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# ON NORM ATTAINING LIPSCHITZ MAPS BETWEEN BANACH SPACES

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ABSTRACT. If $X$ is an asymptotically uniformly smooth separable Banach space and $f$ is a Lipschitz isomorphism between $X$ and $Y$ which attains its Lipschitz norm at some $x$ in a direction $e$, then there is a point $y \in S_Y$ at which the modulus of asymptotic smoothness of $Y$ is controlled from above by the corresponding modulus of $X$. It follows for instance that if $X$ is a subspace of $c_0$ equipped with the canonical norm and $Y$ has the Kadec-Klee property, then no Lipschitz isomorphism between $X$ and $Y$ can attain its norm in the above sense.

With respect and gratitude, this work is dedicated to the memory of Joram Lindenstrauss (1936-2012).

## 1. Introduction

Let $X$ be a Banach space, and $X^*$ its dual space. The classical Bishop-Phelps theorem asserts that the set of norm attaining elements of $X^*$ is norm-dense in $X^*$. Extensions of this result to operators were considered by Joram Lindenstrauss as early as 1963 [9], who showed for instance that if $X$ is strictly convex then norm-attaining operators from $c_0$ equipped with its canonical norm to $X$ have finite rank. J. Lindenstrauss was also a forerunner is non-linear geometry of Banach spaces [10], a very active field today for which the authoritative book is still [3] (see also [7] for a survey on more recent results, and [1]).

The purpose of this note is to show that Lindenstrauss' linear result has a non-linear counterpart. Indeed, a corollary of the main technical result is that a Lipschitz isomorphism from a space which "looks like" $c_0$ to a space which is "somewhat strictly convex" cannot attain its Lipschitz norm, in a sense that we will explain. Hence, even the greater flexibility allowed by non linearity does not provide norm-attaining objects. This could matter since finding points where a norm is attained is a well-known technique for showing existence results: we refer for instance to D. Preiss' fundamental article [12] where maximal rate of change arguments are used for showing the existence of Fréchet-smooth points for real-valued Lipschitz functions defined on spaces with separable dual.

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## 2. Notation and definitions

Let $f : X \to Y$ be a Lipschitz map from a Banach space $X$ to a Banach space $Y$, equipped with norms that we simply denote $\| \, . \, \|$. Unless otherwise specified, all Banach spaces considered in this note are assumed to be infinite dimensional.

We should first explain various ways in which a Lipschitz function can attain its norm. The Lipschitz norm $Lip(f)$ of $f$ is the smallest constant $\lambda \geq 0$ such that

$$\|f(x) - f(x')\| \leq \lambda \|x - x'\|$$

for all $(x, x') \in X^2$. We say that $f$ attains its Lipschitz norm (in short, its norm) on a couple $(x, x')$ of distinct points of $X$ if $\|f(x) - f(x')\| = Lip(f)\|x - x'\|$.

The derivative of a Lipschitz map $f : X \to Y$ at $x$ in the direction $e \in S_X$ is defined to be

$$f'(x, e) = \lim_{t \to 0^+} t^{-1}(f(x + te) - f(x))$$

when this limit exists in the Banach space $Y$. When the equation $\|f'(x, e)\| = Lip(f)$ is satisfied for some $x \in X$ and $e \in S_X$, we say that $f$ attains its norm at $x$ in the direction $e$. Finally, we say that $f$ attains its norm in a direction $y \in Y$ if $\|y\| = Lip(f)$ and there exists a sequence $(u_n, v_n)$ of pairs of distinct points in $X$ such that

$$\lim_n \, (f(u_n) - f(v_n))/\|u_n - v_n\| = y$$

It is trivial that if the norm of $f$ is attained on a couple $(x, x') \in X^2$, then $f$ attains its norm in the direction $(f(x) - f(x'))$. Compactness shows that if $dim(Y) < \infty$, any Lipschitz map with range in $Y$ attains its norm in some direction, although it does not necessarily attain its norm on a couple (take $f(t) = sin(t)$). It is clear that if $f$ attains its norm at $x$ in the direction $e \in S_X$ then it attains its norm in the direction $y = f'(x, e) \in Y$, and again the converse is false (take $f(t) = (1 + t^2)^{1/2}$).

We refer to [8] for a recent work on possible extensions of the Bishop-Phelps theorem to real-valued Lipschitz functions defined on a Banach space. There is however no overlap between [8] and the present work.

We now recall the definition of the modulus of asymptotic smoothness.

**Definition 2.1.** Let $X$ be a Banach space equipped with the norm $\| \, . \, \|$. If $\|x\| = 1$, $\tau > 0$ and $Y$ is a closed finite-codimensional subspace of $X$, we let

$$\rho(\tau, x, Y) = \sup_{y \in S_Y} \|x + \tau y\| - 1$$

where $S_Y$ denotes the unit sphere of $Y$. Then we let

$$\rho(\tau, x) = \inf_Y \rho(\tau, x, Y)$$

where the infimum is taken over all closed finite-codimensional subspaces. Finally, we let

$$\rho(\tau) = \sup_{x \in S_X} \rho(\tau, x).$$

This function $\rho$ (or $\rho_X$ if the space $X$ needs to be specified) is called the modulus of asymptotic uniform smoothness of $X$. It is sometimes denoted $\overline{\rho}$ to distinguish it from the modulus of uniform smoothness, but this latter notion is not used in this note. A Banach space $X$ is said to be asymptotically uniformly smooth (in short, a. u. s) if

$$\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0.$$

The space $X = c_0$ is a. u. s., and $\rho_{c_0}(\tau) = 0$ for all $\tau \in (0,1]$. More generally, a Banach space $E$ is called asymptotically uniformly flat if there exists $\tau_0 > 0$ such that $\rho_E(\tau_0) = 0$. It is shown in [5] that a separable space has an equivalent asymptotically uniformly flat norm if and only if it is isomorphic to a subspace of $c_0$.

## 3. Results

We first recall a practical way of computing the modulus $\rho$.

**Lemma 3.1.** *Let $X$ be a Banach space with separable dual, $\tau \in (0,1]$ and $x \in S_X$. We let*

$$\eta(\tau, x) = \sup\ [\overline{\lim_{n\to\infty}} \|x + x_n\| - 1]$$

*where the supremum is taken over all sequences $(x_n)$ which converge weakly to 0 and such that $\|x_n\| \le \tau$ for all $n$, and $\eta(\tau) = \sup_{x \in S_X} \eta(\tau, x)$. Then $\eta(\tau, x) = \rho(\tau, x)$ and $\eta(\tau) = \rho(\tau)$ for every $\tau \in (0,1]$.*

*Proof.* Let $(x_n)$ be a sequence which converges weakly to 0 and such that $\|x_n\| \le \tau$ for all $n$. Let $Y \subset X$ be a closed subspace of finite codimension. The distance $d(x_n, Y)$ from $x_n$ to $Y$ tends to 0, so given $\epsilon > 0$, there exists for $n$ large enough $y_n \in Y$ with $\|x_n - y_n\| < \epsilon$. Then $\|y_n\| < \tau + \epsilon$ and

$$\|x + x_n\| - 1 \le \|x + y_n\| - 1 + \|x_n - y_n\| \le \rho(\tau + \epsilon, x, Y) + \epsilon.$$

Since $Y$ of finite codimension is arbitrary, we have for $n$ large enough

$$\|x_n + x\| - 1 \le \rho(\tau + \epsilon, x) + \epsilon$$

and since $\epsilon > 0$ is arbitrary it follows that $\eta(\tau, x) \le \rho(\tau, x)$.

Conversely, we have $\eta(\tau, x) \ge \rho(\tau, x)$. Indeed, let $(x_j^*)$ be a dense sequence in $X^*$, and let

$$Y_n = \bigcap_{j=0}^{n} Ker(x_j^*).$$

Given $\epsilon > 0$, there is $x_n \in Y_n$ such that $\|x_n\| \le \tau$ with

$$\|x + x_n\| - 1 + \epsilon \ge \rho(\tau, x, Y_n) \ge \rho(\tau, x).$$

It is easy to check that the sequence $(x_n)$ weakly converges to 0. Since $\epsilon > 0$ is arbitrary, it follows that $\eta(\tau, x) \ge \rho(\tau, x)$. Hence these two quantities are equal, and the last assertion follows immediately by taking the supremum over $x \in S_X$. □

The following result is Theorem 5.4 in [6], with an actual estimate of the constants. The calculations were left to the reader in [6] since the result is a special case of the previous Theorem 5.3 from [6]. However, these computations are non trivial even in this special case. Moreover we will need the expression of the equivalent norm. For the convenience of the reader, we provide a complete proof below.

**Theorem 3.2.** *Let $X$ and $Y$ be two separable Banach spaces. We assume that $X$ is asymptotically uniformly smooth, and that there exists a Lipschitz-isomorphism $f$ from $X$ onto $Y$. If $\rho_X$ denotes the modulus of asymptotic uniform smoothness of $X$ and $M = Lip(f).Lip(f^{-1})$, there is an asymptotically uniformly smooth equivalent norm on $Y$ whose modulus $\rho_Y$ satisfies*

$$\rho_Y(\tau/4M) \leq 2\rho_X(\tau)$$

*for every $\tau \in (0,1]$.*

*Proof.* Since $X$ is asymptotically uniformly smooth, $X^*$ is separable, and then it follows from [2] that $Y^*$ is separable as well. We may and do assume that $Lip(f) = 1$ and $Lip(f^{-1}) = M$. We define a norm $|\ .\ |^*$ on $Y^*$ by the formula

$$|y^*|^* = \sup\{\frac{|\langle y^*, f(x) - f(x')\rangle|}{\|x - x'\|};\ x \neq x'\}.$$

The above supremum is taken over all pairs $(x, x')$ of distinct points in $X$. Since $f$ is a Lipschitz isomorphism from $X$ onto $Y$, this formula defines an equivalent norm on $Y^*$. We observe moreover that $|\ .\ |^*$ is weak* lower semi-continuous, since it is a supremum of weak* continuous functions. Then the bipolar theorem shows that $|\ .\ |^*$ is the dual norm of an equivalent norm on $Y$, which we denote $|\ .\ |$.

Let us observe that the unit ball of the norm $|\ .\ |$ is the norm-closed convex hull of the vectors $(f((x) - f(x'))/\|x - x'\|$, where $(x, x')$ runs over all pairs of distinct elements of $X$. This means that this norm $|\ .\ |$ is the largest norm on $Y$ for which the map $f$ is 1-Lipschitz.

We claim that this norm satisfies the requested conditions. By Lemma 3.1, we need to show that $\eta_Y(\tau/4M) \leq 2\rho_X(\tau) = 2\rho(\tau)$, where $\eta_Y = \eta$ is obtained from $|\ .\ |$ along the lines of this Lemma. Let $y \in Y$ with $|y| = 1$ and $(y_n)$ a sequence in $Y$ which converges weakly to 0 and such that $|y_n| \leq \tau/4M$ for all $n$. We have to show that

$$\overline{\lim_{n\to\infty}} |y + y_n| - 1 \leq 2\rho(\tau).$$

For all $n$, we pick $y_n^* \in Y^*$ with $|y_n^*|^* = 1$ such that $\langle y_n^*, y + y_n\rangle = |y + y_n|$. We may and do assume that the sequence $(y_n^*)$ is weak* convergent to $y^*$ with $|y^*|^* \leq 1$ and that $\lim |y^* - y_n^*|^* = l$ exists. Pick $\epsilon > 0$ and $x \neq x'$ in $X$ such that

$$\langle y^*, f(x) - f(x')\rangle \geq (1 - \epsilon)|y^*|^* \|x - x'\|.$$

We may and do assume that $x' = -x$ (hence $x \neq 0$) and $f(x') = -f(x)$, and thus

$$\langle y^*, f(x)\rangle \geq (1 - \epsilon)|y^*|^* \|x\|.$$

Pick any $\beta > \rho(\tau)$. By definition of $\rho(\tau)$, there exists a subspace $X_0$ of finite codimension in $X$ such that if $z \in X_0$ and $\|z\| \leq \tau\|x\|$, then

$$\|x + z\| \leq (1 + \beta)\|x\|.$$

Pick $b < \tau\|x\|/2M$ and let $d = \tau\|x\|/2$. Since $f^{-1}$ is $M$-Lipschitz (for the original norm, and thus for the larger norm $|\ .\ |$), we can apply Gorelik's principle ([5], Prop. 2.7) for these values of $b$ and $d$ and conclude that there exists a compact set $K$ such that $bB_{|\ .\ |} \subset K + f(2dB_{X_0})$.

We observe now that the sequence $(y_n^* - y^*)$ converges to 0 uniformly on the compact set $K$. It follows that there exists a sequence $(z_n)$ in $X_0$ such that $\|z_n\| \leq 2d = \tau\|x\|$ and $\lim\langle y_n^* - y^*, f(z_n)\rangle = -bl$.

We set $A_n = \langle y_n^*, f(x) - f(z_n)\rangle$. We have $A_n \leq |y_n^*|^*\|x - z_n\| \leq (1+\beta)\|x\|$. Moreover,

$$A_n = \langle y^*, f(x) - f(z_n)\rangle + \langle y_n^* - y^*, f(x)\rangle - \langle y_n^* - y^*, f(z_n)\rangle$$

and since $(y_n^* - y^*)$ weak* converges to 0 and $f(-x) = -f(x)$, one has

$$A_n = 2\langle y^*, f(x)\rangle - \langle y^*, f(z_n) - f(-x)\rangle + bl + \epsilon(n)$$

with $\lim \epsilon(n) = 0$. Since we have

$$\langle y^*, f(z_n) - f(-x)\rangle \leq |y^*|^*\|z_n + x\| \leq |y^*|^*(1+\beta)\|x\|$$

il follows that

$$A_n \geq 2(1-\epsilon)|y^*|^*\|x\| - |y^*|^*(1+\beta)\|x\| + bl + \epsilon(n).$$

We can now combine the two inequalities on $A_n$ and let $n$ increase to infinity to obtain

$$(1+\beta)\|x\| \geq (1-\beta-2\epsilon)|y^*|^*\|x\| + bl.$$

Playing on $\beta$ and $b$ leads to

$$(1+\rho(\tau))\|x\| \geq (1-\rho(\tau)-2\epsilon)|y^*|^*\|x\| + l\tau\|x\|/2M$$

and since we can divide by $\|x\| \neq 0$ and that $\epsilon > 0$ is arbitrary, it follows that

$$|y^*|^* \leq 1 + \frac{2\rho(\tau)}{1-\rho(\tau)} - \frac{l\tau}{2M(1-\rho(\tau))}.$$

We have

$$|y + y_n| = \langle y_n^*, y + y_n\rangle = \langle y_n^* - y^*, y\rangle + \langle y_n^* - y^*, y_n\rangle + \langle y^*, y + y_n\rangle$$

and thus

$$\overline{\lim}|y + y_n| \leq (\tau/4M)\lim|y_n^* - y^*|^* + |y^*|^* = \frac{l\tau}{4M} + |y^*|^*.$$

If $\frac{l\tau}{4M} \leq 2\rho(\tau)$, then since $|y^*|^* \leq 1$, it follows that $\overline{\lim}|y + y_n| - 1 \leq 2\rho(\tau)$. If $\frac{l\tau}{4M} > 2\rho(\tau)$, then

$$|y^*|^* \leq 1 - \frac{l\tau}{4M(1-\rho(\tau))} \leq 1 - \frac{l\tau}{4M}$$

and thus $\overline{\lim}|y + y_n| \leq 1$. Hence in both cases we have

$$\overline{\lim_{n\to\infty}}|y + y_n| - 1 \leq 2\rho(\tau)$$

and this concludes the proof. □

**Remark 3.3.** The renorming $|\,.\,|$ constructed in the above proof can also be defined with the techniques of [4]. Indeed, if $f : X \to Y$ is a Lipschitz-isomorphism which maps 0 to 0 and $\overline{f} : \mathcal{F}(X) \to Y$ is its canonical extension to a quotient map from the free space $\mathcal{F}(X)$ onto $Y$ (see Lemma 2.5 in [4]), then the norm $|\,.\,|$ is the quotient norm of the canonical norm of $\mathcal{F}(X)$ obtained through $\overline{f}$. Along these lines, let us recall that the correspondance $f \to \overline{f}$ is an isometry from $Lip_0(X, Y)$ onto the space $L(\mathcal{F}(X), Y)$ of linear operators from $\mathcal{F}(X)$ to $Y$, hence some norm attainment statements have linear translations. For instance, $f$ attains its norm on a couple $(x, x')$ of distinct points if and only if $\overline{f}$ attains its operator norm on a molecule $(\delta(x) - \delta(x'))/\|x - x'\|$ of $\mathcal{F}(X)$ (see [13]).

The following observation is the main technical result of this note. It relies heavily on Theorem 3.2 and its proof. In the statement of Theorem 3.4, the moduli of asymptotic smoothness of the Banach spaces $X$ and $Y$ are computed with respect to the original norms of these spaces. We use the notation of Definition 2.1.

**Theorem 3.4.** *Let $X$ and $Y$ be separable Banach spaces. We assume that $X$ is asymptotically uniformly smooth, and that there exists a Lipschitz isomorphism from $X$ onto $Y$ which attains its norm in some direction $y \in Y$. Then there is a constant $C > 0$ such that $\rho_Y(y, \tau/C) \leq 2\rho_X(\tau)$ for all $\tau \in (0, 1]$.*

*Proof.* We may and do assume that $Lip(f) = 1$. We denote by $\| \, . \, \|$ the original norm on the space $Y$. Then $1 = \|y\| \leq |y|$, where $| \, . \, |$ denotes the equivalent norm on $Y$ constructed in Theorem 3.2. Moreover $|y| \leq 1$ since $y = \lim_n (f(u_n) - f(v_n))/\|u_n - v_n\|$. Hence $\|y\| = |y| = 1$. Since $\|z\| \leq |z|$ for all $z \in Y$, Theorem 3.2 implies that $\rho_Y(y, \tau/4M) \leq 2\rho_X(\tau)$ for all $\tau \in (0, 1]$, where $M = Lip(f^{-1})$. □

Note that the assumptions of Theorem 3.4 are satisfied in particular if $f$ attains its norm at some point $x \in X$ in the direction $e \in S_X$, or on a pair of distinct points $(x, x') \in X^2$. This result implies, as we will see now, that the attainment of the Lipschitz norm is a quite restrictive condition. Note that it is an open question to know if two separable Banach spaces which are Lipschitz isomorphic are actually linearly isomorphic. However the following corollary is of isometric nature, and applies to a Banach space equipped with two equivalent norms. The assumption made on $Y$ below is usually called the Kadec-Klee property.

**Corollary 3.5.** *Let $E$ be a separable asymptotically uniformly flat Banach space. Let $Y$ be a Banach space such the the weak and norm topologies agree on the unit sphere of $Y$. Let $f$ be a Lipschitz isomorphism between $E$ and $Y$. Then there is no $y \in Y$ such that $f$ attains its norm in the direction $y$.*

*Proof.* Assume otherwise. Since $Y$ is Lipschitz-isomorphic to $E$, its dual $Y^*$ is separable ([2], see also Theorem 2.1 in [5]). By Theorem 3.4 there exists $y \in S_Y$ and $\tau_0 > 0$ such that $\rho_Y(y, \tau_0) = 0$. Hence by Lemma 3.1, if $(y_n)$ is a weakly-null sequence in $Y$ with $\|y_n\| = \tau_0$, then $\lim \|y + y_n\| = 1$. If we let now

$$z_n = (y + y_n)/\|y + y_n\|$$

the sequence $(z_n)$ is contained in $S_Y$ and it converges weakly but not in norm to $y$, which contradicts our assumption. □

**Remark 3.6.** The assumptions of Corollary 3.5 are somewhat optimal. It fails to hold if $E$ is merely assumed to be a.u.s. Indeed, if $1 < p < \infty$, the space $l_p$ equipped with its canonical norm is a.u.s. and uniformly convex, and $f = Id$ serves as a counterexample. On the other hand, Corollary 3.5 applies if $E$ is asymptotically uniformly flat and $Y$ is locally uniformly rotund. But it fails if $Y$ is simply assumed to be strictly convex. To check this, we observe that if $T$ is a one-to-one continuous linear map from $c_0$ into $l_2$, then $T$ maps weakly null sequences in $c_0$ to norm null sequences, and thus the equivalent norm on $c_0$ given by

$$|x| = \|x\|_\infty + \|T(x)\|_2$$

is asymptotically uniformly flat and strictly convex. But again, $f = Id$ trivially attains its Lipschitz norm.

However, when $E$ is a subspace of $c_0$ equipped with the restriction of the canonical norm, Lipschitz maps with strictly convex range cannot satisfy the strongest form of norm-attainment. I am grateful to A. Naor who suggested the use of geodesics in this context [11].

**Proposition 3.7.** *Let $E$ be an infinite dimensional subspace of $c_0$ equipped with the restriction of the canonical norm. Let $Y$ be a strictly convex Banach space. Then no one-to-one Lipschitz map from $E$ into $Y$ attains its norm on a pair of distinct points $(x, x') \in E^2$.*

*Proof.* Assume that a one-to-one Lipschitz map $f$ from $E$ to $Y$ attains its norm on a couple $(x, x') \in E^2$. Then the usual mid-point argument shows that a geodesic between $x$ and $x'$ (that is, the range of a 1-Lipschitz map $g$ from $[0, \|x' - x\|]$ to $E$ such that $g(0) = x$ and $g(\|x' - x\|) = x'$) is mapped to a geodesic between $f(x)$ and $f(x')$. We note now that if $Y$ is strictly convex, any couple of points in $Y$ is connected by a unique geodesic, namely the linear segment between these points.

Since $f$ is one-to-one, Proposition 3.7 will be shown if we prove that if $E$ is an infinite dimensional subspace of $c_0$ equipped with the restriction of the canonical norm, and $x \neq x'$ are distinct points in $E$, there are at least two geodesics between $x$ and $x'$. For doing so, we may and do assume that $x' = 0$ and $\|x\| = 1$. Since $E$ is infinite-dimensional, its unit ball $B_E$ contains no extreme point: indeed if $x \in S_E$, the set $F = \{n \in \mathbb{N};\ |x(n)| > 1/2\}$ is finite, and if $y \in E$ if such that $y(n) = 0$ for all $n \in F$ and $\|y\| \leq 1/2$, then $\|x + y\| = \|x - y\| = 1$. If $z$ is any point on the segment $[(x - y)/2, (x + y)/2]$ the map $g_z : [0, 1] \to E$ such that $g(0) = 0$, $g(1/2) = z$ and $g(1) = x$ which is affine on $[0, 1/2]$ and $[1/2, 1]$ is a geodesic between 0 and $x$. This concludes the proof. □

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