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THE CONVEXITY PRINCIPLE FOR SET-VALUED MAPPINGS AND APPLICATION TO LOCAL PROGRAMMING

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Dedicated to Professor Masao Fukushima on the occasion of his 75th birthday

Abstract: In the papers ([14, 15]), Polyak has established a convexity principle, stated that the image of a small ball by a $C^{1,1}-$smooth mapping between Hilbert spaces is convex. This convexity principle has some interesting applications in optimization and control theory. In this note, we give an extension of this principle to weakly convex multifunctions and its applications to local programming.

Key words: *convexity, image, inverse multifunction, weak convexity, local programming*

Mathematics Subject Classification: *49J52, 49J53, 52A05, 90C26*

1 Introduction and Preliminaries

In 2001, Polyak gave a convexity principle which is stated precisely as follows.

Theorem 1.1 ([14, 15]). *Let $f : X \to Y$ be a nonlinear differentiable mapping between the two Hilbert spaces X, Y such that its derivative f' is Lipschitz on a ball $B(a, r)$ with Lipschitz constant L. Suppose that $f'(a)$ is surjective. Then there exists a positive real ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, the image of the ball $B(a, \varepsilon)$, $f(B(a, \varepsilon))$ is convex.*

In ([14, 15]), this convexity principle has various interesting applications in optimization and in some control problems. In [18], Uderzo has extented this principle to the case of multifunctions which are the sum of a convex multifunction and a $C^{1,1}-$mapping.

In this short note, firstly, we establish an inverse multifunction theorem. By using this result, we obtain an extension of this principle to the case of weakly convex multifunctions. Some applications of this result to local programming with weakly convex data are reported.

We recall some basic notions from nonsmooth analysis (see, e.g., books [1, 2, 7, 12]). For a Banach space X, the open and closed balls entered at $x \in X$ with radius $r > 0$ are denoted respectively by $B(x, r), B[x, r]$, while B_X denotes the unit ball in X.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued function defined on a Banach space X. The effective domain and the epigraph of f are denoted by $\operatorname{Dom} f := \{x \in X : f(x) < +\infty\}$;

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$\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} : \; f(x) \le \alpha\}$, respectively. The *Fréchet subdifferential* $\partial f(x)$ of f at $x \in \operatorname{Dom} f$ is defined as

$$\partial f(x) := \left\{ x^* \in X^* : \; \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \ge 0 \right\}$$
$$\text{and } \partial f(x) := \emptyset \text{ if } \; f(x) = +\infty.$$

The *Fréchet normal cone* to a subset $C \subseteq X$ at some point $x \in C$ is defined as the Fréchet subdifferential of the indicator function δ_C of C at the same point:

$$N_C(x) := \partial \delta_C(x) = \left\{ x^* \in X^* : \; \limsup_{z \to x} \frac{\langle x^*, z - x \rangle}{\|z - x\|} \le 0 \right\}.$$

When X is Asplund, i.e., when every continuous convex function defined on X is generically Fréchet differentiable, the Fréchet subdifferential enjoys a fuzzy sum rule (see, e.g., [2]): For any $\varepsilon > 0$, for $x \in \operatorname{Dom} f_1 \cap \operatorname{Dom} f_2$, provided f_1, f_2 are lower semicontinuous and one of them is locally Lipschitz around x, one has

$$\partial(f_1 + f_2)(x) \subseteq \bigcup \{ \quad \partial f_1(x_1) + \partial f_2(x_2) + \varepsilon \mathbb{B}_{X^*} : \; (x_i, f(x_i)) \in \mathbb{B}((x, f(x)), \varepsilon), \; i = 1, 2\}. \tag{1.1}$$

Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces X, Y. Throughout we assume that F is a closed multifuntion, that is, the graph of F, $\operatorname{gph} F$, is closed in $X \times Y$. As usual, denote by

$$\operatorname{Dom} F := \{x \in X : \;\; F(x) \ne \emptyset\}$$

the effective domain of F, and the inverse of F : $F^{-1} : Y \rightrightarrows X$, defined by

$$F^{-1}(y) = \{x \in X : \;\; y \in F(x)\}, \;\; y \in X.$$

The *contingent derivative* of F at a given point $(x, y) \in \operatorname{gph} F$ denoted by $DF(x, y) : X \rightrightarrows Y$ and is defined by, for $u \in X$,

$$DF(x, y)(u) := \{ \quad v \in Y : \;\; \exists (u_n) \to u, \; (v_n) \to v, \; t_n \to 0^+, \; \text{with } (x + t_n u_n, y + t_n v_n) \in \operatorname{gph} F\}.$$

Note that for all $(x, y) \in \operatorname{gph} F$, $DF(x, y)$ is a positive homogeneous set-valued mapping. Also recall the *inner norm* of a positive homogeneous set-valued mapping $H : X \rightrightarrows Y$ is

$$\|H\| := \sup_{\|u\| \le 1} \inf\{\|v\| : \;\; v \in H(u), \;\; u \in X\}.$$

Here, a convention $\inf \emptyset = +\infty$ is used.

Finally, for a multifunction $F : X \rightrightarrows Y$, the *Fréchet coderivative* of F at a point $(x, y) \in \operatorname{gph} F$, refers to a multifunction $DF^*(x, y) : Y^* \rightrightarrows X^*$ and defined as

$$DF^*(x, y)(y^*) := \{x^* : \;\; (x^*, -y^*) \in N_{\operatorname{gph} F}(x, y)\}, \;\; y^* \in Y^*,$$

for every $(x, y) \in \operatorname{gph} F$. Associated to a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, define the multifunction $F : X \rightrightarrows R$ by $\operatorname{gph} F = \operatorname{epi} f$, that is, $F(x) = [f(x), +\infty)$, $x \in \mathbb{R}^n$. One has

$$x^* \in \partial f(x) \iff x^* \in DF^*(x, f(x))(1).$$

Finally, recall the notion of weak convexity ([6, 8, 16, 17, 11, 19]). An extended real valued function f defined on X is said to be *weakly convex* around a point $a \in \operatorname{Dom} f$, if there are $\rho > 0, \varepsilon > 0$ such that any $x_1, x_2 \in B[\bar{x}, \varepsilon]$, any $t \in [0,1]$, one has

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) + \frac{\rho}{2}t(1-t)\|x_1 - x_2\|^2.$$

Note that when f is weakly convex around a (with respect to ρ, ε as above), for any $x \in B[\bar{x}, \varepsilon]$, $x^* \in \partial f(x)$, one has

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \frac{\rho}{2}\|y - x\|^2, \ \ \forall y \in B[a, \varepsilon]. \tag{1.2}$$

Definition 1.2. A multifunction $F : X \rightrightarrows Y$ between Banach spaces X, Y is called weakly convex around $\bar{x} \in X$ if there exist $\rho > 0$, $\varepsilon > 0$ such that for any $x_1, x_2 \in B(\bar{x}, \varepsilon)$, any $t \in [0, 1]$, one has

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + \frac{\rho}{2}t(1-t)\|x_1 - x_2\|^2 B_Y.$$

For example (see, e.g., [11]), a mapping $f : X \to Y$ being of $C^{1,1}-$class around a point $\bar{x} \in X$, is weakly convex around this point; the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is a weakly convex function around $\bar{x} \in X$ *iff* the associated multifunction $F : \operatorname{gph} F = \operatorname{epi} f$, is weakly convex around this point (with the same constants).

2 An Inverse Multifunction Theorem

The inverse function theorem stated in the following is a set-valued version of the one by Ekeland in [4] (see, for the earlier version, e.g., in [5]).

Theorem 2.1. *Let X, Y be Banach spaces and let $F : X \rightrightarrows Y$ be a closed multifunction (i.e., its graph is a closed set in $X \times Y$). For given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and reals $R_1, R_2, M > 0$, suppose that*

$$\sup\{\|DF(x,y)^{-1}\| : \ \ (x,y) \in \operatorname{gph} F \cap (B(\bar{x}, R_1) \times B(\bar{y}, R_2))\} < M. \tag{2.1}$$

Let parameters $r_1, r_2, r_3 > 0$ such that

$$r_1 + Mr_3 < R_1, \ \ r_2 + r_3 < R_2. \tag{2.2}$$

Then for all $a \in B[\bar{x}, r_1]$; $b \in F(a) \cap B[\bar{y}, r_2]$, and $y \in Y$ with

$$\|y - b\| \leq r_3, \tag{2.3}$$

there exists $z \in X$ such that

$$z \in F^{-1}(y) \ \ \text{and} \ \ \|z - a\| \leq M\|y - b\|. \tag{2.4}$$

Proof. For given $y \in Y$ satisfying (2.7), take any $M_1 < M$ such that $MM_1^{-1}\|y - b\| < r_3$ and

$$\sup\{\|DF(x,y)^{-1}\| : \ \ (x,y) \in \operatorname{gph} F \cap (B(\bar{x}, R_1) \times B(\bar{y}, R_2))\} < M_1.$$

Consider the following norm in the product space $X \times Y$:

$$\|(u,v)\| = \max\{\|u\|, M_1\|v\|\}, \ \ (u,v) \in X \times Y,$$

and define the extended real valued function $\varphi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(u,v) := \|y - v\| + \delta_{\operatorname{gph} F}(u,v), \quad (u,v) \in X \times Y,$$

where $\delta_{\operatorname{gph} F}$ stands for the indicator function of $\operatorname{gph} F$. As

$$\varphi(a,b) = \|y - b\| \leq \inf_{(u,v)\in X\times Y} \varphi(u,v) + \|y - b\|,$$

by applying the Ekeland Variational Principle, there is $(z,w) \in X \times Y$ such that

(i) $\|(z,w) - (a,b)\| \leq M\|y - b\|$;

(ii) $\varphi(z,w) \leq \varphi(a,b) = \|y - b\|$;

(iii) $\varphi(z,w) \leq \varphi(u,v) + M^{-1}\|(u,v) - (z,w)\|$, for all $(u,v) \in X \times Y$.

By (i), $\|z - a\| \leq M\|y - b\| \leq Mr_3$, and $\|w - b\| \leq MM_1^{-1}\|y - b\| \leq MM_1^{-1}r_3$, therefore

$$\|z - \bar{x}\| \leq \|z - a\| + \|a - \bar{x}\| \leq Mr_3 + r_1 < R_1;$$

$$\|w - \bar{y}\| \leq \|w - b\| + \|b - \bar{y}\| \leq MM_1^{-1}r_3 + r_2 < R_2.$$

Relation (ii) implies $(z,w) \in \operatorname{gph} F$. We shall show that $w = y$, so $z \in F^{-1}(y)$. Indeed, assume to contrary that $w \neq y$, then in view of (2.1), there is $u \in X$, satisfying

$$u \in DF(z,w)^{-1}(y - w) \text{ and } \|u\| < M_1\|y - w\|.$$

Thus there are sequences $(u_n) \to u$, $(v_n) \to y - w$ and $(t_n) \to 0^+$ with $(z + t_n u_n, w + t_n v_n) \in \operatorname{gph} F$, for all $n \in \mathbb{N}$. From relation (iii), one has

$$\|y - w\| \leq \|y - w - t_n v_n\| + M^{-1} t_n \|(u_n, v_n)\|.$$

By making use of the following inequality

$$\begin{aligned} \|y - w - t_n v_n\| &= \|(1 - t_n)(y - w) + t_n(y - w - v_n)\| \\ &\leq (1 - t_n)\|y - w\| + t_n\|y - w - v_n\|, \end{aligned}$$

one derives that

$$\|y - w\| \leq \|y - w - v_n\| + M^{-1}\|(u_n, v_n)\|.$$

By letting $n \to \infty$, one obtains

$$0 < \|y - w\| \leq M^{-1} \max\{\|u\|, M_1\|y - w\|\} < \|y - w\|,$$

a contradiction. Thus $w = y$, showing the conclusion

$$z \in F^{-1}(y) \text{ and } \|z - a\| \leq M\|y - b\|.$$

□

When X, Y are Asplund spaces, one establish a following version of inverse function theorem, in which, a criteria based on the coderivatives is used, instead of the contingent derivative. Its proof follows the same schema as in the one of the preceding theorem, but here alternatively, the fuzzy sum rule (1.1) for Fréchet subdifferentials is made of use (see also, [10] for a related result), so it is omitted.

Theorem 2.2. *Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. For given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and reals $R_1, R_2, m > 0$, suppose that*

$$\inf\{d_*(0, DF^*(x,y)(S_{Y^*})) : \ (x,y) \in \operatorname{gph} F \cap (B(\bar{x}, R_1) \times B(\bar{y}, R_2))\} > m, \tag{2.5}$$

where, d_* (from a point to a set) stands for the distance in X^* associated to the dual norm, and S_{Y^*} denotes the unit sphere in Y^*.

Let parameters $r_1, r_2, r_3 > 0$ such that

$$r_1 + m^{-1} r_3 < R_1, \ \ r_2 + r_3 < R_2. \tag{2.6}$$

Then for all $a \in B[\bar{x}, r_1]$; $b \in F(a) \cap B[\bar{y}, r_2]$, and $y \in Y$ with

$$\|y - b\| \le r_3, \tag{2.7}$$

there exists $z \in X$ such that

$$z \in F^{-1}(y) \ \text{ and } \ \|z - a\| \le m^{-1}\|y - b\|. \tag{2.8}$$

3 The Convexity Principle for Weakly Convex Multifunctions

The main result is stated as follows, asserting the convexity of an image of a small ball by a weakly convex multifunction from a Hilbert space to a Banach space. This result is regarded as a set-valued version of the convexity principle by Polyak ([14, 15]).

Theorem 3.1. *Let X be a Hilbert space and let Y be a Banach space. Let $F : X \rightrightarrows Y$ be a closed multifunction. For given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and reals $R_1, R_2, M > 0$, suppose that the following two conditions are satisfied.*

(i) *either*

$$\sup\{\|DF(x,y)^{-1}\| : \ (x,y) \in \operatorname{gph} F \cap (B(\bar{x}, R_1) \times B(\bar{y}, R_2))\} < M, \tag{3.1}$$

or the space Y is assumed to be an Asplund space and

$$\inf\{d_*(0, DF^*(x,y)(S_{Y^*})) : \ (x,y) \in \operatorname{gph} F \cap (B(\bar{x}, R_1) \times B(\bar{y}, R_2))\} > m. \tag{3.2}$$

(ii) *The multifunction F is $\rho-$weakly convex on $B(\bar{x}, R_1)$, that is, for any $x_1, x_2 \in B(\bar{x}, R_1)$, any $t \in [0,1]$, one has*

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + \frac{\rho}{2} t(1-t)\|x_1 - x_2\|^2 B_Y.$$

Then for $\varepsilon_1, \varepsilon_2 > 0$ satisfying for $\alpha = M$ if (3.1) holds, and $\alpha = m^{-1}$ if (3.2) holds,

$$\varepsilon_1 \le 1/(\alpha\rho), \ \varepsilon_1 + \frac{\alpha\rho}{2}\varepsilon_1^2 < R_1, \ \rho\varepsilon_1^2 + \varepsilon_2 < R_2, \tag{3.3}$$

the set $F(B[\bar{x}, \varepsilon_1]) \cap B[\bar{y}, \varepsilon_2]$ is convex.

Proof. Let $x_1, x_2 \in B[\bar{x}, \varepsilon_1]$, $t \in [0,1]$, and $y_1 \in F(x_1) \cap B[\bar{y}, \varepsilon_2]$, $y_2 \in F(y_2) \cap B[\bar{y}, \varepsilon_2]$. For $y := y_t = ty_1 + (1-t)y_2$, by (ii), there is $v \in F(tx_1 + (1-t)x_2)$ such that

$$\|y - v\| \le \frac{\rho}{2}\|x_1 - x_2\|^2 t(1-t) \le \frac{\rho}{8}\|x_1 - x_2\|^2 \le \frac{\rho}{2}\varepsilon_1^2.$$

Thus

$$\|v-\bar{y}\| \le \|v-y\| + \|y-\bar{y}\| \le \frac{\rho}{2}\varepsilon_1^2 + \varepsilon_2.$$

In virtue of Theorem 2.1, with parameters $r_1 := \varepsilon_1$, $r_2 := \rho\varepsilon_1^2/2 + \varepsilon_2$, $r_3 := \rho\varepsilon_1^2/2$, there exists $u \in F^{-1}(y)$, such that

$$\|tx_1 + (1-t)x_2 - u\| \le M\|y-v\| \le \frac{M\rho}{2} t(1-t)\|x_1 - x_2\|^2.$$

As

$$\begin{aligned} \|tx_1 + (1-t)x_2 - \bar{x}\|^2 &= t\|x_1-\bar{x}\|^2 + (1-t)\|x_2-\bar{x}\|^2 - t(1-t)\|x_1-x_2\|^2 \\ &\le \varepsilon_1^2 - t(1-t)\|x_1-x_2\|^2, \end{aligned}$$

one has from the two relations above,

$$\begin{aligned} \|u-\bar{x}\| &\le \|tx_1+(1-t)x_2-u\| + \|tx_1+(1-t)x_2-\bar{x}\| \\ &\le \tfrac{M\rho}{2} t(1-t)\|x_1-x_2\|^2 + \left(\varepsilon_1^2 - t(1-t)\|x_1-x_2\|^2\right)^{1/2}. \end{aligned}$$

Since $\varepsilon_1 \le 1/(M\rho)$, it is straightforward to derive that

$$\frac{M\rho}{2} t(1-t)\|x_1-x_2\|^2 + \left(\varepsilon_1^2 - t(1-t)\|x_1-x_2\|^2\right)^{1/2} \le \varepsilon_1,$$

so $u \in B[\bar{x}, \varepsilon_1]$, hence, $y = ty_1 + (1-t)y_2 \in F(B[\bar{x},\varepsilon_1]) \cap B[\bar{y},\varepsilon_2]$, which shows that $F(B[\bar{x},\varepsilon_1]) \cap B[\bar{y},\varepsilon_2]$ is a convex set. □

Remark 3.2. Observe from the proof of the above theorem, one sees that, for $x_1, x_2 \in B[\bar{x},\varepsilon_1]$, $t \in [0,1]$, and $y_1 \in F(x_1) \cap B[\bar{y},\varepsilon_2]$, $y_2 \in F(y_2) \cap B[\bar{y},\varepsilon_2]$, then there exists $u \in B[\bar{x},\varepsilon_1]$ such that

$$\|tx_1 + (1-t)x_2 - u\| \le \frac{M\rho}{2} t(1-t)\|x_1-x_2\|^2, \;\; ty_1 + (1-t)y_2 \in F(u).$$

4 Application: Local Programming

Consider the optimization problem with a set-valued mapping constraint of the form:

$$\begin{aligned} &\min f_0(x) \\ &x \in \mathbb{R}^n, 0 \in G(x), \end{aligned} \tag{4.1}$$

where, $f_0 : \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a closed set-valued mapping. As in ([13, 14]), for a given feasible point $a \in \mathbb{R}^n$, and some $\varepsilon > 0$, associated to it, consider its local version with added constraint: $\|x-a\| \le \varepsilon$.

$$\begin{aligned} &\min f_0(x) \\ &x \in \mathbb{R}^n, 0 \in G(x), \\ &\|x-a\| \le \varepsilon. \end{aligned} \tag{4.2}$$

Define the multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m+1}$ by

$$F(x) = [f_0(x), +\infty] \times G(x), \; x \in \mathbb{R}^n. \tag{4.3}$$

The Lagrange function of (4.1) is of the form:

$$L(x,\lambda,y^*) := \lambda f_0(x) + \inf_{y\in G(x)} \langle y^*, y\rangle, \;\; (x,\lambda,y^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m. \tag{4.4}$$

Theorem 4.1. *Assume that the multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m+1}$ defined by (4.3) is weakly convex around a given $a \in \mathbb{R}^n$ with $0 \in F(a)$, and that for some $r > 0$,*

$$\inf\{d_*(0, DF^*(x,y)(S_{\mathbb{R}^{m+1}})) : \ (x,y) \in \operatorname{gph} F \cap (B(\bar{a}, r) \times \mathbb{R}^{m+1})\} > 0. \tag{4.5}$$

Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in]0, \varepsilon^*[$, for a solution $x^* \in B[a,\varepsilon]$ of problem (4.2), there exists $(\lambda^*, y^*) \in \mathbb{R} \times \mathbb{R}^m \setminus \{(0, 0_{\mathbb{R}^m})\}$ with $\lambda^* \geq 0$, such that

$$L(x, \lambda^*, y^*) \geq L(x^*, \lambda^*, y^*), \ \ \forall x \in B[a, \varepsilon], \tag{4.6}$$

and

$$\inf_{y \in G(x^*)} \langle y^*, y \rangle = 0. \tag{4.7}$$

Proof. By Theorem 3.1, there is $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the image set $F(B[a,\varepsilon]) \subseteq \mathbb{R}^{m+1}$ is a closed convex set. For x^* being a solution of (4.2), the point $(f_0(x^*), 0_{\mathbb{R}^m})$ is a boundary point of $F(B[a,\varepsilon])$. Hence by the separating theorem (for convex sets), we can find $(\lambda^*, y^*) \in \mathbb{R} \times \mathbb{R}^m \setminus \{(0, 0_{\mathbb{R}^m})\}$ such that

$$\langle (\lambda^*, y^*), (\alpha, y) \rangle \geq \langle (\lambda^*, y^*), (f_0(x^*), 0) \rangle, \ \forall (\alpha, y) \in F(B[a,\varepsilon]).$$

Therefore, one has $\lambda^* \geq 0$, and

$$\lambda^* \alpha + \langle y^*, y \rangle \geq \lambda^* f_0(x^*), \ \forall (\alpha, y) \in F(B[a,\varepsilon]),$$

and this follows directly (4.6) and (4.7). □

Consider the standard mathematical programming:

$$\begin{aligned} &\min f_0(x), \ x \in \mathbb{R}^n, \\ &f_i(x) \leq 0, \ i = 1, ..., p \\ &f_i(x) = 0, \ i = p+1, ..., m, \end{aligned} \tag{4.8}$$

and its local version with respect to a feasible point a and $\varepsilon > 0$,

$$\begin{aligned} &\min f_0(x), \ x \in \mathbb{R}^n, \\ &f_i(x) \leq 0, \ i = 1, ..., p \\ &f_i(x) = 0, \ i = p+1, ..., m, \\ &\|x - a\| \leq \varepsilon. \end{aligned} \tag{4.9}$$

where, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, are lower semicontinuous functions. Obviously, (4.9) is a particular case of (4.2) with the multifunction $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$G(x) = \Pi_{i=1}^{p} [f_i(x), +\infty] \times \{(f_{p+1}(x)), ..., f_m(x))\}, \ x \in \mathbb{R}^n. \tag{4.10}$$

Assume that f_i, $i = 0, ..., p$, are weakly convex functions around a, and f_i, $i = p+1, ..., m$, are $C^{1,1}$ functions around a. So G and F are weakly convex multifunctions around a. For points $x \in \mathbb{R}^n$ (near a), and for

$$(x, y) := (x, \alpha_0, \alpha_1, ..., \alpha_p, f_{p+1}(x), ..., f_m(x)) \in \operatorname{gph} F,$$

By a straightforward computation, the coderivative mapping $DF^*(x,y)$ is given as follows.

For $y^* = (y_1^*, ..., y_m^*) \in \mathbb{R}^m$,

$$DF^*(x,y)(y^*) = \emptyset \text{ if either } y_i^* < 0 \text{ or } y_i^*(f_i(x) - \alpha_i) \neq 0,$$
$$\text{for some } i \in \{0, ..., p\}, \ \text{ otherwise,}$$
$$DF^*(x,y)(y^*) = \left\{ x^* : \ x^* \in \sum_{i=0}^{m} y_i^* \partial f_i(x) \right\}. \tag{4.11}$$

Denote by

$$m(a) = \inf \left\{ \|x^*\| : \begin{array}{l} x^* \in \sum_{i=0}^m y_i^* \partial f_i(a), \ \|y^*\| = \|(y_0^*, ..., y_m^*)\| = 1 \\ y_i^* \geq 0, \ y_i^* f_i(a) = 0 \text{ for } i = 0, ..., p. \end{array} \right\} \tag{4.12}$$

The Lagrangian of problem (4.8) is defined by:

$$L(x, \lambda, y) = \lambda f_0(x) + \sum_{i=1}^{m} y_i f_i(x). \tag{4.13}$$

From Theorem 4.1, one obtains the following one which generalizes ([14], Theo. 4.1, and 4.2).

Theorem 4.2. *With above assumptions, if $m(a) > 0$, then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in]0, \varepsilon^*[$, a solution $x^* \in B[a, \varepsilon]$ of problem (4.2) exists, is unique, lines on the boundary of $B[a, \varepsilon] : \|x^* - a\| = \varepsilon$, and that there exists $(\lambda^*, y^*) \in \mathbb{R} \times \mathbb{R}^m \setminus \{(0, 0_{\mathbb{R}^m})\}$ with $\lambda^* \geq 0$, such that*

$$L(x, \lambda^*, y^*) \geq L(x^*, \lambda^*, y^*), \ \ \forall x \in B[a, \varepsilon], \tag{4.14}$$

and

$$y_i^* \geq 0, \ y_i^* f_i(x^*) = 0, \ \textit{for } i = 1, ..., p. \tag{4.15}$$

Moreover, suppose further the following regularity condition is satisfied: for any $\varepsilon > 0$, any $\delta = (\delta_1, \delta_2, ..., \delta_m) \in \mathbb{R}^m$, with $\delta_i = 1$ for $i = 1, ..., p$; $|\delta_i| = 1$ for $i = p+1, ..., m$, there exists $x_\delta \in B[a, \varepsilon]$ such that

$$\delta_i f_i(x_\delta) < 0, \ \ i = 1, ..., m, \tag{4.16}$$

then one can take $\lambda = 1$, and in this case, the two conditions (4.14), (4.15) are necessary and sufficient condition of optimality for local programming (4.9).

Proof. Note that obviously in this particular situation, condition (4.5) in Theorem 4.1 is equivalent to $m(a) > 0$. So by virtue of Theorem 4.1, there exists $(\lambda^*, y^*) \in \mathbb{R} \times \mathbb{R}^m \setminus \{(0, 0_{\mathbb{R}^m})\}$ with $\lambda^* \geq 0$, such that

$$\lambda^* f_0(x) + \inf_{y \in G(x)} \langle y^*, y \rangle \geq \lambda^* f_0(x^*), \ \ \forall x \in B[a, \varepsilon],$$

and $\inf_{y \in G(x^*)} \langle y^*, y \rangle = 0$, where G defined by (4.10). The second relation implies directly $y_i^* \geq 0$, $y_i^* f_i(x^*) = 0$ for $i = 1, ..., p$. Therefore, the first implies, for all $x \in B[a, \varepsilon]$,

$$\begin{aligned} L(x, \lambda^*, y^*) &= \lambda^* f_0(x) + \sum_{i=1}^{m} y_i^* f_i(x) \\ &= \lambda^* f_0(x) + \inf_{y \in G(x)} \langle y^*, y \rangle \geq \lambda_0^* f(x^*) = L(x^*, \lambda^*, y^*). \end{aligned}$$

Suppose now (4.16) holds, if $\lambda^* = 0$ then by taking δ, x_δ as in 4.16 with $\delta_i = \text{sign} y_i^*$, it is derived to

$$L(x_\delta, \lambda^*, y^*) = \sum_{i=1}^{m} y_i^* f_i(x_\delta) < 0 = L(x^*, \lambda^*, y^*),$$

a contradiction. So $\lambda^* \neq 0$ and therefore one can take $\lambda^* = 1$. For this case, the optimal sufficiency of conditions (4.14) and (4.15), is obvious. □

Lemma 4.3. *Assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a weakly convex lower semicontinuous function around point $a \in \operatorname{Dom} f$. If $0 \notin \partial f(a)$, then there exist $\alpha, \varepsilon > 0$ such that for all $x^* \in B[a, \varepsilon]$, all $y^* \in \partial f(x^*)$ one has*

$$\langle y^*, x - x^* \rangle \leq f(x) - f(x^*) + \alpha \|y^*\| \|x - x^*\|^2, \text{ for all } x \in B[a, \varepsilon].$$

Proof. As $0 \notin f(a)$, for the associated weakly convex multifunction $F(x) = [f(x), +\infty)$, there is $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $F(B[a, \varepsilon])$ is a convex set. For given $0 < \varepsilon < \varepsilon^*$, such that f being weakly convex with some constant $\rho > 0$, on $B[a, \varepsilon]$; for all $x^* \in B[a, \varepsilon]$, $y^* \in \partial f(x^*)$, one has

$$\langle y^*, x - x^* \rangle \leq f(x) - f(x^*) + \frac{\rho}{2} \|x - x^*\|^2, \text{ for all } x \in B[a, \varepsilon].$$

For $x \in B[a, \varepsilon] \cap \operatorname{Dom} f$, $t \in]0, 1[$, as was noticed in Remark 3.2, for some $M > 0$, there is $u \in B[a, \varepsilon]$ with

$$\|tx + (1-t)x^* - u\| \leq \frac{M\rho}{2} t(1-t) \|x - x^*\|^2;$$
$$f(u) \leq t f(x) + (1-t) f(x^*).$$

Hence, by virtue of above inequalities,

$$\begin{aligned}
&\langle y^*, t(x - x^*) \rangle = \langle y^*, u - x^* \rangle + \langle y^*, x^* + t(x - x^*) - u^* \rangle \\
&\leq \langle y^*, u - x^* \rangle + \|y^*\| \|x^* + t(x - x^*) - u^*\| \\
&\leq f(u) - f(x^*) + \frac{\rho}{2} \|u - x^*\|^2 + \frac{M\rho}{2} t(1-t) \|y^*\| \|x - x^*\|^2 \\
&\leq t(f(x) - f(x^*)) + t^2 \|x - x^*\|^2 \left(1 + M\rho \|x - x^*\|/2\right)^2 + \frac{M\rho}{2} t(1-t) \|y^*\| \|x - x^*\|^2.
\end{aligned}$$

By deviding the inequality by $t > 0$, then letting $t \to 0$, one obtains

$$\langle y^*, x - x^* \rangle \leq f(x) - f(x^*) + \alpha \|y^*\| \|x - x^*\|^2, \ \alpha := M\rho/2.$$

□

Lemma 4.4. *Assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a weakly convex lower semicontinuous function around point $a \in \operatorname{Dom} f$, and $0 \notin \partial f(a)$. Then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the problem:*

$$\min f(x) : \ x \in \mathbb{R}^n, \ \|x - a\| \leq \varepsilon,$$

attains a unique solution x^* with $\|x^* - a\| = \varepsilon$ and the necessary and sufficient condition holds: There is $y^* \in \partial f(x^*)$ such that

$$x^* = a - \varepsilon \frac{y^*}{\|y^*\|}.$$

Proof. Let $x^* \in B[a,\varepsilon]$ be a solution of the problem. As $0 \notin \partial f(a)$, when ε is enough small, $0 \notin \partial f(x^*)$, so $\|x^* - a\| = \varepsilon$. By the standard necessary optimal condition, there exists $y^* \in \partial f(x^*)$ such that $x^* = a - \varepsilon \frac{y^*}{\|y^*\|}$. To show this is a sufficient condition, for $\varepsilon^* > 0$ enough small $0 < \varepsilon < \varepsilon^*$ such as $F(B[a,\varepsilon])$ is convex, where, $F(x) := [f(x), +\infty]$. By the preceding lemma, for some $\alpha > 0$,

$$\langle y^*, x - x^* \rangle \leq f(x) - f(x^*) + \alpha \|y^*\| \|x - x^*\|^2, \text{ for all } x \in B[a,\varepsilon].$$

Thus for any $x \in B[a,\varepsilon]$,

$$\varepsilon^{-1}\|y^*\|\langle a - x^*, x - x^* \rangle \leq f(x) - f(x^*) + \alpha \|y^*\| \|x - x^*\|^2. \tag{4.17}$$

On the other hand, since

$$\|a - x^*\|^2 = \|a - x\|^2 + 2\langle a - x, x - x^* \rangle + \|x - x^*\|^2,$$

and $\|a - x\| \leq \varepsilon = \|a - x^*\|$,

$$\langle a - x, x - x^* \rangle \geq -\|x - x^*\|^2/2,$$

which implies

$$\langle a - x^*, x - x^* \rangle = \langle a - x, x - x^* \rangle + \|x - x^*\|^2 \geq \|x - x^*\|^2/2.$$

So, by (4.17),

$$(\varepsilon^{-1} - \alpha)\|y^*\| \|x - x^*\|^2 \leq f(x) - f(x^*).$$

Therefore x^* is a strongly optimal solution when $0 < \varepsilon < 1/\alpha$, and consequently the solution x^* is unique. □

Theorem 4.2 together with the above two lemmas yield the following optimal necessary and sufficient condition for local programming (4.9).

Theorem 4.5. *Suppose that $m(a) > 0$ and the regularity condition (4.16) in Theorem 4.2 holds. Then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, a point $x^* \in B[a,\varepsilon]$ is a optimal solution of local programming (4.9) if and only if there is $y^* \in \mathbb{R}^m$ satisfying*

$$y_i^* \geq 0, \; y_i^* f_i(x^*) = 0, \; \textit{for } i = 1, ..., p;$$
$$x^* = a - \varepsilon \frac{u^*}{\|u^*\|}, \;\; u^* \in \partial L(x^*, 1, y^*).$$

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