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# STP METHOD FOR SOLVING THE MINIMAL NORM TOEPLITZ SOLUTIONS OF $\mathbf{MX - \widetilde{X}N = GY + R}$*

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**Abstract:** This paper will consider the nonhomogeneous Yakubovich-(conjugate) quaternion matrix equations $MX - \widetilde{X}N = GY + R$, where $\widetilde{X}$ is $X$ or the $\{i, j, k\}$-conjugate of $X$. The STP method for solving the minimal norm least squares lower(upper) triangular Toeplitz solution and the minimal norm least squares lower(upper) triangular Toeplitz $\{i, j, k\}$-conjugate solutions of the above equation are given, and the expressions of the above solutions of these equations. In addition, we also give the necessary and sufficient conditions and expressions for the existence of (anti)self-conjugate solutions of the corresponding conjugate matrix equations.

**Key words:** *Toeplitz matrix, least squares solution, STP method, real vector representation*

**Mathematics Subject Classification:** *15A06, 15A21*

## 1 Introduction

The semi-tensor product (STP) of matrices is originally proposed by Cheng [5], which is a generalization of traditional matrix product. As a new matrix product, it is not limited by matrix dimensions. That is to say, we can use STP to calculate the product of two real matrices $M$ and $N(M \in R^{m\times n}, N \in R^{p\times q})$. At present, it has been widely used in Boolean networks [10], graph coloring [18], cryptography [11], and other fields. This paper will study the application of STP in quaternion linear system.

Quaternion linear systems are widely used in many fields, such as control theory, signal and color image processing, quantum physics [9, 1, 12, 17]. In recent years, many scholars have studied the equations with different algebraic structures by different methods, and obtained many valuable results, see [6, 16, 15]. In [6], the explicit expressions of least square solution are obtained by using the real vector representation of quaternion matrix. In [16], the explicit solutions to the quaternion matrix equations $XF - AX = C$ and $XF - A\widetilde{X} = C$ are established by using complex representation. The authors in [15] derived the necessary and sufficient conditions for the existence of solutions of the equations $AX^\star - XB = CY + D$ and $X - AX^\star B = CY + D$.

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In recent years, Toeplitz matrix has become a special kind of matrix in scientific research, like

$$\begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ t_1 & s_0 & s_1 & \dots & s_{n-2} \\ t_2 & t_1 & s_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s_1 \\ t_{n-1} & \dots & t_2 & t_1 & s_0 \end{pmatrix}.$$

It is widely used in many scientific fields, such as digital signal processing, digital image processing, numerical analysis, numerical solution of differential equations [14]. In digital image processing, the process of image degradation is equivalent to linear transformation of the original image matrix by transfer function and noise, while the process of image restoration is equivalent to transforming the least square problem into the inversion of Toeplitz matrix when the transfer function is separable [19]. In addition, the expressions of special solutions of constant coefficient linear differential equations and difference equations are given by using the upper triangular Toeplitz matrix, which brings great convenience to solve constant coefficient linear differential equations and difference equations [7, 8]. Therefore, the study of Toeplitz matrix has important application value.

In this article, we will study the minimal norm least squares lower(upper) Toeplitz $i\{j,k\}$-conjugate solutions to the following nonhomogeneous Yakubovich-(conjugate) quaternion matrix equation

$$MX - \widetilde{X}N = GY + R, \ \widetilde{X} = \{X, X^i, X^j, X^k\}. \tag{1.1}$$

**Problem 1.** Suppose $M, \ N, \ G, \ R \in Q^{n\times n}$, and denote

$$L = \{W \mid W = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q^{n\times n}_{LT}), \ \|MX - XN - GY - R\| = min\}.$$

Find out $W_{Q_{LT}} \in L$, which satisfies

$$\|W_{Q_{LT}}\| = \min_{W\in L} \|W\|.$$

Then, $W_{Q_{LT}}$ is named as the minimal norm least squares lower triangular Toeplitz solution of Eq.(3.1). If $min = 0$, $W_{Q_{LT}}$ is called the minimal norm lower triangular Toeplitz solution of Eq.(3.1).

**Problem 2.** Suppose $M, \ N, \ G, \ R \in Q^{n\times n}$, and denote

$$U = \{W \mid W = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q^{n\times n}_{UT}), \ \|MX - XN - GY - R\| = min\}.$$

Find out $W_{Q_{UT}} \in U$, which satisfies

$$\|W_{Q_{UT}}\| = \min_{W\in U} \|W\|.$$

Then, $W_{Q_{UT}}$ is named as the minimal norm least squares upper triangular Toeplitz solution of Eq.(3.1). If $min = 0$, $W_{Q_{UT}}$ is called the minimal norm upper triangular Toeplitz solution of Eq.(3.1).

**Problem 3.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$L^i = \{W^i \mid W^i = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}),\ \|MX - X^i N - GY - R\| = min\}.$$

Find out $W^i_{Q_{LT}} \in L^i$, which satisfies

$$\|W^i_{Q_{LT}}\| = \min_{W^i \in L^i} \|W^i\|.$$

Then, $W^i_{Q_{LT}}$ is named as the minimal norm least squares lower triangular Toeplitz $i$-conjugate solution of Eq.(3.32). If $min = 0$, $W^i_{Q_{LT}}$ is called the minimal norm lower triangular Toeplitz $i$-conjugate solution of Eq.(3.32).

**Problem 4.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$U^i = \{W^i \mid W^i = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}),\ \|MX - X^i N - GY - R\| = min\}.$$

Find out $W^i_{Q_{UT}} \in U^i$, which satisfies

$$\|W^i_{Q_{UT}}\| = \min_{W^i \in U^i} \|W^i\|.$$

Then, $W^i_{Q_{UT}}$ is named as the minimal norm least squares upper triangular Toeplitz $i$-conjugate solution of Eq.(3.32). If $min = 0$, $W^i_{Q_{UT}}$ is called the minimal norm upper triangular Toeplitz $i$-conjugate solution of Eq.(3.32).

**Problem 5.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$L^j = \{W^j \mid W^j = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}),\ \|MX - X^j N - GY - R\| = min\}.$$

Find out $W^j_{Q_{LT}} \in L^j$, which satisfies

$$\|W^j_{Q_{LT}}\| = \min_{W^j \in L^j} \|W^j\|.$$

Then, $W^j_{Q_{LT}}$ is named as the minimal norm least squares lower triangular Toeplitz $j$-conjugate solution of Eq.(3.60). If $min = 0$, $W^j_{Q_{LT}}$ is called the minimal norm lower triangular Toeplitz $j$-conjugate solution of Eq.(3.60).

**Problem 6.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$U^j = \{W^j \mid W^j = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}),\ \|MX - X^j N - GY - R\| = min\}.$$

Find out $W^j_{Q_{UT}} \in U^j$, which satisfies

$$\|W^j_{Q_{UT}}\| = \min_{W^j \in U^j} \|W^j\|.$$

Then, $W^j_{Q_{UT}}$ is named as the minimal norm least squares upper triangular Toeplitz $j$-conjugate solution of Eq.(3.60). If $min = 0$, $W^j_{Q_{UT}}$ is called the minimal norm upper triangular Toeplitz $j$-conjugate solution of Eq.(3.60).

**Problem 7.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$L^k = \{W^k \mid W^k = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}),\ \|MX - X^k N - GY - R\| = min\}.$$

Find out $W_{Q_{LT}}^k \in L^k$, which satisfies

$$\|W_{Q_{LT}}^k\| = \min_{W^k \in L^k} \|W^k\|.$$

Then, $W_{Q_{LT}}^k$ is named as the minimal norm least squares lower triangular Toeplitz $k$-conjugate solution of Eq.(3.74). If $min = 0$, $W_{Q_{LT}}^k$ is called the minimal norm lower triangular Toeplitz $k$-conjugate solution of Eq.(3.74).

**Problem 8.** Suppose $M,\ N,\ G,\ R \in Q^{n\times n}$, and denote

$$U^k = \{W^k \mid W^k = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}),\ \|MX - X^k N - GY - R\| = min\}.$$

Find out $W_{Q_{UT}}^k \in U^k$, which satisfies

$$\|W_{Q_{UT}}^k\| = \min_{W^k \in U^k} \|W^k\|.$$

Then, $W_{Q_{UT}}^k$ is named as the minimal norm least squares upper triangular Toeplitz $k$-conjugate solution of Eq.(3.74). If $min = 0$, $W_{Q_{UT}}^k$ is called the minimal norm upper triangular Toeplitz $k$-conjugate solution of Eq.(3.74).

The structure of this paper is as follows. In Section 2, we review some basic knowledge used in this paper. In Section 3, we study the minimal norm least squares lower(upper) triangular Toeplitz solution and the minimal norm least squares lower(upper) triangular Toeplitz $i\{j,k\}$-conjugate solutions to nonhomogeneous Yakubovich-(conjugate) quaternion matrix equation Eq.(1.1) by using STP method. In Section 4, we discuss the self-conjugate and anti-self-conjugate solutions of the studied quaternion conjugate matrix equation. Finally, in Section 5, a brief summary of this paper is given.

The symbols used in this article are explained as follows. $R/Q$ are denoted by the set of the real/quaternion field. $R^n$ stands for the set of all real column vectors with order $n$. $R^{m\times n}/Q^{m\times n}$ are denoted by the set of all real/quaternion matrices with order $m\times n$. $Q_{LT}^{n\times n}/Q_{UT}^{n\times n}$ stand for the set of all the lower/upper triangle Toeplitz quaternion matrices with order $n\times n$. $(A)IQ_{LT}^{n\times n}/(A)IQ_{UT}^{n\times n}$ stand for the set of all $n\times n$ (anti)I-self-conjugate the lower/upper triangle Toeplitz quaternion matrices. $(A)JQ_{LT}^{n\times n}/(A)JQ_{UT}^{n\times n}$ stand for the set of all $n\times n$ (anti)J-self-conjugate the lower/upper triangle Toeplitz quaternion matrices. $(A)KQ_{LT}^{n\times n}/(A)KQ_{UT}^{n\times n}$ stand for the set of all $n\times n$ (anti)K-self-conjugate the lower/upper triangle Toeplitz quaternion matrices. $A^i, A^j, A^k$ stand for $\{i,j,k\}$-conjugate matrix of $A$, respectively. $I_n$ is denoted by unit matrix with order $n$. $\delta_n^i$ stands for the $i$-th column of unit matrix $I_n$. $A^T/A^+$ are denoted by the transpose/ $MP$ inverse of matrix $A$. $\otimes/\ltimes$ are denoted by the Kronecker product/the semi-tensor product of matrices. $\|\cdot\|$ stands for the Frobenius norm of a matrix or Euclidean norm of a vector.

## 2 Preliminaries

**Definition 2.1** ([20])**.** Let $a \in Q$, $M \in Q^{m\times n}$, the norm of quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$ , and the Frobenius norm of matrix $M = M_0 + M_1 i + M_2 j + M_3 k$ are defined separately as

$$\|a\| = \sqrt{\|a_0\|^2 + \|a_1\|^2 + \|a_2\|^2 + \|a_3\|^2}, \tag{2.1}$$

and

$$\|M\| = \sqrt{\|M_0\|^2 + \|M_1\|^2 + \|M_2\|^2 + \|M_3\|^2}. \tag{2.2}$$

**Definition 2.2** ([6])**.** Let $M = (M^{st}) \in Q^{m\times n}$, and $M^{st} = M_1^{st} + M_2^{st}i + M_3^{st}j + M_4^{st}k$, where $M_1^{st}$, $M_2^{st}$, $M_3^{st}$, $M_4^{st} \in R^{m\times n}$. Denote

$$(M^i)^{st} = M_1^{st} + M_2^{st}i - M_3^{st}j - M_4^{st}k, \tag{2.3}$$

$$(M^j)^{st} = M_1^{st} - M_2^{st}i + M_3^{st}j - M_4^{st}k, \tag{2.4}$$

$$(M^k)^{st} = M_1^{st} - M_2^{st}i - M_3^{st}j + M_4^{st}k, \tag{2.5}$$

then $M^i = (M^i)^{st}$ is defined $i$-conjugate matrix of $M$, $M^j = (M^j)^{st}$ is defined $j$-conjugate matrix of $M$, $M^k = (M^k)^{st}$ is defined $k$-conjugate matrix of $M$. If $M^i = M$, we call $M$ $i$-self-conjugate matrix. If $M^i = -M$, we call $M$ anti-$i$-self-conjugate matrix. The representation of $\{j, k\}$-conjugates matrix is similar to that of $i$-conjugates matrix.

**Definition 2.3** ([2])**.** Let $M \in R^{m\times n}$, $N \in R^{p\times q}$, the semi-tensor product of $M$ and $N$ denoted by

$$M \ltimes N = (M \otimes I_{t/n})(N \otimes I_{t/p}), \tag{2.6}$$

where $t = lcm(n, p)$ is the least common multiple of $n$ and $p$.

**Remark 2.4.** If $n = p$, obviously, there is $M \ltimes N = MN$.

**Example.** Suppose

$$M = \left[\begin{array}{cc:cc} 2 & -2 & -1 & 1 \\ \hdashline 1 & 0 & 3 & -3 \\ \hdashline -2 & -3 & 2 & 1 \end{array}\right], N = \left[\begin{array}{c:c} -2 & 1 \\ \hdashline -3 & 2 \end{array}\right].$$

First, one can block matrix $M$ and $N$ into

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \end{bmatrix}, N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

Then,

$$\begin{aligned} M \ltimes N &= (M \otimes I_1)(N \otimes I_2) \\ &= \begin{bmatrix} 2 & -2 & -1 & 1 \\ 1 & 0 & 3 & -3 \\ -2 & -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ -3 & 0 & 2 & 0 \\ 0 & -3 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ -11 & 9 & 7 & -6 \\ -2 & 3 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} M_{11} \ltimes N_{11} + M_{12} \ltimes N_{21} & M_{11} \ltimes N_{12} + M_{12} \ltimes N_{22} \\ M_{21} \ltimes N_{11} + M_{22} \ltimes N_{21} & M_{21} \ltimes N_{12} + M_{22} \ltimes N_{22} \\ M_{31} \ltimes N_{11} + M_{32} \ltimes N_{21} & M_{31} \ltimes N_{12} + M_{32} \ltimes N_{22} \end{bmatrix}. \end{aligned}$$

**Lemma 2.5** ([4])**.** *Suppose $F,\ G,\ H$ are quaternion matrices, $\lambda,\ \mu \in R$, then*

(1) *(Associative rule)*

$$(F \ltimes G) \ltimes H = F \ltimes (G \ltimes H). \tag{2.7}$$

(2) *(Distributive rule)*

$$F \ltimes (\lambda G \pm \mu H) = \lambda F \ltimes G \pm \mu F \ltimes H, \tag{2.8a}$$

$$(\lambda F \pm \mu G) \ltimes H = \lambda F \ltimes H \pm \mu G \ltimes H. \tag{2.8b}$$

(3) *Let $\omega \in R^m$, $\sigma \in R^n$, then*

$$\omega \ltimes \sigma = \omega \otimes \sigma. \tag{2.9}$$

**Lemma 2.6** ([4])**.** *Let $\omega \in R^t$, $M \in R^{m\times n}$, then*

$$\omega \ltimes M = (I_t \otimes M) \ltimes \omega, \tag{2.10a}$$

$$M \ltimes \omega^T = \omega^T \ltimes (I_t \otimes M). \tag{2.10b}$$

**Definition 2.7** ([3])**.** Suppose that $W_{[m,n]} \in R^{mn\times mn}$ is defined as the swap matrix,

$$\begin{aligned} W_{[m,n]} &= [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \dots, I_n \otimes \delta_m^m] \\ &= \delta_{mn}[1, \dots, (n-1)m+1, \dots, m, \dots, nm], \end{aligned} \tag{2.11}$$

and $\delta_k[i_1, \dots, i_s]$ is abbreviation of $[\delta_k^{i_1}, \dots, \delta_k^{i_s}]$.

**Remark 2.8.** Especially, when $m = n$, one can denote $W_{[n]} := W_{[n,n]}$.

**Lemma 2.9** ([3])**.** *Let $\alpha \in R^m$ and $\beta \in R^n$, then*

$$W_{[m,n]} \ltimes (\alpha \ltimes \beta) = \beta \ltimes \alpha, \tag{2.12a}$$

$$(\alpha^T \ltimes \beta^T) \ltimes W_{[m,n]} = \beta^T \ltimes \alpha^T. \tag{2.12b}$$

**Definition 2.10** ([3])**.** Let $\Omega_i (i = 0, 1, \dots, n)$ be vector spaces. The mapping $\Phi : \Omega_1 \times \Omega_2 \times \dots \Omega_n \to \Omega_0$ is named as a multilinear mapping. If $dim(\Omega_i) = k_i$, where $(\delta_{k_i}^1, \delta_{k_i}^2, \dots, \delta_{k_i}^{k_i})$ is the basis of $\Omega_i$, denote

$$\Phi(\delta_{k_1}^{j_1}, \delta_{k_2}^{j_2}, \dots, \delta_{k_n}^{j_n}) = \textstyle\sum_{s=1}^{k_0} \lambda_s^{j_1,j_2,\dots,j_n} \delta_{k_0}^s, \tag{2.13}$$

where $j_t = 1, \dots, k_t, \quad t = 1, \dots, n$, then

$$\{\lambda_s^{j_1,j_2,\dots,j_n} | j_t = 1, \dots, k_t, \quad t = 1, \dots, n;\ s = 1, \dots, k_0\}, \tag{2.14}$$

is named as structure constants of $\Phi$.

$$M_\Phi = \begin{pmatrix} \lambda_1^{11\dots1} & \dots & \lambda_1^{11\dots k_n} & \dots & \lambda_1^{k_1k_2\dots k_n} \\ \lambda_2^{11\dots1} & \dots & \lambda_2^{11\dots k_n} & \dots & \lambda_2^{k_1k_2\dots k_n} \\ \vdots & & \vdots & & \vdots \\ \lambda_{k_0}^{11\dots1} & \dots & \lambda_{k_0}^{11\dots k_n} & \dots & \lambda_{k_0}^{k_1k_2\dots k_n} \end{pmatrix}, \tag{2.15}$$

and $M_\Phi$ is named as the structure matrix of $\Phi$.

Next, we will give some conclusions of real linear system of matrix equations and real vector representation of quarernion matrix.

**Lemma 2.11** ([13]). *The linear system of equations $Ax = b$ with $A \in R^{m\times n}$ and $b \in R^m$, has a solution $x \in R^n$ if and only if*

$$AA^{\dagger}b = b. \tag{2.16}$$

*In that case it has the general solution*

$$x = A^{\dagger}b + (I - A^{\dagger}A)y, \tag{2.17}$$

*where $y \in R^n$ is an arbitrary vector.*

**Lemma 2.12** ([13]). *The least squares solutions of linear system $Ax = b$ with $A \in R^{m\times n}$ and $b \in R^m$, can be represented as*

$$x = A^{\dagger}b + (I - A^{\dagger}A)y, \tag{2.18}$$

*where $y \in R^n$ is an arbitrary vector. The minimal norm least squares solution of the linear system $Ax = b$ is $A^{\dagger}b$.*

**Definition 2.13** ([6]). Let $a = a_0 + a_1 i + a_2 j + a_3 k \in Q$, denote

$$v^R(a) = (a_0, a_1, a_2, a_3)^T, \tag{2.19}$$

$v^R(a)$ is named as the real staking form of $a$.

**Lemma 2.14** ([6]). *Let $a, b \in Q$, then*

$$v^R(ab) = M_Q \ltimes v^R(a) \ltimes v^R(b), \tag{2.20}$$

*where the structure matrix $M_Q$ of multiplication of quaternion can be expressed as*

$$M_Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

**Definition 2.15** ([6]). Let $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_n)^T$ be quaternion vectors, where $a_i, b_i \in Q, (i = 1, 2, \dots, n)$. Denote

$$v^R(a) = \begin{pmatrix} v^R(a_1) \\ v^R(a_2) \\ \vdots \\ v^R(a_n) \end{pmatrix}, \ v^R(b) = \begin{pmatrix} v^R(b_1) \\ v^R(b_2) \\ \vdots \\ v^R(b_n) \end{pmatrix}, \tag{2.21}$$

in which $v^R(a)$ and $v^R(b)$ are named as the real staking form of quaternion vector $a$ and $b$, respectively.

**Definition 2.16** ([6])**.** Let $M \in Q^{m\times n}$, we denote

$$v_c^R(M) = \begin{pmatrix} v^R(Col_1(M)) \\ v^R(Col_2(M)) \\ \vdots \\ v^R(Col_n(M)) \end{pmatrix} = \begin{pmatrix} v^R(M_{11}) \\ \vdots \\ v^R(M_{m1}) \\ \vdots \\ v^R(M_{1n}) \\ \vdots \\ v^R(M_{mn}) \end{pmatrix}, v_r^R(M) = \begin{pmatrix} v^R(Row_1(M)) \\ v^R(Row_2(M)) \\ \vdots \\ v^R(Row_m(M)) \end{pmatrix} = \begin{pmatrix} v^R(M_{11}) \\ \vdots \\ v^R(M_{1n}) \\ \vdots \\ v^R(M_{m1}) \\ \vdots \\ v^R(M_{mn}) \end{pmatrix}, \tag{2.22}$$

$v_c^R(M)/v_r^R(M)$ are named as the real column/row stacking form of $M$.

**Remark 2.17.** In addition, $v_c^R(M)$ and $v_r^R(M)$ can be transformed into each other. Then we have

$$W_{[m,n]} \ltimes v_r^R(M) = v_c^R(M), \tag{2.23a}$$

$$W_{[n,m]} \ltimes v_c^R(M) = v_r^R(M). \tag{2.23b}$$

**Lemma 2.18** ([6])**.** *Let $a = (a_1, a_2, \dots, a_n)$, $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ and $b = (b_1, b_2, \dots, b_n)^T$ be quaternion vectors, $\alpha \in R$, then we have*

$$v^R(a + \widetilde{a}) = v^R(a) + v^R(\widetilde{a}), \tag{2.24}$$

$$v^R(\alpha a) = \alpha v^R(a), \tag{2.25}$$

$$v^R(ab) = M_Q \ltimes \textstyle\sum_{i=1}^n \{(\delta_n^i)^T \ltimes [I_{4n} \otimes (\delta_n^j)^T]\} \ltimes v^R(a) \ltimes v^R(b). \tag{2.26}$$

**Lemma 2.19** ([6])**.** *If $M \in Q^{m\times n}, \widetilde{M} \in Q^{m\times n}$, $N \in Q^{n\times p}$, $\alpha \in R$, then we have*

$$v_r^R(M + \widetilde{M}) = v_r^R(M) + v_r^R(\widetilde{M}),\ v_c^R(M + \widetilde{M}) = v_c^R(M) + v_c^R(\widetilde{M}), \tag{2.27}$$

$$v_r^R(\alpha M) = \alpha v_r^R(M),\ v_c^R(\alpha M) = \alpha v_c^R(M), \tag{2.28}$$

$$\|M\| = \|v_r^R(M)\| = \|v_c^R(M)\|, \tag{2.29}$$

$$v_c^R(MN) = G \ltimes v_r^R(M) \ltimes v_c^R(N),\ v_r^R(MN) = G' \ltimes v_r^R(M) \ltimes v_c^R(N). \tag{2.30}$$

*where*

$$G = \begin{pmatrix} H \ltimes (\delta_m^1)^T \ltimes [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ H \ltimes (\delta_m^m)^T \ltimes [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ H \ltimes (\delta_m^1)^T \ltimes [I_{4mn} \otimes (\delta_p^p)^T] \\ \vdots \\ H \ltimes (\delta_m^m)^T \ltimes [I_{4mn} \otimes (\delta_p^p)^T] \end{pmatrix},\ G' = \begin{pmatrix} H \ltimes (\delta_m^1)^T \ltimes [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ H \ltimes (\delta_m^1)^T \ltimes [I_{4mn} \otimes (\delta_p^p)^T] \\ \vdots \\ H \ltimes (\delta_m^m)^T \ltimes [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ H \ltimes (\delta_m^m)^T \ltimes [I_{4mn} \otimes (\delta_p^p)^T] \end{pmatrix}, \tag{2.31}$$

*and*

$$H = M_Q \ltimes \textstyle\sum_{i=1}^n \{(\delta_n^i)^T \ltimes [I_{4n} \otimes (\delta_n^i)^T]\}. \tag{2.32}$$

# 3 The Main Results of Problems 1-8

In this part, we consider the minimal norm least squares Toeplitz solutions of quaternion matrix equation (1.1). Part one, we discuss the minimal norm least squares lower triangular Toeplitz solution and the minimal norm least squares upper triangular Toeplitz solution to equation (3.1). In order to reduce the complexity of the problem, firstly, according to the structural characteristics of the lower(upper) triangular toeplitz matrix, the independent elements are extracted and arrange them in the real column stacking form, denoted as $v_s^R(X)$, it can be proved that $v_s^R(X)$ and $v_c^R(X)$ have the following relationship. Part two, according to the properties of $\{i,j,k\}$-conjugate matrix, the relationship between the real column stacking form and $\{i,j,k\}$-conjugate matrix is given, we research separately the minimal norm least squares lower(upper) triangular Toeplitz $\{i,j,k\}$-conjugate solutions of equation (3.31).

## 3.1 The main results of Problems 1-2

First, we will begin to study the following equation

$$MX - XN = GY + R. \tag{3.1}$$

**Theorem 3.1.** *Let $X = [X_1, X_2, \ldots, X_n] \in Q_{LT}^{n\times n}$, $Y = [Y_1, Y_2, \ldots, Y_n] \in Q_{LT}^{n\times n}$, then*

$$v_c^R(X) = Kv_s^R(X),\ v_c^R(Y) = Kv_s^R(Y), \tag{3.2}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, K = \begin{pmatrix} K_1 \\ \vdots \\ K_m \\ \vdots \\ K_n \end{pmatrix}, K_m = \begin{pmatrix} K_{1m} \\ \vdots \\ K_{rm} \\ \vdots \\ K_{nm} \end{pmatrix}, m = 1, 2, \ldots n, \tag{3.3}$$

*and*

$$K_{rm} = \begin{cases} 0_{4\times 4n}, & r < m, \\ \left(\delta_n^{r-m+1}\right)^T \otimes I_4, & r \geq m. \end{cases} \tag{3.4}$$

*Proof.* Let $X = (X_1, X_2, \ldots, X_n)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, by Definition 2.16 and $K_m v_s^R(X) = v^R(X_m)$, $K_m v_s^R(Y) = v^R(Y_m)$ $(1 \leq m \leq n)$, so we can obtain

$$v_c^R(X) = \begin{pmatrix} v^R(X_1) \\ v^R(X_2) \\ \vdots \\ v^R(X_n) \end{pmatrix} = \begin{pmatrix} K_1 v_s^R(X) \\ K_2 v_s^R(X) \\ \vdots \\ K_n v_s^R(X) \end{pmatrix} = Kv_s^R(X), \tag{3.5}$$

$$v_c^R(Y) = \begin{pmatrix} v^R(Y_1) \\ v^R(Y_2) \\ \vdots \\ v^R(Y_n) \end{pmatrix} = \begin{pmatrix} K_1 v_s^R(Y) \\ K_2 v_s^R(Y) \\ \vdots \\ K_n v_s^R(Y) \end{pmatrix} = Kv_s^R(Y). \tag{3.6}$$

Therefore, the proof process is as above. □

**Theorem 3.2.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$O = [G_1 \ltimes v_r^R(M) \ltimes K - G_1^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes K, -G_1^{''} \ltimes v_r^R(G) \ltimes K], \quad (3.7)$$

*where $G_1$, $G_1^{'}$, $G_1^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. Then the set $L$ of Problem 1 can be represented as*

$$L = \{W|W = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = O^+ v_c^R(R) + (I_{8n} - O^+O)y, \ \forall y \in R^{8n}\}, \quad (3.8)$$

*and the minimal norm least squares lower triangular Toeplitz solution $\psi_L$ satisfies*

$$\psi_L = O^+ v_c^R(R). \quad (3.9)$$

*Proof.* By Remark 2.17 and Lemma 2.19, we can get

$$\begin{aligned}
&\|MX - XN - GY - R\| \\
&= \|v_c^R(MX - XN - GY - R)\| \\
&= \|v_c^R(MX) - v_c^R(XN) - v_c^R(GY) - v_c^R(R)\| \\
&= \|G_1 \ltimes v_r^R(M) \ltimes v_c^R(X) - G_1^{'} \ltimes v_r^R(X) \ltimes v_c^R(N) \\
&\quad - G_1^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_1 \ltimes v_r^R(M) \ltimes v_c^R(X) - G_1^{'} \ltimes W_{[n]} \ltimes v_c^R(X) \ltimes v_c^R(N) \\
&\quad - G_1^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_1 \ltimes v_r^R(A) \ltimes v_c^R(X) - G_1^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes v_c^R(X) \\
&\quad - G_1^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_1 \ltimes v_r^R(A) \ltimes K v_s^R(X) - G_1^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes K v_s^R(X) \\
&\quad - G_1^{''} \ltimes v_r^R(G) \ltimes K v_s^R(Y) - v_c^R(R)\| \\
&= \|[G_1 \ltimes v_r^R(A) \ltimes K - G_1^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes K, -G_1^{''} \ltimes v_r^R(G) \ltimes K] \\
&\quad \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix} - v_c^R(R)\| \\
&= \|O\psi - v_c^R(R)\| \ .
\end{aligned} \quad (3.10)$$

Therefore

$$\|MX - XN - GY - R\| = min \quad (3.11)$$

if and only if

$$\|O\psi - v_c^R(R)\| = min. \quad (3.12)$$

For the real matrix equation

$$O\psi = v_c^R(R). \quad (3.13)$$

According to Lemma 2.12, its least squares lower triangular Toeplitz solutions can be written as

$$\psi = O^+ v_c^R(R) + (I_{8n} - O^+O)y, \ \forall y \in R^{8n}. \quad (3.14)$$

Hence we can obtain the formula (3.8).

Notice

$$\underset{W_{Q_{LT}}\in Q_{LT}^{2n}}{min}||W_{Q_{LT}}|| \Longleftrightarrow \underset{\psi_L\in R^{8n}}{min}||\psi_L||. \tag{3.15}$$

Then, by the prior proof, $\psi_L \in L$ of Eq.(3.1) satisfies

$$\psi_L = O^+ v_c^R(R). \tag{3.16}$$

Thus, the formula (3.9) holds. □

**Corollary 3.3.** *Suppose that* $M, N, G, R \in Q^{n\times n}$, $O$ *and* $\psi$ *are the same as Theorem 3.2. Then Eq.*(3.1) *has a solution over* $Q_{LT}^{n\times n}$ *if and only if*

$$(OO^+ - I_{4n^2})v_c^R(R) = 0. \tag{3.17}$$

*If* (3.17) *holds, the solution set of Eq.*(3.1) *over* $Q_{LT}^{n\times n}$ *can be represented as*

$$\widetilde{L} = \{W|W = \begin{pmatrix} X \\ Y \end{pmatrix}(X, Y \in Q_{LT}^{n\times n}), \psi = O^+ v_c^R(R) + (I_{8n} - O^+O)y,\ \forall y \in R^{8n}\}. \tag{3.18}$$

*Moreover, the minimal norm lower triangular Toeplitz solution over* $Q_{LT}^{n\times n}$ *satisfies*

$$\psi_L = O^+ v_c^R(R). \tag{3.19}$$

*Proof.* Eq.(3.1) has a solution over $\psi_L \in Q_{LT}^{n\times n}$ if and only if

$$\|MX - XN - GY - R\| = 0. \tag{3.20}$$

Combined Theorem 3.2 with $OO^+O = O$, we can obtain

$$\begin{aligned}\|MX - XN - GY - R\| &= \|O\psi - v_c^R(R)\| = \|OO^+O\psi - v_c^R(R)\| \\ &= \|OO^+v_c^R(R) - v_c^R(R)\| = \|(OO^+ - I_{4n^2})v_c^R(R)\|.\end{aligned} \tag{3.21}$$

So, it can be derived

$$\begin{aligned}\|MX - XN - GY - R\| = 0 &\Leftrightarrow \|(OO^+ - I_{4n^2})v_c^R(R)\| = 0 \\ &\Leftrightarrow (OO^+ - I_{4n^2})v_c^R(R) = 0.\end{aligned} \tag{3.22}$$

Therefore, it can be concluded that Eq.(3.1) is compatible and its solution satisfies

$$O\psi = v_c^R(R). \tag{3.23}$$

Moreover, by Lemma 2.11, its solution $W_{Q_{LT}}$ satisfies

$$\psi = O^+ v_c^R(R) + (I_{8n} - O^+O)y. \tag{3.24}$$

And then we can get the minimal norm lower triangular Toeplitz solution $\psi_L$ satisfies

$$\psi_L = O^+ v_c^R(R). \tag{3.25}$$

Therefore, we finish the proof. □

In the followng, we will to solve Problem 2. The process of proof is similar to the above conclusion, and we omit it here.

**Theorem 3.4.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{O} = [\widetilde{G_1} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_1'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \widetilde{K}, -\widetilde{G_1''} \ltimes v_r^R(G) \ltimes \widetilde{K}], \quad (3.26)$$

*where $\widetilde{G_1}$, $\widetilde{G_1'}$, $\widetilde{G_1''}$ have the same structure as $G$, $G'$ in Lemma 2.19 excepting for the dimension, and $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. Then the set $U$ of Problem 2 can be represented as*

$$U = \{W | W = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}), \psi = \widetilde{O}^+ v_c^R(R) + (I_{8n} - \widetilde{O}^+\widetilde{O})y, \ \forall y \in R^{8n}\}. \quad (3.27)$$

*In the case, the minimal norm least squares upper triangular Toeplitz solution $\psi_U$ of Eq.*(3.1) *satisfies*

$$\psi_U = \widetilde{O}^+ v_c^R(R). \quad (3.28)$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \quad (3.29)$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \le m. \end{cases} \quad (3.30)$$

## 3.2 The main results of Problems 3-8

According to the method proposed in the subsection 3.1, we will continue to solve Problems 3-8. First of all, based on the properties of $\{i, j, k\}$-conjugate quaternion matrix, we give the relationship between $\{i, j, k\}$-conjugate quaternion matrix equation

$$MX - \widetilde{X}N = GY + N, \ \widetilde{X} = \{X^i, X^j, X^k\}, \quad (3.31)$$

and its real column stacking form of matrix equation. Next, we construct the expressions of minimal norm least square $\{i, j, k\}$-conjugate solutions.

Now, we start with studying the related property of $i$-conjugate matrix and give the related theorems and conclusions.

$$MX - X^i N = GY + N. \quad (3.32)$$

**Theorem 3.5.** *Let $X = (x^{ij})_{n\times n} \in Q_{LT}^{n\times n}$ with $x^{st} = x_1^{st} + x_2^{st}i + x_3^{st}j + x_4^{st}k$, then*

$$v_c^R(X^i) = \rho v_c^R(X), \quad (3.33)$$

*where*

$$\rho = I_{n^2} \otimes R_4^{(1)} \quad and \quad R_4^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.34)$$

*Proof.* By using Definition 2.2, we can get

$$
\begin{aligned}
v_c^R(X^i) &= \begin{pmatrix} R_4^{(1)} v^R(x^{11}) \\ \vdots \\ R_4^{(1)} v^R(x^{n1}) \\ \vdots \\ R_4^{(1)} v^R(x^{nn}) \end{pmatrix} \\
&= \begin{pmatrix} R_4^{(1)}[(\delta_{n^2}^{11})^T \ltimes v_c^R(X)] \\ \vdots \\ R_4^{(1)}[(\delta_{n^2}^{n1})^T \ltimes v_c^R(X)] \\ \vdots \\ R_4^{(1)}[(\delta_{n^2}^{nn})^T \ltimes v_c^R(X)] \end{pmatrix} = \begin{pmatrix} R_4^{(1)} \ltimes (\delta_{n^2}^{11})^T \ltimes v_c^R(X) \\ \vdots \\ R_4^{(1)} \ltimes (\delta_{n^2}^{n1})^T \ltimes v_c^R(X) \\ \vdots \\ R_4^{(1)} \ltimes (\delta_{n^2}^{nn})^T \ltimes v_c^R(X) \end{pmatrix} \\
&= \begin{pmatrix} (\delta_{n^2}^{11})^T \otimes R_4^{(1)} \\ \vdots \\ (\delta_{n^2}^{n1})^T \otimes R_4^{(1)} \\ \vdots \\ (\delta_{n^2}^{nn})^T \otimes R_4^{(1)} \end{pmatrix} v_c^R(X) = (I_{n^2} \otimes R_4^{(1)}) v_c^R(X).
\end{aligned}
\tag{3.35}
$$

Thus, the proof is finished. □

According to the previous conclusions, we now resolve problems 3-4.

**Theorem 3.6.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$P = [G_2 \ltimes v_r^R(M) \ltimes K - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho \ltimes K, -G_2^{''} \ltimes v_r^R(G) \ltimes K], \tag{3.36}$$

*where $G_2$, $G_2^{'}$, $G_2^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, $K$ is the same as Theorem 3.1, and $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. Then the set $L^i$ of Problem 3 can be represented as*

$$L^i = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = P^+ v_c^R(R) + (I_{8n} - P^+ P)y,\ \forall y \in R^{8n}\}. \tag{3.37}$$

*In this case, the minimal norm least squares lower triangular Toeplitz i-conjugate solution $\psi_{L^i}$ satisfies*

$$\psi_{L^i} = P^+ v_c^R(R). \tag{3.38}$$

*Proof.* By Remark 2.8, Lemma 2.19 and Theorem 3.5, we can get

$$\begin{aligned}
&\|MX - X^iN - GY - R\| \\
&= \|v_c^R(MX - X^iN - GY - R)\| \\
&= \|v_c^R(MX) - v_c^R(X^iN) - v_c^R(GY) - v_c^R(R)\| \\
&= \|G_2 \ltimes v_r^R(M) \ltimes v_c^R(X) - G_2^{'} \ltimes v_r^R(X^i) \ltimes v_c^R(N) \\
&\quad -G_2^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_2 \ltimes v_r^R(M) \ltimes v_c^R(X) - G_2^{'} \ltimes W_{[n]} \ltimes v_c^R(X^i) \ltimes v_c^R(N) \\
&\quad -G_2^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_2 \ltimes v_r^R(A) \ltimes v_c^R(X) - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes v_c^R(X^i) \\
&\quad -G_2^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \qquad (3.39)\\
&= \|G_2 \ltimes v_r^R(A) \ltimes v_c^R(X) - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho \ltimes v_c^R(X) \\
&\quad -G_2^{''} \ltimes v_r^R(G) \ltimes v_c^R(Y) - v_c^R(R)\| \\
&= \|G_2 \ltimes v_r^R(A) \ltimes Kv_s^R(X) - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho \ltimes Kv_s^R(X) \\
&\quad -G_2^{''} \ltimes v_r^R(G) \ltimes Kv_s^R(Y) - v_c^R(R)\| \\
&= \|[G_2 \ltimes v_r^R(A) \ltimes K - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho \ltimes K, -G_2^{''} \ltimes v_r^R(G) \ltimes K] \\
&\quad \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix} - v_c^R(R)\| \\
&= \|P\psi - v_c^R(R)\| \ .
\end{aligned}$$

Therefore

$$\|MX - X^iN - GY - R\| = min \tag{3.40}$$

if and only if

$$\|P\psi - v_c^R(R)\| = min. \tag{3.41}$$

For the real matrix equation

$$P\psi = v_c^R(R). \tag{3.42}$$

According to Lemma 2.12, its least squares lower triangular Toeplitz $i$-conjugate solutions can be written as

$$\psi = P^+v_c^R(R) + (I_{8n} - P^+P)y, \ \forall y \in R^{8n}. \tag{3.43}$$

Hence we can obtain the formula (3.37).

Notice

$$\min_{W_{Q^i_{LT}} \in Q^{2n}_{LT}} ||W_{Q^i_{LT}}|| \Longleftrightarrow \min_{\psi_{L^i} \in R^{8n}} ||\psi_{L^i}||. \tag{3.44}$$

Then, by the prior proof, $\psi_{L^i} \in L^i$ of Eq.(3.32) satisfies

$$\psi_{L^i} = P^+v_c^R(R). \tag{3.45}$$

Thus, the formula (3.38) holds. □

**Corollary 3.7.** *Let $M, N, G, R \in Q^{n\times n}$, $P$ and $\psi$ be the same as Theorem 3.6. Then Eq.*(3.32) *has a solution over $Q^{n\times n}_{LT}$ if and only if*

$$(PP^+ - I_{4n^2})v_c^R(R) = 0. \tag{3.46}$$

*If* (3.46) *holds, the solution set of Eq.(3.32) over* $Q_{LT}^{n\times n}$ *can be represented as*

$$\widetilde{L^i} = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = P^+ v_c^R(R) + (I_{8n} - P^+P)y,\ \forall y \in R^{8n}\}. \quad (3.47)$$

*Moreover, the minimal norm lower triangular Toeplitz* $i$*-conjugate solution over* $Q_{LT}^{n\times n}$ *satisfies*

$$\psi_{L^i} = P^+ v_c^R(R). \quad (3.48)$$

*Proof.* Eq.(3.32) has a solution over $\psi_{L^i} \in Q_{LT}^{n\times n}$ if and only if

$$\|MX - X^iN - GY - R\| = 0. \quad (3.49)$$

Combined Theorem 3.6 with $PP^+P = P$, we can obtain

$$\begin{aligned} \|MX - X^iN - GY - R\| &= \|P\psi - v_c^R(R)\| = \|PP^+P\psi - v_c^R(R)\| \\ &= \|PP^+v_c^R(R) - v_c^R(R)\| = \|(PP^+ - I_{4n^2})v_c^R(R)\|. \end{aligned} \quad (3.50)$$

So, it can be derived

$$\begin{aligned} \|MX - X^iN - GY - R\| = 0 &\Leftrightarrow \|(PP^+ - I_{4n^2})v_c^R(R)\| = 0 \\ &\Leftrightarrow (PP^+ - I_{4n^2})v_c^R(R) = 0. \end{aligned} \quad (3.51)$$

Therefore, it can be concluded that Eq.(3.32) is compatible and its solution satisfies

$$P\psi = v_c^R(R). \quad (3.52)$$

Moreover, by Lemma 2.11, its solution $W_{Q_{LT}^i}$ satisfies

$$\psi = P^+v_c^R(R) + (I_{8n} - P^+P)y. \quad (3.53)$$

And then we can get the minimal norm lower triangular Toeplitz $i$-conjugate solution $\psi_{L^i}$ satisfies

$$\psi_{L^i} = P^+ v_c^R(R). \quad (3.54)$$

Therefore, we complete the proof. □

**Theorem 3.8.** *Let* $M, N, G, R \in Q^{n\times n}$*, denote*

$$\widetilde{P} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho \ltimes \widetilde{K}, -\widetilde{G_2''} \ltimes v_r^R(G) \ltimes \widetilde{K}], \quad (3.55)$$

*where* $\widetilde{G_2}$*,* $\widetilde{G_2'}$*,* $\widetilde{G_2''}$ *have the same structure as* $G$*,* $G'$ *in Lemma 2.19 excepting for the dimension, and* $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$*. Then the set* $U^i$ *of Problem 4 can be represented as*

$$U^i = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}), \psi = \widetilde{P}^+ v_c^R(R) + (I_{8n} - \widetilde{P}^+\widetilde{P})y,\ \forall y \in R^{8n}\}. \quad (3.56)$$

*Moreover, the minimal norm least squares upper triangular Toeplitz* $i$*-conjugate solution* $\psi_{U^i}$ *satisfies*

$$\psi_{U^i} = \widetilde{P}^+ v_c^R(R), \quad (3.57)$$

*in which*

$$v_s^R(X)=\begin{pmatrix} v^R(X_1)\\ \vdots \\ v^R(X_m)\\ \vdots \\ v^R(X_n)\end{pmatrix}, v_s^R(Y)=\begin{pmatrix} v^R(Y_1)\\ \vdots \\ v^R(Y_m)\\ \vdots \\ v^R(Y_n)\end{pmatrix}, \widetilde{K}=\begin{pmatrix} \widetilde{K_1}\\ \vdots \\ \widetilde{K_m}\\ \vdots \\ \widetilde{K_n}\end{pmatrix}, \widetilde{K_m}=\begin{pmatrix} \widetilde{K_{1m}}\\ \vdots \\ \widetilde{K_{rm}}\\ \vdots \\ \widetilde{K_{nm}}\end{pmatrix}, m=1,2,\dots n, \tag{3.58}$$

*and*

$$\widetilde{K_{rm}}=\begin{cases} 0_{4\times 4n}, & r>m,\\ \left(\delta_n^{m-r+1}\right)^T\otimes I_4, & r\le m.\end{cases} \tag{3.59}$$

By analogy with Problems 3-4, we will consider the related property of {j,k}-conjugate quaternion matrix to solve Problems 5-8. The solution process of {j,k}-conjugate is similar to {i}-conjugate, so we only give its corresponding theorem and corollary, and the proof is omitted.

Now, we start to consider the related property of $j$-conjugate quaternion matrix to solve Problems 5-6.

$$MX-X^jN=GY+N. \tag{3.60}$$

**Theorem 3.9.** *Let $X=(x^{ij})_{n\times n}\in Q_{LT}^{n\times n}$ with $x^{st}=x_1^{st}+x_2^{st}i+x_3^{st}j+x_4^{st}k$, then*

$$v_c^R(X^j)=\theta v_c^R(X), \tag{3.61}$$

*where*

$$\rho=I_{n^2}\otimes R_4^{(2)} \quad and \quad R_4^{(2)}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}. \tag{3.62}$$

**Theorem 3.10.** *Let $M,N,G,R\in Q^{n\times n}$, denote*

$$S=[G_3\ltimes v_r^R(M)\ltimes K-G_3^{'}\ltimes W_{[n]}\ltimes W_{[4n^2]}\ltimes v_c^R(N)\ltimes\theta\ltimes K,-G_3^{''}\ltimes v_r^R(G)\ltimes K], \tag{3.63}$$

*where $G_3$, $G_3^{'}$, $G_3^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, $K$ is the same as Theorem 3.1, and $\psi=\begin{bmatrix} v_s^R(X)\\ v_s^R(Y)\end{bmatrix}$. Then the set $L^j$ of Problem 5 can be represented as*

$$L^j=\{W^j|W^j=\begin{pmatrix}X\\Y\end{pmatrix}(X,Y\in Q_{LT}^{n\times n}),\psi=S^+v_c^R(R)+(I_{8n}-S^+S)y,\ \forall y\in R^{8n}\}. \tag{3.64}$$

*In addition, the minimal norm least squares lower triangular Toeplitz $j$-conjugate solution $\psi_{L^j}$ satisfies*

$$\psi_{L^j}=S^+v_c^R(R). \tag{3.65}$$

**Corollary 3.11.** *Let $M, N, G, R \in Q^{n\times n}$, $S$ and $\psi$ be the same as Theorem 3.10. Then Eq.*(3.60) *has a solution over $Q_{LT}^{n\times n}$ if and only if*

$$(SS^{+} - I_{4n^2})v_c^R(R) = 0. \tag{3.66}$$

*If* (3.66) *holds, the solution set of Eq.*(3.60) *over $Q_{LT}^{n\times n}$ can be represented as*

$$\widetilde{L^j} = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = S^{+}v_c^R(R) + (I_{8n} - S^{+}S)y,\ \forall y \in R^{8n}\}. \tag{3.67}$$

*Moreover, the minimal norm lower triangular Toeplitz $j$-conjugate solution over $Q_{LT}^{n\times n}$ satisfies*

$$\psi_{L^j} = S^{+}v_c^R(R). \tag{3.68}$$

Similarly, according to the solution process of Problem 5, and we can obtain the following theorems to solve Problem 6.

**Theorem 3.12.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{S} = [\widetilde{G_3} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_3'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \theta \ltimes \widetilde{K}, -\widetilde{G_3''} \ltimes v_r^R(G) \ltimes \widetilde{K}], \tag{3.69}$$

*where $\widetilde{G_3}$, $\widetilde{G_3'}$, $\widetilde{G_3''}$ have the same structure as $G$, $G'$ in Lemma 2.19 excepting for the dimension, and $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. Then the set $U^j$ of Problem 6 can be represented as*

$$U^j = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}), \psi = \widetilde{S}^{+}v_c^R(R) + (I_{8n} - \widetilde{S}^{+}\widetilde{S})y,\ \forall y \in R^{8n}\}, \tag{3.70}$$

*and the minimal norm least squares upper triangular Toeplitz $j$-conjugate solution $\psi_{U^j}$ satisfies*

$$\psi_{U^j} = \widetilde{S}^{+}v_c^R(R), \tag{3.71}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \dots n, \tag{3.72}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \le m. \end{cases} \tag{3.73}$$

By analogy with Problems 5-6, we start to consider the related property of $k$-conjugate matrix to solve Problems 7-8.

$$MX - X^k N = GY + N. \tag{3.74}$$

**Theorem 3.13.** *Let* $X = (x^{ij})_{n\times n} \in Q_{LT}^{n\times n}$ *with* $x^{st} = x_1^{st} + x_2^{st} i + x_3^{st} j + x_4^{st} k$*, then*

$$v_c^R(X^k) = \eta v_c^R(X), \tag{3.75}$$

*where*

$$\eta = I_{n^2} \otimes R_4^{(3)} \quad and \quad R_4^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.76}$$

**Theorem 3.14.** *Let* $M, N, G, R \in Q^{n\times n}$*, denote*

$$T = [G_4 \ltimes v_r^R(M) \ltimes K - G_4' \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \eta \ltimes K, -G_4'' \ltimes v_r^R(G) \ltimes K], \tag{3.77}$$

*where* $G_4$, $G_4'$, $G_4''$ *have the same structure as* $G$, $G'$ *in Lemma 2.19 excepting for the dimension,* $K$ *is the same as Theorem 3.1, and* $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. *Then the set* $L^k$ *of Problem 7 can be represented as*

$$L^k = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = T^+ v_c^R(R) + (I_{8n} - T^+T)y, \ \forall y \in R^{8n}\}, \tag{3.78}$$

*and the minimal norm least squares lower triangular Toeplitz k-conjugate solution* $\psi_{L^k}$ *satisfies*

$$\psi_{L^k} = T^+ v_c^R(R). \tag{3.79}$$

**Corollary 3.15.** *Let* $M, N, G, R \in Q^{n\times n}$, $T$ *and* $\psi$ *be the same as Theorem 3.14. Then Eq.*(3.74) *has a solution over* $Q_{LT}^{n\times n}$ *if and only if*

$$(TT^+ - I_{4n^2}) v_c^R(R) = 0. \tag{3.80}$$

*If* (3.80) *holds, the solution set of Eq.*(3.74) *over* $Q_{LT}^{n\times n}$ *can be represented as*

$$\widetilde{L^k} = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{LT}^{n\times n}), \psi = T^+ v_c^R(R) + (I_{8n} - T^+T)y, \ \forall y \in R^{8n}\}. \tag{3.81}$$

*Moreover, the minimal norm lower triangular Toeplitz k-conjugate solution over* $Q_{LT}^{n\times n}$ *satisfies*

$$\psi_{L^k} = T^+ v_c^R(R). \tag{3.82}$$

Similarly, according to the solution process of Problem 7, and we can obtain the following theorems to solve Problem 8.

**Theorem 3.16.** *Let* $M, N, G, R \in Q^{n\times n}$*, denote*

$$\widetilde{T} = [\widetilde{G_4} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_4'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \eta \ltimes \widetilde{K}, -\widetilde{G_4''} \ltimes v_r^R(G) \ltimes \widetilde{K}], \tag{3.83}$$

*where* $\widetilde{G_4}$, $\widetilde{G_4'}$, $\widetilde{G_4''}$ *have the same structure as* $G$, $G'$ *in Lemma 2.19 excepting for the dimension, and* $\psi = \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix}$. *Then the set* $U^k$ *of Problem 8 can be represented as*

$$U^k = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix} (X, Y \in Q_{UT}^{n\times n}), \psi = \widetilde{T}^+ v_c^R(R) + (I_{8n} - \widetilde{T}^+\widetilde{T})y, \ \forall y \in R^{8n}\}. \tag{3.84}$$

*In this case, the minimal norm least squares upper triangular Toeplitz k-conjugate solution $\psi_{U^k}$ satisfies*

$$\psi_{U^k} = \widetilde{T}^{+} v_c^R(R), \tag{3.85}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{3.86}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \leq m. \end{cases} \tag{3.87}$$

## 4 Special Cases

In this part, we study some special cases of Problems 3-8, and give the minimal norm least squares lower(upper) triangular Toeplitz (anti)$\{i, j, k\}$-self-conjugate solutions of the studied quaternion matrix equation.

### 4.1 $i$-self-conjugate and anti-$i$-self-conjugate solutions

Based on the characteristics of $i$-self-conjugate matrix, by adjusting some elements of $R_4^{(1)}$ in Theorem 3.5, we can obtain the following Theorem 4.1. Then, we use Theorem 3.1 to extract Independent elements and reduce the calculation scale, and then derive the following related theorems.

**Theorem 4.1.** *Suppose that $X = (x^{ij}) \in IQ_{LT}^{n\times n}$ with $x^{st} = x_1^{st} + x_2^{st} i + x_3^{st} j + x_4^{st} k$, denote*

$$v_t^R(X) = [x_1^{11}, x_2^{11}, \ldots, x_1^{n1}, x_2^{n1}]^T. \tag{4.1}$$

*then the relation*

$$v_c^R(X^i) = \rho' v_c^R(X) = \rho' K v_s^R(X) = \rho' K V v_t^R(X) \tag{4.2}$$

*holds, in which*

$$\rho' = I_{n^2} \otimes (R_4^{(1)})', \quad (R_4^{(1)})' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \delta_{2n^2}[1, 2, 5, 6, \ldots, 4n-2]. \tag{4.3}$$

**Theorem 4.2.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$P' = [G_2 \ltimes v_r^R(M) \ltimes K - G_2' \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho' \ltimes K, -G_2'' \ltimes v_r^R(G) \ltimes K]V, \tag{4.4}$$

*where* $G_2,\ G_2^{'},\ G_2^{''}$ *have the same structure as* $G,\ G^{'}$ *in Lemma 2.19 excepting for the dimension,* $K$ *is the same as in Theorem 3.1, and* $\psi^i = \begin{bmatrix} v_t^R(X) \\ v_t^R(Y) \end{bmatrix}$. *Then the set* $I_L$ *of solution can be written as*

$$I_L = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in IQ_{LT}^{n\times n}), \psi^i = (P^{'})^+ v_c^R(R) + (I_{4n} - (P^{'})^+(P^{'}))y,\ \forall y \in R^{4n}\}. \tag{4.5}$$

*Moreover, the minimal norm least squares lower triangular Toeplitz i-self-conjugate solution* $\psi_{I_L}$ *satisfies*

$$\psi_{I_L} = (P^{'})^+ v_c^R(R). \tag{4.6}$$

**Corollary 4.3.** *Let* $M, N, G, R \in Q^{n\times n}$, $P^{'}$ *and* $\psi^i$ *be the same as Theorem 4.2. Then Eq.*(3.32) *has a lower triangular Toeplitz i-self-conjugate solution if and only if*

$$((P^{'})(P^{'})^+ - I_{4n^2})v_c^R(R) = 0. \tag{4.7}$$

*If* (4.7) *holds, the lower triangular Toeplitz i-self-conjugate solution set of Eq.*(3.32) *can be represented as*

$$\begin{aligned} \widetilde{I_L} = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in IQ_{LT}^{n\times n}), \\ \psi^i = (P^{'})^+ v_c^R(R) + (I_{4n} - (P^{'})^+(P^{'}))y,\ \forall y \in R^{4n}\}. \end{aligned} \tag{4.8}$$

*Moreover, the minimal norm lower triangular Toeplitz i-self-conjugate solution satisfies*

$$\psi_{I_L} = (P^{'})^+ v_c^R(R). \tag{4.9}$$

**Theorem 4.4.** *Let* $M, N, G, R \in Q^{n\times n}$, *denote*

$$\widetilde{P^{'}} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2^{'}} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho^{'} \ltimes \widetilde{K}, -\widetilde{G_2^{''}} \ltimes v_r^R(G) \ltimes \widetilde{K}]V, \tag{4.10}$$

*where* $\widetilde{G_2},\ \widetilde{G_2^{'}},\ \widetilde{G_2^{''}}$ *have the same structure as* $G,\ G^{'}$ *in Lemma 2.19 excepting for the dimension, and* $\psi^i = \begin{bmatrix} v_t^R(X) \\ v_t^R(Y) \end{bmatrix}$. *Then the set* $I_U$ *of solution can be written as*

$$I_U = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in IQ_{UT}^{n\times n}), \psi^i = \widetilde{P^{'}}^+ v_c^R(R) + (I_{4n} - \widetilde{P^{'}}^+ \widetilde{P^{'}})y,\ \forall y \in R^{4n}\}. \tag{4.11}$$

*Moreover, the minimal norm least squares upper triangular Toeplitz i-self-conjugate solution* $\psi_{I_U}$ *satisfies*

$$\psi_{I_U} = \widetilde{P^{'}}^+ v_c^R(R), \tag{4.12}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{4.13}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \leq m. \end{cases} \tag{4.14}$$

In order to research anti-$i$-self-conjugate solution, according to the characteristics of anti-$i$-self-conjugate matrix, by adjusting some elements of $R_4^{(1)}$ in Theorem 3.5, we can get the following Theorem 4.5. Meanwhile, by combining the above conclusions with the Theorem 3.1, we extract independent elements and reduce the calculation scale, then derive the following theorems.

**Theorem 4.5.** *Suppose that $X = (x^{ij}) \in AIQ_{LT}^{n\times n}$ with $x^{st} = x_1^{st} + x_2^{st}i + x_3^{st}j + x_4^{st}k$, denote*

$$v_{t'}^R(X) = [x_3^{11}, x_4^{11}, \ldots, x_3^{n1}, x_4^{n1}]^T. \tag{4.15}$$

*then the relation*

$$v_c^R(X^i) = \rho^{''} v_c^R(X) = \rho^{''} K v_s^R(X) = \rho^{''} K V^{'} v_{t'}^R(X) \tag{4.16}$$

*in which*

$$\rho^{''} = I_{n^2} \otimes (R_4^{(1)})^{''}, \quad (R_4^{(1)})^{''} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ \ and \ \ V^{'} = \delta_{2n^2}[3, 4, 7, 8, \ldots, 4n]. \tag{4.17}$$

**Theorem 4.6.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$P^{''} = [G_2 \ltimes v_r^R(M) \ltimes K - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho^{''} \ltimes K, -G_2^{''} \ltimes v_r^R(G) \ltimes K]V^{'}, \tag{4.18}$$

*where $G_2$, $G_2^{'}$, $G_2^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, $K$ is the same as Theorem 3.1, and $\psi^{i^{'}} = \begin{bmatrix} v_{t'}^R(X) \\ v_{t'}^R(Y) \end{bmatrix}$. Then the set $AI_L$ of solution can be written as*

$$AI_L = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AIQ_{LT}^{n\times n}), \psi^{i^{'}} = (P^{''})^+ v_c^R(R) + (I_{4n} - (P^{''})^+(P^{''}))y, \forall y \in R^{4n}\}. \tag{4.19}$$

*Moreover, the minimal norm least squares lower triangular Toeplitz anti-i-self-conjugate solution $\psi_{AI_L}$ satisfies*

$$\psi_{AI_L} = (P^{''})^+ v_c^R(R). \tag{4.20}$$

**Corollary 4.7.** *Let $M, N, G, R \in Q^{n\times n}$, $P^{''}$ and $\psi^{i^{'}}$ be the same as Theorem 4.6. Then Eq.(3.32) has a lower triangular Toeplitz anti-i-self-conjugate solution if and only if*

$$((P^{''})(P^{''})^+ - I_{4n^2}) v_c^R(R) = 0. \tag{4.21}$$

*If (4.21) holds, then the lower triangular Toeplitz anti-i-self-conjugate solution set of Eq.(3.32) can be represented as*

$$\widetilde{AI_L} = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AIQ_{LT}^{n\times n}), \psi^{i^{'}} = (P^{''})^+ v_c^R(R) + (I_{4n} - (P^{''})^+(P^{''}))y, \forall y \in R^{4n}\}. \tag{4.22}$$

*Moreover, the minimal norm lower triangular Toeplitz anti-i-self-conjugate solution satisfies*

$$\psi_{AI_L} = (P^{''})^{+} v_c^R(R). \tag{4.23}$$

**Theorem 4.8.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{P^{''}} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2^{'}} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho^{''} \ltimes \widetilde{K}, -\widetilde{G_2^{''}} \ltimes v_r^R(G) \ltimes \widetilde{K}]V^{'}, \tag{4.24}$$

*where $\widetilde{G_2}$, $\widetilde{G_2^{'}}$, $\widetilde{G_2^{''}}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi^{i^{'}} = \begin{bmatrix} v_{t^{'}}^R(X) \\ v_{t^{'}}^R(Y) \end{bmatrix}$. Then the set $AI_U$ of solution can be written as*

$$AI_U = \{W^i | W^i = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AIQ_{UT}^{n\times n}), \psi^{i^{'}} = \widetilde{P^{''}}^{+} v_c^R(R) + (I_{4n} - \widetilde{P^{''}}^{+}\widetilde{P^{''}})y,\ \forall y \in R^{4n}\}, \tag{4.25}$$

*and the minimal norm least squares upper triangular Toeplitz anti-i-self-conjugate solution $\psi_{AI_U}$ satisfies*

$$\psi_{AI_U} = \widetilde{P^{''}}^{+} v_c^R(R), \tag{4.26}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{4.27}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \leq m. \end{cases} \tag{4.28}$$

## 4.2 $j$-self-conjugate and anti-$j$-self-conjugate solutions

Similar to the solution process of (anti)$i$-self-conjugate solution, according to the properties of (anti)$j$-self-conjugate matrix, we can get the relevant (anti)$j$-self-conjugate solution.

**Theorem 4.9.** *Suppose that $X = (x^{ij}) \in JQ_{LT}^{n\times n}$ with $x^{st} = x_1^{st} + x_2^{st} i + x_3^{st} j + x_4^{st} k$, denote*

$$v_a^R(X) = [x_1^{11}, x_3^{11}, \ldots, x_1^{n1}, x_3^{n1}]^T. \tag{4.29}$$

*then the relation*

$$v_c^R(X^j) = \theta^{'} v_c^R(X) = \theta^{'} K v_s^R(X) = \theta^{'} K H v_t^R(X) \tag{4.30}$$

*holds, in which*

$$\theta' = I_{n^2} \otimes (R_4^{(2)})^{'}, \quad (R_4^{(2)})^{'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H = \delta_{2n^2}[1, 3, 5, 7, \ldots, 4n-1]. \tag{4.31}$$

**Theorem 4.10.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$S^{'} = [G_2 \ltimes v_r^R(M) \ltimes K - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \theta^{'} \ltimes K, -G_2^{''} \ltimes v_r^R(G) \ltimes K]H, \tag{4.32}$$

*where $G_2$, $G_2^{'}$, $G_2^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, $K$ is the same as Theorem 3.1, and $\psi^j = \begin{bmatrix} v_a^R(X) \\ v_a^R(Y) \end{bmatrix}$. Then the set $J_L$ of solution can be written as*

$$J_L = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in JQ_{LT}^{n\times n}), \psi^j = (S^{'})^+ v_c^R(R) + (I_{4n} - (S^{'})^+(S^{'}))y, \ \forall y \in R^{4n}\}, \tag{4.33}$$

*and the minimal norm least squares lower triangular Toeplitz $j$-self-conjugate solution $\psi_{J_L}$ satisfies*

$$\psi_{J_L} = (S^{'})^+ v_c^R(R). \tag{4.34}$$

**Corollary 4.11.** *Let $M, N, G, R \in Q^{n\times n}$, $S^{'}$ and $\psi^j$ be the same as Theorem 4.10. Then Eq.(3.60) has a lower triangular Toeplitz $j$-self-conjugate solution if and only if*

$$((S^{'})(S^{'})^+ - I_{4n^2}) v_c^R(R) = 0. \tag{4.35}$$

*If (4.35) holds, the lower triangular Toeplitz $j$-self-conjugate solution set of Eq.(3.60) can be represented as*

$$\widetilde{J_L} = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in JQ_{LT}^{n\times n}), \psi^i = (S^{'})^+ v_c^R(R) + (I_{4n} - (S^{'})^+(S^{'}))y, \ \forall y \in R^{4n}\}. \tag{4.36}$$

*Moreover, the minimal norm lower triangular Toeplitz $j$-self-conjugate solution satisfies*

$$\psi_{J_L} = (S^{'})^+ v_c^R(R). \tag{4.37}$$

**Theorem 4.12.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{S^{'}} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2^{'}} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \rho^{'} \ltimes \widetilde{K}, -\widetilde{G_2^{''}} \ltimes v_r^R(G) \ltimes \widetilde{K}]H, \tag{4.38}$$

*where $\widetilde{G_2}$, $\widetilde{G_2^{'}}$, $\widetilde{G_2^{''}}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi^j = \begin{bmatrix} v_a^R(X) \\ v_a^R(Y) \end{bmatrix}$. Then the set $J_U$ of solution can be written as*

$$J_U = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in JQ_{UT}^{n\times n}), \psi^j = \widetilde{S^{'}}^+ v_c^R(R) + (I_{4n} - \widetilde{S^{'}}^+ \widetilde{S^{'}})y, \ \forall y \in R^{4n}\}. \tag{4.39}$$

*Moreover, the minimal norm least squares upper triangular Toeplitz $j$-self-conjugate solution $\psi_{J_U}$ satisfies*

$$\psi_{J_U} = \widetilde{S^{'}}^+ v_c^R(R), \tag{4.40}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{4.41}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \le m. \end{cases} \tag{4.42}$$

In order to research anti-$j$-self-conjugate solution, according to the characteristics of anti-$j$-self-conjugate matrix, by adjusting some elements of $R_4^{(2)}$ in Theorem 3.9, we can get the following Theorem 4.13. Meanwhile, by combining the above conclusions with the Theorem 3.1, we extract independent elements and reduce the calculation scale, then derive the following theorems..

**Theorem 4.13.** *Suppose that* $X = (x^{ij}) \in AJQ_{LT}^{n\times n}$ *with* $x^{st} = x_1^{st} + x_2^{st}i + x_3^{st}j + x_4^{st}k$, *denote*

$$v_{a'}^R(X) = [x_2^{11}, x_4^{11}, \dots, x_2^{n1}, x_4^{n1}]^T. \tag{4.43}$$

*then the relation*

$$v_c^R(X^j) = \theta'' v_c^R(X) = \theta'' K v_s^R(X) = \theta'' K H' v_{t'}^R(X) \tag{4.44}$$

*in which*

$$\theta'' = I_{n^2} \otimes (R_4^{(2)})'', \quad (R_4^{(2)})'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad H' = \delta_{2n^2}[2, 4, 6, 8, \dots, 4n]. \tag{4.45}$$

**Theorem 4.14.** *Let* $M, N, G, R \in Q^{n\times n}$, *denote*

$$S'' = [G_2 \ltimes v_r^R(M) \ltimes K - G_2' \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \theta'' \ltimes K, -G_2'' \ltimes v_r^R(G) \ltimes K]H', \tag{4.46}$$

*where* $G_2$, $G_2'$, $G_2''$ *have the same structure as* $G$, $G'$ *in Lemma 2.19 excepting for the dimension,* $K$ *is the same as Theorem 3.1, and* $\psi^{j'} = \begin{bmatrix} v_{a'}^R(X) \\ v_{a'}^R(Y) \end{bmatrix}$. *Then the set* $AJ_L$ *of solution can be written as*

$$AJ_L = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AJQ_{LT}^{n\times n}), \psi^{j'} = (S'')^+ v_c^R(R) + (I_{4n} - (S'')^+(S''))y, \forall y \in R^{4n}\}, \tag{4.47}$$

*and the minimal norm least squares lower triangular Toeplitz anti-*$j$*-self-conjugate solution* $\psi_{AJ_L}$ *satisfies*

$$\psi_{AJ_L} = (S'')^+ v_c^R(R). \tag{4.48}$$

**Corollary 4.15.** *Let* $M, N, G, R \in Q^{n\times n}$, $S''$ *and* $\psi^{j'}$ *be the same as Theorem 4.14. Then Eq.*(3.60) *has a lower triangular Toeplitz anti-*$j$*-self-conjugate solution if and only if*

$$((S'')(S'')^+ - I_{4n^2}) v_c^R(R) = 0. \tag{4.49}$$

*If* (4.49) *holds, the lower triangular Toeplitz anti-*$j$*-self-conjugate solution set of Eq.*(3.60) *can be represented as*

$$\widetilde{AJ_L} = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AJQ_{LT}^{n\times n}), \psi^{j'} = (S'')^+ v_c^R(R) + (I_{4n} - (S'')^+(S''))y, \forall y \in R^{4n}\}. \tag{4.50}$$

*Moreover, the minimal norm lower triangular Toeplitz anti-$j$-self-conjugate solution satisfies*

$$\psi_{AJ_L} = (S^{''})^{+} v_c^R(R). \tag{4.51}$$

**Theorem 4.16.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{S''} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \theta^{''} \ltimes \widetilde{K}, -\widetilde{G_2''} \ltimes v_r^R(G) \ltimes \widetilde{K}]H^{'}, \tag{4.52}$$

*where $\widetilde{G_2}$, $\widetilde{G_2'}$, $\widetilde{G_2''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi^{j^{'}} = \begin{bmatrix} v_{a'}^R(X) \\ v_{a'}^R(Y) \end{bmatrix}$. Then the set $AJ_U$ of solution can be written as*

$$AJ_U = \{W^j | W^j = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AJQ_{UT}^{n\times n}), \psi^{j^{'}} = \widetilde{S''}^{+} v_c^R(R) + (I_{4n} - \widetilde{S''}^{+}\widetilde{S''})y, \ \forall y \in R^{4n}\}, \tag{4.53}$$

*and the minimal norm least squares upper triangular Toeplitz anti-$i$-self-conjugate solution $\psi_{AJ_U}$ satisfies*

$$\psi_{AJ_U} = \widetilde{S''}^{+} v_c^R(R), \tag{4.54}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{4.55}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \le m. \end{cases} \tag{4.56}$$

## 4.3 $k$-self-conjugate and anti-$k$-self-conjugate solutions

Similar to the solution process of (anti)$i$-self-conjugate solution, according to the properties of (anti)$k$-self-conjugate matrix, we can get the relevant (anti)$k$-self-conjugate solution.

**Theorem 4.17.** *Suppose that $X = (x^{ij}) \in KQ_{LT}^{n\times n}$ with $x^{st} = x_1^{st} + x_2^{st}i + x_3^{st}j + x_4^{st}k$, denote*

$$v_b^R(X) = [x_1^{11}, x_4^{11}, \ldots, x_1^{n1}, x_4^{n1}]^T. \tag{4.57}$$

*then the relation*

$$v_c^R(X^k) = \eta^{'} v_c^R(X) = \eta^{'} K v_s^R(X) = \eta^{'} K J v_b^R(X) \tag{4.58}$$

*holds, in which*

$$\eta^{'} = I_{n^2} \otimes (R_4^{(3)})^{'}, \quad (R_4^{(3)})^{'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \delta_{2n^2}[1, 4, 5, 8, \ldots, 4n]. \tag{4.59}$$

**Theorem 4.18.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$T^{'} = [G_2 \ltimes v_r^R(M) \ltimes K - G_2^{'} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \eta^{'} \ltimes K, -G_2^{''} \ltimes v_r^R(G) \ltimes K]J, \tag{4.60}$$

*where $G_2$, $G_2^{'}$, $G_2^{''}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, $K$ is the same as Theorem 3.1, and $\psi^k = \begin{bmatrix} v_b^R(X) \\ v_b^R(Y) \end{bmatrix}$. Then the set $K_L$ of solution can be written as*

$$K_L = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in KQ_{LT}^{n\times n}), \psi^k = (T^{'})^{+} v_c^R(R) + (I_{4n} - (T^{'})^{+}(T^{'}))y, \forall y \in R^{4n}\}, \tag{4.61}$$

*and the minimal norm least squares lower triangular Toeplitz k-self-conjugate solution $\psi_{K_L}$ satisfies*

$$\psi_{K_L} = (T^{'})^{+} v_c^R(R). \tag{4.62}$$

**Corollary 4.19.** *Let $M, N, G, R \in Q^{n\times n}$, $T^{'}$ and $\psi^k$ be the same as Theorem 4.18. Then Eq.*(3.74) *has a lower triangular Toeplitz k-self-conjugate solution if and only if*

$$((T^{'})(T^{'})^{+} - I_{4n^2}) v_c^R(R) = 0. \tag{4.63}$$

*If* (4.63) *holds, the lower triangular Toeplitz k-self-conjugate solution set of Eq.*(3.74) *can be represented as*

$$\widetilde{K_L} = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in KQ_{LT}^{n\times n}), \psi^k = (T^{'})^{+} v_c^R(R) + (I_{4n} - (T^{'})^{+}(T^{'}))y, \ \forall y \in R^{4n}\}. \tag{4.64}$$

*Moreover, the minimal norm lower triangular Toeplitz k-self-conjugate solution satisfies*

$$\psi_{K_L} = (T^{'})^{+} v_c^R(R). \tag{4.65}$$

**Theorem 4.20.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{T^{'}} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2^{'}} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \eta^{'} \ltimes \widetilde{K}, -\widetilde{G_2^{''}} \ltimes v_r^R(G) \ltimes \widetilde{K}]J, \tag{4.66}$$

*where $\widetilde{G_2}$, $\widetilde{G_2^{'}}$, $\widetilde{G_2^{''}}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi^k = \begin{bmatrix} v_b^R(X) \\ v_b^R(Y) \end{bmatrix}$. Then the set $K_U$ of solution can be written as*

$$K_U = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in KQ_{UT}^{n\times n}), \psi^k = \widetilde{T^{'}}^{+} v_c^R(R) + (I_{4n} - \widetilde{T^{'}}^{+} \widetilde{T^{'}})y, \forall y \in R^{4n}\}, \tag{4.67}$$

*and the minimal norm least squares upper triangular Toeplitz k-self-conjugate solution $\psi_{K_U}$ satisfies*

$$\psi_{K_U} = \widetilde{T^{'}}^{+} v_c^R(R), \tag{4.68}$$

*in which*

$$v_s^R(X)=\begin{pmatrix} v^R(X_1)\\ \vdots \\ v^R(X_m)\\ \vdots \\ v^R(X_n)\end{pmatrix}, v_s^R(Y)=\begin{pmatrix} v^R(Y_1)\\ \vdots \\ v^R(Y_m)\\ \vdots \\ v^R(Y_n)\end{pmatrix}, \widetilde{K}=\begin{pmatrix} \widetilde{K_1}\\ \vdots \\ \widetilde{K_m}\\ \vdots \\ \widetilde{K_n}\end{pmatrix}, \widetilde{K_m}=\begin{pmatrix} \widetilde{K_{1m}}\\ \vdots \\ \widetilde{K_{rm}}\\ \vdots \\ \widetilde{K_{nm}}\end{pmatrix}, m=1,2,\dots n, \tag{4.69}$$

*and*

$$\widetilde{K_{rm}}=\begin{cases} 0_{4\times 4n}, & r>m,\\ \left(\delta_n^{m-r+1}\right)^T\otimes I_4, & r\le m.\end{cases} \tag{4.70}$$

In order to research anti-$k$-self-conjugate solution, according to the characteristics of anti-$i$-self-conjugate matrix, by adjusting some elements of $R_4^{(3)}$ in Theorem 3.13, we can get the following Theorem 4.21. Meanwhile, by combining the above conclusions with the Theorem 3.1, we extract independent elements and reduce the calculation scale, then derive the following theorems.

**Theorem 4.21.** *Suppose that* $X=(x^{ij})\in AKQ_{LT}^{n\times n}$ *with* $x^{st}=x_1^{st}+x_2^{st}i+x_3^{st}j+x_4^{st}k$, *denote*

$$v_{a'}^R(X)=[x_2^{11},x_3^{11},\dots,x_2^{n1},x_3^{n1}]^T. \tag{4.71}$$

*then the relation*

$$v_c^R(X^k)=\eta^{''}v_c^R(X)=\eta^{''}Kv_s^R(X)=\eta^{''}KV^{'}v_{b'}^R(X) \tag{4.72}$$

*holds, in which*

$$\eta^{''}=I_{n^2}\otimes(R_4^{(3)})^{''},\quad (R_4^{(3)})^{''}=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\quad \text{and}\quad J^{'}=\delta_{2n^2}[2,3,6,7,\dots,4n-1]. \tag{4.73}$$

**Theorem 4.22.** *Let* $M,N,G,R\in Q^{n\times n}$, *denote*

$$T^{''}=[G_2\ltimes v_r^R(M)\ltimes K-G_2^{'}\ltimes W_{[n]}\ltimes W_{[4n^2]}\ltimes v_c^R(N)\ltimes\eta^{''}\ltimes K,-G_2^{''}\ltimes v_r^R(G)\ltimes K]J^{'}, \tag{4.74}$$

*where* $G_2$, $G_2^{'}$, $G_2^{''}$ *have the same structure as* $G$, $G^{'}$ *in Lemma 2.19 excepting for the dimension,* $K$ *is the same as Theorem 3.1, and* $\psi^{k^{'}}=\begin{bmatrix} v_{b'}^R(X)\\ v_{b'}^R(Y)\end{bmatrix}$. *Then the set* $AK_L$ *of solution can be written as*

$$AK_L=\{W^k|W^k=\begin{pmatrix}X\\Y\end{pmatrix},(X,Y\in AKQ_{LT}^{n\times n}),\psi^{k^{'}}=(T^{''})^{+}v_c^R(R)+(I_{4n}-(T^{''})^{+}(T^{''}))y,\forall y\in R^{4n}\}, \tag{4.75}$$

*and the minimal norm least squares lower triangular Toeplitz anti-k-self-conjugate solution* $\psi_{AK_L}$ *satisfies*

$$\psi_{AK_L}=(T^{''})^{+}v_c^R(R). \tag{4.76}$$

**Corollary 4.23.** *Let $M, N, G, R \in Q^{n\times n}$, $T^{''}$ and $\psi^{k^{'}}$ be the same as Theorem 4.22. Then Eq.(3.74) has a lower triangular Toeplitz anti-k-self-conjugate solution if and only if*

$$((T^{''})(T^{''})^{+} - I_{4n^2})v_c^R(R) = 0. \tag{4.77}$$

*If (4.77) holds, the lower triangular Toeplitz anti-k-self-conjugate solution set of Eq.(3.74) can be represented as*

$$\begin{aligned}\widetilde{AK_L} = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AKQ_{LT}^{n\times n}),\\ \psi^{k^{'}} = (T^{''})^{+}v_c^R(R) + (I_{4n} - (T^{''})^{+}(T^{''}))y, \forall y \in R^{4n}\}.\end{aligned} \tag{4.78}$$

*Moreover, the minimal norm lower triangular Toeplitz anti-k-self-conjugate solution satisfies*

$$\psi_{AK_L} = (T^{''})^{+}v_c^R(R). \tag{4.79}$$

**Theorem 4.24.** *Let $M, N, G, R \in Q^{n\times n}$, denote*

$$\widetilde{T^{''}} = [\widetilde{G_2} \ltimes v_r^R(M) \ltimes \widetilde{K} - \widetilde{G_2^{'}} \ltimes W_{[n]} \ltimes W_{[4n^2]} \ltimes v_c^R(N) \ltimes \eta^{''} \ltimes \widetilde{K}, -\widetilde{G_2^{''}} \ltimes v_r^R(G) \ltimes \widetilde{K}]J^{'}, \tag{4.80}$$

*where $\widetilde{G_2}$, $\widetilde{G_2^{'}}$, $\widetilde{G_2^{''}}$ have the same structure as $G$, $G^{'}$ in Lemma 2.19 excepting for the dimension, and $\psi^{k^{'}} = \begin{bmatrix} v_{b^{'}}^R(X) \\ v_{b^{'}}^R(Y) \end{bmatrix}$. Then the set $AK_U$ of solution can be written as*

$$AK_U = \{W^k | W^k = \begin{pmatrix} X \\ Y \end{pmatrix}, (X, Y \in AKQ_{UT}^{n\times n}), \psi^{k^{'}} = \widetilde{T^{''}}^{+} v_c^R(R) + (I_{4n} - \widetilde{T^{''}}^{+}\widetilde{T^{''}})y, \ \forall y \in R^{4n}\}. \tag{4.81}$$

*Moreover, the minimal norm least squares upper triangular Toeplitz anti-k-self-conjugate solution $\psi_{AK_U}$ satisfies*

$$\psi_{AK_U} = \widetilde{T^{''}}^{+} v_c^R(R), \tag{4.82}$$

*in which*

$$v_s^R(X) = \begin{pmatrix} v^R(X_1) \\ \vdots \\ v^R(X_m) \\ \vdots \\ v^R(X_n) \end{pmatrix}, v_s^R(Y) = \begin{pmatrix} v^R(Y_1) \\ \vdots \\ v^R(Y_m) \\ \vdots \\ v^R(Y_n) \end{pmatrix}, \widetilde{K} = \begin{pmatrix} \widetilde{K_1} \\ \vdots \\ \widetilde{K_m} \\ \vdots \\ \widetilde{K_n} \end{pmatrix}, \widetilde{K_m} = \begin{pmatrix} \widetilde{K_{1m}} \\ \vdots \\ \widetilde{K_{rm}} \\ \vdots \\ \widetilde{K_{nm}} \end{pmatrix}, m = 1, 2, \ldots n, \tag{4.83}$$

*and*

$$\widetilde{K_{rm}} = \begin{cases} 0_{4\times 4n}, & r > m, \\ \left(\delta_n^{m-r+1}\right)^T \otimes I_4, & r \leq m. \end{cases} \tag{4.84}$$

## 5 Conclusion Remarks

In this paper, we study the least squares problems of non-homogeneous Yakubovich-(conjugate) quaternion matrix equation (1.1). According to the structural characteristics of the lower(upper) triangular toeplitz matrix, the problem of solving the quaternion matrix equation is transformed into the corresponding problem in the real number field by using STP method and real vector representation. We investigate the minimal norm least squares lower(upper) triangular toeplitz solution of (1.1) and the minimal norm least squares lower(upper) triangular toeplitz $\{i, j, k\}$-conjugate solutions of its corresponding quaternion matrix equation, and the necessary and sufficient conditions and expressions for the existence of solutions are derived. In addition, we also study the minimal norm least squares lower(upper) triangular toeplitz self-conjugate solutions and anti-self-conjugate solutions of the studied quaternion conjugate matrix equation. This method can be used to solve different quaternion linear systems, for example, Stein quaternion matrix equation.

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