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THE BOX CONVOLUTION AND THE DILWORTH TRUNCATION OF BISUBMODULAR FUNCTIONS*

Satoru Fujishige† and Sachin B. Patkar‡

Dedicated to Masao Fukushima on the occasion of his 75th birthday

Abstract: For a signed ring family $\mathcal{F} \subseteq 3^E$ (closed with respect to the reduced union and intersection) and for a bisubmodular function $f : \mathcal{F} \to \mathbb{R}$ with $(\emptyset, \emptyset) \in \mathcal{F}$ and $f(\emptyset, \emptyset) = 0$, the bisubmodular polyhedron associated with $(\mathcal{F}, f)$ is given by

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbb{R}^E, \forall (X,Y) \in \mathcal{F} : x(X,Y) \leq f(X,Y)\},$$

where $x(X,Y) = \sum_{e \in X} x(e) - \sum_{e \in Y} x(e)$. We define the convolution of a bisubmodular function f and a special bisubmodular function w called a box-bisubmodular function determined by upper and lower bound vectors w^+ and w^-. We show that the convolution is a bisubmodular function, too. The bisubmodular polyhedron associated with the convolution is shown to be the intersection of $\mathrm{P}_*(f)$ and the box determined by the upper and lower bound vectors w^+ and w^-. This also generalizes some known min-max results on bisubmodular functions and ordinary submodular functions. Moreover, we consider the Dilworth truncation of bisubmodular functions, which generalizes the Dilworth truncation of submodular functions.

Key words: *bisubmodular functions, box convolution, Dilworth truncation, submodular functions*

Mathematics Subject Classification: *90C27, 90C47, 90C57*

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1 Introduction

A bisubmodular function is a generalization of an ordinary submodular function and some of the results on submodular functions can naturally be generalized to those on bisubmodular functions (see [3, 4, 6, 7, 9, 10, 11, 13, 14, 23, 25, 26, 27]). A characterization of b-matching degree-sequence polyhedra is nicely given by means of bisubmodular functions in [12]. Also, a min-max theorem with respect to the ℓ_1 norm for bisubmodular polyhedra is given in [18] as a generalization of a min-max relation shown in [12].

The convolution of a submodular function and a modular function plays a fundamental role in the theory of submodular functions (see [15, 17]). In the present paper we consider

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the convolution of a bisubmodular function and a so-called box-bisubmodular function and reveal its implications. The result implies the min-max theorem given by W. H. Cunningham and J. Green-Krótki [12] associated with b-matching degree-sequence polyhedra as well as the well-known min-max relations concerning a vector reduction of polymatroids and submodular systems ([15, 17]). We also consider the Dilworth truncation of bisubmodular functions which generalizes that of submodular functions.

In Section 2 we give some basic definitions about bisubmodular functions. Section 3 treats the convolution of bisubmodular functions and gives a formula for the convolution and its implications. In Section 4 we consider the Dilworth truncation of bisubmodular functions.

2 Basic Definitions

For a finite nonempty set E define

$$3^E = \{(X,Y) \mid X, Y \subseteq E, X \cap Y = \emptyset\}. \tag{2.1}$$

Note that each element $(X,Y) \in 3^E$ can be made one-to-one correspond to its characteristic vector $\chi_{(X,Y)} \in \{0, \pm 1\}^E$, where

$$\chi_{(X,Y)}(e) = \begin{cases} 1 & \text{if } e \in X \\ -1 & \text{if } e \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (e \in E). \tag{2.2}$$

We call an element of 3^E a *signed set*. For any $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$ we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Also we write $(X_1, Y_1) \sqsubset (X_2, Y_2)$ if $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ and $(X_1, Y_1) \neq (X_2, Y_2)$. The binary relation $\sqsubseteq$ is a partial order on 3^E. We call $(\emptyset, \emptyset) \in 3^E$ the *null signed set*.

We consider two binary operations $\sqcup$ (*reduced union*) and $\sqcap$ (*intersection*) on 3^E defined as follows. For any $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$,

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)), \tag{2.3}$$

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2). \tag{2.4}$$

Let $\mathcal{F}$ be a family of signed sets in 3^E that is closed with respect to the reduced union $\sqcup$ and the intersection $\sqcap$. We call such a family $\mathcal{F}$ of signed sets a *signed ring family*. A function $f : \mathcal{F} \to \mathbb{R}$ on a signed ring family $\mathcal{F}$ is a *bisubmodular function* if for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ we have

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)). \tag{2.5}$$

It should be noted that we have the following equations:

$$\chi_{(X_1,Y_1)} + \chi_{(X_2,Y_2)} = \chi_{(X_1,Y_1)\sqcup(X_2,Y_2)} + \chi_{(X_1,Y_1)\sqcap(X_2,Y_2)} \tag{2.6}$$

and for any $x \in \mathbb{R}^E$

$$x(X_1, Y_1) + x(X_2, Y_2) = x((X_1, Y_1) \sqcup (X_2, Y_2)) + x((X_1, Y_1) \sqcap (X_2, Y_2)), \tag{2.7}$$

where for any $X \subseteq E$ $x(X) = \sum_{e \in X} x(e)$, $x(\emptyset) = 0$, and for any $(X, Y) \in 3^E$

$$x(X,Y) = x(X) - x(Y). \tag{2.8}$$

In the following we assume that $(\emptyset, \emptyset) \in \mathcal{F}$ and $f(\emptyset, \emptyset) = 0$. Then the pair $(\mathcal{F}, f)$ is called a *bisubmodular system* on E (see [3, 4, 6, 7, 17, 18]). When $\mathcal{F} = 3^E$, a bisubmodular system is called a polypseudomatroid ([11, 23]) (also see [17, Sec. 3.5(b)] and [3] for properties of bisubmodular functions and related concepts).

It should be noted that the argument throughout this paper is valid when $\mathbb{R}$ is any totally ordered additive group such as the sets of reals, rationals, and integers.

The *bisubmodular polyhedron* $\mathrm{P}_*(f)$ associated with the bisubmodular system $(\mathcal{F}, f)$ on E is given by

$$\mathrm{P}_*(f) = \{x \in \mathbb{R}^E \mid \forall (X, Y) \in \mathcal{F} : x(X, Y) \leq f(X, Y)\}. \tag{2.9}$$

It is known that we have $\mathrm{P}_*(f) \neq \emptyset$ for every bisubmodular system $(\mathcal{F}, f)$ on E (see [2, 17]).

3 The Box Convolution of Bisubmodular Functions

Let $(\mathcal{F}, f)$ be a bisubmodular system on E and suppose that we are given two vectors $w^+ \in \mathbb{R}^E$ and $w^- \in \mathbb{R}^E$ with $w^+ \geq w^-$. A *box-bisubmodular function* $w : 3^E \to \mathbb{R}$ is defined in terms of such two vectors w^+ and w^- as follows. We define for each $(X, Y) \in 3^E$

$$w(X, Y) = w^+(X) - w^-(Y). \tag{3.1}$$

We can easily see that the function $w : 3^E \to \mathbb{R}$ is bisubmodular and that its associated bisubmodular polyhedron is a box in $\mathbb{R}^E$ given by

$$\mathrm{P}_*(w) = \{x \in \mathbb{R}^E \mid w^- \leq x \leq w^+\}. \tag{3.2}$$

We define the *convolution*, denoted by $f \circ w$, of the bisubmodular function f and the box-bisubmodular function w determined by upper and lower bound vectors w^+ and w^- as follows. For any $(X, Y) \in 3^E$,

$$f \circ w(X, Y) = \min\{f(\hat{X}, \hat{Y}) + w(X \setminus \hat{X}, Y \setminus \hat{Y}) + w(\hat{Y} \setminus Y, \hat{X} \setminus X) \mid (\hat{X}, \hat{Y}) \in \mathcal{F}\}. \tag{3.3}$$

Here, note that we do not impose restrictions such as $\hat{X} \subseteq X$ and $\hat{Y} \subseteq Y$. Equation (3.3) can be rewritten as

$$\begin{aligned} f \circ w(X, Y) &= \min\{f(\hat{X}, \hat{Y}) + w^+(X \setminus \hat{X}) - w^-(Y \setminus \hat{Y}) \\ &\qquad + w^+(\hat{Y} \setminus Y) - w^-(\hat{X} \setminus X) \mid (\hat{X}, \hat{Y}) \in \mathcal{F}\} \\ &= \min\{f(\hat{X}, \hat{Y}) + w^+((X \cup \hat{X}) \setminus \hat{X}) - w^-(Y \setminus (Y \cap \hat{Y})) \\ &\qquad + w^+(\hat{Y} \setminus (Y \cap \hat{Y})) - w^-((X \cup \hat{X}) \setminus X) \mid (\hat{X}, \hat{Y}) \in \mathcal{F}\} \\ &= \min\{f(\hat{X}, \hat{Y}) + w^+((X \cup \hat{X}) \setminus \hat{X}) - w^-((X \cup \hat{X}) \setminus X) \\ &\qquad + w^+(\hat{Y} \setminus (Y \cap \hat{Y})) - w^-(Y \setminus (Y \cap \hat{Y})) \mid (\hat{X}, \hat{Y}) \in \mathcal{F}\} \\ &= \min\{f(\hat{X}, \hat{Y}) + w^+(\tilde{X} \setminus \hat{X}) - w^-(\tilde{X} \setminus X) + w^+(\hat{Y} \setminus \tilde{Y}) - w^-(Y \setminus \tilde{Y}) \\ &\qquad \mid (\hat{X}, \hat{Y}) \in \mathcal{F}, \tilde{X} \supseteq X \cup \hat{X}, \tilde{Y} \subseteq Y \cap \hat{Y}\}, \end{aligned} \tag{3.4}$$

where the last equality is due to the non-negativity of the difference vector $w^+ - w^-$.

We use the following lemma to show the bisubmodularity of $f \circ w$.

Lemma 3.1. *Let x be a vector in $\mathbb{R}^E$ and A, B, C, D be subsets of E such that $A \supseteq B$ and $C \supseteq D$. Then we have*

$$x(A \setminus B) + x(C \setminus D) = x((A \cup C) \setminus (B \cup D)) + x((A \cap C) \setminus (B \cap D)), \tag{3.5}$$

$$x(A \setminus B) + x(C \setminus D) = x((A \setminus D) \setminus (B \setminus C))) + x((C \setminus B) \setminus (D \setminus A)). \tag{3.6}$$

Proof. The validity of (3.5) and (3.6) can easily be seen by drawing the Venn diagram of the four sets A, B, C, D with $A \supseteq B$ and $C \supseteq D$. $\square$

Now, we have the following theorem.

Theorem 3.2. *The function $f \circ w : 3^E \to \mathbb{R}$ defined by* (3.3) *is a bisubmodular function.*

Proof. It follows from (3.4) that for any $(X,Y), (V,W) \in 3^E$ there exist $X_i, Y_i, V_i, W_i \subseteq E$ $(i = 1, 2)$ with

$$(X_1, Y_1),\ (V_1, W_1) \in \mathcal{F}, \tag{3.7}$$

$$X_2 \supseteq X \cup X_1, \quad Y_2 \subseteq Y \cap Y_1, \quad V_2 \supseteq V \cup V_1, \quad W_2 \subseteq W \cap W_1 \tag{3.8}$$

such that

$$f \circ w(X,Y) = f(X_1, Y_1) + w(X_2 \setminus X_1, Y \setminus Y_2) + w(Y_1 \setminus Y_2, X_2 \setminus X), \tag{3.9}$$

$$f \circ w(V,W) = f(V_1, W_1) + w(V_2 \setminus V_1, W \setminus W_2) + w(W_1 \setminus W_2, V_2 \setminus V). \tag{3.10}$$

From (3.8) and (3.5) in Lemma 3.1 we have

$$w^+(X_2 \setminus X_1) + w^+(V_2 \setminus V_1) = w^+((X_2 \cup V_2) \setminus (X_1 \cup V_1)) + w^+((X_2 \cap V_2) \setminus (X_1 \cap V_1)), \tag{3.11}$$

$$w^+(Y_1 \setminus Y_2) + w^+(W_1 \setminus W_2) = w^+((Y_1 \cup W_1) \setminus (Y_2 \cup W_2)) + w^+((Y_1 \cap W_1) \setminus (Y_2 \cap W_2)), \tag{3.12}$$

$$w^-(Y \setminus Y_2) + w^-(W \setminus W_2) = w^-((Y \cup W) \setminus (Y_2 \cup W_2)) + w^-((Y \cap W) \setminus (Y_2 \cap W_2)), \tag{3.13}$$

$$w^-(X_2 \setminus X) + w^-(V_2 \setminus V) = w^-((X_2 \cup V_2) \setminus (X \cup V)) + w^-((X_2 \cap V_2) \setminus (X \cap V)). \tag{3.14}$$

Moreover, since

$$X_2 \cup V_2 \supseteq X_1 \cup V_1, \qquad X_2 \cap V_2 \supseteq X_1 \cap V_1, \tag{3.15}$$

$$Y_1 \cup W_1 \supseteq Y_2 \cup W_2, \qquad Y_1 \cap W_1 \supseteq Y_2 \cap W_2, \tag{3.16}$$

$$Y \cup W \supseteq Y_2 \cup W_2, \qquad Y \cap W \supseteq Y_2 \cap W_2, \tag{3.17}$$

$$X_2 \cup V_2 \supseteq X \cup V, \qquad X_2 \cap V_2 \supseteq X \cap V, \tag{3.18}$$

we have from (3.6) in Lemma 3.1

$$\begin{aligned} &w^+((X_2 \cup V_2) \setminus (X_1 \cup V_1)) + w^+(Y_1 \cup W_1) \setminus (Y_2 \cup W_2)) \\ &= w^+(((X_2 \cup V_2) \setminus (Y_2 \cup W_2)) \setminus ((X_1 \cup V_1) \setminus (Y_1 \cup W_1))) \\ &\quad + w^+(((Y_1 \cup W_1) \setminus ((X_1 \cup V_1)) \setminus ((Y_2 \cup W_2) \setminus (X_2 \cup V_2)))), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &w^+((X_2 \cap V_2) \setminus (X_1 \cap V_1)) + w^+(Y_1 \cap W_1) \setminus (Y_2 \cap W_2)) \\ &= w^+(((X_2 \cap V_2) \setminus (Y_2 \cap W_2)) \setminus (X_1 \cap V_1)) \\ &\quad + w^+((Y_1 \cap W_1) \setminus ((Y_2 \cap W_2) \setminus (X_2 \cap V_2))), \end{aligned} \tag{3.20}$$

$$\begin{aligned} &w^-((Y \cup W) \setminus (Y_2 \cup W_2)) + w^-(X_2 \cup V_2) \setminus (X \cup V)) \\ &= w^-(((Y \cup W) \setminus (X \cup V)) \setminus ((Y_2 \cup W_2) \setminus (X_2 \cup V_2))) \\ &\quad + w^-(((X_2 \cup V_2) \setminus (Y_2 \cup W_2)) \setminus ((X \cup V) \setminus (Y \cup W))), \end{aligned} \tag{3.21}$$

$$\begin{aligned}
&w^-((Y\cap W)\setminus(Y_2\cap W_2))+w^-(X_2\cap V_2)\setminus(X\cap V))\\
&=w^-((Y\cap W)\setminus((Y_2\cap W_2)\setminus(X_2\cap V_2)))\\
&\quad +w^-(((X_2\cap V_2)\setminus(Y_2\cap W_2))\setminus(X\cap V)). \qquad (3.22)
\end{aligned}$$

Also, by the bisubmodularity of f we have

$$f(X_1,Y_1)+f(V_1,W_1)\ge f((X_1\cup V_1)\setminus(Y_1\cup W_1),(Y_1\cup W_1)\setminus(X_1\cup V_1))+f(X_1\cap V_1,Y_1\cap W_1). \qquad (3.23)$$

Combining (3.9)–(3.23), we have

$$\begin{aligned}
&f\circ w(X,Y)+f\circ w(V,W)\\
&\ge f((X_1\cup V_1)\setminus(Y_1\cup W_1),(Y_1\cup W_1)\setminus(X_1\cup V_1))\\
&\quad +w^+(((X_2\cup V_2)\setminus(Y_2\cup W_2))\setminus((X_1\cup V_1)\setminus(Y_1\cup W_1)))\\
&\quad -w^-(((X_2\cup V_2)\setminus(Y_2\cup W_2))\setminus((X\cup V)\setminus(Y\cup W)))\\
&\quad +w^+(((Y_1\cup W_1)\setminus(X_1\cup V_1))\setminus((Y_2\cup W_2)\setminus(X_2\cup V_2)))\\
&\quad -w^-(((Y\cup W)\setminus(X\cup V))\setminus((Y_2\cup W_2)\setminus(X_2\cup V_2)))\\
&\quad +f(X_1\cap V_1,Y_1\cap W_1)\\
&\quad +w^+(((X_2\cap V_2)\setminus(Y_2\cap W_2))\setminus(X_1\cap V_1))\\
&\quad -w^-(((X_2\cap V_2)\setminus(Y_2\cap W_2))\setminus(X\cap V))\\
&\quad +w^+((Y_1\cap W_1)\setminus((Y_2\cap W_2)\setminus(X_2\cap V_2)))\\
&\quad -w^-((Y\cap W)\setminus((Y_2\cap W_2)\setminus(X_2\cap V_2))), \qquad (3.24)
\end{aligned}$$

where

$$(X_2\cup V_2)\setminus(Y_2\cup W_2)\supseteq((X\cup V)\setminus(Y\cup W))\cup((X_1\cup V_1)\setminus(Y_1\cup W_1)), \qquad (3.25)$$

$$(Y_2\cup W_2)\setminus(X_2\cup V_2)\subseteq((Y\cup W)\setminus(X\cup V))\cap((Y_1\cup W_1)\setminus(X_1\cup V_1)), \qquad (3.26)$$

$$(X_2\cap V_2)\setminus(Y_2\cap W_2)\supseteq(X\cap V)\cup(X_1\cap V_1), \qquad (3.27)$$

$$(Y_2\cap W_2)\setminus(X_2\cap V_2)\subseteq(Y\cap W)\cup(Y_1\cap W_1). \qquad (3.28)$$

From (3.24)–(3.28) and (3.4) we have the following inequality.

$$f\circ w(X,Y)+f\circ w(V,W)\ge f\circ w((X,Y)\sqcup(V,W))+f\circ w((X,Y)\sqcap(V,W)). \qquad (3.29)$$

This completes the proof of the present theorem. □

Remark 3.3. Theorem 3.2 is valid without the assumption that $\mathrm{P}_*(f)\cap\mathrm{P}_*(w)\neq\emptyset$ (which will be imposed in Theorem 3.4), so that we may have $f\circ w(\emptyset,\emptyset)<0$ here. It should be noted that if $Q=\{x\in\mathbb{R}^E\mid\forall(X,Y)\in\mathcal{F}\setminus\{(\emptyset,\emptyset)\}: x(X,Y)\le f\circ w(X,Y)\}$ is nonempty, then Q is the polyhedron of the *Dilworth truncation* of $f\circ w$, which will be discussed in Section 4 (see Theorem 4.2).

Moreover, we have the following theorem.

Theorem 3.4. *Suppose that* $\mathrm{P}_*(f)\cap\mathrm{P}_*(w)\neq\emptyset$*. Then we have* $f\circ w(\emptyset,\emptyset)=0$ *and the bisubmodular polyhedron associated with the convolution* $f\circ w$ *is given by*

$$\begin{aligned}
\mathrm{P}_*(f\circ w)&=\mathrm{P}_*(f)\cap\mathrm{P}_*(w)\\
&=\{x\mid x\in\mathrm{P}_*(f),\ w^-\le x\le w^+\}. \qquad (3.30)
\end{aligned}$$

Proof. If $\mathrm{P}_*(f) \cap \mathrm{P}_*(w) \neq \emptyset$, then for any vector $x \in \mathrm{P}_*(f) \cap \mathrm{P}_*(w)$ and any $(X, Y) \in \mathcal{F}$ we have

$$x(X,Y) \leq f(X,Y), \quad x(Y) \leq w^+(Y), \quad x(X) \geq w^-(X). \tag{3.31}$$

Hence, for any $(X, Y) \in \mathcal{F}$,

$$f(X,Y) + w^+(Y) - w^-(X) \geq x(X,Y) + x(Y) - x(X) = 0. \tag{3.32}$$

It follows from (3.3) and (3.32) that

$$f \circ w(\emptyset, \emptyset) = \min\left\{ f(\hat{X}, \hat{Y}) + w^+(\hat{Y}) - w^-(\hat{X}) \mid (\hat{X}, \hat{Y}) \in \mathcal{F} \right\} = 0. \tag{3.33}$$

Therefore, the bisubmodular polyhedron $\mathrm{P}_*(f \circ w)$ is well defined.

For any $(X_0, Y_0) \in \mathcal{F}$ the inequality

$$x(X_0, Y_0) \leq f \circ w(X_0, Y_0) \tag{3.34}$$

is implied by the system of inequalities

$$x(X,Y) \leq f(X,Y) \qquad ((X,Y) \in \mathcal{F}), \tag{3.35}$$

$$w^-(e) \leq x(e) \leq w^+(e) \qquad (e \in E) \tag{3.36}$$

due to the definition (3.3) of $f \circ w$. Moreover, it follows from (3.3) that

$$f \circ w(X,Y) \leq f(X,Y) \qquad ((X,Y) \in \mathcal{F}), \tag{3.37}$$

$$f \circ w(X, \emptyset) \leq w^+(X) \qquad (X \subseteq E), \tag{3.38}$$

$$f \circ w(\emptyset, Y) \leq -w^-(Y) \qquad (Y \subseteq E). \tag{3.39}$$

Hence, we have (3.30). □

The above argument is valid even if the upper-bound vector w^+ has components equal to $+\infty$ and the lower-bound vector w^- has components equal to $-\infty$ (i.e., $w^+ \in (\mathbb{R} \cup \{+\infty\})^E$ and $w^- \in (\mathbb{R} \cup \{-\infty\})^E$). In such a case the domain of the convolution $f \circ w$ is a signed ring family including $\mathcal{F}$ and being possibly a strict subset of 3^E.

Remark 3.5. Our arguments throughout the present paper hold for any totally ordered additive group. Hence Theorem 3.4 implies the following integrality property:

- When $\mathbb{R}$ is the set of reals, f is integer-valued, and w^+ and w^- are integral vectors allowing $\pm\infty$ components, the box-convolution $f \circ w$ is an integer-valued bisubmodular function and there exists an integral vector in $\mathrm{P}_*(f \circ w) \subseteq \mathbb{R}^E$.

The integrality property of bisubmodular functions is discussed in relation to *integrally convex functions* in [24].

For any given $v \in \mathbb{R}^E$ and $S \subseteq E$ define $w^+ \in (\mathbb{R} \cup \{+\infty\})^E$ and $w^- \in (\mathbb{R} \cup \{-\infty\})^E$ by

$$w^-(e) = \begin{cases} v(e) & (e \in S) \\ -\infty & (e \in E \setminus S) \end{cases}, \qquad w^+(e) = \begin{cases} v(e) & (e \in E \setminus S) \\ +\infty & (e \in S) \end{cases}. \tag{3.40}$$

Also define the partial order $\leq_S$ on $\mathbb{R}^E$ by $x \leq_S y \Leftrightarrow x(e) \geq y(e)$ $(e \in S)$ and $x(e) \leq y(e)$ $(e \in E \setminus S)$. Then we have the following corollary.

Corollary 3.6. *Given any $v \in \mathbb{R}^E$ and $S \subseteq E$, for w^- and w^+ defined by* (3.40) *we have*

$$\begin{aligned} \mathrm{P}_*(f \circ w) &= \mathrm{P}_*(f) \cap \mathrm{P}_*(w) \\ &= \{x \mid x \in \mathrm{P}_*(f),\ x \leq_S v\}. \end{aligned} \tag{3.41}$$

In particular (when $S = \emptyset$, i.e., $w^- \in \{-\infty\}^E$), we have the following corollary due to Cunningham and Green-Krótki [12] (for $\mathcal{F} = 3^E$).

Corollary 3.7 (Cunningham–Green-Krótki)**.** *For any $w^+ \in \mathbb{R}^E$ such that $\{x \in \mathrm{P}_*(f) \mid x \leq w^+\} \neq \emptyset$ we have*

$$\max\{x(E) \mid x \in \mathrm{P}_*(f),\ x \leq w^+\} = \min\{f(\hat{X}, \hat{Y}) + w^+(E \setminus \hat{X}) + w^+(\hat{Y}) \mid (\hat{X}, \hat{Y}) \in \mathcal{F}\}. \tag{3.42}$$

Proof. The present theorem follows from Theorem 3.4 and Corollary 3.6 with $S = E$. □

The feasibility condition, $\mathrm{P}_*(f) \cap \mathrm{P}_*(w) \neq \emptyset$, appearing in Theorem 3.4 can also be expressed as follows. We need some definitions from [17, Sec. 3.5(b)]. A signed set $(S,T) \in 3^E$ with $S \cup T = \emptyset$ is called an *orthant* of $\mathbb{R}^E$. For each orthant (S,T) define $2^{(S,T)} = \{(X,Y) \in 3^E \mid (X,Y) \sqsubseteq (S,T)\}$ and

$$\mathrm{P}_{(S,T)}(f) = \{x \in \mathbb{R}^E \mid \forall (X,Y) \in \mathcal{F} \cap 2^{(S,T)} : x(X,Y) \leq f(X,Y)\}. \tag{3.43}$$

For any polyhedron $Q \subseteq \mathbb{R}^E$ and $U \subseteq E$ define a *reflection* of Q by U as

$$Q|U = \{x \mid y \in \mathbb{R}^E, \forall e \in U : x(e) = -y(e),\ \forall e \in E \setminus U : x(e) = y(e)\}. \tag{3.44}$$

The reflection of $\mathrm{P}_{(S,T)}(f)$ by T is a submodular polyhedron associated with the ordinary submodular set function $\bar{f}$ defined by $\bar{f}(X) = f(S \cap X, T \cap X)$ for $X \subseteq E$ with $(S \cap X, T \cap X) \in \mathcal{F}$. It is known that $\mathrm{P}_*(f)$ is equal to the intersection of all $\mathrm{P}_{(S,T)}(f)$ for all orthants (S,T).

Theorem 3.8. *Suppose that we are given a bisubmodular system $(\mathcal{F}, f)$ on E and two vectors $w^+, w^- \in \mathbb{R}^E$ with $w^- \leq w^+$. We have $\mathrm{P}_*(f) \cap \mathrm{P}_*(w) \neq \emptyset$ if and only if for every orthant (S,T) of $\mathbb{R}^E$ we have*

$$w_{(S,T)} \in \mathrm{P}_{(S,T)}(f), \tag{3.45}$$

where $w_{(S,T)} \in \mathbb{R}^E$ is defined by

$$w_{(S,T)}(e) = \begin{cases} w^-(e) & (e \in S) \\ -w^+(e) & (e \in T) \end{cases} \qquad (e \in E). \tag{3.46}$$

Proof. Note that for every orthant (S,T) the vector $w_{(S,T)}$ is the minimum vector in the reflected box $\mathrm{P}_*(w)|T$. Hence the "only if" part is easy. So we show the "if" part in the following.

Suppose that (3.45) holds for every orthant (S,T), which is equivalent to the following system of inequalities, due to (3.43).

$$w^-(X) - w^+(Y) \leq f(X,Y) \qquad ((X,Y) \in (\mathcal{F} \cap 2^{(S,T)})) \tag{3.47}$$

for all orthants (S,T). It follows from (3.3) and (3.47) that

$$f \circ w(\emptyset, \emptyset) = \min\{f(X,Y) + w^+(Y) - w^-(X) \mid (X,Y) \in \mathcal{F}\} \geq 0, \tag{3.48}$$

where the last inequality holds with equality since $(X,Y) = (\emptyset, \emptyset) \in \mathcal{F}$. Hence, from Theorem 3.2 and (3.48) we have a bisubmodular system $(3^E, f \circ w)$, so that $\mathrm{P}_*(f) \cap \mathrm{P}_*(w) = \mathrm{P}_*(f \circ w) \neq \emptyset$. □

Remark 3.9. When $f(E,\emptyset)+f(\emptyset,E)=0$, the bisubmodular polyhedron $\mathrm{P}_*(f)$ becomes a base polyhedron lying on the hyperplane $x(E)=f(E,\emptyset)(=-f(\emptyset,E))$ (see, e.g., [17]). In this case we need (3.45) (or (3.47)) only for two orthants $(S,T)\in\{(E,\emptyset),(\emptyset,E)\}$ in order to guarantee $\mathrm{P}_*(f)\cap\mathrm{P}_*(w)\neq\emptyset$ (see [17, Theorem 3.8] and for matroid base polytopes in [22]).

Bisubmodular/submodular functions and their associated polyhedra have very recently drawn much attention in the field of algebraic geometry and combinatorics (see, e.g., [1, 8] and [16, 19] for box convolution).

4 The Dilworth Truncation of Bisubmodular Functions

Let $\mathcal{F}\subseteq 3^E$ be a signed ring family with $(\emptyset,\emptyset)\in\mathcal{F}$ and $f:\mathcal{F}\to\mathbb{R}$ be a bisubmodular function. In this section we do not assume $f(\emptyset,\emptyset)=0$. If $f(\emptyset,\emptyset)\geq 0$, then re-defining $f(\emptyset,\emptyset)=0$, we obtain a bisubmodular function $f:\mathcal{F}\to\mathbb{R}$ again. Hence we consider the case where $f(\emptyset,\emptyset)<0$ in the sequel. In this case the system of linear inequalities

$$x(X,Y)(=x(X)-x(Y))\leq f(X,Y) \qquad ((X,Y)\in\mathcal{F}\setminus\{(\emptyset,\emptyset)\}) \tag{4.1}$$

for $x\in\mathbb{R}^E$ is possibly inconsistent. So we impose the following assumption:

(A) (4.1) is consistent, i.e., there exists a feasible solution $x\in\mathbb{R}^E$ for (4.1).

We call the set of distinct signed sets (X_i,Y_i) $(i=1,2,\ldots,k)$ in $\mathcal{F}\setminus\{(\emptyset,\emptyset)\}$ a *reduced partition* of $(X,Y)\in\mathcal{F}$ if we have

$$(X_i,Y_i)\sqcap(X_j,Y_j)=(\emptyset,\emptyset) \qquad (i,j=1,2,\ldots,k;\ i\neq j), \tag{4.2}$$

$$(X_i,Y_i)\sqcap(X,Y)\neq(\emptyset,\emptyset) \qquad (i=1,2,\ldots,k), \tag{4.3}$$

$$(X,Y)=(X_1,Y_1)\sqcup(X_2,Y_2)\sqcup\cdots\sqcup(X_k,Y_k). \tag{4.4}$$

Here, it should be noted that the reduced union $\sqcup$ is not associative in general but that under the condition (4.2) the right-hand side of (4.4) does not depend on the order of the reduced-union operations. (We also define a reduced partition of the null set $(\emptyset,\emptyset)$ by (4.2) and (4.4) without imposing (4.3).) Put $I=\{1,2,\ldots,k\}$. Under conditions (4.2)–(4.4) the following three statements hold:

(a) For each $e\in X$ there uniquely exists $i^*\in I$ such that $e\in X_{i^*}$ and for each $e\in Y$ there uniquely exists $j^*\in I$ such that $e\in Y_{i^*}$.

(b) For each $i\in I$ we have $X\cap Y_i=\emptyset$ and $Y\cap X_i=\emptyset$.

(c) $\cup_{i\in I}(X_i\setminus X)=\cup_{i\in I}(Y_i\setminus Y)$, where the both set unions are disjoint set unions.

(Here (a) follows from (4.2)–(4.4). For (b), if there exists $i\in I$ and $e\in E$ such that $e\in X\cap Y_i$, then there must exist distinct $i_1,i_2\in I$ such that $e\in X_{i_1}\cap X_{i_2}$ (due to (4.4)), which contradicts (4.2). For (c), because of (4.2) and (4.4), for every $e\in E\setminus(X\cup Y)$ we have $|\{i\in I\mid e\in X_i\}|=|\{i\in I\mid e\in Y_i\}|=1$ or 0. Hence (c) follows from (4.4).) Equation (4.4) becomes

$$\begin{aligned}(X,Y)=((X_1\cup X_2\cup\cdots\cup X_k)&\setminus(Y_1\cup Y_2\cup\cdots\cup Y_k),\\ (Y_1\cup Y_2\cup\cdots\cup Y_k)&\setminus(X_1\cup X_2\cup\cdots\cup X_k)).\end{aligned} \tag{4.5}$$

We call a reduced partition $\{(X_i, Y_i) \mid i = 1, 2, \ldots, k\}$ of (X, Y) a *partition* of (X, Y) if

$$(X_i, Y_i) \sqsubseteq (X, Y) \qquad (i = 1, 2, \ldots, k). \tag{4.6}$$

For a reduced partition $\{(X_i, Y_i) \mid i = 1, 2, \ldots, k\}$ of non-null $(X, Y) \in \mathcal{F}$ define

$$\begin{aligned}(\hat{X}_i, \hat{Y}_i) &= (X, Y) \sqcap (X_i, Y_i) \\ &= (X_i \setminus (Y_1 \cup Y_2 \cup \cdots \cup Y_k), Y_i \setminus (X_1 \cup X_2 \cup \cdots \cup X_k))\end{aligned} \tag{4.7}$$

for $i = 1, 2, \ldots, k$. The collection of the signed sets $(\hat{X}_i, \hat{Y}_i)$ $(i = 1, 2, \ldots, k)$ forms a partition of (X, Y). Then, because of (4.2)–(4.4) we have

$$\begin{aligned}
& f(X, Y) + \sum_{i=1}^{k} f(X_i, Y_i) \\
& \geq f((X, Y) \sqcap (X_1, Y_1)) + f((X, Y) \sqcup (X_1, Y_1)) + \sum_{i=2}^{k} f(X_i, Y_i) \\
& = f(\hat{X}_1, \hat{Y}_1) + f((X, Y) \sqcup (X_1, Y_1)) + \sum_{i=2}^{k} f(X_i, Y_i) \\
& \geq f(\hat{X}_1, \hat{Y}_1) + f(((X, Y) \sqcup (X_1, Y_1)) \sqcap (X_2, Y_2)) \\
& \qquad + f(((X, Y) \sqcup (X_1, Y_1)) \sqcup (X_2, Y_2)) + \sum_{i=3}^{k} f(X_i, Y_i) \\
& = f(\hat{X}_1, \hat{Y}_1) + f(\hat{X}_2, \hat{Y}_2) + f((X, Y) \sqcup ((X_1, Y_1) \sqcup (X_2, Y_2))) + \sum_{i=3}^{k} f(X_i, Y_i) \\
& \vdots \\
& \geq \sum_{i=1}^{k} f(\hat{X}_i, \hat{Y}_i) + f((X, Y) \sqcup ((X_1, Y_1) \sqcup (X_2, Y_2) \sqcup \cdots \sqcup (X_k, Y_k))) \\
& = \sum_{i=1}^{k} f(\hat{X}_i, \hat{Y}_i) + f(X, Y),
\end{aligned} \tag{4.8}$$

where note that we have $((X, Y) \sqcup (X_1, Y_1)) \sqcap (X_2, Y_2) = (\hat{X}_2, \hat{Y}_2)$ since $\{(X_i, Y_i) \mid i = 1, 2, \ldots, k\}$ is a reduced partition of (X, Y).

Consequently, we have from (4.8)

$$\sum_{i=1}^{k} f(X_i, Y_i) \geq \sum_{i=1}^{\hat{k}} f(\hat{X}_i, \hat{Y}_i). \tag{4.9}$$

It follows from (4.9) that for any non-null $(X, Y) \in \mathcal{F}$

$$\begin{aligned}
& \min\Big\{ \sum_{i \in I} f(X_i, Y_i) \mid \{(X_i, Y_i) \mid i \in I\} : \text{a reduced partition of } (X, Y)\Big\} \\
& = \min\Big\{ \sum_{i \in I} f(X_i, Y_i) \mid \{(X_i, Y_i) \mid i \in I\} : \text{a partition of } (X, Y)\Big\}.
\end{aligned} \tag{4.10}$$

Now, we define the *Dilworth truncation*, denoted by $\hat{f}$, of f as follows, For each nun-null $(X, Y) \in \mathcal{F}$

$$\hat{f}(X,Y) = \min\Big\{ \sum_{i\in I} f(X_i, Y_i) \mid \{(X_i,Y_i) \mid i \in I\} : \text{a partition of } \ (X,Y)\Big\} \tag{4.11}$$

and we also define $\hat{f}(\emptyset, \emptyset) = 0$.

We show the following theorem. This partially answers a problem posed by Liqun Qi in [27].

Theorem 4.1. *The Dilworth truncation $\hat{f}$ is a bisubmodular function on $\mathcal{F}$.*

Proof. Suppose that for any $(X,Y), (V,W) \in \mathcal{F} \setminus \{(\emptyset,\emptyset)\}$ we have

$$\hat{f}(X,Y) = \sum_{i\in I} f(X_i,Y_i), \qquad \hat{f}(V,W) = \sum_{i\in J} f(V_j,W_j), \tag{4.12}$$

where $\{(X_i, Y_i) \mid i \in I\}$ and $\{(V_j, W_j) \mid j \in J\}$ are, respectively, partitions of (X, Y) and (V, W). Also, suppose that $I = \{1, 2, \ldots, k\}$ and $J = \{1, 2, \ldots, \ell\}$. Now we have

$$\begin{aligned} \hat{f}(X,Y) + \hat{f}(V,W) &= \sum_{i=1}^{k} f(X_i,Y_i) + \sum_{j=1}^{\ell} f(V_j,W_j) \\ &= f(X_1,Y_1) + \sum_{j=1}^{\ell} f(V_j,W_j) + \sum_{i=2}^{k} f(X_i,Y_i). \end{aligned} \tag{4.13}$$

Let us assume without loss of generality that $(X_1, Y_1) \sqcap (V_j, W_j) \neq (\emptyset, \emptyset)$ for $j = 1, 2, \ldots, p$ with $0 \le p \le \ell$. Then the first two terms on the right-hand side of (4.13) is transformed as follows.

$$\begin{aligned} & f(X_1,Y_1) + \sum_{j=1}^{\ell} f(V_j,W_j) \\ & \ge f((X_1,Y_1) \sqcup (V_1,W_1)) + \sum_{j=2}^{\ell} f(V_j,W_j) + f((X_1,Y_1) \sqcap (V_1,W_1)) \\ & \vdots \\ & \ge f((X_1,Y_1) \sqcup (\cup_{j=1}^{p} V_j, \cup_{j=1}^{p} W_j)) + \sum_{j=p+1}^{\ell} f(V_j,W_j) \\ & \qquad + \sum_{j=1}^{p} f((X_1,Y_1) \sqcap (V_j,W_j)), \end{aligned} \tag{4.14}$$

where use is made of the fact that whenever $(A, B), (C_1, D_1), (C_2, D_2) \in 3^E$ and $(C_1 \cup D_1) \cap (C_2 \cup D_2) = \emptyset$, we have

$$((A,B) \sqcup (C_1,D_1)) \sqcup (C_2,D_2) = (A,B) \sqcup (C_1 \cup C_2, D_1 \cup D_2). \tag{4.15}$$

We note that for the first two terms on the right-hand side of the above expression (4.14) we have the sum of the values of f on the blocks of a "reduced partition" of $(X_1, Y_1) \sqcup (V, W)$ (if it is non-null) and for the third term the sum of the values of f on the blocks of a "partition" of $(X_1, Y_1) \sqcap (V, W)$ (if it is non-null).

Proceeding from (4.13) in the same manner as in (4.14), for $i = 2, \ldots, k$ we combine at every stage (X_i, Y_i) with the blocks of the currently generated "reduced partition" of $(\cup_{t=1}^{i-1} X_t, \cup_{t=1}^{i-1} Y_t) \sqcup (V, W)$. In the end the expression

$$\sum_{i=1}^{k} f(X_i, Y_i) + \sum_{j=1}^{\ell} f(V_j, W_j) \tag{4.16}$$

is transformed into an expression in which we have the sum of the values of f on the blocks of a "reduced partition" of $(\cup_{i=1}^{k} X_i, \cup_{i=1}^{k} Y_i) \sqcup (V, W) = (X, Y) \sqcup (V, W)$ and the sum of the values of f on the blocks of a "partition" of $(\cup_{i=1}^{k} X_i, \cup_{i=1}^{k} Y_i) \sqcap (V, W) = (X, Y) \sqcap (V, W)$ (if it is non-null). It follows from (4.9) and the definition (4.11) of $\hat{f}$ that

$$\hat{f}(X, Y) + \hat{f}(V, W) \geq \hat{f}((X, Y) \sqcup (V, W)) + \hat{f}((X, Y) \sqcap (V, W)). \tag{4.17}$$

This establishes the bisubmodularity of the Dilworth truncation $\hat{f}$. □

The bisubmodular polyhedron $\mathrm{P}_*(\hat{f})$ associated with the Dilworth truncation $\hat{f}$ is related to the original f as follows.

Theorem 4.2.

$$\mathrm{P}_*(\hat{f}) = \{x \in \mathbb{R}^E \mid \forall (X, Y) \in \mathcal{F} \setminus \{(\emptyset, \emptyset)\} : x(X, Y) \leq f(X, Y)\}. \tag{4.18}$$

Moreover, for each $(X, Y) \in \mathcal{F}$,

$$\hat{f}(X, Y) = \max\{x(X, Y) \mid x \in \mathrm{P}_*(\hat{f})\}. \tag{4.19}$$

Proof. For each $(X_0, Y_0) \in \mathcal{F} \setminus \{(\emptyset, \emptyset)\}$ the inequality

$$x(X_0, Y_0) \leq \hat{f}(X_0, Y_0) \tag{4.20}$$

is implied by the system of inequalities

$$x(X, Y) \leq f(X, Y) \qquad ((X, Y) \in \mathcal{F} \setminus \{(\emptyset, \emptyset)\}) \tag{4.21}$$

since (4.20) is obtained by adding both sides of inequalities chosen appropriately from among (4.21) according to the way of the construction of $\hat{f}(X_0, Y_0)$ in terms of $f(X, Y)$ $((X, Y) \in \mathcal{F} \setminus \{(\emptyset, \emptyset)\})$ as shown in the proof of Theorem 4.1. Also note that the domain of $\hat{f}$ is equal to $\mathcal{F}$ and we have $\hat{f}(X, Y) \leq f(X, Y)$ for all non-null $(X, Y) \in \mathcal{F}$. Therefore, the present theorem follows from Theorem 4.1 and the well-known fact that every inequality $x(X, Y) \leq \hat{f}(X, Y)$ is tight for the bisubmodular polyhedron $\mathrm{P}_*(\hat{f})$ associated with the bisubmodular system $(\mathcal{F}, \hat{f})$ (see, e.g., [2, 3]). □

Remark 4.3. A family $\mathcal{F} \subseteq 3^E$ is called an *intersecting family* if for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$, $(X_1, Y_1) \sqcap (X_2, Y_2) \neq \emptyset$ implies $(X_1, Y_1) \sqcup (X_2, Y_2)$, $(X_1, Y_1) \sqcap (X_2, Y_2) \in \mathcal{F}$. Also, a function f on an intersecting family $\mathcal{F} \subseteq 3^E$ is called an *intersecting-bisubmodular* function if for each intersecting pair $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ (i.e., $(X_1 Y_1) \sqcap (X_2, Y_2) \neq (\emptyset, \emptyset)$) we have the bisubmodularity inequality (2.5). Note that the arguments in the present section are also valid *mutatis mutandis* if we consider intersecting-bisubmodular functions satisfying Assumption **(A)**. Another extension of the Dilworth truncation and the intersection of two bisubmodular polyhedra are also investigated in [21].

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Satoru Fujishige
Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502, Japan
E-mail address: fujishig@kurims.kyoto-u.ac.jp

Sachin B. Patkar
Department of Electrical Engineering
Indian Institute of Bombay, Powai, Mumbai 400 076, India
E-mail address: patkar@ee.iitb.ac.in