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STRUCTURE OF SOLUTIONS OF OPTIMAL CONTROL PROBLEMS ON LARGE INTERVALS: A SURVEY OF RECENT RESULTS

ALEXANDER J. ZASLAVSKI

Abstract. In this paper we survey our recent results on the structure of approximate solutions of autonomous optimal control problems. We discuss turnpike properties of approximate solutions as well as their structure in regions close to the endpoints of the time intervals.

1. Introduction

The growing significance of the study of (approximate) solutions of variational and optimal control problems defined on infinite intervals and on sufficiently large intervals has been further realized in the recent years [2, 4–9, 12–18, 21, 26, 28, 29, 32, 33, 35, 36, 38, 39, 54, 55]. This is due not only to theoretical achievements in this area, but also because of numerous applications to engineering [1, 11, 24, 39], models of economic dynamics [10, 11, 19, 23, 27, 31, 34, 37, 39, 40, 54, 55], the game theory [20,22,39,48,54], models of solid-state physics [3] and the theory of thermodynamical equilibrium for materials [25, 30]. In this paper we survey our recent results on the structure of approximate solutions of autonomous optimal control problems. We discuss turnpike properties of approximate solutions as well as their structure in regions close to the endpoints of the time intervals.

In the first part of the paper we analyze the structure of approximate solutions of an autonomous nonconcave discrete-time optimal control system with a compact metric space of states, on large intervals, describing a general model of economic dynamics [40, 43, 44, 46, 47, 49, 51, 54, 56, 58, 59]. We consider optimal control systems which are discrete-time analogs of Lagrange problems in the calculus of variations. These systems are described by a bounded upper semicontinuous objective function which determines an optimality criterion. We also discuss optimal control systems which are discrete-time analogs of Bolza problems in the calculus of variations. These systems are described by a pair of objective functions which determines an optimality criterion.

In the second part of the paper we study the structure of approximate solutions of autonomous variational problems with a lower semicontinuous extended-valued

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integrand considered in [41, 42, 45, 50, 54, 55, 57, 60] and discuss our recent results obtained for Lagrange and Bolza problems. We show that the turnpike phenomenon holds for approximate solutions of these variational problems on large intervals and study the structure of approximate solutions in regions close to the endpoints of the time intervals.

2. Discrete-time optimal control problems

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \geq 0$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$, where integers T_1, T_2 satisfy $0 \leq T_1 < T_2$, is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [T_1, T_2 - 1]$.

We analyze the problem

$$(P1) \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \ x_0 = z_1, \ x_T = z_2$$

which was considered in [43, 54], the problem

$$(P2) \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \ x_0 = z_1$$

which was studied in [40, 54] and the problem

$$(P3) \qquad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega,$$

considered in [53], where $T \geq 1$ is an integer, $z_1, z_2 \in X$ and the objective functin $v : \Omega \to R^1$ is bounded and upper semicontinuous. These optimal control problems are discrete-time analogs of Lagrange problems in the calculus of variations.

It should be mentioned that these optimal control problems describe a general model of economic dynamics. For this model the set X is the space of states, v is a utility function and $v(x_t, x_{t+1})$ evaluates consumption at moment t. The interest in discrete-time optimal problems of types (P1)-(P3) also stems from the study of various optimization problems which can be reduced to them [3, 23–25, 30, 39]. Optimization problems of the types (P1)-(P3) with $\Omega = X \times X$ were considered in [39].

In [40,43,44,46,47,49,53,54] we analyzed a turnpike phenomenon for the approximate solutions of problems (P1)-(P3) which is independent of the length of the interval T, for all sufficiently large intervals. The turnpike phenomenon holds if the approximate solutions of the optimal control problems are determined mainly by the objective function v, and are essentially independent of T and z_1, z_2. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [37]).

Problems (P1), (P2) and (P3) were analyzed in [40, 43, 53, 54], where we showed, under certain assumptions, that the turnpike property holds and that the turnpike $\bar{x}$ is a unique maximizer of the optimization problem $v(x, x) \to \max$, $(x, x) \in \Omega$. Namely, we considered a collections of (v)-good programs which are approximate solutions of the corresponding infinite horizon optimal control problem associated with the objective function v. It was shown that the turnpike property holds and $\bar{x}$ is the turnpike if the following asymptotic turnpike property holds: the all (v)-good programs converge to $\bar{x}$.

It [53] we showed that the asymptotic turnpike property holds for most cost functions in the sense of Baire category. In other words, the asymptotic turnpike property holds for a generic (typical) cost function.

In this paper we discuss the structure of approximate solutions of the problems $(P2)$ and $(P3)$ in regions close to the endpoints of the time intervals and present the results obtained in [56]. The results of [56] show that in regions close to the right endpoint T of the time interval approximate solutions are determined only by the objective function, and are essentially independent of the choice of interval and endpoint value z_1. For the problems $(P3)$, approximate solutions are determined only by the objective function also in regions close to the left endpoint 0 of the time interval.

More precisely, we define $\bar{\Omega} = \{(y, x) \in X \times X : (x, y) \in \Omega\}$ and $\bar{v}(y, x) = v(x, y)$ for all $(x, y) \in \Omega$ and consider the collection $\mathcal{P}(\bar{v})$ of all solutions of a corresponding infinite horizon optimal control problem associated with the pair $(\bar{v}, \bar{\Omega})$. For given $\epsilon > 0$ and an integer $\tau \geq 1$, we show that if T is large enough and $\{x_t\}_{t=0}^T$ is an approximate solution of the problem $(P2)$, then $\rho(x_{T-t}, y_t) \leq \epsilon$ for all integers $t \in [0, \tau]$, where $\{y_t\}_{t=0}^\infty \in \mathcal{P}(\bar{v})$.

Moreover, using the Baire category approach, we show that for most objective functions v the set $\mathcal{P}(\bar{v})$ is a singleton.

We also discuss the results on the structure of solutions of optimal control systems obtained in [58,59] which are discrete-time analogs of Bolza problems in the calculus of variations. These systems are described by a pair of objective functions which determines an optimality criterion.

3. The turnpike results for discrete-time Lagrange problems

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$. We denote by $\mathcal{M}(\Omega)$ the set of all bounded functions $u : \Omega \to R^1$. For every function $w \in \mathcal{M}(\Omega)$ define

$$\|w\| = \sup\{|w(x, y)| : (x, y) \in \Omega\}.$$

For each $x, y \in X$, each integer $T \geq 1$ and each $u \in \mathcal{M}(\Omega)$ set

$$\sigma(u, T, x) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^T \text{ is an } (\Omega) - \text{program and } x_0 = x\},$$

$$\sigma(u, T, x, y) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) :$$
$$\{x_i\}_{i=0}^T \text{ is an } (\Omega) - \text{program and } x_0 = x,\ x_T = y\},$$

$$\sigma(u, T) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^T \text{ is an } (\Omega) - \text{program}\}.$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

For every pair of points $x, y \in X$, every pair of nonnegative integers T_1, T_2 which satisfies $T_1 < T_2$ and every finite sequence of functions $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$ define

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_1} = x\},$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_1} = x,\ x_{T_2} = y\},$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :\ \{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program}\},$$

$$\widehat{\sigma}(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, y) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_2} = y\}.$$

Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function and that there exist a point $\bar{x}_v \in X$ and real positive constants $\bar{c}_v$ and $\bar{r}_v$ such that the following assumptions hold.

(A1) $\{(x, y) \in X \times X :\ \rho(x, \bar{x}_v),\ \rho(y, \bar{x}_v) \le \bar{r}_v\} \subset \Omega$ and the function v is continuous at the point $(\bar{x}_v, \bar{x}_v)$.

(A2) $\sigma(v, T) \le Tv(\bar{x}_v, \bar{x}_v) + \bar{c}_v$ for all integers $T \ge 1$.

Clearly, for every positive integer T and every (Ω)-program $\{x_t\}_{t=0}^{T}$, we have

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \le \sigma(v, T) \le Tv(\bar{x}_v, \bar{x}_v) + \bar{c}_v.$$

The relation above easily implies the following result.

Proposition 3.1. *For every (Ω)-program $\{x_t\}_{t=0}^{\infty}$ either the sequence*

$$\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v, \bar{x}_v)\}_{T=1}^{\infty}$$

is bounded or $\lim_{T\to\infty}[\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v, \bar{x}_v)] = -\infty$.

We say that an (Ω)-program $\{x_t\}_{t=0}^{\infty}$ is (v, Ω)-good if the sequence

$$\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v, \bar{x}_v)\}_{T=1}^{\infty}$$

is bounded.

We suppose that the following assumption holds:

(A3) (the asymptotic turnpike property or, briefly, (ATP)) For every (v, Ω)-good program $\{x_t\}_{t=0}^{\infty}$, $\lim_{t\to\infty} \rho(x_t, \bar{x}_v) = 0$.

Assumptions (A1) and (A3) imply that $\|v\| > 0$ if $(\bar{x}_v, \bar{x}_v)$ is not an isolated point of the metric space $X \times X$.

Several examples of optimal control problems which satisfy assumptions (A1)-(A3) are discussed in [40, 43, 54].

In this paper we denote by $\mathrm{Card}(A)$ the cardinality of a set A and suppose that the sum over empty set is zero.

It is easy to see that for every pair of nonnegative integers T_1, T_2 for which $T_1 < T_2$, every finite sequence of functions $\{w_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$ and every pair of points $x, y \in X$ for which $\rho(x, \bar{x}_v), \rho(y, \bar{x}_v) \le \bar{r}_v$ the value $\sigma(\{w_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y)$ is finite.

Let T be a positive integer. We denote by Y_T the collection of all $x \in X$ for which there is an (Ω)-program $\{x_t\}_{t=0}^{T}$ such that $x_0 = \bar{x}_v$ and $x_T = x$ and denote by $\bar{Y}_T$ the collection of all points $x \in X$ for which there is an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfying that $x_0 = x$ and $x_T = \bar{x}_v$.

The following two theorems stated below were established in [47]. They show that the turnpike phenomenon holds for approximate solutions of the optimal control problems of the types (P1) and (P2) with objective functions u_t, $t = 0, \dots, T-1$ belonging to a small neighborhood of v.

Theorem 3.2. *Let $\epsilon \in (0, \bar{r}_v)$, L_0 be a positive integer and M_0 be a positive number. Then there exist a natural number L and a positive number $\delta < \epsilon$ such that for every natural number $T > 2L$, every finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ for which*

$$\|u_t - v\| \le \delta,\ t = 0 \dots T-1,$$

and every (Ω)-program $\{x_t\}_{t=0}^{T}$ for which

$$x_0 \in \bar{Y}_{L_0},\ x_T \in Y_{L_0},$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for every nonnegative integer $\tau \le T-L$ there exist integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}_v) \le \epsilon,\ t = \tau_1, \dots, \tau_2.$$

Moreover, if $\rho(x_0, \bar{x}_v) \le \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}_v) \le \delta$, then $\tau_2 = T$.

Theorem 3.3. *Let a positive number $\epsilon < \bar{r}_v$, L_0 be a positive integer and M_0 be a positive number. Then there exist a positive integer L and a positive number $\delta < \epsilon$ such that for every natural number $T > 2L$, every finite sequence $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ for which*

$$\|u_t - v\| \le \delta,\ t = 0 \dots, T-1$$

and every (Ω)*-program* $\{x_t\}_{t=0}^T$ *for which*

$$x_0 \in \bar{Y}_{L_0}, \ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for every integer $\tau \in [0, T-L]$ *there exist a pair of integers* $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ *such that*

$$\rho(x_t, \bar{x}_v) \leq \epsilon, \ t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}_v) \leq \delta$, *then* $\tau_1 = 0$.

The next theorem obtained in [53] establishes the turnpike property for approximate solutions of the optimal control problems of the type (P3).

Theorem 3.4. *Let a positive number* $\epsilon < \bar{r}_v$ *and* M *be a positive number. Then there exist a positive number* $\delta < \min\{1, M\}$ *and a positive integer* L *such that the following assertions hold.*

1. Assume that a natural number $T \geq L$, *a finite sequence of functions* $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ *and an* (Ω)*-program* $\{x_t\}_{t=0}^T$ *satisfy*

$$\|u_t - v\| \leq \delta, \ t = 0, \dots, T-1, \tag{3.1}$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M. \tag{3.2}$$

Then

$$Card(\{t \in \{0, \dots, T\} : \ \rho(x_t, \bar{x}_v) > \epsilon\}) < L.$$

2. Assume that a natural number $T \geq 2L$, *a finite sequence of functions* $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ *and that an* (Ω)*-program* $\{x_t\}_{t=0}^T$ *satisfy (3.1), (3.2) and*

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta. \tag{3.3}$$

Then there exists a pair of integers $\tau_1 \in [0, L]$ *and* $\tau_2 \in [T-L, T]$ *such that*

$$\rho(x_t, \bar{x}_v) \leq \epsilon, \ t = \tau_1, \dots, \tau_2.$$

Moreover, if $\rho(x_0, \bar{x}_v) \leq \delta$, *then* $\tau_1 = 0$ *and if* $\rho(x_T, \bar{x}_v) \leq \delta$, *then* $\tau_2 = T$.

3. Assume that a sequence of functions $\{u_t\}_{t=0}^\infty \subset \mathcal{M}(\Omega)$ *and that an* (Ω)*-program* $\{x_t\}_{t=0}^\infty$ *satisfy*

$$\|u_t - v\| \leq \delta \text{ for all integers } t \geq 0,$$

$$\limsup_{T \to \infty} [\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T)] > -M.$$

Then the inequality

$$Card(\{t \text{ is a nonnegative integer such that } \rho(x_t, \bar{x}_v) > \epsilon\}) < L$$

is true.

4. Assume that T is a positive integer, a finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$, an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfy (3.1), (3.2), (3.3) and that pair of nonnegative integers T_1, T_2 satisfies $T_1 < T_2 \le T$. Then

$$\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - (4L+2)(2\|v\|+2) - M - 1.$$

It is clear that Assertions 1 and 2 establish the turnpike phenomenon for approximate solutions of the problem (P3). Assertion 3 shows that the turnpike phenomenon holds for approximate solutions of the corresponding infinite horizon problem. Moreover, they also show that the turnpike phenomenon is stable under small perturbations of the objective function v.

In this paper we use a well-known notion of an overtaking optimal program [39,54].

We say that an (Ω)-program $\{x_t\}_{t=0}^{\infty}$ is (v,Ω)-overtaking optimal if for every (Ω)-program $\{y_t\}_{t=0}^{\infty}$ for which $y_0 = x_0$ the relation

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} [v(y_t, y_{t+1}) - v(x_t, x_{t+1})] \le 0$$

is valid.

The following result obtained in [40] establishes the existence of an overtaking optimal program.

Theorem 3.5. *Assume that a point $x \in X$ and that there exists a (v,Ω)-good program $\{x_t\}_{t=0}^{\infty}$ for which that $x_0 = x$. Then there exists an (v,Ω)-overtaking optimal program $\{x_t^*\}_{t=0}^{\infty}$ satisfying $x_0^* = x$.*

The following result was proved in [40]. It provides necessary and sufficient conditions for overtaking optimality.

Theorem 3.6. *Assume that $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program and that there exists a (v,Ω)-good program $\{y_t\}_{t=0}^{\infty}$ satisfying that $y_0 = x_0$. Then the program $\{x_t\}_{t=0}^{\infty}$ is (v,Ω)-overtaking optimal if and only if the following conditions hold:*

(i) $\lim_{t\to\infty} \rho(x_t, \bar{x}_v) = 0$;

(ii) for every positive integer T and every (Ω)-program $\{y_t\}_{t=0}^{T}$ for which $y_0 = x_0$, $y_T = x_T$ the relation $\sum_{t=0}^{T-1} v(y_t, y_{t+1}) \le \sum_{t=0}^{T-1} v(x_t, x_{t+1})$ is valid.

The next two results were proved in [56]. They show the uniform convergence of overtaking optimal programs to $\bar{x}_v$.

Theorem 3.7. *Let L_0 be a positive integer and ϵ be a positive number. Then there exists a positive integer T_0 such that for every (v,Ω)-overtaking optimal program $\{x_t\}_{t=0}^{\infty}$ which satisfies $x_0 \in \bar{Y}_{L_0}$ the relation $\rho(x_t, \bar{x}_v) \le \epsilon$ is valid for all natural numbers $t \ge T_0$.*

Theorem 3.8. *Let ϵ be a positive number. Then there exists a positive number δ such that for every (v,Ω)-overtaking optimal program $\{x_t\}_{t=0}^{\infty}$ which satisfies $\rho(x_0, \bar{x}_v) \le \delta$ the inequality $\rho(x_t, \bar{x}_v) \le \epsilon$ is valid for all nonnegative integers t.*

It is easy to see that $(\mathcal{M}(\Omega), \|\cdot\|)$ is a Banach space. We denote by $\mathcal{M}_0(\Omega)$ the collection of all upper semicontinuous functions $v \in \mathcal{M}(\Omega)$ such that there exist a point $\bar{x}_v \in X$ and a pair of numbers $\bar{c}_v > 0$ and $\bar{r}_v \in (0,1)$ such that

$$\{(x,y) \in X \times X : \ \rho(x,\bar{x}_v),\ \rho(y,\bar{x}_v) \le \bar{r}_v\} \subset \Omega,$$
$$v \text{ is continuous at } (\bar{x}_v,\bar{x}_v),$$
$$\sigma(v,T) \le Tv(\bar{x}_v,\bar{x}_v) + \bar{c}_v \text{ for all integers } T \ge 1.$$

In other words, $\mathcal{M}_0(\Omega)$ is the collection of all upper semicontinuous functions $v \in \mathcal{M}(\Omega)$ such that assumptions (A1) and (A2) are true with some $\bar{x}_v \in X$, $\bar{r}_v \in (0,1)$, $\bar{c}_v > 0$. We associate with every function $v \in \mathcal{M}_0(\Omega)$ the triplet $(\bar{x}_v, \bar{c}_v, \bar{r}_v)$.

We denote by $\mathcal{M}_{c,0}(\Omega)$ the collection of all continuous functions $v \in \mathcal{M}_0(\Omega)$ and denote by $\bar{\mathcal{M}}_{c,0}(\Omega)$ and $\bar{\mathcal{M}}_0(\Omega)$ the closure of subspaces $\mathcal{M}_{c,0}(\Omega)$ and $\mathcal{M}_0(\Omega)$ in $\mathcal{M}(\Omega)$, respectively.

The sets $\bar{\mathcal{M}}_{c,0}(\Omega)$ and $\bar{\mathcal{M}}_0(\Omega)$ are equipped with the metric d which induced by the norm $\|\cdot\|$: $d(u_1,u_2) = \|u_1 - u_2\|$, $u_1, u_2 \in \bar{\mathcal{M}}_0(\Omega)$.

For every function $u \in \bar{\mathcal{M}}_0(\Omega)$ and every positive number r define

$$B_d(u,r) = \{w \in \bar{\mathcal{M}}_0(\Omega) : \ \|u - w\| < r\}.$$

We denote by $\mathcal{M}_*(\Omega)$ the collection of all functions $v \in \mathcal{M}_0(\Omega)$ such that for every (v,Ω)-good program $\{x_i\}_{i=0}^\infty$, we have

$$\lim_{i\to\infty} \rho(x_i, \bar{x}_v) = 0.$$

Define

$$\mathcal{M}_{c*}(\Omega) = \mathcal{M}_*(\Omega) \cap \mathcal{M}_c(\Omega).$$

The next result was obtained in [53].

Theorem 3.9. *$\mathcal{M}_*(\Omega)$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_0(\Omega)$ and $\mathcal{M}_{c*}(\Omega)$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_{c,0}(\Omega)$.*

4. Preliminaries

We use the notation, definitions and assumptions introduced in Section 3. Let $v \in \mathcal{M}(\Omega)$ be an upper semicontinuous function. Suppose that $\bar{x}_v \in X$, $\bar{r}_v \in (0,1)$, $\bar{c}_v > 0$ and that assumptions (A1), (A2) and (A3) hold.

For every positive number M we denote by X_M the collection of all points $x \in X$ such that there exists a (Ω)-program $\{x_t\}_{t=0}^\infty$ satisfying $x_0 = x$ and that for all positive integers T the relation

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v, \bar{x}_v) \ge -M$$

is true. It is clear that $\cup\{X_M : \ M \in (0,\infty)\}$ is the collection of all $x \in X$ for which there is a (v,Ω)-good program $\{x_t\}_{t=0}^\infty$ such that $x_0 = x$.

The next proposition follows from the boundedness of the function v.

Proposition 4.1. *Let T be a positive integer. Then there is a positive number M for which $\bar{Y}_T \subset X_M$.*

The next auxiliary proposition is true.

Proposition 4.2. *Let M be a positive number. Then there exists a positive integer T for which $X_M \subset \bar{Y}_T$.*

For every point $x \in X \setminus \cup\{X_M : M \in (0,\infty)\}$ put

$$\pi^v(x) = -\infty.$$

Let

$$x \in \cup\{X_M : M \in (0,\infty)\}.$$

Set

$$\pi^v(x) = \sup\{\limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) :$$

$$\{x_t\}_{t=0}^{\infty} \text{ is an } (\Omega) - \text{program such that } x_0 = x\}.$$

Evidently,

$$-\infty < \pi^v(x) \le \bar{c}_v$$

and

$$\pi^v(x) = \sup\{\limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) :$$

$$\{x_t\}_{t=0}^{\infty} \text{ is an } (v,\Omega) - \text{good program such that } x_0 = x\}.$$

We denote by $\mathcal{P}(v,x)$ the collection of all (v,Ω)-overtaking optimal programs $\{x_t\}_{t=0}^{\infty}$ satisfying $x_0 = x$. Theorem 3.5 implies that the collection $\mathcal{P}(v,x)$ is nonempty.

It should be mentioned that the function $\pi^v(x)$, $x \in X$ which plays an important role in our study of the structure of approximate solutions on large intervals in the regions close to the endpoints.

The next two propositions easily follow from the definitions above.

Proposition 4.3. *1. Let $\{x_t\}_{t=0}^{\infty}$ be a (v,Ω)-good program. Then for every non-negative integer t the inequality*

$$\pi^v(x_t) \ge v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) + \pi^v(x_{t+1})$$

is valid.

2. Let T be a positive integer and $\{x_t\}_{t=0}^{T}$ be an (Ω)-program satisfying $\pi^v(x_T) > -\infty$. Then the inequality

$$\pi^v(x_t) \ge v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) + \pi^v(x_{t+1})$$

holds for all integers $t = 0, \dots, T-1$.

Proposition 4.4. *Let $x \in \cup\{X_M : M \in (0,\infty)\}$ and $\{x_t\}_{t=0}^{\infty}$ be a (v,Ω)-overtaking optimal program satisfying $x_0 = x$. Then the equality*

$$\pi^v(x) = \limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v))$$

is valid.

Corollary 4.5. *Let $\{x_t\}_{t=0}^\infty$ be a (v,Ω)-overtaking optimal and (v,Ω)-good program. Then for every nonnegative integer t the equality*

$$\pi^v(x_t) = v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) + \pi^v(x_{t+1})$$

is valid.

Define

$$\sup(\pi^v) = \sup\{\pi^v(z) : \ z \in \cup\{X_M : \ M \in (0,\infty)\}\},$$
$$X_v = \{x \in \cup\{X_M : \ M \in (0,\infty)\} : \ \pi^v(x) \geq \sup(\pi^v) - 1\}.$$

The following propositions were obtained in [56].

Proposition 4.6. $\pi^v(\bar{x}_v) = 0$.

Proposition 4.7. *The function π^v is finite-valued in a neighborhood of the point $\bar{x}_v$ and continuous at the point $\bar{x}_v$.*

Proposition 4.8. *Let $x_0 \in \cup\{X_M : \ M \in (0,\infty)\}$ and $\{x_t\}_{t=0}^\infty \in \mathcal{P}(v, x_0)$. Then the equality*

$$\pi^v(x_0) = \lim_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v))$$

is valid.

Proposition 4.9. *There exists a positive integer L_v for which $X_v \subset \bar{Y}_{L_v}$.*

Proposition 4.10. *The function $\pi^v : X \to R^1 \cup \{-\infty\}$ is upper semicontinuous.*

Define

$$\mathcal{D}(v) = \{x \in X : \ \pi^v(x) = \sup(\pi^v)\}.$$

It is clear that the set $\mathcal{D}(v)$ is nonempty and closed subset of the metric space X. The next results are obtained in [56].

Proposition 4.11. *Let $\{x_t\}_{t=0}^\infty$ be a (v,Ω)-good program such that for every nonnegative integers t the equality*

$$v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) = \pi^v(x_t) - \pi^v(x_{t+1})$$

is valid. Then $\{x_t\}_{t=0}^\infty$ is a (v,Ω)-overtaking optimal program.

Proposition 4.12. *For every positive number ϵ there is a positive integer T_ϵ such that for every point $z \in \mathcal{D}(v)$ and every (Ω)-program $\{x_t\}_{t=0}^\infty \in \mathcal{P}(v, z)$ the inequality $\rho(x_t, \bar{x}_v) \leq \epsilon$ is true for every natural number $t \geq T_\epsilon$.*

In order to analyze the structure of approximate solutions of the problems (P2) and (P3) on large intervals in the regions close to the endpoints we introduce the following notation and definitions.

Define

$$\bar{\Omega} = \{(x,y) \in X \times X : \ (y,x) \in \Omega\}.$$

Evidently, $\bar{\Omega}$ is a nonempty closed subset of the metric space $X \times X$ and

$$\{(x,y) \in X \times X : \ \rho(x, \bar{x}_v), \ \rho(y, \bar{x}_v) \leq \bar{r}_v\} \subset \bar{\Omega}.$$

Then $\mathcal{M}(\bar{\Omega})$ is the collection of all bounded functions $u : \bar{\Omega} \to R^1$ with

$$\|u\| = \sup\{|u(z)| : \ z \in \bar{\Omega}\}.$$

For every function $u \in \mathcal{M}(\Omega)$ define a functin $\bar{u} \in \mathcal{M}(\bar{\Omega})$ as follows:

$$\bar{u}(x,y) = u(y,x),\ (x,y) \in \bar{\Omega}.$$

Evidently, $u \to \bar{u},\ u \in \mathcal{M}(\Omega)$ is a linear invertible isometry operator.

Let $T_1 < T_2$ be nonnegative integers and let a finite sequence $\{x_t\}_{t=T_1}^{T_2}$ be an (Ω)-program. Define a finite sequence $\{\bar{x}_t\}_{t=T_1}^{T_2} \subset X$ as follows:

$$\bar{x}_t = x_{T_2-t+T_1},\ t = T_1, \dots, T_2.$$

Evidently, $\{\bar{x}_t\}_{t=T_1}^{T_2}$ is an $(\bar{\Omega})$-program.

Assume that a finite sequence of functions $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$. It is clear that

$$\sum_{t=T_1}^{T_2-1} \bar{u}_{T_2-t+T_1-1}(\bar{x}_t, \bar{x}_{t+1}) = \sum_{t=T_1}^{T_2-1} u_{T_2-t+T_1-1}(x_{T_2-t+T_1-1}, x_{T_2-t+T_1})$$

$$= \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}).$$

The equality above implies the following proposition.

Proposition 4.13. *Let T be a natural number, M be a nonnegative number, a finite sequence of functions*

$$\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$$

and let $\{x_t^{(i)}\}_{t=0}^{T}$, $i = 1, 2$ be (Ω)-programs. Then the inequality

$$\sum_{t=0}^{T-1} u_t(x_t^{(1)}, x_{t+1}^{(1)}) \ge \sum_{t=0}^{T-1} u_t(x_t^{(2)}, x_{t+1}^{(2)}) - M$$

is valid if and only if

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t^{(1)}, \bar{x}_{t+1}^{(1)}) \ge \sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t^{(2)}, \bar{x}_{t+1}^{(2)}) - M.$$

The next result easily follows from Proposition 4.13.

Proposition 4.14. *Let T be a positive integer, M be a nonnegative number, a finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and let $\{x_t\}_{t=0}^{T}$ be an (Ω)-program. Then $\{\bar{x}_t\}_{t=0}^{T}$ is an $(\bar{\Omega})$-program and the following assertions are true*

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \ge \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \ge \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_0, \bar{x}_T) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \widehat{\sigma}(\{u_t\}_{t=0}^{T-1}, 0, T, x_T) - M$, *then*

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_0) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M$, *then*

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \widehat{\sigma}(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_T) - M.$$

The following proposition was proved in [56].

Proposition 4.15. *Let* $v \in \mathcal{M}(\Omega)$ *be an upper semicontinuous function. Suppose that* $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ *and that assumptions (A1), (A2) and (A3) hold. Then the function* $\bar{v}$ *is upper semicontinuous,*

$$\{(x, y) \in X \times X : \ \rho(x, \bar{x}_v), \ \rho(y, \bar{x}_v) \leq \bar{r}_v\} \subset \bar{\Omega},$$

the function $\bar{v}$ *is continuous at the point* $(\bar{x}_v, \bar{x}_v)$,

$$\sigma(\bar{v}, T) \leq T\bar{v}(\bar{x}_v, \bar{x}_v) + \bar{c}_v \text{ for all integers } T \geq 1$$

and for every $(\bar{v}, \bar{\Omega})$*-good program* $\{x_t\}_{t=0}^{\infty}$ *the equality*

$$\lim_{t \to \infty} \rho(x_t, \bar{x}_v) = 0$$

holds.

Proposition 4.15 implies that, if a function $v \in \mathcal{M}(\Omega)$ is upper semicontinuous and satisfies assumptions (A1)-(A3), then the function $\bar{v}$ is also upper semicontinuous and satisfies assumptions (A1)-(A3). Therefore all the results stated above for the pair (v, Ω) are also valid for the pair $(\bar{v}, \bar{\Omega})$.

5. Structure of solutions of Lagrange problems in the regions close to the endpoints

We use the notation, definitions and assumptions introduced in Sections 3 and 4. The results of this session were obtained in [56].

The following result describes the structure of approximate solutions of the problems of the type (P2) in the regions close to the right endpoints.

Theorem 5.1. *Suppose that a function* $v \in \mathcal{M}(\Omega)$ *is upper semicontinuous, a point* $\bar{x}_v \in X$, $\bar{r}_v$, $\bar{c}_v$ *are positive numbers and that assumptions (A1), (A2) and (A3) hold. Let* L_0, τ_0 *be positive integers and let* ϵ *be a positive number. Then there exist a positive number* δ *and a natural number* $T_0 \geq \tau_0$ *such that for every natural number* $T \geq T_0$, *every finite sequence of functions* $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ *for which*

$$\|u_t - v\| \leq \delta, \ t = 0 \ldots, T-1$$

and every (Ω)*-program* $\{x_t\}_{t=0}^{T}$ *for which*

$$x_0 \in \bar{Y}_{L_0}, \ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta$$

there exists an $(\bar{\Omega})$*-program*

$$\{x_t^*\}_{t=0}^\infty \in \cup\{\mathcal{P}(\bar{v}, z) : z \in \mathcal{D}(\bar{v})\}$$

which satisfies

$$\rho(x_{T-t}, x_t^*) \le \epsilon, \ t = 0, \dots, \tau_0.$$

Recall that $\cup\{\mathcal{P}(\bar{v}, z) : z \in \mathcal{D}(\bar{v})\}$ is the collection of all $(\bar{v}, \bar{\Omega})$-overtaking optimal programs $\{x_t^*\}_{t=0}^\infty$ for which that x_0^* is the maximizer of the function $\pi^{\bar{v}}$.

The next theorem describes the structure of approximate solutions of the problems of the type (P3) on large intervals in the regions close to the endpoints.

Theorem 5.2. *Suppose that a function* $v \in \mathcal{M}(\Omega)$ *is upper semicontinuous, a point* $\bar{x}_v \in X$, $\bar{r}_v$, $\bar{c}_v$ *are positive numbers and that assumptions (A1), (A2) and (A3) hold. Let* τ_0 *be a positive integer and let* ϵ *be a positive number. Then there exist a positive number* δ *and a natural number* $T_0 \ge \tau_0$ *such that for every natural number* $T \ge T_0$, *every finite sequence of functions* $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ *for which*

$$\|u_t - v\| \le \delta, \ t = 0 \dots, T-1$$

and every (Ω)*-program* $\{x_t\}_{t=0}^T$ *for which*

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - \delta$$

there exist an (Ω)*-program*

$$\{y_t^*\}_{t=0}^\infty \in \cup\{\mathcal{P}(v, z) : z \in \mathcal{D}(v)\}$$

and an $(\bar{\Omega})$*-program*

$$\{x_t^*\}_{t=0}^\infty \in \cup\{\mathcal{P}(\bar{v}, z) : z \in \mathcal{D}(\bar{v})\}$$

which satisfy for all integers $t = 0, \dots, \tau_0$,

$$\rho(x_{T-t}, x_t^*) \le \epsilon, \ \rho(x_t, y_t^*) \le \epsilon.$$

Proposition 5.3. *Suppose that a function* $v \in \mathcal{M}(\Omega)$ *is upper semicontinuous, a point* $\bar{x}_v \in X$, $\bar{r}_v$, $\bar{c}_v$ *are positive numbers and that assumptions (A1), (A2) and (A3) are valid. Let* τ_0 *be a positive integer and let* ϵ *be a positive number. Then there exist a positive number* δ *and a natural number* $T_0 \ge \tau_0$ *such that for every function* $u \in B_d(v, r) \cap \mathcal{M}_*(\Omega)$ *the following assertions are true:*

for every sequence $\{x_t\}_{t=0}^\infty \in \cup\{\mathcal{P}(u, z) : z \in \mathcal{D}(u)\}$ *there exists a sequence*

$$\{y_t\}_{t=0}^\infty \in \cup\{\mathcal{P}(v, z) : z \in \mathcal{D}(v)\}$$

such that the inequality $\rho(x_t, y_t) \le \epsilon$ *holds for all* $t = 0, \dots, \tau_0$*;*

for every sequence $\{x_t\}_{t=0}^\infty \in \cup\{\mathcal{P}(\bar{u}, z) : z \in \mathcal{D}(\bar{u})\}$ *there exists a sequence*

$$\{y_t\}_{t=0}^\infty \in \cup\{\mathcal{P}(\bar{v}, z) : z \in \mathcal{D}(\bar{v})\}$$

such that the inequality $\rho(x_t, y_t) \le \epsilon$ *is valid for all integers* $t = 0, \dots, \tau_0$.

Note that the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is a linear isometry which has the inverse. It is clear that

$$\bar{v} \in \mathcal{M}_0(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_0(\Omega),$$

$$\bar{v} \in \mathcal{M}_{c,0}(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_{c,0}(\Omega),$$

$$\bar{v} \in \mathcal{M}_*(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_*(\Omega),$$
$$\bar{v} \in \mathcal{M}_{c*}(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_{c*}(\Omega),$$
$$\bar{v} \in \bar{\mathcal{M}}_0(\bar{\Omega}) \text{ if and only if } v \in \bar{\mathcal{M}}_0(\Omega),$$
$$\bar{v} \in \bar{\mathcal{M}}_{c,0}(\bar{\Omega}) \text{ if and only if } v \in \bar{\mathcal{M}}_{c,0}(\Omega).$$

The following generic result establishes that for most objective functions v the collections $\cup\{\mathcal{P}(v,z) : z \in \mathcal{D}(v)\}$ and $\cup\{\mathcal{P}(\bar{v},z) : z \in \mathcal{D}(\bar{v})\}$ are singletons. In this case approximate solutions of the problems of the types (P2) and (P3) on large intervals have a simple structure in the regions close to the endpoints.

Theorem 5.4. *Let $\mathfrak{M}$ be either $\bar{\mathcal{M}}_0(\Omega)$ or $\bar{\mathcal{M}}_{c,0}(\Omega)$. Then there exists a subset $\mathcal{F} \subset \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for every function $v \in \mathcal{F}$ there exists a unique pair of points $z, \bar{z} \in X$ for which*

$$\pi^v(z) = \sup(\pi^v),\ \pi^{\bar{v}}(\bar{z}) = \sup(\pi^{\bar{v}})$$

and there exist a unique (v,Ω)-overtaking optimal program $\{z_t\}_{t=0}^\infty$ for which $z_0 = z$ and a unique $(\bar{v},\bar{\Omega})$-overtaking optimal program $\{\widehat{z}_t\}_{t=0}^\infty$ for which $\widehat{z}_0 = \bar{z}$.

6. The first class of discrete-time Bolza problems

We use the notation, definitions and assumptions introduced in Sections 3, 4 and 5.

For every nonempty set Y and every function $h : Y \to R^1 \cup \{-\infty\}$ set

$$\sup(h) = \sup\{h(y) :\ y \in Y\}.$$

We denote by $\mathcal{M}(X)$ the collection of all bounded functions $h : X \to R^1$. For every function $h \in \mathcal{M}(X)$ put

$$\|h\| = \sup\{|h(x)| :\ x \in X\}.$$

It is clear that $(\mathcal{M}(X), \|\cdot\|)$ is a Banach space. For every pair of functions $h_1, h_2 \in \mathcal{M}(X)$ put

$$d_X(h_1,h_2) = \|h_1 - h_2\|.$$

For every point $x \in X$, every pair of nonnegative integers $T_1 < T_2$, every finite sequence of functions $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$ and every function $h \in \mathcal{M}(X)$ we consider the problem

$$(P4) \qquad \sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) + h(x_{T_2}) \to \max,\ \{(x_t,x_{t+1})\}_{t=0}^{T-1} \subset \Omega,\ x_{T_1} = x$$

and define

$$\sigma(h, \{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t,x_{t+1}) + h(x_{T_2}) :$$
$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega)-\text{program and } x_{T_1} = x\}.$$

For every point $x \in X$, every pair of nonnegative integers $T_1 < T_2$, every function $u \in \mathcal{M}(\Omega)$ and every function $h \in \mathcal{M}(X)$ define

$$\sigma(h, u, T_1, T_2, x) = \sigma(h, \{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) \text{ where } u_t = u,\ t = T_1, \dots, T_2 - 1.$$

In this section we analyze the structure of approximate solutions of problems of the type (P4) on large intervals and present the results obtained in [59]. The next three results establish the turnpike properties of the approximate solutions.

Theorem 6.1. *Let $\epsilon \in (0, \bar{r}_v)$ and M be a positive number. Then there are a positive number $\delta < \min\{1, M\}$ and a positive integer L such that for every natural number $T \geq L$, every finite sequence $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$, every function $h \in \mathcal{M}(X)$ and every (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfying*

$$\|h\| \leq M, \ \|u_t - v\| \leq \delta, \ t = 0, \dots, T-1,$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) + h(x_T) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T) - M$$

the inequality

$$Card(\{t \in \{0, \dots, T\} : \ \rho(x_t, \bar{x}_v) > \epsilon\}) < L$$

is valid.

Theorem 6.2. *Let a positive number $\epsilon < \bar{r}_v$, L_0 be a natural number integer and M_0 be a positive number. Then there exist a positive integer L and a positive number $\delta < \epsilon$ such that for every natural number $T > 2L$, every sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ for which*

$$\|u_t - v\| \leq \delta, \ t = 0 \dots, T-1,$$

every function $h \in \mathcal{M}(X)$ for which

$$\|h\| \leq M_0$$

and every (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfying

$$x_0 \in \bar{Y}_{L_0},$$

$$h(x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for every integer $\tau \in [0, T-L]$ there exists a pair of integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ for which

$$\rho(x_t, \bar{x}_v) \leq \epsilon, \ t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}_v) \leq \delta$, then $\tau_1 = 0$.

The next turnpike result easily follows from Theorem 6.2.

Theorem 6.3. *Let a positive number $\epsilon < \bar{r}_v$, $L_0 \geq 1$ be an integer and $M_0 > 0$. Then there exist an integer $L \geq 1$ and a number $\delta \in (0, \epsilon)$ such that for each integer $T > 2L$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying*

$$\|u_t - v\| \leq \delta, \ t = 0 \dots, T-1,$$

each $h \in \mathcal{M}(X)$ satisfying

$$\|h\| \leq M_0$$

and each (Ω)*-program* $\{x_t\}_{t=0}^{T}$ *which satisfies*

$$x_0 \in \bar{Y}_{L_0},$$

$$h(x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta$$

there exist integers $\tau_1 \in [0, L]$, $\tau_2 \in [T - L, T]$ *such that*

$$\rho(x_t, \bar{x}_v) \leq \epsilon, \ t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}_v) \leq \delta$, *then* $\tau_1 = 0$.

The next theorem describes the structure of approximate solutions of the problems of the type (P4) on large intervals in the regions close to the right endpoints.

Theorem 6.4. *Suppose that a function* $g \in \mathcal{M}(X)$ *and a function* $v \in \mathcal{M}(\Omega)$ *are upper semicontinuous, a point* $\bar{x}_v \in X$, *numbers* $\bar{r}_v$, $\bar{c}_v$ *are positive and that assumptions (A1), (A2) and (A3) are valid. Let* L_0, τ_0 *be natural numbers and* $\epsilon > 0$, $M_0 > 1$. *Then there exist a positive number* δ *and a natural number* $T_0 \geq \tau_0$ *such that for every natural number* $T \geq T_0$, *every function* $h \in \mathcal{M}(X)$ *for which*

$$\|h - g\| \leq \delta,$$

every sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ *for which*

$$\|u_t - v\| \leq \delta, \ t = 0 \dots, T - 1$$

and every (Ω)*-program* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$x_0 \in \bar{Y}_{L_0},$$

$$h(x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0,$$

$$\sum_{t=\tau}^{\tau+T_0-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+T_0-1}, \tau, \tau + T_0, x_\tau, x_{\tau+T_0}) - \delta$$

for every integer $\tau \in \{0, \dots, T - T_0\}$,

$$h(x_T) + \sum_{t=T-T_0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=T-T_0}^{T-1}, T - T_0, T, x_{T-T_0}) - \delta$$

there exists an $(\bar{v}, \bar{\Omega})$*-overtaking optimal program* $\{x_t^*\}_{t=0}^{\infty}$ *which satisfies*

$$(\pi^{\bar{v}} + g)(x_0^*) = \sup(\pi^{\bar{v}} + g),$$

$$\rho(x_{T-t}, x_t^*) \leq \epsilon, \ t = 0, \dots, \tau_0.$$

Let $g \in \mathcal{M}(X)$ and $v \in \mathcal{M}(\Omega)$ be as in the statement of Theorem 6.4 and satisfy all the assumptions posed there. Then the function $\pi^v + g : X \to R^1 \cup \{-\infty\}$ is upper semicontinuous, bounded from above for which $(\pi^v + g)(\bar{x}_v) = g(\bar{x}_v)$ is finite. Hence there is a point $x \in X$ which satisfies

$$(\pi^v + g)(x) = \sup(\pi^v + g).$$

We denote by $\mathcal{M}_u(X)$ the collection of all upper semicontinuous functions belonging to the set $\mathcal{M}(X)$ and denote by $\mathcal{M}_c(X)$ the collection of all continuous functions belonging to the set $\mathcal{M}(X)$. It is clear that $\mathcal{M}_u(X)$ and $\mathcal{M}_c(X)$ are closed subsets of the topological space $\mathcal{M}(X)$. We consider the complete metric spaces $\mathcal{M}_u(X)$ and $\mathcal{M}_c(X)$ equipped with the metric d_X.

In order to state our next result we need the following notion of porosity [51].

Let (Y, d) be a complete metric space. Denote by $B_Y(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. We say that a set $E \subset Y$ is porous (with respect to d) if there exist constants $\alpha \in (0, 1]$ and $r_0 > 0$ such that for every number $r \in (0, r_0]$ and every point $y \in Y$ there exists a point $z \in Y$ such that

$$B_Y(z, \alpha r) \subset B_Y(y, r) \setminus E.$$

We say that a subset of the space Y is σ-porous (with respect to d) if it is a countable union of porous (with respect to d) subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category. To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_Y(z, s) \subset B_Y(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

The discussion of the porosity notion and the corresponding references can be found in [51]. Theorem 5.9 of [51] and Theorem 6.4 imply the following result.

Theorem 6.5. *Suppose that a function $v \in \mathcal{M}(\Omega)$ is upper semicontinuous, a point $\bar{x}_v \in X$, numbers $\bar{r}_v$, $\bar{c}_v$ are positive and that assumptions (A1), (A2) and (A3) hold. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c(X)$ such that the set $\mathcal{M}_c(X) \setminus \mathcal{F}$ is σ-porous in $\mathcal{M}_c(X)$ and that for every function $g \in \mathcal{F}$ the following assertions hold.*

1. There exists a unique point $x_g \in X$ which satisfies

$$\{x \in X :\ (\pi^{\bar{v}} + g)(x) = \sup(\pi^{\bar{v}} + g)\} = \{x_g\}.$$

2. Let L_0, τ_0 be natural numbers and $\epsilon > 0$, $M_0 > 1$. Then there exist a positive number δ and a natural number $T_0 \geq \tau_0$ such that for every natural number $T \geq T_0$, every function $h \in \mathcal{M}(X)$ for which

$$\|h - g\| \leq \delta,$$

every finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ such that

$$\|u_t - v\| \leq \delta,\ t = 0 \ldots, T - 1$$

and every (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfying

$$x_0 \in \bar{Y}_{L_0},$$

$$h(x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0,$$

$$\sum_{t=\tau}^{\tau+T_0-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+T_0-1}, \tau, \tau + T_0, x_\tau, x_{\tau+T_0}) - \delta$$

for every integer $\tau \in \{0, \dots, T - T_0\}$,

$$h(x_T) + \sum_{t=T-T_0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(h, \{u_t\}_{t=T-T_0}^{T-1}, T - T_0, T, x_{T-T_0}) - \delta$$

there exists an $(\bar{v}, \bar{\Omega})$*-overtaking optimal program* $\{x_t^*\}_{t=0}^{\infty}$ *which satisfies*

$$x_0^* = x_g,$$
$$\rho(x_{T-t}, x_t^*) \le \epsilon,\ t = 0, \dots, \tau_0.$$

The following result establishes that given a function $g \in \mathcal{M}_u(X)$, for a generic objective function v (in the sense of the Baire category) the exists a unique pair of a (v, Ω)-overtaking optimal program $\{z_t\}_{t=0}^{\infty}$ and a $(\bar{v}, \bar{\Omega})$-overtaking optimal program $\{\widehat{z}_t\}_{t=0}^{\infty}$ which satisfy

$$(\pi^v + g)(z_0) = \sup(\pi^v + g),$$
$$(\pi^{\bar{v}} + g)(\widehat{z}_0) = \sup(\pi^{\bar{v}} + g).$$

In this case approximate solutions of the problems of the types (P4) in the regions close to the right endpoints have a simple structure.

Theorem 6.6. *Let* $\mathfrak{M}$ *be either* $\bar{\mathcal{M}}_0(\Omega)$ *or* $\bar{\mathcal{M}}_{c,0}(\Omega)$ *and let* $g \in \mathcal{M}_u(X)$. *Then there exists a set* $\mathcal{F} \subset \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ *which is a countable intersection of open everywhere dense subsets of* $\mathfrak{M}$ *such that for every function* $v \in \mathcal{F}$ *there exists a unique pair of points* $z, \bar{z} \in X$ *such that*

$$(g + \pi^v)(z) = \sup(g + \pi^v),\ (g + \pi^{\bar{v}})(\bar{z}) = \sup(g + \pi^{\bar{v}})$$

and there exist a unique (v, Ω)*-overtaking optimal program* $\{z_t\}_{t=0}^{\infty}$ *such that* $z_0 = z$ *and a unique* $(\bar{v}, \bar{\Omega})$*-overtaking optimal program* $\{\widehat{z}_t\}_{t=0}^{\infty}$ *such that* $\widehat{z}_0 = \bar{z}$.

Let $v \in \mathcal{M}_*(\Omega)$ and $g \in \mathcal{M}_u(X)$. Define

$$\mathcal{D}(g, v) = \{z \in X :\ (g + \pi^v)(z) = \sup(g + \pi^v)\}$$

and denote by $\tilde{\mathcal{P}}(g, v)$ the collection of all (v, Ω)-overtaking optimal programs $\{z_t\}_{t=0}^{\infty}$ such that $z_0 \in \mathcal{D}(g, v)$.

Proposition 6.7. *Suppose that a function* $g \in \mathcal{M}_u(X)$ *and that a function* $v \in \mathcal{M}(\Omega)$ *are upper semicontinuous, a point* $\bar{x}_v \in X$, *numbers* $\bar{r}_v$, $\bar{c}_v$ *are positive and that assumptions (A1), (A2) and (A3) are valid. Let* τ_0 *be a natural number and* ϵ *be a positive number. Then there exist a positive number* δ *and a natural number* $T_0 \ge \tau_0$ *such that for every function* $u \in B_d(v, r) \cap \mathcal{M}_*(\Omega)$ *and every function* $h \in \mathcal{M}_u(X)$ *for which* $\|h - g\| \le \delta$ *the following properties are true:*

for every sequence $\{x_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(h, u)$ *there exists a sequence* $\{y_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(g, v)$ *which satisfies*

$$\rho(x_t, y_t) \le \epsilon$$

for all integers $t = 0, \dots, \tau_0$;

for every sequence $\{x_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(h, \bar{u})$ *there exists a sequence* $\{y_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(g, \bar{v})$ *which satisfies*

$$\rho(x_t, y_t) \le \epsilon$$

for all integers $t = 0, \dots, \tau_0$.

The following theorem is extension of Theorem 6.6.

Theorem 6.8. *Let $\mathfrak{M}$ be either $\bar{\mathcal{M}}_0(\Omega)$ or $\bar{\mathcal{M}}_{c,0}(\Omega)$ and $\mathfrak{A}$ be either $\mathcal{M}_u(X)$ or $\mathcal{M}_c(X)$. Then there exists a set $\mathcal{F} \subset (\mathfrak{M} \cap \mathcal{M}_*(\Omega)) \times \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M} \times \mathfrak{A}$ such that for every pair of functions $(v, g) \in \mathcal{F}$ there exists a unique pair of sequences $\{x_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(g, v)$ and $\{\bar{x}_t\}_{t=0}^{\infty} \in \tilde{\mathcal{P}}(\bar{g}, v)$.*

7. The second class of discrete-time Bolza problems

We use the notation, definitions and assumptions introduced in Sections 3, 4 and 5 and analyze the following problem

$$(P5) \qquad g(x_0, x_T) + \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega,$$

where T is a positive integer number and $v : \Omega \to R^1$ and $g : X \times X \to R^1$ are bounded upper semicontinuous objective functions.

For every nonempty set Y, every nonempty subset $C \subset Y$ and every function $h : Y \to R^1 \cup \{-\infty\}$ define

$$\sup(h) = \sup\{h(y) : \ y \in Y\}, \ \sup(h; C) = \sup\{h(y) : \ y \in C\}.$$

If (X_i, ρ_i), $i = 1, 2$ are metric spaces, then the product $X_1 \times X_2$ is equipped with the metric

$$\rho_1(x_1, y_1) + \rho_2(x_2, y_2) \text{ for all } (x_1, x_2), \ (y_1, y_2) \in X_1 \times X_2.$$

We denote by $\mathcal{M}(X \times X)$ the collection of all bounded functions $h : X \times X \to R^1$. For every function $h \in \mathcal{M}(X \times X)$ put

$$\|h\| = \sup\{|h(x, y)| : \ x, y \in X\}.$$

It is not difficult to see that $(\mathcal{M}(X \times X), \|\cdot\|)$ is a Banach space.

For every pair of nonnegative integers $T_1 < T_2$, every finite sequence of functions $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$ and every function $h \in \mathcal{M}(X \times X)$ we study the problem

$$\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + h(x_{T_1}, x_{T_2}) \to \max, \ \{(x_t, x_{t+1})\}_{t=T_1}^{T_2-1} \subset \Omega$$

and define

$$\sigma(h, \{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + h(x_{T_1}, x_{T_2}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega)-\text{program}\}.$$

For every pair of nonnegative integers $T_1 < T_2$, every function $u \in \mathcal{M}(\Omega)$ and every function $h \in \mathcal{M}(X \times X)$ define

$$\sigma(h, u, T_1, T_2) = \sigma(h, \{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) \text{ where } u_t = u, \ t = T_1, \dots, T_2 - 1.$$

We denote by $\mathcal{M}_u(X \times X)$ the collection of all upper semicontinuous functions belonging to the set $\mathcal{M}(X \times X)$ and denote by $\mathcal{M}_c(X \times X)$ the collection of all continuous functions belonging to the set $\mathcal{M}(X \times X)$. We consider the complete metric spaces $\mathcal{M}_u(X \times X)$ and $\mathcal{M}_c(X \times X)$ equipped with the metric

$$d_{X\times X}(h_1, h_2) = \|h_1 - h_2\|, \ h_1, h_2 \in \mathcal{M}_u(X \times X).$$

The next result obtained in [58] shows that the turnpike phenomenon holds for approximate solutions of problems (P5) on large intervals.

Theorem 7.1. *Suppose that a function $v \in \mathcal{M}_*(\Omega)$ is an upper semicontinuous, a point $\bar{x}_v \in X$, real numbers $\bar{r}_v \in (0,1)$, $\bar{c}_v > 0$ and that assumptions (A1), (A2) and (A3) hold. Let $\epsilon \in (0, \bar{r}_v)$ and M be a positive number. Then there exist a natural number L and a positive number $\delta < \epsilon$ such that for every natural number $T > 2L$, every sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ for which*

$$\|u_t - v\| \le \delta, \ t = 0 \ldots, T-1,$$

every function $h \in \mathcal{M}(X \times X)$ for which

$$\|h\| \le M$$

and every (Ω)-program $\{x_t\}_{t=0}^{T}$ such that

$$h(x_0, x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T) - M,$$

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for every integer $\tau \in [0, T-L]$ there exists a pair of integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}_v) \le \epsilon, \ t = \tau_1, \ldots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}_v) \le \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}_v) \le \delta$, then $\tau_2 = T$.

It is easy to see that for every function $v \in \mathcal{M}_*(\Omega)$ and every function $g \in \mathcal{M}_u(X \times X)$, the function

$$(\xi, \eta) \in \pi^v(\xi) + \pi^{\bar{v}}(\eta) + g(\xi, \eta), \ \xi, \eta \in X$$

is upper semicontinuous and bounded from above and has a maximizer.

The following theorem obtained in [58], describes the structure of approximate solutions of the problems (P5) on large intervals in the regions close to the right endpoints.

Theorem 7.2. *Suppose that a function $g \in \mathcal{M}(X \times X)$ and a function $v \in \mathcal{M}(\Omega)$ are upper semicontinuous, a point $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2) and (A3) are valid.. Let τ_0 be a natural number and ϵ be a positive number. Then there exist a positive number δ and a natural number $T_0 \ge \tau_0$ such that for every natural number $T \ge T_0$, every function $h \in \mathcal{M}(X \times X)$ for which $\|h - g\| \le \delta$, every finite sequence of function $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ such that*

$$\|u_t - v\| \le \delta, \ t = 0 \ldots, T-1$$

and every (Ω)*-program* $\{x_t\}_{t=0}^T$ *for which*

$$h(x_0, x_T) + \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(h, \{u_t\}_{t=0}^{T-1}, 0, T) - \delta$$

there exists an (v, Ω)*-overtaking optimal and* (v, Ω)*-good program* $\{x_t^*\}_{t=0}^\infty$ *and an* $(\bar{v}, \bar{\Omega})$*-overtaking optimal and* $(\bar{v}, \bar{\Omega})$*-good program* $\{\bar{x}_t^*\}_{t=0}^\infty$ *which satisfy*

$$\pi^v(x_0^*) + \pi^{\bar{v}}(\bar{x}_0^*) + g(x_0^*, \bar{x}_0^*) \geq \pi^v(\xi) + \pi^{\bar{v}}(\eta) + g(\xi, \eta) \text{ for all } \xi, \eta \in X$$

and for all $t = 0, \dots, \tau_0$, $\rho(x_t, x_t^*) \leq \epsilon$ *and* $\rho(x_{T-t}, \bar{x}_t^*) \leq \epsilon$.

Let $v \in \mathcal{M}_*(\Omega)$ and $g \in \mathcal{M}_u(X \times X)$ be given. We denote by $\mathcal{L}(g, v)$ the collection of all pairs of sequences $\{x_t\}_{t=0}^\infty$, $\{\bar{x}_t\}_{t=0}^\infty \subset X$ such that $\{x_t\}_{t=0}^\infty$ is a (v, Ω)-overtaking optimal and (v, Ω)-good program and $\{\bar{x}_t\}_{t=0}^\infty$ is a $(\bar{v}, \bar{\Omega})$-overtaking optimal and $(\bar{v}, \bar{\Omega})$-good program satisfying

$$\pi^v(x_0) + \pi^{\bar{v}}(\bar{x}_0) + g(x_0, \bar{x}_0) \geq \pi^v(\xi) + \pi^{\bar{v}}(\eta) + g(\xi, \eta) \text{ for all } \xi, \eta \in X.$$

It is clear the set $\mathcal{L}(g, v)$ is nonempty.

The next stability theorem was obtained in [58].

Theorem 7.3. *Suppose that a function* $g \in \mathcal{M}_u(X \times X)$*, a function* $v \in \mathcal{M}(\Omega)$ *is upper semicontinuous, a point* $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ *and that assumptions (A1), (A2) and (A3) hold. Let* τ_0 *be a natural number and* ϵ *be a positive number. Then there exists a positive number* δ *such that for every function* $u \in B_d(v, \delta) \cap \mathcal{M}_*(\Omega)$ *and every function* $h \in \mathcal{M}_u(X \times X)$ *for which* $\|h - g\| \leq \delta$ *the following assertion is valid:*

for every pair of sequence $(\{x_t\}_{t=0}^\infty, \{y_t\}_{t=0}^\infty) \in \mathcal{L}(h, u)$ *there exists a pair of sequences* $(\{x_t^*\}_{t=0}^\infty, \{y_t^*\}_{t=0}^\infty) \in \mathcal{L}(g, v)$ *which satisfies for all integers* $t = 0, \dots, \tau_0$,

$$\rho(x_t, x_t^*) \leq \epsilon, \; \rho(y_t, y_t^*) \leq \epsilon.$$

The following result obtained in [58] shows, for a typical (in the sense of the Baire category) pair of funnctions (v, g), that the set $\mathcal{L}(g, v)$ is a singleton. In this case approximate solutions of problems (P5) on large intervals have a simple structure in the regions close to the endpoints.

Theorem 7.4. *Let* $\mathfrak{M}$ *be either* $\bar{\mathcal{M}}_0(\Omega)$ *or* $\bar{\mathcal{M}}_{c,0}(\Omega)$ *and let* $\mathfrak{A}$ *be either* $\mathcal{M}_u(X \times X)$ *or* $\mathcal{M}_c(X \times X)$*. Then there exists a set* $\mathcal{F} \subset (\mathfrak{M} \cap \mathcal{M}_*(\Omega)) \times \mathfrak{A}$ *which is a countable intersection of open everywhere dense subsets of* $\mathfrak{M} \times \mathfrak{A}$ *such that for each* $(v, g) \in \mathcal{F}$ *the set* $\mathcal{L}(g, v)$ *is a singleton.*

8. Variational problems with extended-valued integrands

We analyze the following variational problems with extended-valued integrands:

$$(P_1) \qquad \int_0^T f(v(t), v'(t))dt \to \min,$$

$v : [0, T] \to R^n$ is an absolutely continuous (a. c.) function such that

$$v(0) = x, \; v(T) = y;$$

$$(P_2) \qquad \int_0^T f(v(t), v'(t))dt \to \min,$$

$$v : [0, T] \to R^n \text{ is an a. c. function such that } v(0) = x;$$

$$(P_3) \qquad \int_0^T f(v(t), v'(t))dt \to \min$$

$$v : [0, T] \to R^n \text{ is an a. c. function;}$$

$$(P_4) \qquad \int_0^T f(v(t), v'(t))dt + g(v(T)) \to \min,$$

$$v : [0, T] \to R^n \text{ is an a. c. function such that } v(0) = x;$$

$$(P_5) \qquad \int_0^T f(v(t), v'(t))dt + h(v(0), v(T)) \to \min,$$

$$v : [0, T] \to R^n \text{ is an a. c. function,}$$

where the points $x, y \in R^n$ and T is a positive number. Here R^n is the n-dimensional Euclidean space with the Euclidean norm $|\cdot|$, $f : R^n \times R^n \to R^1 \cup \{\infty\}$ is an extended-valued and lower semicontinuous integrand and $g : R^n \to R^1$ and $h : R^n \times R^n \to R^1$ are lower semicontinuous functions which are bounded on bounded sets. We are interested in the structure of approximate solutions of these problems considered on large intervals which was studied in [41, 42, 45, 50, 54, 57, 60]. We discuss our recent results obtained for Lagrange problems $(P_1) - (P_3)$ and their extension of these results for Bolza problems (P_4) and (P_5). Our results provide a full description of the structure of approximate solutions of variational problems on large intervals.

In [41,42,45,50] we studied the turnpike phenomenon for approximate solutions of problems (P_1) and (P_2) which is independent of the length of the interval T, for all sufficiently large intervals. To have this phenomenon means that the approximate solutions are determined mainly by the integrand, and are essentially independent of the choice of time interval and data, except in regions close to the endpoints of the time interval.

The problems (P_1) and (P_2) were studied in [41,45] where it was established, under certain assumptions, that the turnpike phenomenon holds and that the turnpike $\bar{x}$ is a unique minimizer of the optimization problem $f(x, 0) \to \min$, $x \in R^n$.

The structure of approximate solutions of the problems (P_2) and (P_3), in regions close to the endpoints of the time intervals, was analyzed in [57]. It was shown there that in regions close to the right endpoint T of the time interval these approximate solutions are determined only by the integrand, and are essentially independent of the choice of interval and endpoint value x. For the problems (P_3), approximate solutions are determined only by the integrand function also in regions close to the left endpoint 0 of the time interval. In [60] we extend these results to approximate solutions of problems (P_4) and (P_5).

In this section we discuss the turnpike results for problems $(P_1) - (P_3)$.

We denote by $\text{mes}(E)$ the Lebesgue measure of a Lebesgue measurable set $E \subset R^1$, denote by $|\cdot|$ the Euclidean norm of the n-dimensional Euclidean space R^n and by $\langle \cdot, \cdot \rangle$ the inner product of R^n. For every function $f : X \to R^1 \cup \{\infty\}$, where the set X is a nonempty, define

$$\mathrm{dom}(f) = \{x \in X : \; f(x) < \infty\}.$$

Let a be a positive number, $\psi : [0, \infty) \to [0, \infty)$ be an increasing function satisfying

$$\lim_{t \to \infty} \psi(t) = \infty$$

and let $f : R^n \times R^n \to R^1 \cup \{\infty\}$ be a lower semicontinuous function such that the set

$$\mathrm{dom}(f) = \{(x, y) \in R^n \times R^n : \; f(x, y) < \infty\}$$

is nonempty, convex and closed and which satisfies

$$f(x, y) \geq \max\{\psi(|x|), \; \psi(|y|)|y|\} - a \text{ for each } x, y \in R^n.$$

We say that a function v defined on an infinite subinterval of R^1 with values in R^n is absolutely continuous (a. c.) if v is absolutely continuous on every finite subinterval of its domain.

For every pair of points $x, y \in R^n$ and every positive number T set

$$\sigma(f, T, x) = \inf\{\int_0^T f(v(t), v'(t))dt : \; v : [0, T] \to R^n$$
$$\text{is an a. c. function satisfying } v(0) = x\},$$

$$\sigma(f, T, x, y) = \inf\{\int_0^T f(v(t), v'(t))dt : \; v : [0, T] \to R^n$$
$$\text{is an a. c. function satisfying } v(0) = x, \; v(T) = y\},$$

$$\sigma(f, T) = \inf\{\int_0^T f(v(t), v'(t))dt : \; v : [0, T] \to R^n \text{ is an a. c. function}\},$$

$$\widehat{\sigma}(f, T, y) = \inf\{\int_0^T f(v(t), v'(t))dt : \; v : [0, T] \to R^n$$
$$\text{is an a. c. function satisfying } v(T) = y\}.$$

(Here we assume that infimum over an empty set is infinity.)

We suppose that there exists a point $\bar{x} \in R^n$ such that

$$f(\bar{x}, 0) \leq f(x, 0) \text{ for each } x \in R^n$$

and that the following assumptions hold:

(A1) $(\bar{x}, 0)$ is an interior point of the set $\mathrm{dom}(f)$ and the function f is continuous at the point $(\bar{x}, 0)$;

(A2) for every positive number M there exists a positive number c_M which satisfies

$$\sigma(f, T, x) \geq T f(\bar{x}, 0) - c_M$$

for every point $x \in R^n$ such that $|x| \leq M$ and every positive number $T > 0$;

(A3) for every point $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1 \cup \{\infty\}$ is convex.

It follows from assumption (A2) that for every a.c. function $v : [0, \infty) \to R^n$ the function

$$T \to \int_0^T f(v(t), v'(t))dt - T f(\bar{x}, 0), \; T \in (0, \infty)$$

is bounded from below.

It should be mentioned that the inequality above and assumptions (A1)-(A3) are common in the literature and hold for many infinite horizon optimal control problems. In particular, we need this inequality and assumption (A2) in the cases when the problems (P_1) and (P_2) possess the turnpike property and the point $\bar{x}$ is its turnpike. Assumption (A2) means that the constant function $\bar{v}(t) = \bar{x}$, $t \in [0, \infty)$ is an approximate solution of the infinite horizon variational problem with the integrand f related to the problems (P_1) and (P_2).

An a. c. function $v : [0, \infty) \to R^n$ is called (f)-good [39, 54] if

$$\sup\{|\int_0^T f(v(t), v'(t))dt - Tf(\bar{x}, 0)| : \ T \in (0, \infty)\} < \infty.$$

In our study we use the next proposition which is proved in [41] (see also Proposition 3.1 of [54]).

Proposition 8.1. *Let $v : [0, \infty) \to R^n$ be an a. c. function. Then either the function v is (f)-good or*

$$\int_0^T f(v(t), v'(t))dt - Tf(\bar{x}, 0) \to \infty \text{ as } T \to \infty.$$

Moreover, if the function v is (f)-good, then $\sup\{|v(t)| : \ t \in [0, \infty)\} < \infty$.

For every pair of real numbers $T_2 > T_1$ and every a. c. function $v : [T_1, T_2] \to R^n$ set

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t))dt$$

and for every number $T \in [T_1, T_2]$ put $I^f(T, T, v) = 0$.

For every positive real number M denote by $X_{M,f}$ the collection of all points $x \in R^n$ for which $|x| \leq M$ and there exists an a. c. function $v : [0, \infty) \to R^n$ such that

$$v(0) = x, \ I^f(0, T, v) - Tf(\bar{x}, 0) \leq M \text{ for each } T \in (0, \infty).$$

It is clear that $\cup\{X_{M,f} : \ M \in (0, \infty)\}$ is the collection of all $x \in X$ for which there exists an (f)-good function $v : [0, \infty) \to R^n$ such that $v(0) = x$.

We suppose that the following assumption holds:

(A4) (the asymptotic turnpike property) for every (f)-good function $v : [0, \infty) \to R^n$ the equality $\lim_{t\to\infty} |v(t) - \bar{x}| = 0$ is true.

Examples of extended-valued integrands f satisfying assumptions (A1)-(A4) can be found in [41, 54].

The next theorem which was proved in [41] (see also Theorem 3.2 of [54]) establishes the turnpike property for approximate solutions of problem (P_2) on large intervals.

Theorem 8.2. *Let ϵ, M be positive numbers. Then there exist a natural number L and a positive number δ such that for every $T > 2L$ and every a. c. function $v : [0, T] \to R^n$ for which*

$$v(0) \in X_{M,f} \text{ and } I^f(0, T, v) \leq \sigma(f, T, v(0)) + \delta$$

there exist a pair of numbers $\tau_1 \in [0, L]$ *and* $\tau_2 \in [T - L, T]$ *such that*

$$|v(t) - \bar{x}| \le \epsilon \text{ for all } t \in [\tau_1, \tau_2]$$

and if $|v(0) - \bar{x}| \le \delta$, *then* $\tau_1 = 0$.

Let M be a positive real number. Denote by $Y_{M,f}$ the collection of all points $x \in R^n$ for which there are $T \in (0, M]$ and an a. c. function $v : [0, T] \to R^n$ such that $v(0) = \bar{x}$, $v(T) = x$ and $I^f(0, T, v) \le M$.

The following result obtained in [50] (see also Theorem 3.5 of [54]) shows that the turnpike phenomenon holds for approximate solutions of problem (P_1).

Theorem 8.3. *Let* $\epsilon, M_0, M_1 > 0$. *Then there exist numbers* $L,\ \delta > 0$ *such that for each number* $T > 2L$, *each point* $z_0 \in X_{M_0,f}$ *and each point* $z_1 \in Y_{M_1,f}$, *the value* $\sigma(f, T, z_0, z_1)$ *is finite and for each a. c. function* $v : [0, T] \to R^n$ *which satisfies*

$$v(0) = z_0,\ v(T) = z_1,\ I^f(0, T, v) \le \sigma(f, T, z_0, z_1) + \delta$$

there exists a pair of numbers $\tau_1 \in [0, L], \tau_2 \in [T - L, T]$ *such that*

$$|v(t) - \bar{x}| \le \epsilon,\ t \in [\tau_1, \tau_2].$$

Moreover if $|v(0) - \bar{x}| \le \delta$, *then* $\tau_1 = 0$ *and if* $|v(T) - \bar{x}| \le \delta$, *then* $\tau_2 = T$.

In the sequel we use a notion of an overtaking optimal function which plays an important role in the turnpike theory and the infinite horizon optimal control [39,54].

An a. c. function $v : [0, \infty) \to R^n$ is called (f)-overtaking optimal if for each a. c. function $u : [0, \infty) \to R^n$ satisfying $u(0) = v(0)$ the inequality

$$\limsup_{T \to \infty} [I^f(0, T, v) - I^f(0, T, u)] \le 0$$

holds.

The following existence result was obtained in [41] (see also Theorem 3.3 of [54].)

Theorem 8.4. *Assume that* $x \in R^n$ *and that there exists an* (f)*-good function* $v : [0, \infty) \to R^n$ *satisfying* $v(0) = x$. *Then there exists an* (f)*-overtaking optimal function* $u_* : [0, \infty) \to R^n$ *such that* $u_*(0) = x$.

In view of assumption (A1) there exists a real number $\bar{r} \in (0, 1)$ such that:

$$\Omega_0 := \{(x, y) \in R^n \times R^n :\ |x - \bar{x}| \le \bar{r} \text{ and } |y| \le \bar{r}\} \subset \ \mathrm{dom}(f),$$

$$\Delta_0 := \sup\{|f(z_1, z_2)| :\ (z_1, z_2) \in \Omega_0\} < \infty.$$

Evidently, the value $\sigma(f, T, x, y)$ is finite for every real number $T \ge 1$ and every pair of points $x, y \in R^n$ satisfying $|x - \bar{x}|,\ |y - \bar{x}| \le \bar{r}/2$.

Let M be a positive number. Denote by $\bar{Y}_{M,f}$ the collection of all points $x \in R^n$ satisfying $|x| \le M$ and such that there exist $T \in (0, M]$ and an a. c. function $v : [0, T] \to R^n$ which satisfy $v(0) = x$, $v(T) = \bar{x}$ and $I^f(0, T, v) \le M$.

It is not difficult to see that the next proposition is valid.

Proposition 8.5. *For every positive number* M *there exists a positive number* M_0 *for which* $\bar{Y}_{M,f} \subset X_{M_0,f}$.

The next useful proposition was proved in [57].

Proposition 8.6. *For every positive number M there exists a positive number M_0 for which that $X_{M,f} \subset \bar{Y}_{M_0,f}$.*

We say that an a. c. function $v : [0,\infty) \to R^n$ is (f)-minimal [39, 54] if for every pair of nonnegative numbers $T_2 > T_1$ and every a. c. function $u : [T_1, T_2] \to R^n$ for which $u(T_i) = v(T_i)$, $i = 1, 2$ the inequality

$$\int_{T_1}^{T_2} f(v(t), v'(t))dt \leq \int_{T_1}^{T_2} f(u(t), u'(t))dt$$

is valid.

The following result which is proved in [45] (see also Theorem 3.32 of [54]) establishes the equivalence of the optimality notions introduced above.

Theorem 8.7. *Assume that a point $x \in R^n$ and that there exists an (f)-good function $\tilde{v} : [0,\infty) \to R^n$ such that $\tilde{v}(0) = x$. Let $v : [0,\infty) \to R^n$ be an a. c. function satisfying $v(0) = x$. Then the following conditions are equivalent:*

(i) the function v is (f)-overtaking optimal; (ii) the function v is (f)-good and (f)-minimal; (iii) the function v is (f)-minimal and $\lim_{t\to\infty} v(t) = \bar{x}$; (iv) the function v is (f)-minimal and $\liminf_{t\to\infty} |v(t) - \bar{x}| = 0$.

The following two results which is proved in [45] (see also Theorems 3.33 and 3.34 of [54]) describe the asymptotic behavior of overtaking optimal functions.

Theorem 8.8. *Let ϵ be a positive number. Then there exists a positive number δ such that:*

(i) For each point $x \in R^n$ for which $|x - \bar{x}| \leq \delta$ there exists an (f)-overtaking optimal and (f)-good function $v : [0,\infty) \to R^n$ which satisfies $v(0) = x$.

(ii) For every (f)-overtaking optimal function $v : [0,\infty) \to R^n$ for which $|v(0) - \bar{x}| \leq \delta$, the relation $|v(t) - \bar{x}| \leq \epsilon$ is valid for every nonnegative numbers t.

Theorem 8.9. *Let ϵ, M be positive numbers. Then there exists a positive number L such that for every point $x \in X_{M,f}$ and every (f)-overtaking optimal function $v : [0,\infty) \to R^n$ for which $v(0) = x$ the relation $|v(t) - \bar{x}| \leq \epsilon$ holds for every $t \in [L,\infty)$.*

The next turnpike theorem for approximate solutions of the problems of the type (P_3) on large intervals was proved in [57].

Theorem 8.10. *Let ϵ be positive number. Then there exist positive numbers L, δ such that for every $T > 2L$ and every a. c. function $v : [0,T] \to R^n$ for which*

$$I^f(0,T,v) \leq \sigma(f,T) + \delta$$

there exists a pair of numbers $\tau_1 \in [0,L], \tau_2 \in [T-L,T]$ such that

$$|v(t) - \bar{x}| \leq \epsilon,\ t \in [\tau_1, \tau_2].$$

Moreover if $|v(0) - \bar{x}| \leq \delta$, then $\tau_1 = 0$ and if $|v(T) - \bar{x}| \leq \delta$, then $\tau_2 = T$.

9. Structure of solutions of Lagrange problems in the regions close to the endpoints

We use the notation, definitions and assumptions introduced in Section 8. For every point $x \in R^n \setminus \cup\{X_{M,f} : M \in (0,\infty)\}$ put

$$\pi^f(x) = \infty.$$

Let a point

$$x \in \cup\{X_{M,f} : M \in (0,\infty)\}$$

be given. We denote by $\Lambda(f,x)$ the collection of all (f)-overtaking optimal functions $v : [0,\infty) \to R^n$ satisfying $v(0) = x$. It is clear that the collection $\Lambda(f,x)$ is nonempty and that every function belonging to $\Lambda(f,x)$ is an (f)-good function. Set

$$\pi^f(x) = \liminf_{T\to\infty}[I^f(0,T,v) - Tf(\bar{x},0)],$$

where $v \in \Lambda(f,x)$. It is clear that $\pi^f(x)$ does not depend on the choice of the function v. By assumption (A2), $\pi^f(x)$ is finite.

The function π^f plays an important role in our analysis of the structure of solutions of Lagrange problems in the regions close to the endpoints.

The next result easily follows from our definitions.

Proposition 9.1. *1. Let $v : [0,\infty) \to R^n$ be an (f)-good function. Then*

$$\pi^f(v(0)) \le \liminf_{T\to\infty}[I^f(0,T,v) - Tf(\bar{x},0)]$$

and for every pair of nonnegative numbers $S > T$,

$$\pi^f(v(T)) \le I^f(T,S,v) - (S-T)f(\bar{x},0) + \pi^f(v(S)).$$

2. Let $S > T \ge 0$ and $v : [0,S] \to R^n$ be an a. c. function satisfying $\pi^f(v(T)), \pi^f(v(S)) < \infty$. Then the inequality

$$\pi^f(v(T)) \le I^f(T,S,v) - (S-T)f(\bar{x},0) + \pi^f(v(S))$$

is valid.

It is not difficult to see that the next proposition is valid.

Proposition 9.2. *Let $v : [0,\infty) \to R^n$ be an (f)-overtaking optimal and (f)-good function. Then for every pair of nonnegative numbers $S > T$ the equality*

$$\pi^f(v(T)) = I^f(T,S,v) - (S-T)f(\bar{x},0) + \pi^f(v(S))$$

holds.

The next three propositions were obtained in [57].

Proposition 9.3. $\pi^f(\bar{x}) = 0$.

Proposition 9.4. *The function π^f is finite-valued in a neighborhood of the $\bar{x}$ and continuous at the point $\bar{x}$.*

Proposition 9.5. *For every positive number M the set $\{x \in R^n : \pi^f(x) \le M\}$ is bounded.*

(Here we assume that an empty set is bounded.)

Define

$$\inf(\pi^f) = \inf\{\pi^f(z) :\ z \in R^n\}.$$

Assumption (A2) and Proposition 9.5 imply that $\inf(\pi^f)$ is finite. Define

$$X_f = \{x \in R^n :\ \pi^f(x) \le \inf(\pi^f) + 1\}.$$

The next two propositions were also obtained in [57].

Proposition 9.6. *Assume that a point $x \in \cup\{X_{M,f} :\ M \in (0,\infty)\}$ and that a function $v \in \Lambda(f,x)$. Then the equality*

$$\pi^f(x) = \lim_{T\to\infty}[I^f(0,T,v) - Tf(\bar{x},0)]$$

holds.

Proposition 9.7. *There exists a positive number M for which $X_f \subset X_{M,f}$.*

The next proposition easily follows from Propositions 8.6 and 9.7.

Proposition 9.8. *There exists a positive number L for which $X_f \subset \bar{Y}_{L,f}$.*

The following result was obtained in [57].

Proposition 9.9. *The function $\pi^f : R^n \to R^1 \cup \{\infty\}$ is lower semicontinuous.*

Define

$$\mathcal{D}(f) = \{x \in R^n :\ \pi^f(x) = \inf(\pi^f)\}.$$

In view of Propositions 9.5 and 9.9, the set $\mathcal{D}(f)$ is nonempty, bounded and closed subset of R^n. The next two results were obtained in [57].

Proposition 9.10. *Let $v : [0,\infty) \to R^n$ be an (f)-good function such that for every positive number T the equality*

$$I^f(0,T,v) - Tf(\bar{x},0) = \pi^f(v(0)) - \pi^f(v(T))$$

holds. Then the function v is (f)-overtaking optimal.

Proposition 9.11. *For every positive number ϵ there exists a positive number T_ϵ such that for every point $z \in \mathcal{D}(f)$ and every function $v \in \Lambda(f,z)$ the inequality $|v(t) - \bar{x}| \le \epsilon$ is valid for every number $t \ge T_\epsilon$.*

In order to analyze the structure of approximate solutions of the problems (P_2) and (P_3) on large intervals, in the regions close to the endpoints, we need to introduce the following notation and definitions.

Define a function $\bar{f} : R^n \times R^n \to R^1 \cup \{\infty\}$ as follows:

$$\bar{f}(x,y) = f(x,-y) \text{ for all } x,y \in R^n.$$

Evidently,

$$\text{dom}(\bar{f}) = \{(x,y) \in R^n \times R^n :\ (x,-y) \in \text{dom}(f)\},$$

$\text{dom}(\bar{f})$ is a nonempty closed convex set, $\bar{f}$ is a lower semicontinuous function such that

$$\bar{f}(x,y) \ge \max\{\psi(|x|),\ \psi(|y|)|y|\} - a \text{ for each } x,y \in R^n.$$

The notation introduced for the function f is also used for the function $\bar{f}$. Namely, for every pair of real numbers $T_2 > T_1$ and every a. c. function $v : [T_1, T_2] \to R^n$ define

$$I^{\bar{f}}(T_1, T_2, v) = \int_{T_1}^{T_2} \bar{f}(v(t), v'(t))dt$$

and for every pair of points $x, y \in R^n$ and every positive number T set

$$\sigma(\bar{f}, T, x) = \inf\{I^{\bar{f}}(0, T, v) :\ v : [0, T] \to R^n$$

$$\text{is an a. c. function satisfying } v(0) = x\},$$

$$\sigma(\bar{f}, T, x, y) = \inf\{I^{\bar{f}}(0, T, v) :\ v : [0, T] \to R^n$$

$$\text{is an a. c. function satisfying } v(0) = x,\ v(T) = y\},$$

$$\sigma(\bar{f}, T) = \inf\{I^{\bar{f}}(0, T, v) :\ v : [0, T] \to R^n \text{ is an a. c. function}\},$$

$$\widehat{\sigma}(\bar{f}, T, y) = \inf\{I^{\bar{f}}(0, T, v) :\ v : [0, T] \to R^n$$

$$\text{is an a. c. function satisfying } v(T) = y\}.$$

Let $v : [0, T] \to R^n$ be an a. c. function. Define

$$\bar{v}(t) = v(T - t),\ t \in [0, T].$$

It is clear that

$$\int_0^T \bar{f}(\bar{v}(t), \bar{v}'(t))dt = \int_0^T f(\bar{v}(t), -\bar{v}'(t))dt$$

$$= \int_0^T f(v(T - t), v'(T - t))dt = \int_0^T f(v(t), v'(t))dt.$$

It is easy to see that for all points $x \in R^n$,

$$\bar{f}(\bar{x}, 0) = f(\bar{x}, 0) \le f(x, 0) = \bar{f}(x, 0),$$

$(\bar{x}, 0)$ is an interior point of the set $\mathrm{dom}(\bar{f})$ and the function $\bar{f}$ is continuous at the point $(\bar{x}, 0)$. Therefore assumption (A1) is valid for the function $\bar{f}$. Evidently, for every point $x \in R^n$ the function $\bar{f}(x, 0) : R^n \to R^1 \cup \{\infty\}$ is convex. Thus assumption (A3) holds for the function $\bar{f}$. We can easily obtain the next proposition.

Proposition 9.12. *Let T be a positive number, M be a nonnegative number and $v_i : [0, T] \to R^n$, $i = 1, 2$ be a. c. functions. Then the inequality*

$$I^f(0, T, v_1) \ge I^f(0, T, v_2) - M$$

is valid if and only if the inequality

$$I^{\bar{f}}(0, T, \bar{v}_1) \ge I^{\bar{f}}(0, T, \bar{v}_2) - M$$

holds.

The next proposition easily follows from Proposition 9.12.

Proposition 9.13. *Let T be a positive number and $v : [0,T] \to R^n$ be an a. c. function. Then the following assertions hold:*

$$I^f(0,T,v) \le \sigma(f,T) + M \text{ if and only if}$$

$$I^{\bar{f}}(0,T,\bar{v}) \le \sigma(\bar{f},T) + M;$$

$$I^f(0,T,v) \le \sigma(f,T,v(0),v(T)) + M \text{ if and only if}$$

$$I^{\bar{f}}(0,T,\bar{v}) \le \sigma(\bar{f},T,\bar{v}(0),\bar{v}(T)) + M;$$

$$I^f(0,T,v) \le \widehat{\sigma}(f,T,v(T)) + M \text{ if and only if } I^{\bar{f}}(0,T,\bar{v}) \le \sigma(\bar{f},T,\bar{v}(0)) + M;$$

$$I^f(0,T,v) \le \sigma(f,T,v(0)) + M \text{ if and only if } I^{\bar{f}}(0,T,\bar{v}) \le \widehat{\sigma}(\bar{f},T,\bar{v}(T)) + M.$$

The following proposition was obtained in [57].

Proposition 9.14. *1. For every positive number M there exists a positive number c_M such that $\sigma(\bar{f},T,x) \ge T\bar{f}(\bar{x},0) - c_M$ for every point $x \in R^n$ such that $|x| \le M$ and every positive number T.*

2. For every $(\bar{f})$-good function $v : [0,\infty) \to R^n$ the equality $\lim_{t\to\infty} v(t) = \bar{x}$ holds.

Proposition 9.14 implies that $\bar{f}$ satisfies assumptions (A2) and (A4). We have already mentioned that assumptions (A1) and (A3) hold for the function $\bar{f}$. Therefore all the results stated above for the function f are also true for the function $\bar{f}$.

The next two results which were established in [57] describe the structure of approximate solutions of the problems (P_2) and (P_3) on large intervals in the regions close to the endpoints respectively, .

Theorem 9.15. *Let L_0, τ_0 and ϵ be positive numbers. Then there exist numbers $\delta > 0$ and $T_0 > \tau_0$ such that for every number $T \ge T_0$ and every a. c. function $v : [0,T] \to R^n$ for which*

$$v(0) \in \bar{Y}_{L_0,f},\ I^f(0,T,v) \le \sigma(f,T,v(0)) + \delta$$

there exists an $(\bar{f})$-overtaking optimal function $v^* : [0,\infty) \to R^n$ which satisfies $v^*(0) \in \mathcal{D}(\bar{f})$ and $|v(T-t) - v^*(t)| \le \epsilon$ for all $t \in [0,\tau_0]$.

Theorem 9.16. *Let τ_0 and ϵ be positive numbers. Then there exist numbers $\delta > 0$ and $T_0 > \tau_0$ such that for every $T \ge T_0$ and every a. c. function $v : [0,T] \to R^n$ for which*

$$I^f(0,T,v) \le \sigma(f,T) + \delta$$

there exist an (f)-overtaking optimal function $u^* : [0,\infty) \to R^n$ and an $(\bar{f})$-overtaking optimal function $v^* : [0,\infty) \to R^n$ which satisfies $u^*(0) \in \mathcal{D}(f)$, $v^*(0) \in \mathcal{D}(\bar{f})$ and such that for all $t \in [0,\tau_0]$,

$$|v(t) - u^*(t)| \le \epsilon \text{ and } |v(T-t) - v^*(t)| \le \epsilon.$$

10. The Bolza problem (P_4)

We use the notation, definitions and assumptions introduced in Sections 8 and 9.

For every nonempty set Y and every function $h : Y \to R^1 \cup \{\infty\}$ set

$$\inf(h) = \inf\{h(y) : \ y \in Y\}.$$

Let a_0 be a positive number. We denote by $\mathfrak{A}(R^n)$ the collection of all lower semicontinuous functions $h : R^n \to R^1$ which are bounded on bounded subsets of R^n and which satisfy

$$h(z) \ge -a_0 \text{ for all } z \in R^n.$$

For simplicity, we put $\mathfrak{A} = \mathfrak{A}(R^n)$. The space $\mathfrak{A}$ is equipped with the uniformity determined by the base

$$E(N, \epsilon) = \{(h_1, h_2) \in \mathfrak{A} \times \mathfrak{A} :$$
$$|h_1(z) - h_2(z)| \le \epsilon \text{ for all } z \in R^n \text{ satisfying } |z| \le N\},$$

where $N, \epsilon > 0$. It is not difficult to see that the uniform space $\mathfrak{A}$ is metrizable and complete.

For every point $x \in R^n$, every pair of nonnegative numbers $T_1 < T_2$ and every function $h \in \mathfrak{A}$ set

$$\sigma(f, h, T_1, T_2, x) = \inf\{\int_{T_1}^{T_2} f(v(t), v'(t))dt + h(v(T_2)) :$$
$$v : [T_1, T_2] \to R^n \text{ is an a. c. function satisfying } v(T_1) = x\}.$$

(Here we assume that the infimum over an empty set is ∞.)

In view of assumption (A1), there exists a number $\bar{r} \in (0, 1)$ such that

$$\{(z_1, z_2) \in R^n \times R^n : \ |z_1 - \bar{x}| \le \bar{r} \text{ and } |z_2| \le \bar{r}\} \subset \text{ dom}(f),$$
$$|f(z_1, z_2) - f(\bar{x}, 0)| \le 1 \text{ for each } (z_1, z_2) \in R^n \times R^n$$
$$\text{satisfying } |z_1 - \bar{x}| \le \bar{r}, \ |z_2| \le \bar{r}.$$

The next theorem established in [60] shows that the turnpike phenomenon holds for approximate solutions of problems (P_4) on large intervals.

Theorem 10.1. *Let $\epsilon, M > 0$. Then there exist a number $L \ge 1$ and a positive number δ such that for every number $T > 2L$, every function $h \in \mathfrak{A}$ for which $|h(\bar{x})| \le M$ and every a. c. function $v : [0, T] \to R^n$ such that*

$$v(0) \in X_{M,f} \text{ and } I^f(0, T, v) + h(v(T)) \le \sigma(f, h, 0, T, v(0)) + \delta$$

there exists a pair of real numbers $\tau_1 \in [0, L]$ and $\tau_2 \in [T - L, T]$ such that

$$|v(t) - \bar{x}| \le \epsilon \text{ for all } t \in [\tau_1, \tau_2]$$

and if $|v(0) - \bar{x}| \le \delta$, then $\tau_1 = 0$.

Let $g \in \mathfrak{A}$ be given. In view of Propositions 9.5 and 9.9, the function $\pi^f + g : R^n \to R^1 \cup \{\infty\}$ is lower semicontinuous and bounded from below and satisfies

$$\lim_{|z| \to \infty} (\pi^f + g)(z) = \infty$$

and

$$\mathcal{D}(f, g) := \{x \in R^n : \ (\pi^f + g)(x) = \inf(\pi^f + g)\} \ne \emptyset.$$

The next theorem established in [60] describes the structure of approximate solutions of Bolza problems of the type (P_4) on large intervals in the regions close to the endpoints.

Theorem 10.2. *Let $g \in \mathfrak{A}$ and L_0, τ_0, ϵ be positive numbers. Then there exist numbers $\delta > 0$, $T_0 > \tau_0$ and a neighborhood $\mathcal{U}$ of the function g in the space $\mathfrak{A}$ such that for every number $T \geq T_0$, every function $h \in \mathcal{U}$ and every a. c. function $v : [0,T] \to R^n$ satisfying*

$$v(0) \in \bar{Y}_{L_0,f},\ I^f(0,T,v) + h(v(T)) \leq \sigma(f,h,0,T,v(0)) + \delta$$

there exists an $(\bar{f})$-overtaking optimal function $v^* : [0,\infty) \to R^n$ such that $v^*(0) \in \mathcal{D}(\bar{f}, g)$ and $|v(T-t) - v^*(t)| \leq \epsilon$ for all $t \in [0,\tau_0]$.

11. The Bolza problem (P_5)

We use the notation, definitions and assumptions introduced in Sections 8, 9 and 10.

Let a_0 be a positive number. We denote by $\mathfrak{A}(R^n \times R^n)$ the collection of all lower semicontinuous functions $h : R^n \times R^n \to R^1$ which are bounded on bounded subsets of $R^n \times R^n$ and satisfy

$$h(z_1, z_2) \geq -a_0 \text{ for all } z_1, z_2 \in R^n.$$

For simplicity, we put $\mathfrak{A} = \mathfrak{A}(R^n \times R^n)$. The space $\mathfrak{A}$ is equipped with the uniformity determined by the base

$$E(N,\epsilon) = \{(h_1,h_2) \in \mathfrak{A} \times \mathfrak{A} :$$

$$|h_1(z) - h_2(z)| \leq \epsilon \text{ for all } z \in R^n \times R^n \text{ satisfying } |z| \leq N\},$$

where N, ϵ are positive numbers. It is clear that the uniform space $\mathfrak{A}$ is metrizable and complete.

For every pair of nonnegative numbers $T_1 < T_2$ and every function $h \in \mathfrak{A}$ set

$$\sigma(f,h,T_1,T_2) = \inf\{\int_{T_1}^{T_2} f(v(t), v'(t))dt + h(v(T_1),(v(T_2)) :$$

$$v : [T_1,T_2] \to R^n \text{ is an a. c. function}\}.$$

(Here we assume that the infimum over an empty set is ∞.)

Assumption (A1) implies that there exists a number $\bar{r} \in (0,1)$ such that

$$\Omega_0 := \{(z_1,z_2) \in R^n \times R^n : \ |z_1 - \bar{x}| \leq \bar{r} \text{ and } |z_2| \leq \bar{r}\} \subset \ \mathrm{dom}(f),$$

$$|f(z_1,z_2) - f(\bar{x},0)| \leq 1 \text{ for each } (z_1,z_2) \in \Omega_0.$$

The next theorem established in [60] shows that the turnpike phenomenon holds for approximate solutions of problems (P_5) on large intervals.

Theorem 11.1. *Let $\epsilon > 0$, $M > \bar{r}_v$. Then there exist a number $L \geq 1$ and a positive number δ such that for every number $T > 2L$, every function $h \in \mathfrak{A}$ for which $|h(\bar{x},\bar{x})| \leq M$ and every a. c. function $v : [0,T] \to R^n$ such that*

$$I^f(0,T,v) + h(v(0),v(T)) \leq \sigma(f,h,0,T) + \delta$$

there exists a pair of real numbers $\tau_1 \in [0, L]$ *and* $\tau_2 \in [T - L, T]$ *such that*

$$|v(t) - \bar{x}| \le \epsilon \text{ for all } t \in [\tau_1, \tau_2].$$

Moreover, if $|v(0) - \bar{x}| \le \delta$*, then* $\tau_1 = 0$ *and if* $|v(T) - \bar{x}| \le \delta$*, then* $\tau_2 = T$.

Let $h \in \mathfrak{A}$ be given. It follows from Propositions 9.5 and 9.9 that the function

$$\pi^f(z_1) + \pi^{\bar{f}}(z_2) + h(z_1, z_2),\ z_1, z_2 \in R^n$$

is lower semicontinuous and bounded from below, satisfies

$$\lim_{|(z_1,z_2)|\to\infty} (\pi^f(z_1) + \pi^{\bar{f}}(z_2) + h(z_1, z_2)) = \infty$$

and the set

$$\mathcal{D}(f, h) := \{(x, y) \in R^n \times R^n :$$
$$\pi^f(x) + \pi^{\bar{f}}(y) + h(x, y) \le \pi^f(\xi) + \pi^{\bar{f}}(\eta) + h(\xi, \eta)$$
$$\text{for all } \xi, \eta \in R^n\}$$

is nonempty and closed.

The next theorem is obtained in [60]. It describes the structure of approximate solutions of Bolza problems of the type (P_5) on large intervals in the regions close to the endpoints.

Theorem 11.2. *Let* $g \in \mathfrak{A}$ *and* τ_0, ϵ *be positive numbers. Then there exist numbers* $\delta > 0$, $T_0 > \tau_0$ *and a neighborhood* $\mathcal{U}$ *of the function* g *in the space* $\mathfrak{A}$ *such that for every number* $T \ge T_0$*, every function* $h \in \mathcal{U}$ *and every a. c. function* $v : [0, T] \to R^n$ *for which*

$$I^f(0, T, v) + h(v(0), v(T)) \le \sigma(f, h, 0, T) + \delta$$

there exist an (f)*-overtaking optimal function* $v_1^* : [0, \infty) \to R^n$ *and an* $(\bar{f})$*-overtaking optimal function* $v_2^* : [0, \infty) \to R^n$ *such that*

$$\pi^f(v_1^*(0)) + \pi^{\bar{f}}(v_2^*(0)) + g(v_1^*(0), v_2^*(0)) \le \pi^f(\xi_1) + \pi^{\bar{f}}(\xi_2) + g(\xi_1, \xi_2)$$
$$\text{for all } \xi_1, \xi_2 \in R^n$$

and that for all $t \in [0, \tau_0]$,

$$|v(t) - v_1^*(t)| \le \epsilon,\ |v(T - t) - v_2^*(t)| \le \epsilon.$$

References

[1] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[2] S. M. Aseev and V. M. Veliov, *Maximum principle for infinite-horizon optimal control problems with dominating discount*, Dynamics of Continuous, Discrete and Impulsive Systems, SERIES B **19** (2012), 43–63.

[3] S. Aubry and P. Y. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions I*, Physica D **8** (1983), 381–422.

[4] J. Baumeister, A. Leitao and G. N. Silva, *On the value function for nonautonomous optimal control problem with infinite horizon*, Systems Control Lett. **56** (2007), 188–196.

[5] J. Blot, *Infinite-horizon Pontryagin principles without invertibility*, J. Nonlinear Convex Anal. **10** (2009), 177–189.

[6] J. Blot and P. Cartigny, *Optimality in infinite-horizon variational problems under sign conditions*, J. Optim. Theory Appl. **106** (2000), 411–419.

[7] J. Blot and N. Hayek, *Infinite-horizon optimal control in the discrete-time framework*, SpringerBriefs in Optimization, New York, 2014.
[8] I. Bright, *A reduction of topological infinite-horizon optimization to periodic optimization in a class of compact 2-manifolds*, J. Math. Anal. Appl. **394** (2012), 84–101.
[9] D. A. Carlson, *The existence of catching-up optimal solutions for a class of infinite horizon optimal control problems with time delay,* SIAM J. Control Optim. **28** (1990), 402–422.
[10] D. A. Carlson, A. Haurie and A. Jabrane, *Existence of overtaking solutions to infinite dimensional control problems on unbounded time intervals*, SIAM J. Control Optim. **25** (1987), 1517–541.
[11] D. A. Carlson, A. Haurie and A. Leizarowitz, *Infinite Horizon Optimal Control*, Berlin, Springer-Verlag, 1991.
[12] P. Cartigny and P. Michel, *On a sufficient transversality condition for infinite horizon optimal control problems*, Automatica J. IFAC **39** (2003), 1007–1010.
[13] T. Damm, L. Grune, M. Stieler and K. Worthmann, *An exponential turnpike theorem for dissipative discrete time optimal control problems*, SIAM Journal on Control and Optimization **52** (2014), 1935–1957.
[14] V. A. De Oliveira and G. N. Silva, *Optimality conditions for infinite horizon control problems with state constraints*, Nonlinear Analysis **71** (2009), 1788–1795.
[15] V. Gaitsgory, L. Grune and N. Thatcher, *Stabilization with discounted optimal control*, Systems and Control Letters **82** (2015), 91–98.
[16] V. Gaitsgory, S. Rossomakhine and N. Thatcher, *Approximate solution of the HJB inequality related to the infinite horizon optimal control problem with discounting*, Dynamics Contin. Discrete Impuls. Syst., Series B **19** (2012), 65–92.
[17] N. Hayek, *Infinite horizon multiobjective optimal control problems in the discrete time case*, Optimization **60** (2011), 509–529.
[18] H. Jasso-Fuentes and O. Hernandez-Lerma, *Characterizations of overtaking optimality for controlled diffusion processes*, Appl. Math. Optim. **57** (2008), 349–369.
[19] M. Ali Khan and A. J. Zaslavski, *On two classical turnpike results for the Robinson-Solow-Srinivisan (RSS) model*, Adv. Math. Econom. **13** (2010), 47–97.
[20] D. V. Khlopin, *Necessary conditions of overtaking equilibrium for infinite horizon differential games*, Mat. Teor. Igr Pril. **5** (2013), 105–136.
[21] D. V. Khlopin, *Necessity of vanishing shadow price in infinite horizon control problems*, J. Dyn. Control Syst. **19** (2013), 519–552.
[22] V. Kolokoltsov and W. Yang, *The turnpike theorems for Markov games*, Dynamic Games Appl.s **2** (2012), 294–312.
[23] A. Leizarowitz, *Infinite horizon autonomous systems with unbounded cost*, Appl. Math. Opt. **13** (1985), 19–43.
[24] A. Leizarowitz, *Tracking nonperiodic trajectories with the overtaking criterion*, Appl. Math. Opt. **14** (1986), 155–171.
[25] A. Leizarowitz and V. J. Mizel, *One dimensional infinite horizon variational problems arising in continuum mechanics*, Arch. Rational Mech. Anal. **106** (1989), 161–194.
[26] V. Lykina, S. Pickenhain and M. Wagner, *Different interpretations of the improper integral objective in an infinite horizon control problem*, J. Math. Anal. Appl. **340** (2008), 498–510.
[27] V. L. Makarov and A. M. Rubinov, *Mathematical theory of economic dynamics and equilibria*, New York, Springer-Verlag, 1977.
[28] A. B. Malinowska, N. Martins and D. F. M. Torres, *Transversality conditions for infinite horizon variational problems on time scales*, Optimization Lett. **5** (2011), 41–53.
[29] M. Mammadov, *Turnpike theorem for an infinite horizon optimal control problem with time delay*, SIAM J. Control Optim. **52** (2014), 420–438.
[30] M. Marcus and A. J. Zaslavski, *The structure of extremals of a class of second order variational problems*, Ann. Inst. H. Poincaré, Anal. non Lineare **16** (1999), 593–629.
[31] L. W. McKenzie, *Turnpike theory*, Econometrica **44** (1976), 841–866.
[32] B. S. Mordukhovich, *Optimal control and feedback design of state-constrained parabolic systems in uncertainly conditions*, Appl. Analysis **90** (2011), 1075–1109.

[33] B. S. Mordukhovich and I. Shvartsman, *Optimization and feedback control of constrained parabolic systems under uncertain perturbations*, in Optimal Control, Stabilization and Nonsmooth Analysis, Lecture Notes Control Inform. Sci., Springer, 2004, pp. 121–132.

[34] E. Ocana Anaya, P. Cartigny and P. Loisel, *Singular infinite horizon calculus of variations. Applications to fisheries management*, J. Nonlinear Convex Anal. **10** (2009), 157–176.

[35] S. Pickenhain, V. Lykina and M. Wagner, *On the lower semicontinuity of functionals involving Lebesgue or improper Riemann integrals in infinite horizon optimal control problems*, Control Cybernet. **37** (2008), 451–468.

[36] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim. **51** (2013), 4242–4273.

[37] P. A. Samuelson, *A catenary turnpike theorem involving consumption and the golden rule*, Amer. Econom. Rev. **55** (1965), 486–496.

[38] E. Trelat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, J. Differ. Equ. **218** (2015), 81–114.

[39] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, New York, 2006.

[40] A. J. Zaslavski, *Turnpike results for a discrete-time optimal control systems arising in economic dynamics*, Nonlinear Anal. **67** (2007), 2024–2049.

[41] A. J. Zaslavski, *A turnpike result for a class of problems of the calculus of variations with extended-valued integrands*, J. Convex Analysis **15** (2008), 869–890.

[42] A. J. Zaslavski, *Structure of approximate solutions of variational problems with extended-valued convex integrands*, ESAIM: Control, Optim. Calculus Vari. **15** (2009), 872–894.

[43] A. J. Zaslavski, *Two turnpike results for a discrete-time optimal control systems*, Nonlinear Anal. **71** (2009), 902–909.

[44] A. J. Zaslavski, *Stability of a turnpike phenomenon for a discrete-time optimal control systems*, J. Optim. Theory Appl. **145** (2010), 597–612.

[45] A. J. Zaslavski, *Optimal solutions for a class of infinite horizon variational problems with extended-valued integrands,* Optimization **59** (2010), 181–197.

[46] A. J. Zaslavski, *Turnpike properties of approximate solutions for discrete-time control systems,* Comm. Math. Anal. **11** (2011), 36–45.

[47] A. J. Zaslavski, *Structure of approximate solutions for a class of optimal control systems*, J. Math. Appl. **34** (2011), 1–14.

[48] A. J. Zaslavski, *The existence and structure of approximate solutions of dynamic discrete time zero-sum games*, J. Nonlinear Convex Anal. **12** (2011), 49–68.

[49] A. J. Zaslavski, *Stability of a turnpike phenomenon for a class of optimal control systems in metric spaces*, Numer. Algebra, Control Optim. **1** (2011), 245–260.

[50] A. J. Zaslavski, *Two turnpike results for a continuous-time optimal control systems*, Proceedings of an international conference, Complex Analysis and Dynamical Systems IV: function theory and optimization, vol. 553, 2011, pp. 305–317.

[51] A. J. Zaslavski, *Nonconvex Optimal Control and Variational Problems*, Springer Optimization and Its Applications, New York, 2013.

[52] A. J. Zaslavski, *Structure of Solutions of Variational Problems,* Springer Briefs in Optimization, New York, 2013.

[53] A. J. Zaslavski, *Turnpike properties of approximate solutions of nonconcave discrete-time optimal control problems*, J. Convex Anal. **21** (2014), 681–701.

[54] A. J. Zaslavski, *Turnpike Phenomenon and Infinite Horizon Optimal Control*, Springer Optimization and Its Applications, New York, 2014.

[55] A. J. Zaslavski, *Turnpike properties for nonconcave problems*, Advances in Mathematical Economics **18** (2014), 101–134.

[56] A. J. Zaslavski, *Structure of solutions of discrete time optimal control problems in the regions close to the endpoints*, Set-Valued Variational Anal. **22** (2014), 809–842.

[57] A. J. Zaslavski, *Structure of solutions of variational problems with extended-valued integrands in the regions close to the endpoints*, Set-Valued Variational Anal. **23** (2015), 249–275.

[58] A. J. Zaslavski, *Structure of approximate solutions of discrete time optimal control Bolza problems on large intervals*, Nonlinear Anal. **123-124** (2015), 23-55.

[59] A. J. Zaslavski, *Discrete time optimal control problems on large intervals*, Advan. Math. Econ. **19** (2015), 91–135.

[60] A. J. Zaslavski, *Bolza variational problems with extended-valued integrands on large intervals*, Proceedings of the Special Session on Nonlinear Analysis and Optimization, Joint AMS-IMU Conference, Tel-Aviv, June 2014, Contemporary Mathematics. Nonlinear Analysis and Optimization, in press.

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A. J. Zaslavski
Department of Mathematics, The Technion – Israel Institute of Technology, Technion City, Haifa 32000, Israel
E-mail address: `ajzasl@techunix.technion.ac.il`