $v, w \in \cap F(T_n)$, it is easy to see that

$$\begin{aligned} \|cu_{(n+1)+k} + (1-c)v - w\| &= \|cV_{n+k}u_1 + (1-c)v - w\| \\ &\le A_{n,k} + \left\| V_{n+1}^{n+k}(cV_n u_1 + (1-c)v) - w \right\| \\ &\le A_{n,k} + \|cu_{n+1} + (1-c)v - w\| < \|cu_{n+1} + (1-c)v - w\| + \varepsilon \end{aligned}$$

for all $n > n_0$ and $k \in N$. Then the following holds:

$$\begin{aligned} &\limsup_n \|cu_n + (1-c)v - w\| \\ &\qquad = \limsup_k \|cu_{(n+1)+k} + (1-c)v - w\| \le \|cu_{n+1} + (1-c)v - w\| + \varepsilon. \end{aligned}$$

Furthermore, we have

$$\limsup_n \|cu_n + (1-c)v - w\| \le \liminf_n \|cu_n + (1-c)v - w\| + \varepsilon.$$

Thus, since ε is also arbitrary, $\lim_n \|cu_n + (1-c)v - w\|$ exists for any $c \in [0,1]$. By Lemma 5.1, we have the result $v = w$. □

We know that a mapping of Class (C) satisfies the condition (N_2) with $s = 3$. We prove the following lemma which is connected with the conditions (N_3) and (N_4).

Lemma 5.4. *Let $\{\alpha_n\}$ be a sequence in $[1/2, 1]$. Let C be a convex subset of a Banach space E. Let T be a self–mapping of* Class (C) *on C. For any $n \in N$, set $T_n = \alpha_n T + (1-\alpha_n)I$. Let $u_1 \in C$ and $\{u_n\}$ be a sequence defined by*

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n)u_n = T_n u_n = V_n u_1, \qquad \forall n \in N.$$

Then, the followings hold.

(1) (N_3) *holds. That is,*

$$\|Tu_{n+1} - Tu_n\| \le \alpha_n \|Tu_n - u_n\| = \|u_{n+1} - u_n\|, \quad \forall n \in N.$$

(2) *Suppose E is uniformly convex, $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then each T_n is quasi–nonexpansive and $\cap_n F(T_n) = F(T) \neq \emptyset$.*
Moreover, (N_4) holds. That is, for $c \in [0,1]$ and $z \in \cap_n F(T_n)$,

$$\limsup_n \sup_k \Big(\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| - (1-c)\|V_n u_1 - z\| \Big) \le 0.$$

Proof. We prove (1). For any $n \in N$, we have from $\alpha_n \in [1/2, 1]$ that

$$\begin{aligned} \tfrac{1}{2}\|Tu_n - u_n\| &\le \alpha_n \|Tu_n - u_n\| \\ &= \|\alpha_n T u_n + (1-\alpha_n)u_n - u_n\| = \|u_{n+1} - u_n\|. \end{aligned}$$

Then, by the Condition (C), we have

$$\|Tu_{n+1} - Tu_n\| \le \|u_{n+1} - u_n\| = \alpha_n \|Tu_n - u_n\|, \quad \forall n \in N.$$

We prove (2). Let $z \in F(T)$. Since T satisfies (N_2), we have $\emptyset \neq F(T) \subset A(T)$. That is, T is quasi–nonexpansive. By Lemma 3.3 (d), we have $\lim_n \|Tu_n - u_n\| = 0$.

It is obvious that $F(T_i) = F(T)$ for each $i \in N$ and $\cap_i F(T_i) = F(T)$. Then, each T_i is quasi–nonexpansive with $F(T_i) = F(T)$ from

$$\|T_i x - z\| \le \alpha_i \|Tx - z\| + (1-\alpha_i)\|x - z\| \le \|x - z\|, \quad \forall x \in C,\ i \in N.$$

Since each T_i is quasi–nonexpansive, $\{\|V_n u_1 - z\|\}$ is non–increasing.

In the cases $c = 0$ or $c = 1$, (N_4) obviously holds. We can assume $c \in (0,1)$.

Fix $c \in (0,1)$ arbitrary. For simplicity, we set, for any $n, k \in N$,

$$B_{n,k} = \|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| - (1-c)\|V_n u_1 - z\|.$$

It is easy to see that, for any $n, k \in N$,

$$\begin{aligned} B_{n,k} &\le \|V_{n+k}u_1 - z\| + \|z - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| - (1-c)\|V_n u_1 - z\| \\ &\le \|V_n u_1 - z\| + c\|z - V_n u_1\| - (1-c)\|V_n u_1 - z\| = 2c\|V_n u_1 - z\|. \end{aligned}$$

Set $d_n = \|V_n u_1 - z\|$ for all $n \in N$. It is easy to see that $\{d_n\} = \{\|V_n u_1 - z\|\}$ converges to some $d \in [0,\infty)$. By $B_{n,k} \le 2c\|V_n u_1 - z\|$ for all $n, k \in N$, we have that (N_4) holds if $d = \lim_n \|V_n u_1 - z\| = 0$. We assume $\lim_n \|V_n u_1 - z\| = d \ne 0$, that is, $d_n \ge d > 0$ for all $n \in N$. Since $\lim_n d_n = d > 0$, there are $b > 0$ and $l_1 \in N$ such that $c\|V_n u_1 - z\| \le cb < d$ for all $n > l_1$. Then,

$$\begin{aligned} &\|V_{n+m}u_1 - V_{n+1}^{n+m}(cV_n u_1 + (1-c)z)\| \\ &\qquad \ge \|V_{n+m}u_1 - z\| - \|V_{n+1}^{n+m}(cV_n u_1 + (1-c)z) - z\| \\ &\qquad \ge d - \|(cV_n u_1 + (1-c)z) - z\| \\ &\qquad = d - c\|V_n u_1 - z\| \ge d - cb > 0 \end{aligned}$$

for all $n > l_1$ and $m \in N$. On the other hand, by $\lim_n \|Tu_n - u_n\| = 0$, there is $l_2 \in N$ such that $\|TV_{n+m}u_1 - V_{n+m}u_1\| < d - cb$ for all $n > l_2$ and $m \in N$. Thus, we have

$$\tfrac{1}{2}\|TV_{n+m}u_1 - V_{n+m}u_1\| < d - cb \le \|V_{n+m}u_1 - V_{n+1}^{n+m}(cV_n u_1 + (1-c)z)\|$$

for all $n > l_0 = l_1 + l_2$ and $m \in N$. By the Condition (C), we have that

$$\|TV_{n+m}u_1 - TV_{n+1}^{n+m}(cV_n u_1 + (1-c)z)\| \le \|V_{n+m}u_1 - V_{n+1}^{n+m}(cV_n u_1 + (1-c)z)\|$$

for all $n > l_0$ and $m \in N$. We know that, for any $x, y \in C$,

$$\begin{aligned} \|T_{n+k}x - T_{n+k}y\| &= \|(\alpha_{n+k}Tx + (1-\alpha_{n+k})x) - (\alpha_{n+k}Ty + (1-\alpha_{n+k})y)\| \\ &\le \alpha_{n+k}\|Tx - Ty\| + (1-\alpha_{n+k})\|x - y\|. \end{aligned}$$

This implies that if $\|Tx - Ty\| \le \|x - y\|$ then $\|T_{n+k}x - T_{n+k}y\| \le \|x - y\|$. Then, for $n > l_0$ and $k \in N$, we have that

$$\begin{aligned} &\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1-c)z)\| \\ &\qquad = \|T_{n+k}V_{n+k-1}u_1 - T_{n+k}V_{n+1}^{n+k-1}(cV_n u_1 + (1-c)z)\| \\ &\qquad \le \|V_{n+k-1}u_1 - V_{n+1}^{n+k-1}(cV_n u_1 + (1-c)z)\| \\ &\qquad \le \cdots \\ &\qquad \le \|T_{n+1}V_n u_1 - T_{n+1}(cV_n u_1 + (1-c)z)\| \\ &\qquad \le \|V_n u_1 - (cV_n u_1 + (1-c)z)\| = (1-c)\|V_n u_1 - z\|. \end{aligned}$$

This implies that (N_4) holds. □

6. Weak convergence theorems

In this section, we prove some weak convergence theorems for nonlinear mappings in Banach spaces. In particular, we obtain weak convergence theorems for mappings in Class (C) which are generalizations of Khan and Suzuki [10] and Suzuki [20].

Lemma 6.1. *Let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_n \alpha_n(1-\alpha_n)=\infty$. Let E be a uniformly convex Banach space whose dual E^* has the Kadec–Klee property. Let C be a closed and convex subset of E and let T be a self–mapping on C satisfying $F(T)\neq\emptyset$ and (N_2). Set $T_n=\alpha_n T+(1-\alpha_n)I$ for all $n\in N$. Let $\{u_n\}$ be a sequence defined by $u_1\in C$ and*

$$u_{n+1}=\alpha_n T u_n+(1-\alpha_n)u_n=T_n u_n, \quad \forall n\in N.$$

Suppose that (N_3) and (N_4) hold. Then $\{u_n\}$ converges weakly to some $u\in F(T)$.

Proof. By $F(T)\neq\emptyset$ and (N_2), T is quasi-nonexpansive. Let $v\in F(T)\subset A(T)$. Then, $\|Tx-v\|\le\|x-v\|$ for all $x\in C$. We set $D=\{x\in C:\|x-v\|\le\|u_1-v\|\}$. Obviously, D is bounded, closed and convex and T is a self–mapping on D. It is also obvious that $\cap_n F(T_n)=F(T)\neq\emptyset$, each T_n is quasi-nonexpansive self-mapping on D and $\{u_n\}\subset D$. Since (N_3) holds, by Lemma 3.3 (d), we have $\lim_n\|Tu_n-u_n\|=0$. Since D is weakly compact, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some $u\in D$. By Theorem 4.2, we have $u\in F(T)$. Since (N_4) holds, by Lemma 5.3, any weakly convergent subsequence of $\{u_n\}$ converges weakly to u. Thus $\{u_n\}$ converges weakly to $u\in F(T)$. □

Lemma 6.2. *Let b be a real number belonging to $(0,1)$ and let $\{\alpha_n\}$ be a sequence in $[0,b]$.with $\sum_n \alpha_n=\infty$. Let E be a reflexive Banach space which has the Opial property. Let C be a closed and convex subset of E and let T be a self–mapping on C satisfying $F(T)\neq\emptyset$ and (N_2). Let $\{u_n\}$ be a sequence defined by $u_1\in C$ and*

$$u_{n+1}=\alpha_n T u_n+(1-\alpha_n)u_n, \quad \forall n\in N.$$

Suppose that (N_3) holds. Then $\{u_n\}$ converges weakly to some $u\in F(T)$.

Proof. By $F(T)\neq\emptyset$ and (N_2), T is quasi-nonexpansive. Let $v\in F(T)\subset A(T)$. Then, $\|Tx-v\|\le\|x-v\|$ for all $x\in C$. We set $D=\{x\in C:\|x-v\|\le\|u_1-v\|\}$. Then, D is bounded closed convex and T is a self–mapping on D. Obviously, $\{u_n\}\subset D$. Since (N_3) holds, by Lemma 3.3 (b),(c), we have that $\{\|u_n-v\|\}$ converges and $\lim_n\|Tu_n-u_n\|=0$. Since D is weakly compact, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some $u\in D$. By Lemma 3.4, we have $u\in F(T)$. Recall that $\{\|u_n-v\|\}$ converges for $v\in F(T)$. Then, by Lemma 3.5, any weakly convergent subsequence of $\{u_n\}$ converges weakly to u. Thus $\{u_n\}$ converges weakly to $u\in F(T)$. □

Remark 2. In Lemmas 6.1 and 6.2, assume that C is bounded. Set $\alpha_n=b\in(0,1)$ for $n\in N$. Since (N_2) and (N_3) hold, by Lemma 4.3, we have $\emptyset\neq F(T)\subset A(T)$. Then, we can remove the assumption $F(T)\neq\emptyset$.

The following are weak convergence theorems for mappings in Class (C) which are derived from Lemmas 6.1 and 6.2.

Theorem 6.3. *Let $\{\alpha_n\}$ be a sequence in $[1/2, 1]$ with $\sum_n \alpha_n(1-\alpha_n) = \infty$. Let E be a uniformly convex Banach space whose dual E^* has the Kadec–Klee property. Let C be a bounded, closed and convex subset of E and let T be a self–mapping of* Class(C) *on C. Set $T_n = \alpha_n T + (1-\alpha_n)I$ for $n \in N$. Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n)u_n = T_n u_n, \quad \forall n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Proof. We know that a mapping T of Class (C) satisfies (N_2). Under our assumptions, by Lemma 5.4 (1), we have that (N_3) holds. Since C is bounded, by Remark 2, we know $\emptyset \neq F(T) \subset A(T)$. Then, each T_n is quasi–nonexpansive and $\cap_n F(T_n) = F(T) \neq \emptyset$. By $F(T) \neq \emptyset$ and Lemma 5.4 (2), we have that (N_4) holds. By Lemma 6.1, we have the result. □

Theorem 6.4. *Let b be a real number belonging to $[1/2, 1)$ and let $\{\alpha_n\}$ be a sequence in $[1/2, b]$. Let E be a reflexive Banach space which has the Opial property. Let C be a bounded, closed and convex subset of E and let T be a self–mapping of* Class(C) *on C. Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n)u_n, \quad \forall n \in N.$$

Then $\{u_n\}$ converges weakly to some $u \in F(T)$.

Proof. We note that $\sum_n \alpha_n = \infty$ and a mapping T of Class (C) satisfies (N_2). By Lemma 5.4 (1), we have that (N_3) holds. Since C is bounded, by Remark 2, we know $\emptyset \neq F(T) \subset A(T)$. By Lemma 6.2, we have the result. □

A nonexpansive mapping satisfies all assumptions in Lemmas 6.1 and 6.2. The following theorems are direct consequences of Lemmas 6.1 and 6.2.

Theorem 6.5. *Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_n \alpha_n(1-\alpha_n) = \infty$. Let E be a uniformly convex Banach space whose dual E^* has the Kadec–Klee property. Let C be a closed and convex subset of E and let T be a nonexpansive self–mapping on C with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n)u_n, \quad \forall n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Theorem 6.6. *Let b be a real number belonging to $(0, 1)$ and $\{\alpha_n\}$ be a sequence in $[0, b]$ with $\sum_n \alpha_n = \infty$. Let E be a reflexive Banach space which has the Opial property. Let C be a bounded, closed and convex subset of E and let T be a nonexpansive self–mapping on C. Let $\{u_n\}$ be a sequence defined by $u_1 \in C$ and*

$$u_{n+1} = \alpha_n T u_n + (1-\alpha_n)u_n, \quad \forall n \in N.$$

Then $\{u_n\}$ converges weakly to some $u \in F(T)$.

7. Appendix

In this section, we present a mapping T which does not belong to Class (C), but it has desirable properties in our study.

Example 1. Let R^2 be the 2-dimensional Euclidean space. Let $C = [0,1]^2 \subset R^2$. Let T be a mapping on C defined by

$$T(x_1, x_2) = \left(\tfrac{1}{4}(1+2x_2)x_1 \, , \, x_2\right), \quad \forall (x_1, x_2) \in C.$$

Then, the followings hold:

(1) T is not Class (C) and hence T is not nonexpansive.
(2) T satisfies (N_2).

For simplicity, set $T_n = \frac{1}{2}T + \frac{1}{2}I$ for $n \in N$. Let $\{u_n\}$ be a sequence defined by

$$u_1 \in C, \qquad u_{n+1} = \tfrac{1}{2}Tu_n + \tfrac{1}{2}u_n = T_n u_n = V_n u_1, \quad \forall n \in N.$$

Then,

(3) (N_3) and (N_4) hold.

Proof. It is obvious that C is compact and convex and $F(T) = \{(x_1, x_2) \in C : x_1 = 0\}$. It is also obvious that T is quasi–nonexpansive. Let $x = (x_1, x_2) \in C$ and set $r(x_2) = \frac{1}{4}(1+2x_2)$. Then, $T(x_1, x_2) = (r(x_2)x_1, x_2)$ and $\frac{1}{4} \le r(x_2) \le \frac{3}{4}$.

We show (1). Let $x = (1,0)$ and $y = (1,1) \in C$. Then, $y - x = (0,1)$ and

$$x - Tx = (1,0) - (\tfrac{1}{4}, 0) = (\tfrac{3}{4}, 0), \quad Ty - Tx = (\tfrac{3}{4}, 1) - (\tfrac{1}{4}, 0) = (\tfrac{1}{2}, 1).$$

One can easily see that

$$\tfrac{1}{2}\|x - Tx\| < 1 = \|x - y\|, \quad \|x - y\| = 1 < \|Tx - Ty\|.$$

This implies that T is not in Class (C) and hence T is not nonexpansive.

We show (2), that is, we show that T satisfies (N_2).

Let $y = (y_1, y_2) \in C$ and $x = (x_1, x_2)$. In the case of $r(y_2)y_1 \ge x_1$, it is obvious from $y_1 \ge r(y_2)y_1 \ge x_1$ that $|x_1 - r(y_2)y_1| \le |x_1 - y_1|$. This implies that

$$\begin{aligned}\|x - Ty\|^2 &= |x_1 - r(y_2)y_1|^2 + |x_2 - y_2|^2 \\ &\le |x_1 - y_1|^2 + |x_2 - y_2|^2 \\ &= \|x - y\|^2\end{aligned}$$

and hence

$$\|x - Ty\| \le 4\|Tx - x\| + \|x - y\|.$$

In the case of $r(y_2)y_1 < x_1$, we have from $0 \le r(y_2)y_1 < x_1$ that $0 < x_1 - r(y_2)y_1 \le x_1 \le 4(x_1 - r(x_2)x_1)$. Then, we have

$$4\|x - Tx\| = 4\|(x_1, x_2) - (r(x_2)x_1, x_2)\| = 4\|(x_1 - r(x_2)x_1, 0)\| \ge \|(x_1, 0)\|.$$

This implies that

$$\begin{aligned}\|x - Ty\| &= \|(x_1 - r(y_2)y_1, x_2 - y_2)\| \\ &\le \|(x_1 - r(y_2)y_1, 0)\| + \|(0, x_2 - y_2)\| \\ &\le \|(x_1 - r(y_2)y_1, 0)\| + \|(x_1 - y_1, x_2 - y_2)\| \\ &\le \|(x_1, 0)\| + \|x - y\| \le 4\|Tx - x\| + \|x - y\|.\end{aligned}$$

Thus, T satisfies (N_2) with $s = 4$.

We show (3). Set $u_1 = (x(1), x_2)$ and $M = \frac{1}{2}(1 + r(x_2)) < 1$. We know

$$u_2 = V_1 u_1 = \tfrac{1}{2}(r(x_2)x(1), x_2) + \tfrac{1}{2}(x(1), x_2) = \Big(\tfrac{1}{2}(1 + r(x_2))x(1), x_2\Big).$$

Inductively, we have $u_{n+1} = (x(n+1), x_2)$ and $x(n+1) = M^n x(1)$ for $n \in N$. Then, $\lim_n x(n) = 0$. From the following inequality, it is obvious that (N_3) holds.

$$\begin{aligned}\|Tu_{n+1} - Tu_n\| &= \big\|\big(r(x_2)x(n+1), x_2\big) - \big(r(x_2)x(n), x_2\big)\big\| \\ &= r(x_2)\,\big\|\big(x(n+1) - x(n), 0\big)\big\| \\ &\le \|(x(n+1) - x(n), 0)\| \\ &= \|(x(n+1), x_2) - (x(n), x_2)\| \\ &= \|u_{n+1} - u_n\|.\end{aligned}$$

Let $u_1 = (x(1), x_2)$, $u = (0, v) \in F(T)$ and $c \in [0, 1]$. Fix $n \in N$. Set

$$x = V_n u_1 = (x(n), x_2), \quad y = cV_n u_1 + (1 - c)u.$$

Then, we have

$$y = cV_n u_1 + (1 - c)u = c(x(n), x_2) + (1 - c)(0, v) = (cx(n), y_2),$$

where $y_2 = cx_2 + (1 - c)v$. Set $L = \frac{1}{2}(1 + r(y_2))$. We have $0 < L < 1$. We also have that, for any $k \in N$,

$$\begin{aligned}&V_{n+1}^{n+k}x = \big(M^k x(n), x_2\big), \quad V_{n+1}^{n+k}y = \big(cL^k x(n), y_2\big), \\ &\|V_{n+k}u_1 - V_{n+1}^{n+k}(cV_n u_1 + (1 - c)u)\| \\ &\qquad = \|V_{n+1}^{n+k}x - V_{n+1}^{n+k}y\| = \big\|\big((M^k - cL^k)x(n), x_2 - y_2\big)\big\|.\end{aligned}$$

It is obvious that $\lim_n \sup_k |M^k - cL^k| x(n) = 0$. We know $x_2 - y_2 = (1 - c)(x_2 - v)$ and $V_n u_1 - u = (x(n), x_2 - v)$. Since n is arbitrary, we have

$$\begin{aligned}&\limsup_n \sup_{k\in N} \|V_{n+1}^{n+k}x - V_{n+1}^{n+k}y\| \le \|(0, x_2 - y_2)\| = (1 - c)\|(0, x_2 - v)\|, \\ &(1 - c)\lim_n \|V_n u_1 - u\| = (1 - c)\lim_n \|(x(n), x_2 - v)\| = (1 - c)\|(0, x_2 - v)\|.\end{aligned}$$

Thus, it follows that (N_4) holds. □

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