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PARABOLIC GRADIENT EQUATIONS ON $\mathbb{R}$

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Abstract. We study here the existence and uniqueness of solutions to nonlinear divergence parabolic equations in $\mathbb{R} = (-\infty, +\infty)$ via nonlinear semigroup theory.

1. Introduction

We consider the Cauchy problem

$$\begin{aligned} & y_t(t,x) - (\beta(y_x(t,x)))_x \ni 0, && x \in \mathbb{R} = (-\infty,+\infty), \\ & && t \in [0,T],\ 0 < T < \infty, \\ & y(0,x) = y_0(x), && x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

Here y_t and y_x denote the time and space derivatives of y and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone multivalued graph such that $0 \in \beta(0)$ and $y_0 \in L^1(\mathbb{R})$. This equation arises in the theory of crystal growth (see, e.g., [4], [5], [6], [7]) as well as in image restoring techniques (see [8]). In general, it has not a classical solution and it will be treated here by using the theory of nonlinear semigroups of contractions in $L^1(\mathbb{R})$ in connection with the porous media equation in $L^1(\mathbb{R})$,

$$\begin{aligned} & z_t(t,x) - (\beta(z(t,x)))_{xx} \ni 0, && x \in \mathbb{R},\ t \in (0,T), \\ & z(0,x) = z_0(x) = (y_0)_x(x), && x \in \mathbb{R}, \end{aligned} \tag{1.2}$$

which is formally obtained from (1.1) by differentiating with respect to x and setting $z = y_x$.

On the other hand, for equation (1.2) there is a complete existence theory in $L^1(\mathbb{R})$ because the operator

$$\begin{aligned} \Gamma(z) = \{-\Delta\eta\},\ \forall z \in D(\Gamma) = \{z \in L^1(\mathbb{R});\ \exists \eta \in L^1_{\rm loc}(\mathbb{R}), \\ \eta(x) \in \beta(z(x)) \text{ a.e. } x \in \mathbb{R},\ \Delta\eta \in L^1(\mathbb{R})\} \end{aligned} \tag{1.3}$$

is m-accretive in $L^1(\mathbb{R})$ if $0 \in \operatorname{int} D(\beta)$ (see [1], p. 126).

It should be said, however, that the equivalence of problems (1.1) and (1.2) is quite a delicate matter and is dependent of the smoothness of the initial data y_0. This will be discussed in some details later on.

In the special case

$$\beta(r) = \phi^2(r)r,$$

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where $\phi : \mathbb{R} \to 2^{\mathbb{R}}$ is bounded, it follows that the solution z to (1.2) has the probabilistic representation (see [3])

$$z(t,x) = \text{Law density } \{Y(t)\}, \tag{1.4}$$

where Y is the process defined by the stochastic equation

$$dY = \phi(t, Y(t))dW(t), \quad Y(0) = Y_0, \tag{1.5}$$

and W is a Wiener process. Then, such a representation remains true for $z = y_x$ if y is a solution to (1.1).

Notations. $L^p(\mathbb{R})$, $1 \le p \le \infty$, is the space of Lebesgue p-integrable functions on $\mathbb{R} = (-\infty, +\infty)$ and $L^p_{\rm loc}(\mathbb{R})$ that of p-integrable functions on compact intervals of $\mathbb{R}$. By $C(\mathbb{R})$ denote the space of all continuous and bounded functions on $\mathbb{R}$ and by $BV(\mathbb{R})$ the space of all functions with bounded variations on $\mathbb{R}$. By $W^{1,1}_{\rm loc}(\mathbb{R})$ we denote the space of functions $y \in L^1_{\rm loc}(\mathbb{R})$ with distributional derivative $y' \in L^1_{\rm loc}(\mathbb{R})$. By y'' we shall denote the second order derivative of y again in sense of distributions.

Given a Banach space X with the norm $\|\cdot\|_X$, the operator $A : D(A) \subset X \to X$ is called *accretive* in $X \times X$ if

$$\|(I+\lambda A)^{-1}u - (I+\lambda A)^{-1}v\|_X \le \|u-v\|_X, \tag{1.6}$$

for all $\lambda > 0$ and all $u, v \in \mathbb{R}(I+\lambda A)$. (Here $\mathbb{R}(I+\lambda A)$ is the range of the operator $I+\lambda A$ and I is the identity operator.)

The operator $A : D(A) \subset X \to X$ is said to be *m-accretive* if it is accretive and for all $\lambda > 0$ or, equivalently, for some $\lambda > 0$, $\mathbb{R}(I+\lambda A) = X$.

The multivalued function (graph) $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be maximal monotone if it is m-accretive in $\mathbb{R} \times \mathbb{R}$.

In Sections 2 and 3, we shall study existence for (1.1) in $L^1(\mathbb{R})$. In Section 4, we shall study (1.1) under the additional assumption $R(\beta) = \mathbb{R}$, and in Section 5 for $\beta(r) = \operatorname{sign} r$.

2. The m-accretive operator associated to equation (1.1)

Everywhere in the following we assume that

(i) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with the domain $D(\beta) = \mathbb{R}$ and $0 \in \beta(0)$.

Let X be the Banach space

$$X = \left\{ u \in W^{1,1}_{\rm loc}(\mathbb{R}),\ u' \in L^1(\mathbb{R}),\ \lim_{x\to\infty} u(x) = 0 \right\}$$

with the norm

$$\|u\|_X = \int_{-\infty}^{\infty} |u'(x)|dx. \tag{2.1}$$

Consider on X the operator $A : D(A) \subset X \to X$ defined by

$$\begin{aligned} Ay &= \{-\eta' \in L^1(\mathbb{R});\ \eta(x) \in \beta(y'(x)),\ \text{a.e. } x \in \mathbb{R}\}, \\ D(A) &= \{y \in X;\ \exists \eta \in L^1_{\rm loc}(\mathbb{R}),\ \eta(x) \in \beta(y'(x)), \\ &\qquad \text{a.e. } x \in \mathbb{R},\ \eta' \in L^1(\mathbb{R})\}. \end{aligned} \tag{2.2}$$

(If β is single-valued, then $Ay = -(\beta(y'))'$ with the domain $D(A) = \{y \in X; \beta(y') \in L^1_{\text{loc}}(\mathbb{R}),\ (\beta(y'))' \in L^1(\mathbb{R})\}$.)

Lemma 2.1. *The operator A is m-accretive in $X \times X$.*

Proof. Let $f \in X$ and let $y \in D(A)$ be a solution to $(I + \lambda A)y \ni f$, that is,

$$\begin{aligned} & y - \lambda(\beta(y'))' \ni f \quad \text{in } \mathbb{R}, \\ & y' \in L^1(\mathbb{R}), \quad y(+\infty) = 0,\ (\beta(y'))' \in L^1(\mathbb{R}), \end{aligned} \tag{2.3}$$

in sense of distributions. We set $y' = z$ and get by (2.3)

$$z - \lambda(\beta(z))'' \ni f', \quad \text{a.e. in } \mathbb{R};\ z \in L^1(\mathbb{R}). \tag{2.4}$$

Similarly, if $\bar{f} \in X$ and $(I + \lambda A)\bar{y} \ni \bar{f}$, we get

$$\bar{z} - \lambda(\beta(\bar{z}))'' \ni \bar{f}', \quad \text{a.e. in } \mathbb{R},\ \bar{z} \in L^1(\mathbb{R}). \tag{2.5}$$

Taking into account that the operator Γ defined by (1.3) is accretive in $L^1(\mathbb{R})$, we get by (2.4)–(2.5) that

$$\|y - \bar{y}\|_X = \|y' - \bar{y}'\|_{L^1(\mathbb{R})} = \|z - \bar{z}\|_{L^1(\mathbb{R})} \le \|f' - \bar{f}'\|_{L^1(\mathbb{R})} = \|f - \bar{f}\|_X.$$

Hence A is accretive. On the other hand, by m-accretivity of the operator Γ in $L^1(\mathbb{R})$ it follows that for each $f \in X$ equation (2.4) has a unique solution $z \in L^1(\mathbb{R})$ and so $y(x) = -\int_x^\infty z(s)ds$, $\forall x \in \mathbb{R}$, is a solution to (2.3). Hence $\mathbb{R}(I + \lambda A) = X$, $\forall \lambda > 0$, as claimed.

3. The semi-flow generated by the operator A

In terms of A, equation (1.1) can be written as the infinite dimensional Cauchy problem

$$\begin{aligned} \frac{dy}{dt} + Ay &= 0 \qquad \text{on } (0, T), \\ y(0) &= y_0. \end{aligned} \tag{3.1}$$

By the Crandall-Liggett theorem (see, e.g., [1], p. 131), it follows that $-A$ generates on X a semigroup e^{-tA} of nonlinear contractions, that is, for each $y_0 \in X$ (in particular, for $y_0 \in W^{1,1}(\mathbb{R})$),

$$e^{-tA}y_0 = \lim_{n \to \infty} \left(I + \frac{t}{n}A\right)^{-n} y_0 \text{ in } X,\ \text{uniformly in } t \text{ on } [0, T]. \tag{3.2}$$

Such a function $y(t) = e^{-tA}y_0$ is called *mild* solution to equation (3.1) and, respectively (1.1), and, for each $h > 0$, one has

$$\begin{aligned} y(t) + hAy(t) &= y(t - h) \quad \forall t \ge 0, \\ y(t) &= y_0 \qquad\quad \forall t \le 0. \end{aligned}$$

We have, therefore,

Theorem 3.1. *For every $y_0 \in X$, the Cauchy problem* (1.1) *has a unique mild solution $y \in C([0, T]; X)$ given by the finite difference scheme*

$$y(t) = \lim_{h \to 0} y_h(t),\ \forall t \in [0, T], \tag{3.3}$$

where

$$y_h(t) = y_h^i \text{ for } t \in [ih, (i+1)h),\ i = 1, 2, ..., \left[\tfrac{T}{h}\right], \tag{3.4}$$

$$\begin{aligned} &y_h^i(t) - h(\beta((y_h^i)')' = y_h^{i-1} \text{ in } \mathbb{R},\ i = 1, 2, ..., \left[\tfrac{T}{h}\right],\\ &y_h^0 = y_0. \end{aligned} \tag{3.5}$$

By the definition of X, we have also

$$y_x \in C([0,T]; L^1(\mathbb{R})),\ y \in C([0,T]; L^1(\mathbb{R})) \cap C([0,T] \times \mathbb{R}). \tag{3.6}$$

Remark 3.2. If we set $z_h^i = (y_h^i)'$, we see by (3.5) that

$$\begin{aligned} &z_h^i - h(\beta(z_h^i))'' = z_h^{i-1} \text{ in } \mathbb{R},\\ &z_h^0 = y_0' \end{aligned} \tag{3.7}$$

and $z_h(t) \overset{h\to 0}{\longrightarrow} z(t) = y_x(t)$ in $L^1(\mathbb{R}^n)$. Hence $z = y_x$ is *the "mild" solution to the porous media equation* (1.2).

We may conclude that *at the level of the spaces* $(X, L^1(\mathbb{R}))$*, equations* (1.1)–(1.2) *are equivalent through the transformation* $z = y_x$.

4. Equation (1.1) for $\mathbb{R}(\beta) = \mathbb{R}$

We assume here that β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that

$$R(\beta) = \mathbb{R}, \tag{4.1}$$

where $R(\beta)$ is the range of β.

Let $j : \mathbb{R} \to \mathbb{R}$ be the potential of β, that is, a lower semicontinuous convex function such that $\beta = \partial j$ and $j(0) = 0$. (Here ∂j is the subdifferential of j.) We note that condition (3.2) is equivalent to

$$\lim_{|r|\to\infty} \frac{j(r)}{|r|} = +\infty. \tag{4.1$'$}$$

We have

Theorem 4.1. *Let* $y_0 \in L^2(\mathbb{R})$*. Then there is a unique solution* $y : [0,T] \times \mathbb{R} \to \mathbb{R}$ *which satisfies*

$$\begin{aligned} &y \in C([0,T]; L^2(\mathbb{R})), \quad y_x \in L^\infty(\delta, T; L^2(\mathbb{R})),\\ &y_t \in L^\infty(\delta, T; L^2(\mathbb{R})), \quad \forall \delta \in (0,T), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &y_t(t,x) - (\beta(y_x(t,x)))_x \ni 0, \quad && a.e.\ (t,x) \in (0,T) \times \mathbb{R},\\ &y(0,x) = y_0(x), && a.e.\ x \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Moreover, if $y_0' \in L^1(\mathbb{R})$ *and* $j(y_0') \in L^1(\mathbb{R})$*, then*

$$j(y_x(t,x)) \in W^{1,1}([0,T]; L^1(\mathbb{R})),\ y_t \in L^2((0,T); L^1(\mathbb{R})). \tag{4.4}$$

Finally, if $\exists \eta_0 \in L^2(\mathbb{R})$ *such that* $\eta_0 \in (\beta(y_0'))'$*, then* $y_t \in L^\infty(0,T; L^2(\mathbb{R}))$.

Proof. We define in $H = L^2(\mathbb{R})$ the operator

$$A_0 u = \{-\eta' \in L^2(\mathbb{R});\ \eta(x) \in \beta(u'(x)),\ \text{a.e. } x \in \mathbb{R}\},\ \forall u \in D(A_0),$$
$$D(A_0) = \{u \in L^2(\mathbb{R}),\ u' \in L^1(\mathbb{R})),\ \exists \eta \in L^2(\mathbb{R}),\ \eta(x) \in -(\beta(u'(x)))' \text{ a.e. } x \in \mathbb{R}\}. \tag{4.5}$$

(Here, all the derivatives are taken in sense of distributions, that is, in $\mathcal{D}'(\mathbb{R})$.)

Lemma 4.2. *The operator A_0 is m-accretive in $H \times H$.*

Proof. We consider the (energy) functional $\varphi : H \to \overline{\mathbb{R}} =]-\infty, +\infty]$,

$$\varphi(u) = \begin{cases} \int_{\mathbb{R}} j(u'(x))dx & \text{if } u' \in L^1(\mathbb{R}),\ j(u') \in L^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.6}$$

We set $D(\varphi) = \{u \in H;\ \varphi(u) < +\infty\}$. Clearly, φ is convex and lower semicontinuous on H. Indeed, if $\varphi(u_n) \le C$ and $u_n \to u$ in $L^1(\mathbb{R})$, it follows that for each $\delta > 0$ there is $C_\delta > 0$ such that for any Lebesgue measurable set $Q \subset \mathbb{R}$ with the Lebesgue measure $m(Q) \le C_\delta$, we have

$$\int_Q |u_n'(x)|dx \le \delta,\ \forall n.$$

The latter follows by assumption $(4.1)'$ taking into account that $\varphi(u_n) \le C$ implies that

$$\int_Q |u_n'(x)|dx \le \int_{Q \cap \{x; |u_n'(x)| \ge N\}} + \int_{Q \cap \{x; |u_n'(x)| \le N\}} \le \frac{C}{N} + Nm(Q),\ \forall N > 0.$$

Then, by the Dunford–Pettis compactness criterium, it follows that $\{u_n'\}$ is weakly compact in $L^1(\mathbb{R})$ and, therefore, on a subsequence $u_n' \to u'$ weakly in $L^1(\mathbb{R})$. Since the functional $v \to \int_{\mathbb{R}} j(v)dx$ is, by Fatou's lemma, lower semicontinuous in $L^1(\mathbb{R})$, being convex, it is also weakly lower semicontinuous and so

$$\liminf_{n\to\infty} \int_{\mathbb{R}} j(u_n'(x))dx \ge \int_{\mathbb{R}} j(u'(x))dx.$$

Hence $\varphi(u) \le C$, as claimed.

On the other hand, for each $f \in H$, the equation $u + Au \ni f$, that is,

$$u - (\beta(u'))' \ni f \quad \text{in } \mathcal{D}'(\mathbb{R}),$$
$$u \in L^2(\mathbb{R}),\quad u' \in L^1(\mathbb{R}), \tag{4.7}$$

has a solution. Indeed, we associate with (4.7) the minimization problem

$$\text{Min}\left\{\int_{\mathbb{R}}\left(\frac{1}{2}\,u^2(x) + j(u'(x)) - f(x)u(x)\right)dx; u \in L^2(\mathbb{R}), u' \in L^1(\mathbb{R})\right\}$$
$$= \text{Min}\left\{\frac{1}{2}\,\|u\|^2_{L^2(\mathbb{R})} + \varphi(u) - \langle f, u\rangle_{L^2(\mathbb{R})}\right\}. \tag{4.8}$$

Taking into account that the functional

$$u \longrightarrow \frac{1}{2}\,\|u\|^2_{L^2(\mathbb{R})} + \varphi(u) - \langle f, u\rangle_{L^2(\mathbb{R})}$$

is convex, lower semicontinuous and coercive on $H = L^2(\mathbb{R})$, it follows that there is a unique solution u^* to equation (4.7). Since for such a u^* we have

$$\int_{\mathbb{R}} (u^*v + j'((u^*)')v' - fv)dx \geq 0,$$

for all $v \in L^2(\mathbb{R})$ with $v' \in L^1(\mathbb{R})$, where j' is the directional derivative of j, and that $\partial j = \beta$, it follows that u^* is a solution to (4.7). Hence, $\mathbb{R}(I + A_0) = H$. Moreover, $A_0 = \partial\varphi$ is the subdifferential of φ. Indeed, if $z = -\eta' \in A_0 u$, then, for all $v \in L^2(\mathbb{R})$ with $v' \in L^1(\mathbb{R})$, we have

$$\eta(x)(u'(x) - v'(x)) \geq j(u'(x)) - j(v'(x)), \text{ a.e. } x \in \mathbb{R}$$

and, integrating on $(-\infty, +\infty)$, we get

$$\int_{\mathbb{R}} z(u - v)dx = \int_{\mathbb{R}} \eta(u' - v')dx \geq \varphi(u) - \varphi(v)$$

and, therefore, $A_0 \subset \partial\varphi$. Since A_0 is m-accretive, it is maximal accretive and so this implies $A_0 = \partial\varphi$.

Proof of Theorem 4.1 (continued). By the general existence theory for the Cauchy problem associated to m-accretive nonlinear operators of subdifferential form in Hilbert spaces (see, e.g., [1], p. 157), it follows that, for each $y_0 \in \overline{D(A_0)} = H$, the Cauchy problem

$$\begin{aligned} &\frac{dy(t)}{dt} + A_0 y(t) \ni 0 \quad \text{a.e. } t \in (0, T), \\ &y(0) = y_0, \end{aligned} \tag{4.9}$$

has a unique solution $y \in C([0, T]; H)$ with $\sqrt{t}\,\frac{dy}{dt} \in L^2(0, T; H)$. Moreover, if $y_0 \in D(\varphi)$, then $\frac{dy}{dt} \in L^2(0, T; H)$ and $\varphi(y) \in W^{1,1}([0, T])$, $\forall T > 0$. Finally, if $y_0 \in D(A_0)$, then $\frac{dy}{dt} \in L^\infty(0, T; H)$, that is, $y \in W^{1,\infty}([0, T]; H)$.

This concludes the proof.

Remark 4.3. As in the previous case (see Remark 3.2), we have

$$y(t) = \lim_{h \to 0} y_h(t) \text{ in } L^2(\mathbb{R}) \text{ and uniformly on } [0, T],$$

where

$$y_h(t) = y_h^i \text{ on } (ih, (i + 1)h)$$

and y_h^i is defined by the finite difference scheme (3.4). If we set $z_h^i = (y_h^i)'$, we have

$$\begin{aligned} &z_h^i - h(\beta(z_h^i))'' \ni z_h^{i-1}, \quad i = 1, 2, ..., \\ &z_h^0 = y_0'. \end{aligned} \tag{4.10}$$

Since $y_h'(t) \in L^1(\mathbb{R})$ $\forall t$, we infer that $z_h(t) \in L^1(\mathbb{R})$ for all h, and that, for $y_0 \in L^2(\mathbb{R})$,

$$z_h(t) \to z(t) = y_x(t) \text{ in } H^{-1}(\mathbb{R}) \text{ uniformly on } [0, T],$$

where $z \in C([0, T]; H^{-1}(\mathbb{R}))$ is the solution to the porous media equation (1.2) with the initial condition $z_0 = y_0' \in H^{-1}(\mathbb{R})$.

Remark 4.4. Theorem 4.1 remains true in $\mathbb{R}^d$, $d \geq 1$, for parabolic equations of the form

$$\begin{aligned} &y_t(t,x) - \operatorname{div} \beta(\nabla_x y(t,x)) \ni 0, \quad \forall t \geq 0,\ x \in \mathbb{R}^d, \\ &y(0,x) = y_0(x), \qquad\qquad\qquad\qquad x \in \mathbb{R}^d, \end{aligned} \tag{4.11}$$

where $\beta : \mathbb{R}^d \to \mathbb{R}^d$ is a maximal monotone operator of the form $\beta = \partial j$ and $j : \mathbb{R}^d \to (-\infty, +\infty]$ is a lower semicontinuous convex function satisfying

$$\lim_{|r|_d \to \infty} \frac{j(r)}{|r|_d} = +\infty. \tag{4.12}$$

(Here $|\cdot|_d$ is the Euclidean norm of $\mathbb{R}^d$.) As in the previous case, (4.11) can be rewritten as (4.9), where $H = L^2(\mathbb{R}^d)$ and A_0 is the operator

$$\begin{aligned} A_0 u &= \{-\operatorname{div} \eta \in L^2(\mathbb{R}^d);\ \eta(x) \in \beta(\nabla u(x)),\ \text{a.e. } x \in \mathbb{R}^d\},\ \forall u \in D(A_0), \\ D(A_0) &= \{u \in L^2(\mathbb{R}^d);\ \nabla u \in (L^1(\mathbb{R}^d))^d;\ \exists \eta \in (L^2(\mathbb{R}^d))^d, \\ &\qquad\qquad \eta(x) \in -\beta(\nabla u(x)) \text{ a.e. } x \in \mathbb{R}^d\}. \end{aligned}$$

We omit the details.

5. Equations with singular diffusivity

We shall study here equation (1.1) in the special case

$$\beta(r) = \operatorname{sign} r = \begin{cases} \frac{r}{|r|} & \text{for } r \neq 0, \\ [-1,1] & \text{for } r = 0, \end{cases} \tag{5.1}$$

that is,

$$\begin{aligned} &y_t(t,x) - (\operatorname{sign} y_x(t,x))_x = 0,\ t \in (0,T),\ x \in \mathbb{R}, \\ &y(0,x) = y_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5.2}$$

The corresponding operator A_0 defined by (4.5) is no longer m-accretive in $H = L^2(\mathbb{R})$.

In fact, the corresponding energy functional $\varphi : H \to \overline{\mathbb{R}} =]-\infty, +\infty]$,

$$\varphi(u) = \begin{cases} \int_{\mathbb{R}} |u'(x)| dx & \text{if } u' \in L^1(\mathbb{R}), \\ +\infty & \text{otherwise}, \end{cases} \tag{5.3}$$

is not lower semicontinuous, and its ℓ.s.c. closure $\overline{\varphi} : H \to \overline{\mathbb{R}}$ is given by

$$\overline{\varphi}(u) = \begin{cases} \int_{\mathbb{R}} |Du| & \text{if } u \in BV(\mathbb{R}), \\ +\infty & \text{otherwise}, \end{cases} \tag{5.4}$$

where $\int_{\mathbb{R}} |Du|$ is the total variation of u and $BV(\mathbb{R})$ is the space of functions $u : \mathbb{R} \to \mathbb{R}$ with bounded variation. The function $\overline{\varphi}$ is convex and lower semicontinuous on H and so its subdifferential $\partial\overline{\varphi} = A_1$ is m-accretive in $H \times H$. Then, by the general existence theory, *for each $y_0 \in L^2(\mathbb{R})$ there is a unique $y^* \in C([0,T]; L^2(\mathbb{R}))$ such that*

$$\begin{aligned} &\frac{dy^*}{dt}(t) + \partial\overline{\varphi}(y^*(t)) \ni 0, \quad \text{a.e. } t \in (0,T), \\ &y^*(0) = y_0, \end{aligned} \tag{5.5}$$

$$\sqrt{t}\ \frac{dy^*}{dt} \in L^2((0,T)\times\mathbb{R}). \tag{5.6}$$

If $y_0 \in BV(\mathbb{R})$, *then*

$$\frac{dy^*}{dt} \in L^2((0,T)\times\mathbb{R}). \tag{5.7}$$

However, since $\partial\overline{\varphi}$ is hard to describe, we get an idea of how (5.5) looks like from the following approximating process.

For each $\varepsilon > 0$, we set

$$\begin{aligned} (A_1)_\varepsilon u &= -(|u'|^\varepsilon \operatorname{sign} u')' \\ D(A_1)_\varepsilon) &= \{u \in L^2(\mathbb{R}); u' \in L^1(\mathbb{R}), ((u')^\varepsilon \operatorname{sign}\ u')' \in L^2(\mathbb{R})\}. \end{aligned}$$

In other words, $(A_1)_\varepsilon = \partial\varphi_\varepsilon$, where

$$\varphi_\varepsilon(u) = \begin{cases} \frac{1}{1+\varepsilon}\int_{-\infty}^{\infty} |u'|^{1+\varepsilon}dx & \text{if } u' \in L^{1+\varepsilon}(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.8}$$

By Theorem 4.1, there is a unique solution $u_\varepsilon \in C([0,T]; L^2(\mathbb{R}))$ with $\sqrt{t}\ \frac{\partial u_\varepsilon}{\partial t} \in L^2((0,T)\times\mathbb{R})$ of the equation

$$\begin{aligned} &\frac{\partial y_\varepsilon}{\partial t} - (|(y_\varepsilon)_x|^\varepsilon \operatorname{sign}(y_\varepsilon)_x)_x = 0, \ \text{a.e. } t > 0,\ x \in \mathbb{R}, \\ &y_\varepsilon(0,x) = y_0(x). \end{aligned} \tag{5.9}$$

On the other hand, for each $\lambda > 0$ and $f \in L^2(\mathbb{R})$, the solution $u^\varepsilon \in D((A_0)_\varepsilon)$ to the equation

$$u_\varepsilon - \lambda(|(u_\varepsilon)'|^\varepsilon \operatorname{sign}(u_\varepsilon)')' = f \text{ in } \mathbb{R},$$

or equivalently

$$u_\varepsilon = \arg\min \left\{ \int_{\mathbb{R}} \left(\frac{1}{2}\,|u|^2 + \frac{\lambda}{1+\varepsilon}\,|u'|^{1+\varepsilon} - fu \right) dx \right\},$$

converges in $L^2(\mathbb{R})$ to

$$u = \arg\min_u \left\{ \int_{\mathbb{R}} \left(\frac{1}{2}\,|u|^2 - fu \right) dx + \int_{\mathbb{R}} |Du| \right\}.$$

Indeed, for each $\varepsilon > 0$,

$$\int_{\mathbb{R}} \left(\frac{1}{2}\,|u_\varepsilon|^2 + \frac{\lambda}{1+\varepsilon}\,|(u_\varepsilon)'|^{1+\varepsilon} - fu^\varepsilon \right) dx \le \int_{\mathbb{R}} \left(\frac{1}{2}\,|u|^2 + \frac{\lambda}{1+\varepsilon}\,|u'|^{1+\varepsilon} - fu \right) dx,$$

$\forall u \in D(\overline{\varphi})$, and, letting $\varepsilon \to 0$, we get $u_\varepsilon \to \widetilde{u}$ weakly in $L^2(\mathbb{R})$ and

$$\widetilde{u} = \arg\min_{u \in BV} \left\{ \int_{\mathbb{R}} \left(\frac{1}{2}\,|u|^2 - fu \right) dx + \lambda \int_{\mathbb{R}} |Du| \right\}.$$

Hence

$$\widetilde{u} = (I + \lambda A_1)^{-1} f.$$

Then, by the Trotter–Kato theorem for nonlinear semigroups of contractions (see, e.g., [1], p. 170), we have

$$y_\varepsilon(t) \to y^*(t) \text{ strongly in } C([0,T]; L^2(\mathbb{R})), \tag{5.10}$$

as $\varepsilon \to 0$, where y^* is the solution to (5.5). *In this generalized sense, y^* can be viewed as a solution to* (5.2).

If we set $y_x = z$, we may rewrite (5.2) as the porous media equation

$$z_t - (\text{sign } z)_{xx} = 0, \quad t \geq 0, \qquad (5.11)$$
$$z(0) = y_0' = z_0,$$

which in the above sense, for $z_0 \in M(\mathbb{R})$ (the space of Borelian measures on $\mathbb{R}$), has a solution $z \in C([0,T]; M(\mathbb{R}))$, $\forall T > 0$.

Remark 5.1. The above existence result for (5.2) extends in $\mathbb{R}^d$ mutatis-mutandis to the equation

$$y_t - \text{div}\left(\frac{\nabla y}{|\nabla y|}\right) = 0, \quad x \in \mathbb{R}^d,$$
$$y(0,x) = y_0(x).$$

In this case, $H = L^2(\mathbb{R}^d)$, and $\overline{\varphi} : H \to]-\infty, +\infty]$ is given by

$$\overline{\varphi}(u) = \begin{cases} \int_{\mathbb{R}^d} |Du| & \text{if } u \in BV(\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

where $BV(\mathbb{R}^d)$ is the space of functions with bounded variation on $\mathbb{R}^d$.

This equation is relevant in image restoring techniques (see [2] and [8]).

Remark 5.2. The results of this section extends to the maximal monotone graphs $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ such that, for some $\rho > 0$,

$$\eta r \geq \rho |r|, \quad \forall \eta \in \beta(r), \ r \in \mathbb{R}, \qquad (5.12)$$

or

$$\beta = \partial j \ \text{ and } \ j(r) \geq \rho |r|, \ \forall r \in \mathbb{R}. \qquad (5.13)$$

In this case, $\overline{\varphi}$ is the closure of the functional

$$\varphi(u) = \begin{cases} \int_{\mathbb{R}} j(u')dx, & u' \in L^1(\mathbb{R}), \\ +\infty, & \text{otherwise,} \end{cases}$$

and so

$$\overline{\varphi}(u) \geq \rho \int_{\mathbb{R}} |Du|, \ \forall u \in D(\varphi).$$

We omit the details.

6. An example

The nonlinear parabolic equation

$$y_t - a(y_x)(W'(y_x))_x = 0, \ t \geq 0, \ x \in \mathbb{R}, \qquad (6.1)$$
$$y(0,x) = y_0(x), \ x \in \mathbb{R},$$

where W is a convex function on $\mathbb{R}$, $u(0) = 0$ and a is a given nonnegative continuous function, is relevant in materials sciences as a model of interface evolution of two phases of materials as well as in crystal growth and was studied in [4], [5], [6].

By the formal substitution

$$\beta(y)=\int_0^y a(r)W''(r)dr,\ y\in\mathbb{R}, \tag{6.2}$$

it reduces to (1.1). However, if W'' is not in $L^1_{\rm loc}(\mathbb{R})$, assuming that a is smooth (of class C^1 with $a''\in L^1_{\rm loc}(\mathbb{R})$), ≥ 0 and $W(0)=0$, $0\in\partial W(0)$, we can take β as

$$\beta(y)=a(y)\partial W(y)-a'(y)W(y)+\int_0^y a''(s)W(s)ds \tag{6.3}$$

and so Theorem 3.1 is applicable in this case. Moreover, if

$$\beta(+\infty)=+\infty,\ \beta(-\infty)=\infty, \tag{6.4}$$

then we may apply Theorem 4.1.

Finally, if $W(y)=|y|$ and

$$a(y)\geq\rho,\quad \forall y\in\mathbb{R},\ (-1)^k a^{(k)}\geq 0,\ k=1,2, \tag{6.5}$$

then β satisfies condition (5.13) and so the equation

$$\begin{aligned} &y_t-a(y_x)(\text{sign } y_x)_x=0,\ x\in\mathbb{R},\\ &y(0,x)=y_0(x), \end{aligned} \tag{6.6}$$

has, for each $y_0\in L^2(\mathbb{R})$, a unique solution $y\in C([0,T];L^2(\mathbb{R}))$ in sense of (5.5)–(5.7).

Consequently, the porous media equation

$$\begin{aligned} &z_t-(a(z)(\text{sign } z)_x)_x=0,\quad x\in\mathbb{R},\ t\geq 0,\\ &z(0,x)=y_0'(x), \end{aligned} \tag{6.7}$$

has a (generalized) solution $z\in L^2(0,T;H^{-1}(\mathbb{R}))$.

7. Equation (1.1) with periodic conditions

Theorem 4.1 remains true if the space $L^2(\mathbb{R})$ or $L^1(\mathbb{R})$ is replaced by

$$L^2_\pi(\mathbb{R})=\{u\in L^2_{\rm loc}(\mathbb{R});\ u(x+L)\equiv u(x),\ x\in\mathbb{R}\}$$

with the standard Hilbertian norm. For instance, the operator A_0 defined by (4.5) is replaced in this case by

$$\begin{aligned} A_\pi u &= -(\beta(u'))',\quad u\in D(A_\pi),\\ D(A_\pi) &= \{u\in H_\pi(\mathbb{R});\ u'\in L^1_{\rm loc}(\mathbb{R}),\ (\beta(u'))'\in H_\pi\}. \end{aligned}$$

The details are omitted.

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