Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 20, No. 1, pp. 99–117, 2024

A PENALTY-BASED METHOD FOR SOLVING A DISCRETE HJB COMPLEMENTARITY PROBLEM*

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Abstract: We develop a power penalty-based method for numerically solving a Hamilton-Jacobi-Bellman (HJB) complementarity problem (CP) in $\mathbb{R}^N$ arising in optimal control as well as financial engineering. By penalizing one of the complementarity constraints, the HJB CP is transformed into a nonlinear system. We show that the solution to this nonlinear system converges to that of the HJB CP exponentially when the appropriate coefficient matrix is an M-matrix. A modified Newton-type iterative method is proposed to solve the resulting nonlinear system. Numerical examples are presented to confirm the theoretical findings.

Key words: *HJB equation; Complementarity Problem; Penalty method; Convergence rate*

Mathematics Subject Classification: *65N12, 65K10, 91B28*

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1 Introduction

Many real-world decision problems are governed by Hamilton-Jacobi-Bellman (HJB) equations with a complementarity problem of the following form:

$$\max\left\{\min_{u\in U}\left[\mathcal{L}(u)v-\phi\right], v-\psi\right\}=0 \tag{1.1}$$

defined in a spatial domain of interest in n-dimensions and an interval in time horizon with proper initial and boundary conditions, where u and v are unknown control and state functions, $\mathcal{L}$ denotes a differential operator in both space and time, $U\subset\mathbb{R}^m$ is the set of admissible controls, ϕ and ψ are given functions, and m and n are two positive integers. One example of the above equation is an HJB equation with a state constraint arising from a classic feedback optimal control by a dynamic programming approach, where u and v represent the value and control functions respectively ([6, 10]). Another non-trivial example is the problem of pricing an American-style butterfly spread option under uncertainty volatility model. The value of such an option, denoted as $V(S,t)$, satisfies the following HJB equation ([1, 7]):

$$\min\left\{\max_{\sigma_1\leq\sigma\leq\sigma_2}\left(-V_t-\frac{1}{2}\sigma^2S^2V_{SS}-rSV_S+rV\right), V-V^*\right\}=0 \tag{1.2}$$

*This program is supported by the National Natural Science Foundation of China under grant 12071157, 12026602 and the Natural Science Foundation of Guangdong 2020B1515310013. and by Research Grants Council of Hong Kong (PolyU 152182/19E).

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DOI: https://doi.org/10.61208/pjo-2023-011

for $(S,t) \in (0, S_{\max}) \times [0,T)$ with appropriate boundary and terminal conditions, where S denotes the underling asset price, T is the expiring date, $S_{\max}$ is a sufficiently large positive number, r is the risk-free interest rate, and σ_1 and σ_2 are two positive constants, defining the lower bound and upper bound of the volatility σ.

Eq. (1.1) is defined in infinite dimensions, which, in general, can be hardly solved analytically. Thus, a proper discretization technique needs to be applied to (1.1) so that we can find approximations to its exact solution. The application of a discretization technique to (1.1) yields an HJB equation of the following form:

Problem 1.1. Find $x \in \mathbb{R}^N$ such that

$$\max\left\{\min_{q\in\mathcal{Q}}\left(A(q)x - b(q)\right), x - g\right\} = 0, \tag{1.3}$$

where $\mathcal{Q} \subset \mathbb{R}^M$ is the set of all admissible controls, $A\,(q)$ and $b\,(q)$ are respectively $N \times N$ and $N \times 1$ matrices whose entries depend on $q = (q_1, \ldots, q_N)^\top \in \mathcal{Q}$, and $g \in \mathbb{R}^N$ is a given vector. More specifically, $A(q) := (a_{ij}(q_i))$ and $b(q) := (b_1(q_1), \ldots, b_N(q_N))^\top$.

We now introduce some symbols to be used in the rest of this work. Denote by $\mathcal{M}$ the set of real-valued $N \times N$ matrices, and let $\mathbb{I} := \{1, \ldots, N\}$. Throughout this paper, for every $x, y \in \mathbb{R}^N$, $y \geq x$ means that $y_i \geq x_i$, $\forall i \in \mathbb{I}$. We also denote by $\min\{x, y\}$ (resp. $\max\{x, y\}$) the vector with components $\min(x_i, y_i)$ (resp. $\max\{x_i, y_i\}$). The definitions extend trivially to other relational operators.

Problem 1.1 is also regarded as a Hamilton-Jacobi-Bellman-Issac equation [4]. HJB equations can hardly be solved analytically due to their complex structure. Thus, a numerical method needs to be used for finding approximate solutions to such a problem. However, to the our best our knowledge, there are very few efficient numerical methods for (1.3) in the open literature. In general, the policy iteration method is regarded as the best one to solve Problem 1.1 [3]. This method is a Newton-like method which, however, is far from being efficient and effective as pointed out in [16] and [19]. Thus, it is necessary to develop better numerical methods for the HJB CP.

Equivalently, eq. (1.3) can be rewritten as the following complementarity form:

$$\begin{aligned}
\min_{q\in\mathcal{Q}}\left(A\,(q)\,x - b\,(q)\right) &\leq 0,\\
x - g &\leq 0,\\
(x-g)^\top\left(\min_{q\in\mathcal{Q}}\left(A\,(q)\,x - b\,(q)\right)\right) &= 0.
\end{aligned}$$

This is a standard complementarity form [5, 15]. This equivalent form inspires us to propose a power penalty approach to solving Problem 1.1, since the power penalty method has been well developed to approximate both the standard complementarity problems and HJB equations, see [8, 11, 12, 14, 19], etc.

Before further discussion, we make the following assumptions on $A(q)$ in the rest of this paper:

(A) The matrix $A\,(q)$ is a strictly diagonally dominant M-matrix for every $q \in \mathcal{Q}$., i.e.,

$$a_{ii} > 0,\ a_{ij} \leq 0,\ \text{ for } i \neq j, \text{ and } |a_{ii}| > \sum_{i\neq j} a_{ij}, \quad i,j = 1,2,\ldots,N.$$

(B) $A : \mathcal{Q} \mapsto \mathcal{M}$ and $b : q \mapsto R^N$ are continuous functions.

Remark 1.2. It has been shown in [16] that, under the above assumptions, Problem 1.1 has a unique solution. Moreover, for any $q \in \mathcal{Q}$, both $A(q)$ and $A^{-1}(q)$ can be bounded, since each $A(q)$ is an M-matrix and there are only finitely many compositions that can be assumed. Similarly, $b(q)$ can be bounded as well. In the same way, we can infer that $\|A(q)\| \leq C$ and $\|A^{-1}(q)\| \leq C$ with C a constant.

$\boxed{2}$ Power Penalty Approach

Motivated by the [17], we propose the following problem approximating Problem 1.1:

Problem 2.1. Find $x_\lambda \in \mathbb{R}^N$, such that

$$\min_{q \in \mathcal{Q}} \left(A(q)\, x_\lambda - b(q)\right) + \lambda [x_\lambda - g]_+^{1/k} = 0, \tag{2.1}$$

where $1/k > 0$ is the power of the penalty term $[\cdot]_+$, $\lambda > 1$ is the penalty parameter, $[u]_+ := \max\{u, 0\}$, and for any $y = (y_1, \ldots, y_N)^\top \in \mathbb{R}_+^N$, $y^\alpha = [y_1^\alpha, \ldots, y_N^\alpha]^\top$.

Problem 2.1 is the penalization of Problem 1.1. The second term in (2.1) is used to penalize the part of $x - g$ violating the constraint $x - g \leq 0$. The essence of this penalty approach is to force the constraint $x_\lambda \leq g$ to be satisfied up to a tolerance by taking $\lambda \to \infty$. More specifically, we expect that the solution x_λ of Problem 2.1 converges to that of Problem 1.1. Before we present a detailed convergence analysis of the power penalty approach to Problem 1.1, we first show that (2.1) is uniquely solvable. We start this with by showing solutions to Problem 2.1 are bounded uniformly in λ and k in the following lemma.

Lemma 2.2. *Under Assumptions (A) and (B), the solution to Problem 2.1 satisfies*

$$\|x_\lambda\|_\infty \leq C \tag{2.2}$$

for some constant $C > 0$, independent of λ.

Proof. Define $Q_\lambda = (q_\lambda^1, ..., q_\lambda^N)^\top$ to be such that

$$(Q_\lambda)_i = q_\lambda^i = \underset{Q \in \mathcal{Q}}{\arg\min} \left(A(Q)\, x_\lambda - b(Q)\right)_i \tag{2.3}$$

for all $i = 1, 2, ..., N$. Then, we have

$$A(Q_\lambda)\, x_\lambda - b(Q_\lambda) = \min_{q \in \mathcal{Q}} \left(A(q)\, x_\lambda - b(q)\right). \tag{2.4}$$

Hence, (2.1) becomes

$$A(Q_\lambda)\, x_\lambda - b(Q_\lambda) + \lambda [x_\lambda - g]_+^{1/k} = 0. \tag{2.5}$$

Therefore

$$A(Q_\lambda)\, x_\lambda - b(Q_\lambda) = -\lambda [x_\lambda - g]_+^{1/k} \leq 0,$$

implying $A(Q_\lambda)\, x_\lambda \leq b(Q_\lambda)$.

Note that it follows from Assumption (A) that $A(q)$ is a strictly diagonally dominant M-matrix for every $q \in \mathcal{Q}$. Thus, $A^{-1}(Q_\lambda) > 0$. From this, we immediately get that

$$x_\lambda \leq A^{-1}(Q_\lambda)\, b(Q_\lambda). \tag{2.6}$$

From (2.5) we also see that, for every $i \in \mathbb{I}$, either

$$\left(A\left(Q_\lambda\right)x_\lambda\right)_i = \left(b\left(Q_\lambda\right)\right)_i,$$

or,

$$\left(x_\lambda\right)_i \geq g_i.$$

Now, we introduce the matrix $A^* \in \mathcal{M}$ whose ith row is $(A(Q_\lambda))_i$ when $\left(A\left(Q_\lambda\right)x_\lambda\right)_i = \left(b\left(Q_\lambda\right)\right)_i$, and that of the identity matrix I when $\left(x_\lambda\right)_i \geq g_i$. We also let b^* be the $N \times 1$ matrix whose ith row is $b\left(Q_\lambda\right)_i$ when $\left(A\left(Q_\lambda\right)x_\lambda\right)_i = \left(b\left(Q_\lambda\right)\right)_i$ and g_i if $\left(x_\lambda\right)_i \geq g_i$. These two matrices satisfy

$$A^* x_\lambda \geq b^*.$$

From its construction, we see that A^* is also a strictly diagonally dominant M-matrix, and thus the above inequality gives

$$x_\lambda \geq \left(A^*\right)^{-1} b^*. \tag{2.7}$$

Combining (2.6) and (2.7), and using the fact that $A(q)$, $A^{-1}(q)$ and $b(q)$ can be bounded for any $q \in \mathcal{Q}$ (see Remark 1.2), we infer that (2.2) holds true for some positive constant, independent of λ. □

Now, we are ready to establish the unique solvability of Problem 2.1 in the following Proposition.

Proposition 2.3. *For any $\lambda > 0$, there exists a unique solution x_λ to Problem 2.1 if Assumptions (A) and (B) are satisfied.*

Proof. For clarity, we omit the subscript λ of x_λ in this proof. Let

$$F(x) := \min_{q \in \mathcal{Q}} \left(A\left(q\right)x - b\left(q\right)\right) + \lambda[x-g]_+^{1/k}.$$

We will show that $F(x) = 0$ has a solution in the bounded region $S := \{x \in \mathbb{R}^N : -Le < x < Le\}$, where $e = (1, ..., 1)^\top \in \mathbb{R}^N$ and L is a sufficiently large positive constant. Clearly, $F = (f_1, \ldots, f_n) : \bar{S} \subset \mathbb{R}^N \mapsto \mathbb{R}^N$ is continuous. To prove this theorem, it suffices to verify that F satisfies all of the conditions of Miranda's theorem .

We first show that $F(x) \neq 0$ for any x on the boundary ∂S of S. More specifically, we will show that $0 \notin F(\partial S)$ when L is sufficiently large. Suppose this is not true, i.e., $0 \in F(\partial S)$. Then, there exists an $x^* \in \partial S$ such that $F(x^*) = 0$. In this case, there must be an $l \in \mathbb{I}$ such that either $x_l^* = L$ or $x_l^* = -L$. From Lemma 2.2 we see that $\|x^*\|_\infty \leq C$ for a constant C since x^* is a solution to (2.5). However, we see $\|x^*\|_\infty > C$ when $L > C$, contradicting $\|x^*\|_\infty \leq C$ in Lemma 2.2. Therefore, $F(x) = 0$ has no solutions on ∂S when $L > C$.

Now, we show, for any $x = (x_1, ..., x_n)^\top \in S$ and $i \in \mathbb{I}$, the inequalities

$$f_i(x_1, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \leq 0$$
$$\text{and} \quad f_i(x_1, \ldots, x_{i-1}, L, x_{i+1}, \ldots, x_n) \geq 0$$

Let $G = \{x \in R^n : |x_i| < L,$ for $1 \leq i \leq n\}$ and suppose the mapping $F = (f_1, \ldots, f_n) : \bar{G} \to R_n$ is continuous on the closure $\bar{G}$ of G such that $F(x) \neq 0$ for x on the boundary ∂G of G, and

- $f_i(x_1, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \leq 0$, for $1 \leq i \leq n$,
- $f_i(x_1, \ldots, x_{i-1}, L, x_{i+1}, \ldots, x_n) \geq 0$, for $1 \leq i \leq n$.

Then, $F(x) = 0$ has a solution in G. See [2].

are also satisfied. In fact, from Assumption (A) we know that the entries of the matrix A satisfy $a_{ii} > 0$, $a_{i,j} \leq 0$ for $i \neq j$, and $a_{ii} - \sum_{i\neq j} |a_{ij}| \geq 0$, for $i, j \in \mathbb{I}$. Hence, combining this property and $\|x\|_\infty \leq C$, we have for $1 \leq i \leq n$

$$\begin{aligned}
&f_i(x_1, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \\
=&\left(A\left(Q_\lambda\right)\left(x + (-L - x_i)e\right) - b\left(Q_\lambda\right)\right)_i + \lambda[-L - g_i]_+^{1/k} \\
=&\sum_{i\neq j} a_{ij}x_i - a_{i,i}L - b_i \\
\leq& - C\left(a_{i,i} - \sum_{i\neq j} a_{ij}\right) - (L - C)a_{i,i} - b_i \leq 0,
\end{aligned}$$

when L is sufficiently large. Similarly, when $L > C$, where C is the constant in (2.2), we have

$$\begin{aligned}
&f_i(x_1, \ldots, x_{i-1}, L, x_{i+1}, \ldots, x_n) \\
=&\left(A\left(Q_\lambda\right)\left(x + (L - x_i)e\right) - b\left(Q_\lambda\right)\right)_i + \lambda[L - g_i]_+^{1/k} \\
=&\sum_{i\neq j} a_{ij}x_i + a_{i,i}L - b_i + \lambda\left[L - g_i\right]^{1/k} \\
\geq& C\left(a_{i,i} - \sum_{i\neq j} a_{ij}\right) + (L - C)a_{i,i} + \lambda\left[L - g_i\right]^{1/k} > 0.
\end{aligned}$$

From the above analysis, we see that all the conditions of Miranda's theorem are satisfied. Hence, the existence of the solution to the penalized Problem 2.1 is proved.

We then show the uniqueness of the solution to the penalized Problem 2.1. To this end, we suppose x_λ and y_λ are two solutions to Problem 2.1. Then,

$$\min_{q\in\mathcal{Q}}\left(A\left(q\right)x_\lambda - b\left(q\right)\right) + \lambda[x_\lambda - g]_+^{1/k} = 0, \tag{2.8}$$

$$\min_{q\in\mathcal{Q}}\left(A\left(q\right)y_\lambda - b\left(q\right)\right) + \lambda[y_\lambda - g]_+^{1/k} = 0. \tag{2.9}$$

As we did in the proof of Lemma 2.2, we define Q^x and Q^y to be such that

$$\begin{aligned}
A\left(Q^x\right)x_\lambda - b\left(Q^x\right) &= \min_{q\in\mathcal{Q}}\left(A\left(q\right)x_\lambda - b\left(q\right)\right), \\
A\left(Q^y\right)y_\lambda - b\left(Q^y\right) &= \min_{q\in\mathcal{Q}}\left(A\left(q\right)y_\lambda - b\left(q\right)\right).
\end{aligned}$$

Thus, it follows from (2.8), (2.9) and the definition of Q^x and Q^y that

$$A\left(Q^y\right)x_\lambda - b\left(Q^y\right) + \lambda[x_\lambda - g]_+^{1/k} \geq A\left(Q^x\right)x_\lambda - b\left(Q^x\right) + \lambda[x_\lambda - g]_+^{1/k} = 0, \tag{2.10}$$

$$A\left(Q^x\right)y_\lambda - b\left(Q^x\right) + \lambda[y_\lambda - g]_+^{1/k} \geq A\left(Q^y\right)y_\lambda - b\left(Q^y\right) + \lambda[y_\lambda - g]_+^{1/k} = 0. \tag{2.11}$$

Manipulating (2.10) and (2.11) gives

$$\begin{aligned}
A\left(Q^y\right)\left(x_\lambda - y_\lambda\right) + \lambda\left([x_\lambda - g]_+^{1/k} - [y_\lambda - g]_+^{1/k}\right) \geq 0, \\
A\left(Q^x\right)\left(x_\lambda - y_\lambda\right) + \lambda\left([x_\lambda - g]_+^{1/k} - [y_\lambda - g]_+^{1/k}\right) \leq 0.
\end{aligned}$$

Define two disjoint nonempty index subsets K_1 and K_2 of $\mathbb{I}$ as follows

$$K_1 = \{i \,|(x_\lambda)_i \leq (y_\lambda)_i \}, \quad K_2 = \{i \,|(x_\lambda)_i > (y_\lambda)_i \}.$$

Then, by virtue of the monotonicity of the operator $[\cdot]_+^{1/k}$, we have

$$(A(Q^y)(x_\lambda - y_\lambda))_i \geq 0, \quad \forall i \in K_1.$$
$$(A(Q^x)(x_\lambda - y_\lambda))_i \leq 0, \quad \forall i \in K_2.$$

Now, introducing a matrix, denoting $A_1^* \in \mathcal{M}$, to be the matrix having the ith row as that of $(A(Q^y))_i$, $i \in K_1$ and of the identity matrix I when $i \in K_2$. Therefore, we have

$$A_1^*(x_\lambda - y_\lambda) \geq 0.$$

Nevertheless, it follows from the definition of M-matric and Assumption (A) that A_1^* is also a strictly diagonally dominant M-matrix. Hence, $(A_1^*)^{-1} > 0$. Therefore, we have on the whole index set $\mathbb{I}$

$$x_\lambda \geq y_\lambda.$$

Conversely, by introducing another matrix, denoting $A_2^* \in \mathcal{M}$, to be the matrix having the ith row as that of $(A(Q^x))_i$, $i \in K_2$ and of the identity matrix I when $i \in K_1$, we also have

$$A_2^*(x_\lambda - y_\lambda) \leq 0.$$

since A_2^* is also a strictly diagonally dominant M-matrix. Therefore, we have on the whole index set $\mathbb{I}$

$$x_\lambda \leq y_\lambda.$$

Thus, the uniqueness of a solution to Problem 2.1 is proved.

□

3 Convergence Analysis

3.1 Monotonic convergence property

Before establishing the monotonic convergence property of the power penalty method, we first give two lemmas. The first lemma is to show that the solution of the penalized Problem 2.1 is always not less than that of the discrete HJB complementary Problem 1.1, componentwisely.

Lemma 3.1. *Let $\lambda > 1$ and $k > 0$. Assume that x_λ and x are the solutions of the penalized Problem 2.1 and that of the discrete HJB complementary Problem 1.1, respectively. Then*

$$x_\lambda \geq x.$$

Proof. Since x is the solution of the discrete HJB complementary Problem 1.1, we always have

$$x - g \leq 0, \quad \min_{q \in \mathcal{Q}} (A(q)x - b(q)) \leq 0. \tag{3.1}$$

Meanwhile, from the fact x_λ is the solution of the penalized Problem 2.1, it follows

$$\min_{q \in \mathcal{Q}} (A(q)x_\lambda - b(q)) = -\lambda[x_\lambda - g]_+^{1/k} \leq 0. \tag{3.2}$$

Define two disjoint non-empty index subsets I_1 and I_2 of $\mathbb{I}$ as follows

$$I_1 = \{i \,|(x_\lambda - g)_i > 0\}, \quad I_2 = \{i \,|(x_\lambda - g)_i \leq 0\}.$$

We have the following two cases.

- For $i \in I_1$, we have
$$(x_\lambda)_i > g_i \geq x_i.$$

- For $i \in I_2$, it follows from (3.2) that $\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_\lambda - b\left(q\right)\right)_i = 0$, which, along with (3.1), further implies that
$$\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_\lambda - b\left(q\right)\right)_i - \min_{q\in\mathcal{Q}} \left(A\left(q\right)x - b\left(q\right)\right)_i \geq 0. \tag{3.3}$$
On the other hand, we have
$$\begin{aligned}
&\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_\lambda - b\left(q\right)\right)_i - \min_{q\in\mathcal{Q}} \left(A\left(q\right)x - b\left(q\right)\right)_i \\
&\quad= \min_{q\in\mathcal{Q}} \left(A\left(q\right)x_\lambda - b\left(q\right)\right)_i + \max_{q\in\mathcal{Q}} \left(-A\left(q\right)x + b(q)\right)_i \\
&\quad= \min_{q\in\mathcal{Q}} \left(A\left(q\right)x_\lambda - b\left(q\right)\right)_i + (-A(\bar{Q})x + b(\bar{Q}))_i \\
&\quad\leq \left(A(\bar{Q})(x_\lambda - x)\right)_i,
\end{aligned}$$
where $\bar{Q} = \arg\max_{q\in\mathcal{Q}} \left(-A(q)x_\lambda + b(q)\right)$. Thus, combining this inequality with (3.3), we eventually have
$$\left(A(\bar{Q})(x_\lambda - x)\right)_i \geq 0, \quad \forall i \in I_2.$$

Now, again introduce a matrix, still denoting $A^* \in \mathcal{M}$, to be the matrix having the ith row as that of the identity matrix I when $i \in I_1$ and of $A(\bar{Q})$ when $i \in I_2$. Therefore, we have

$$A^*\left(x_\lambda - x\right) \geq 0.$$

Since $A(\bar{Q})$ is an M-matrix by Assumption (A), replacing some of its rows with the corresponding rows of the identity matrix should still yield an M-matrix. Thus, A^* is an M-matrix, and we have

$$x_\lambda \geq x,$$

since $(A^*)^{-1} \geq 0$. □

The next lemma is to show that the solution x_λ of the penalized Problem 2.1 is monotonically decreasing with respect to the penalty parameter λ.

Lemma 3.2. *Let $\lambda_2 > \lambda_1 > 1$, and x_{λ_1} and x_{λ_2} be the solutions of Problem 2.1 corresponding to $\lambda = \lambda_1$ and λ_2, respectively. Then*

$$x_{\lambda_1} \geq x_{\lambda_2}.$$

Proof. Let $Q_{\lambda_2} \in \mathcal{Q}$ be the optimal control corresponding to the solution x_{λ_2} to Problem 2.1. We have that

$$\begin{aligned} 0 =& A\left(Q_{\lambda_2}\right) x_{\lambda_2} - b\left(Q_{\lambda_2}\right) + \lambda_2 \left[x_{\lambda_2} - g\right]_+^{1/k} \\ =& \min_{q\in\mathcal{Q}} (A\left(q\right) x_{\lambda_2} - b\left(q\right)) + \lambda_2 \left[x_{\lambda_2} - g\right]_+^{1/k} \\ =& \min_{q\in\mathcal{Q}} (A\left(q\right) x_{\lambda_1} - b\left(q\right)) + \lambda_1 \left[x_{\lambda_1} - g\right]_+^{1/k} \\ \leq& A\left(Q_{\lambda_2}\right) x_{\lambda_1} - b\left(Q_{\lambda_2}\right) + \lambda_2 \left[x_{\lambda_1} - g\right]_+^{1/k}. \end{aligned}$$

This implies that

$$A\left(Q_{\lambda_2}\right)\left(x_{\lambda_2} - x_{\lambda_1}\right) \leq \lambda_2 \left(\left[x_{\lambda_1} - g\right]_+^{1/k} - \left[x_{\lambda_2} - g\right]_+^{1/k} \right). \tag{3.4}$$

Defining two disjoint nonempty index subsets J_1 and J_2 of $\mathbb{I}$ as follows

$$\begin{aligned} J_1 &= \left\{ j \,\middle|\, \left(\left[x_{\lambda_1} - g\right]_+^{1/k}\right)_j \leq \left(\left[x_{\lambda_2} - g\right]_+^{1/k}\right)_j \right\}, \\ J_2 &= \left\{ j \,\middle|\, \left(\left[x_{\lambda_1} - g\right]_+^{1/k}\right)_j > \left(\left[x_{\lambda_2} - g\right]_+^{1/k}\right)_j \right\}. \end{aligned}$$

Using these two index sets we see, from (3.4), that when $j \in J_1$,

$$\left(A\left(Q_{\lambda_2}\right)\left(x_{\lambda_2} - x_{\lambda_1}\right)\right)_j \leq 0,$$

and when $j \in J_2$,

$$\left(\left[x_{\lambda_1} - g\right]_+^{1/k}\right)_j > \left(\left[x_{\lambda_2} - g\right]_+^{1/k}\right)_j \quad \Rightarrow \quad \left(x_{\lambda_1}\right)_j \geq \left(x_{\lambda_2}\right)_j,$$

since $[\cdot]_+$ is a monotonically increasing function.

Now, as in the proof of Lemma 3.1, introduce a matrix $A^* \in \mathcal{M}$ such that its jth row is $\left(A\left(Q_{\lambda_2}\right)\right)_j$ when $j \in J_1$ and $(I)_j$ if $j \in J_2$. Hence, from the above analysis we have

$$A^*\left(x_{\lambda_2} - x_{\lambda_1}\right) \leq 0,$$

from which we obtain

$$x_{\lambda_1} \geq x_{\lambda_2},$$

since A^* is also an M-matrix. □

With the above two lemmas, we now establish the following monotonic convergence result for the power penalty method.

Theorem 3.3. *Let $\{\lambda_m\}, m = 1, 2, \ldots,$ be a monotonically increasing sequence approaching positive infinity as $m \to \infty$. Assume that x_{λ_m} is the solution to Problem 2.1 with $\lambda = \lambda_m$, and x^* is solution to Problem 1.1. Then the sequence $\{x_{\lambda_m}\}$ is monotonically decreasing and convergent to x^*.*

Proof. It follows from Lemmas 3.1 and 3.2 that

$$x_{\lambda_1} \geq x_{\lambda_2} \geq \cdots \geq x_{\lambda_i} \geq \cdots \geq x^*.$$

This implies that there exists an $\hat{x} \in \mathbb{R}^N$ such that

$$\lim_{m\to\infty} x_{\lambda_m} = \hat{x}.$$

We now prove $\hat{x}$ is the solution to Problem 1.1. As x_{λ_m} is the solution of Problem 2.1, there must hold

$$\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_{\lambda_m} - b\left(q\right)\right) = -\lambda_m [x_{\lambda_m} - g]_+^{1/k} \le 0. \tag{3.5}$$

Thus, it follows from Assumption (B) and (3.5) that as $m \to \infty$

$$\min_{q\in\mathcal{Q}} \left(A\left(q\right)\hat{x} - b\left(q\right)\right) = \lim_{m\to\infty} \left(-\lambda_m [x_{\lambda_m} - g]_+^{1/k}\right) \le 0. \tag{3.6}$$

Rewriting (3.5) as

$$[x_{\lambda_m} - g]_+ = \left(-\frac{\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_{\lambda_m} - b\left(q\right)\right)}{\lambda_m}\right)^k,$$

we see that

$$\lim_{m\to\infty} [x_{\lambda_m} - g]_+ = [\hat{x} - g]_+ = \lim_{m\to\infty} \left(-\frac{\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_{\lambda_m} - b\left(q\right)\right)}{\lambda_m}\right)^k = 0, \tag{3.7}$$

since all the terms $A\left(q\right)$, $b\left(q\right)$ and x_{λ_m} are bounded. This implies that $\hat{x} \le g$. (In fact (3.6) also implies $\hat{x} \le g$ has to be satisfied, as otherwise, the right-hand side of (3.5) approaches $-\infty$ while its left-hand side approaches a fixed vector.)

Now, multiplying both side of (3.5) from the left by $(x_{\lambda_m} - g)^\top$ gives

$$\begin{aligned}
&(x_{\lambda_m} - g)^\top \left[\min_{q\in\mathcal{Q}} \left(A\left(q\right)x_{\lambda_m} - b(q)\right)\right] \\
&= -\lambda_m (x_{\lambda_m} - g)^\top [x_{\lambda_m} - g]_+^{1/k} = -\lambda_m \|[x_{\lambda_m} - g]_+^{1+1/k}\|_2,
\end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^N$. Therefore, letting $m \to \infty$ and using (3.7) we have

$$(\hat{x} - g)^\top \left[\min_{q\in\mathcal{Q}} \left(A\left(q\right)\hat{x} - b\left(q\right)\right)\right] = 0.$$

Combining this equation with (3.5) and $\hat{x} \le g$ we see that $\hat{x}$ is a solution to Problem 1.1. Finally, since Problem 1.1 is uniquely solvable, $\hat{x} = x^*$. □

3.2 Exponential convergence rate

To establish the exponential convergence rate of the power penalty approach, we first present an error estimation for the solution to Problem 2.1.

Theorem 3.4. *Assume that x_λ is the solution to Problem 2.1 for every $\lambda > 1$. There exists a constant $C > 0$, independent of λ and x_λ, such that*

$$\left\|\max\left\{\min_{q\in\mathcal{Q}} \left(A(q)x_\lambda - b(q)\right), x_\lambda - g\right\}\right\|_\infty \le \frac{C}{\lambda^k}. \tag{3.8}$$

Proof. Let $C > 0$ be a generic constant, independent of λ and x_λ. It follows from (2.1) that

$$\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q)) = -\lambda[x_\lambda - g]_+^{1/k} \leq 0.$$

Furthermore, for every $j \in \mathbb{I}$, we either have $(x_\lambda - g)_j \leq 0$ and

$$\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j = (A(Q_\lambda)x_\lambda - b(Q_\lambda))_j = 0 \leq \frac{C}{\lambda^k},$$

or $(x_\lambda - g)_j > 0$ and

$$\begin{aligned}\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j &= (A(Q_\lambda)x_\lambda - b(Q_\lambda))_j \\ &= -\lambda(x_\lambda - g)_j^{1/k} < 0.\end{aligned}$$

- For the first case, we have,

$$\begin{aligned}-\frac{C}{\lambda^k} \leq 0 &= \max\left\{\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j, (x_\lambda - g)_j\right\} \\ &= \min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j \leq \frac{C}{\lambda^k},\end{aligned}$$

  where Q_λ is the optimal control defined in (2.3).
- For the second case, it follows from Remark 1.2 and (2.2) that x_λ, $A(Q_\lambda)$ and $b(Q_\lambda)$ are all bounded. Hence, we have

$$-\frac{C}{\lambda^k} \leq 0 < (x_\lambda - g)_j = \frac{(b(Q_\lambda) - A(Q_\lambda)x_\lambda)_j^k}{\lambda^k} \leq \frac{C}{\lambda^k}. \tag{3.9}$$

  Since $\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j < 0$, from (3.9) we have

$$\begin{aligned}-\frac{C}{\lambda^k} &\leq \max\left\{\min_{q\in\mathcal{Q}} (A(q)x_\lambda - b(q))_j, (x_\lambda - g)_j\right\} \\ &= (x_\lambda - g)_j \leq \frac{C}{\lambda^k}.\end{aligned}$$

Combining the above two cases we see that (3.8) is satisfied. □

Now, with Theorem 3.4, we establish in the following theorem the exponential rate of convergence of the solution of Problem 2.1 to that of Problem 1.1 in terms of the penalty parameter.

Theorem 3.5. *Assume that x_λ and x are the solution of Problem 2.1 and that of Problem 1.1, respectively. Then, when λ is sufficiently large, we have*

$$\|x - x_\lambda\|_\infty \leq \frac{C}{\lambda^k}, \tag{3.10}$$

where C is a positive constant, independent of x, x_λ and λ.

Proof. We first define $Q^* \in \mathcal{Q}$ to be such that for

$$A\left(Q^*\right)x - b\left(Q^*\right) = \min_{q\in\mathcal{Q}}\left(A\left(q\right)x - b\left(q\right)\right). \tag{3.11}$$

For $\lambda > 0$, we also define four disjoint nonempty index subsets J_1', J_2', J_3', and J_4' of $\mathbb{I}$ as follows

$$\begin{aligned} J_1' &= \left\{ j \,\middle|\, (x-g)_j = 0 \text{ and } (x_\lambda - g)_j > 0 \right\}, \\ J_2' &= \left\{ j \,\middle|\, (x-g)_j = 0 \text{ and } (x_\lambda - g)_j \leq 0 \right\}, \\ J_3' &= \left\{ j \,\middle|\, (x-g)_j < 0 \text{ and } (x_\lambda - g)_j > 0 \right\}, \\ J_4' &= \left\{ j \,\middle|\, (x-g)_j < 0 \text{ and } (x_\lambda - g)_j \leq 0 \right\}. \end{aligned}$$

We consider the following four cases.

- For $j \in J_1'$, it follows from Remark 1.2 and (2.2) that x_λ, $A\left(Q_\lambda\right)$ and $b\left(Q_\lambda\right)$ are all bounded. Hence,

$$-\frac{C}{\lambda^k} \leq 0 < (x_\lambda - x)_j = (x_\lambda - g)_j = \frac{\left(b\left(Q_\lambda\right) - A\left(Q_\lambda\right)x_\lambda\right)_j^k}{\lambda^k} \leq \frac{C}{\lambda^k}.$$

- For $j \in J_2'$, it follows from (2.1) that

$$(x_\lambda - g)_j \leq 0 \text{ and } \left(A\left(Q_\lambda\right)x_\lambda - b\left(Q_\lambda\right)\right)_j = 0,$$

  hence,

$$\max\left\{\min_{q\in\mathcal{Q}}\left(A(q)x_\lambda - b(q)\right)_j, (x_\lambda - g)_j\right\} = \left(A\left(Q_\lambda\right)x_\lambda - b\left(Q_\lambda\right)\right)_j, \tag{3.12}$$

  with Q_λ defined in (2.3). Thus, we have

$$(x_\lambda - x)_j = (x_\lambda - g)_j \leq \left(A\left(Q_\lambda\right)x_\lambda - b\left(Q_\lambda\right)\right)_j = 0 \leq \frac{C}{\lambda^k}.$$

  Meanwhile, from (3.11), the definition of J_2' and the fact

$$\max\left\{\min_{q\in\mathcal{Q}}\left(A(q)x - b(q)\right)_j, (x-g)_j\right\} = 0,$$

  it follows

$$(x-g)_j = 0 \text{ and } A\left(Q^*\right)x - b\left(Q^*\right)_j \leq 0.$$

  Thus, using (2.1), (2.4), (3.8) and (3.12), we have

$$\begin{aligned} &\left(A\left(Q^*\right)(x_\lambda - x)\right)_j = \left(A\left(Q^*\right)x_\lambda - b\left(Q^*\right)\right)_j \\ &\quad - \left(A\left(Q^*\right)x - b\left(Q^*\right)\right)_j \geq \left(A\left(Q^*\right)x_\lambda - b\left(Q^*\right)\right)_j \\ &\quad \geq \left(A\left(Q_\lambda\right)x_\lambda - b\left(Q_\lambda\right)\right)_j \geq -\frac{C}{\lambda^k}. \end{aligned}$$

- For $j \in J_3'$, we easily obtain

$$(x_\lambda - x)_j = (x_\lambda - g)_j - (x - g)_j \geq (x_\lambda - g)_j \geq 0 \geq -\frac{C}{\lambda^k}.$$

Meanwhile, it follows from (1.3) and (2.1) that

$$(A\,(Q^*)\,x - b\,(Q^*))_j = 0 > (x - g)_j.$$

and

$$\left(A\,(Q_\lambda)\,x_\lambda - b\,(Q_\lambda)\right)_j = -\lambda(x_\lambda - g)_j^{1/k} < 0.$$

Therefore, considering (3.11) we have

$$\begin{aligned}(A\,(Q_\lambda)\,(x_\lambda - x))_j &= (A\,(Q_\lambda)\,x_\lambda - b\,(Q_\lambda))_j - (A\,(Q_\lambda)\,x - b\,(Q_\lambda))_j \\ &\leq -(A\,(Q_\lambda)\,x - b\,(Q_\lambda))_j \leq -(A\,(Q^*)\,x - b\,(Q^*))_j = 0 \leq \frac{C}{\lambda^k}.\end{aligned}$$

- For $j \in J_4'$, it follows from (1.3) and (2.1) that

$$(A\,(Q^*)\,x - b\,(Q^*))_j = \max\left\{(A\,(Q^*)\,x - b\,(Q^*))_j, (x - g)_j\right\} = 0,$$

and

$$(A\,(Q_\lambda)\,x - b\,(Q_\lambda))_j = \max\left\{(A\,(Q_\lambda)\,x_\lambda - b\,(Q_\lambda))_j, (x_\lambda - g)_j\right\} = 0.$$

Hence, combining (2.4) and (3.11) that we get

$$\begin{aligned}(A\,(Q_\lambda)\,(x_\lambda - x))_j &= (A\,(Q_\lambda)\,x_\lambda - b\,(Q_\lambda))_j - (A\,(Q_\lambda)\,x - b\,(Q_\lambda))_j \\ &= -(A\,(Q_\lambda)\,x - b\,(Q_\lambda))_j \\ &\leq -(A\,(Q^*)\,x - b\,(Q^*))_j \\ &= 0 \leq \frac{C}{\lambda^k},\end{aligned}$$

and

$$\begin{aligned}(A\,(Q^*)\,(x_\lambda - x))_j &= (A\,(Q^*)\,x_\lambda - b\,(Q^*))_j - (A\,(Q^*)\,x - b\,(Q^*))_j \\ &= (A\,(Q^*)\,x_\lambda - b\,(Q^*))_j \\ &\geq (A\,(Q_\lambda)\,x_\lambda - b\,(Q_\lambda))_j \\ &= 0 \geq -\frac{C}{\lambda^k}.\end{aligned}$$

Now, we introduce a matrix, denoting A_1^*, whose ith row is that of the identity matrix I when $i \in J_1' \cup J_2'$, and the ith row as that of $A\,(Q_\lambda)$ when $i \in J_3' \cup J_4'$. Similarly, we introduce matrix A_2^* to be the matrix having the ith row as that of the identity matrix I when $i \in J_1' \cup J_3'$, and having the ith row as that of $A\,(Q^*)$ when $i \in J_2' \cup J_4'$.

Summarizing the above results, we obtain that

$$x_\lambda - x^* \leq \frac{C_1 \left\|(A_1^*)^{-1}\right\|_\infty}{\lambda^k}, \text{ and } x_\lambda - x^* \geq -\frac{C_2 \left\|(A_2^*)^{-1}\right\|_\infty}{\lambda^k},$$

where we utilize the fact that both A_1^* and A_2^* are strictly diagonally dominant M-matrices. Considering the boundedness of both A_1^* and A_2^* , we eventually have

$$\|x^* - x_\lambda\|_\infty \leq \frac{C}{\lambda^k},$$

for some constant $C > 0$ independent of λ, x_λ and x^*.

Combining the above four cases we have that (3.10) holds true. □

$\boxed{4}$ Solution of the Nonlinear Penalized System

Due to the nonlinearity of the power penalty term, the analytical solution to the penalized equation (2.1) is generally not available. Moreover, in the case of $k \geq 1$, (2.1) becomes nonsmooth, which makes the classic Newton method inapplicable. To overcome this difficult, we apply the smoothing technique, developed in [14], to smoothing out the penalty term $[z]_+^{1/k}$ with

$$W(z) = \begin{cases} [z]_+^{1/k}, & z \geq \epsilon, \\ (3 - 1/k)\,\epsilon^{1/k-2}[z]_+^2 + (1/k - 2)\,\epsilon^{1/k-3}[z]_+^3, & z < \epsilon, \end{cases}$$

where $0 < \epsilon \ll 1$ is a regularization parameter.

With this smoothing technique, (2.1) becomes

$$\min_{q \in \mathcal{Q}} (A\,(q)\, x_\lambda - b\,(q)) + P(x_\lambda) = 0, \tag{4.1}$$

where $P(x)$ is a vector defined by

$$[P(x_\lambda)]_i = \lambda W([x_\lambda - g]_i)$$

for $i = 1, \ldots, N$. We then design an iterative method for the numerical solution of (4.1). The new method is a combination of the Newton method and the policy iteration method, which is commonly used to numerically solve the nonlinear HJB equations, cf. [9]. The new method yields the following algorithm. (For clarity, we omit the subscript λ of x_λ in the algorithm.)

Algorithm 4.1.

Step 1. Choose $\epsilon, \varepsilon > 0$ sufficiently small; Set $l = 0$ and choose an initial guess x^0 such that $x^0 \leq g$.

Step 2. Find $Q^l = (q_1^l, \ldots, q_N^l)^\top$, such that

$$q_i^l = \arg\min_{q \in \mathcal{Q}} \left(A\,(q)\, x - b\,(q)\right)_i$$

and solve the following linear system for p^{l+1}:

$$\left[A(Q^l) + J_P(x^l)\right] p^{l+1} = b(Q^l) - A(Q^l)x^l - P(x^l),$$

where $J_P(x)$ is the Jacobian matrices of $P(x)$ defined by

$$[J_P(x)]_{ij} = \begin{cases} \lambda W'(x_i - g_i), & i = j, \\ 0, & i \neq j. \end{cases}$$

Step 3. Set $x^{l+1} = x^l + \nu p^{l+1}$, where $0 < \nu < 1$ is a damping parameter determined by the Armijo linear search method.

Step 4. If $\max_{i \in \mathbb{I}} \frac{\left|x_i^{l+1} - x_i^l\right|}{\max\left(1, \left|x_i^{l+1}\right|\right)} < \varepsilon$, then stop. Otherwise, set $l := l + 1$ and go to Step 2.

5 Numerical Experiments

In this section, we present numerical experiments using three different test examples to illustrate the exponential convergence property, effectiveness and efficiency of the power penalty method. The first example is a discrete double obstacle problem, of which the exact solution is known. By comparing the approximate solution and the exact solution, the convergence rate of the power penalty method is carefully examined. The second example is a one-dimensional discrete HJB complementarity problem, which cannot be solved correctly by the classical policy iteration method under some circumstances. We will show that the penalty can overcome this difficulty effectively. The third example is to price American options under uncertain volatility model, which is studied in [17]. After discretization using the method proposed in [17], the pricing problem is formed as a series of large scale of HJB complementarity problems. In this example, we will examine both the convergence property and the usefulness of the power penalty method proposed.

Example 5.1. Consider the following discrete double obstacle problem:

$$\max\left\{\min_{q\in\mathcal{Q}}\left\{A(q)x-b(q),x-f\right\},x-g\right\}=0,$$

where $\mathcal{Q}=\{1,2\}$, $A(q)\in\mathbb{R}^{4\times4}$, $b\in\mathbb{R}^4$, $f=(0,0,0,0)^\top$ and $g=(5,5,5,5)^\top$. The matrices A and b are defined by $A(q_1)=B$, $A(q_2)=I_{4\times4}$, $b(q_1)=d$ and $b(q_2)=f$ with $I_{4\times4}$ the 4×4 identity matrix,

$$B=\begin{bmatrix}1&2&2&2\\2&5&6&6\\2&6&9&10\\2&6&10&13\end{bmatrix}\quad\text{and}\quad d=\begin{bmatrix}11\\30\\50\\100\end{bmatrix}.$$

The above double obstacle problem can be expressed as the following HJB CP

$$\begin{aligned}&x-g\le0,\quad \min_{q\in\mathcal{Q}}\left(A\left(q\right)x-b\left(q\right)\right)\le0,\\&(x-g)^\top\left(\min_{q\in\{q_1,q_2\}}\left(A\left(q\right)x-b\left(q\right)\right)\right)=0.\end{aligned}$$

The exact solution is $x=(1,0,0,5)^\top$. The power penalty approach to this problem is stated as

$$\min_{q\in\mathcal{Q}}\left(A\left(q\right)x_\lambda-b\left(q\right)\right)+\lambda[x_\lambda-g]_+^{1/k}=0.$$

To obtain the numerical convergence rate of the power penalty method, we use Algorithm 4.1 to solve the above nonlinear system by setting $k=1$ and 2, and compare the approximate solutions with the exact solution, respectively. By setting $\lambda=10^2,10^3,10^4$, and 10^5, the l_∞-norms of the errors between the numerical solutions and the exact solution are calculated. Then, the ratios of errors from two consecutive values of λ are presented. All the computed results are listed in Table 1.

The results in Table 1 clearly shows that the numerical rates of convergence are close to $\mathcal{O}(\lambda^{-k})$, which is consistence with the theoretical result in (3.10). Note that the convergence rate in the case of $k=2$ is less than 2. This is due to the effect caused by the application of the smoothing technique (4.1) in Algorithm 4.1. Note that this example is for illustrative purposes rather than for performance of the method.

Table 1: Results computed by the power penalty method. $\varepsilon = 10^{-6}$ and $\epsilon = 10^{-3}$ are chosen in Algorithm 4.1.

$k=1$			$k=2$		
λ_i	$\|x - x_{\lambda_i}\|_\infty$	Rate	λ_i	$\|x - x_{\lambda_i}\|_\infty$	Rate
10^2	6.06×10^{-1}		10^2	2.05×10^{-1}	
10^3	6.54×10^{-2}	0.98	10^3	1.26×10^{-3}	1.99
10^4	6.59×10^{-3}	0.99	10^4	5.64×10^{-5}	1.76
10^5	6.60×10^{-4}	1.00	10^5	1.12×10^{-6}	1.69

It is worth noting that in this example the matrix B is not an M-matrix. However, the exponential convergence property is still observed, which implies that the assumption (A) is only a sufficient condition.

In the next example we will show that the penalty approach to the HJB complementarity problem is more effective than the classical policy iteration method.

Example 5.2. Consider the following one-dimensional ($N = 1$) HJB complementarity problem

$$\max\left\{\min\left\{Ax - b, x - g\right\}, x - h\right\} = 0, \tag{5.1}$$

with $A > 1$ and b, g, h are given constants with $g < h$.

This example is from Remark 5.7 in [3]. Obviously, in the case that $x^* := b/A \in (g, h)$, the solution to the above problem is x^*. It is pointed in [3] that the classical policy iteration method fails if the initial guess $x^0 \notin (g, h)$. However, this difficulty can be overcome by applying the power penalty method. Specifically, the power penalty approach to (5.1) results in the following nonlinear equation

$$\min\left\{Ax_\lambda - b, x_\lambda - g\right\} + \lambda[x_\lambda - h]_+^{1/k} = 0. \tag{5.2}$$

We use Algorithm 4.1 to solve the above equation with the chosen data, starting from an initial guess x^0 satisfying $x^0 < g$, $g < x^0 < h$ and $x^0 > h$. The results are listed in Table 2. From the table, we can see that no matter whether the initial guess belongs to (g, h) or not, the penalty method with both $k = 1$ and $k = 2$ can solve the HJB complementarity problem (5.2) correctly. We also list the solutions from the policy iteration in the last column of Table 2 to show that the policy iteration algorithm fails to solve the problem when x^0 is no in (g, h).

Table 2: Results computed by the power penalty methods and iteration method with different initial guesses. $A = -2$, $b = 1$, $g = -1$ and $h = 1$ are set in (5.2). In Algorithm 4.1, we choose $\varepsilon = 10^{-6}$, $\epsilon = 10^{-3}$ and $\lambda = 100$ for $k = 1$, $\lambda = 10$ for $k = 2$.

initial guess	$k=1$	$k=2$	policy iteration
x^0	x^*	x^*	x^*
-2	-0.5	-0.5	-1
0	-0.5	-0.5	-0.5
2	-0.5	-0.5	1

Example 5.3. Consider an American-style butterfly spread option under uncertainty volatility pricing model in which the option value V satisfies (1.2).

A butterfly option has the nonconvex payoff

$$V^* = \max(S - K_1, 0) - 2\max(S - (K_1 + K_2)/2, 0) + \max(S - K_2, 0).$$

This corresponds to a long position in two calls at strikes K_1 and K_2 respectively, and a short position in one cal at the strike $(K_1 + K_2)/2$. The parameters used for this butterfly option under uncertain volatility model are listed in Table 3.

This example was examined in [17], where the linear penalty method is directly applied, but without any convergence analysis. In this experiment, we will carefully examine the numerical convergence property of both linear and lower order penalty methods, and investigate their difference.

Table 3: Data used to value an American butterfly spread option under uncertain volatility model

Parameter	values
r	0.10
$\sigma_{\min}$	0.15
$\sigma_{\max}$	0.25
K_1	90
K_2	110
T	0.25

To numerically solve the above pricing equation, we first discretize it following a standard fitted finite volume approach in space and a fully implicit scheme in time (cf. [13]), on the uniform mesh with $M = 1600$ space steps and $N = 800$ time steps. With this discretization scheme, the Assumption (A) is guaranteed automatically (see the analysis in [17]). Then, we solve the problem backwards in time starting from the expiry date T. This means that, for every time step, we have to solve a large scale of discrete HJB complementarity problems as given in Problem 1.3, where $A(q)$ represents the system coefficient matrix from the fitted finite volume discretization, $b(q)$ represents the solution vector known from the previous time step, and g represents the payoff vector. The power penalty approach to this problem results in a large scale of discrete nonlinear systems as given in Problem 2.1 at every time step. We then apply Algorithm 4.1 to sequentially solve these nonlinear systems.

In this numerical experiment, by setting $\lambda_i = 10^i$ for $i = 2, 3, 4, 5$, the l_∞-norms of the errors between the numerical solutions with two consecutive λ values are calculated as follows.

$$\|V_{\lambda^i} - V_{\lambda^{i-1}}\|_\infty = \max_{1\le m\le M, 1\le n\le N} |V_{\lambda^i}(S_m, t_n) - V_{\lambda^{i-1}}(S_m, t_n)|.$$

Then, the ratios (denoted by 'Rate' in Table 4) of errors from two consecutive values of λ are presented. We also present the computational time (denoted by 'CPU(s)' in Table 4) for each penalty method.

Columns 'Rate' in Table 4 clearly shows that the numerical convergence rates of the linear ($k = 1$) and lower order penalty method ($k = 2$) with respect to the penalty parameter are respectively close to 1 and 2, which is consistent with the theoretical result $\mathcal{O}(\lambda^{-k})$ in (3.10). Moreover, from the columns 'CPU', we can see that both the linear penalty method ($k = 1$) and the lower order penalty method ($k = 2$) are very effective, since both methods only need very litter computational time. An additional interesting result can be observed from Table 4, that is, under the same level of accuracy, the lower order penalty method ($k = 2$) requires much less penalty parameter than the linear penalty method ($k = 1$) needs. This

Table 4: Results computed by the power penalty method. $\varepsilon = 10^{-6}$ and $\epsilon = 10^{-3}$ are chosen in Algorithm 4.1.

$k=1$				$k=2$			
λ_i	$\|V_{\lambda^i} - V_{\lambda^{i-1}}\|_\infty$	Rate	CPU(s)	λ_i	$\|V_{\lambda^i} - V_{\lambda^{i-1}}\|_\infty$	Rate	CPU(s)
10^2	6.06×10^{-1}		16	10^2	2.05×10^{-1}		21
10^3	6.54×10^{-2}	0.98	18	10^3	1.26×10^{-3}	1.99	23
10^4	6.59×10^{-3}	0.99	21	10^4	5.64×10^{-5}	1.76	25
10^5	6.60×10^{-4}	1.00	23	10^5	1.12×10^{-6}	1.69	29

illustrates that the lower order penalty method is more promising than the classic linear penalty method.

Finally, we plot the option value and risk parameters: delta and gamma, respectively, at the last time step of the butterfly spread option with a particular choice of parameter set. These figures are consistent with those in [17], which again demonstrate the usefulness of the power penalty method.

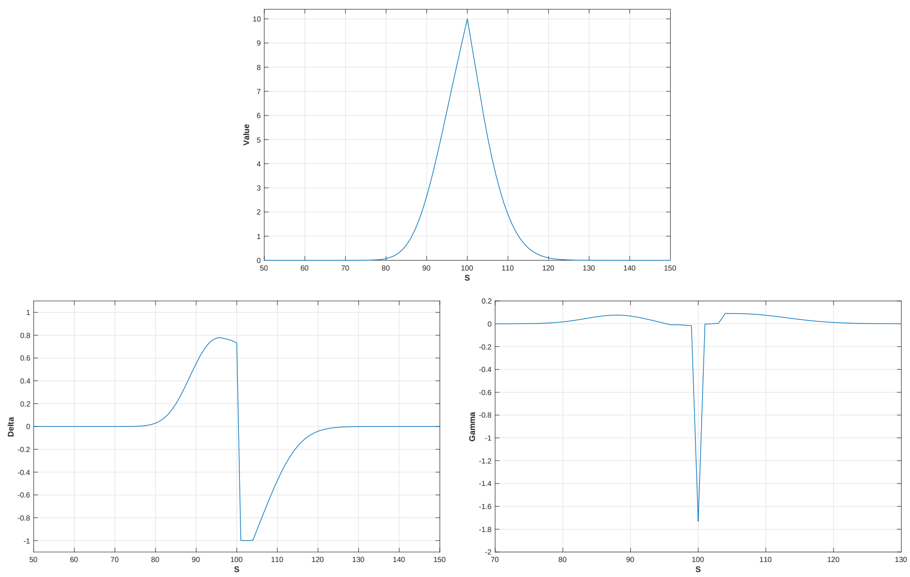

Figure 1: Solution $U(s)$ with $N = 99$. The solution is obtained with the lower order penalty method ($k = 2$). $\varepsilon = 10^{-6}$, $\epsilon = 10^{-3}$ and $\lambda = 1000$ are chosen in Algorithm 4.1.

6 Conclusions

We have developed a solution method to numerically solving the discrete HJB complementarity problem. The new method is based on a power penalty approach to the complementarity problem, where the complementarity condition are penalized and combined into a nonlinear system. Moreover, the unique solvability and exponential convergence rate of the power penalty approach was established. We also presented a combined solution method with a smoothing technique to solve the nonlinear penalized system. Finally, we carried out three numerical experiments to demonstrate the rates of convergence and effectiveness of the power penalty method.

Acknowledgement

The authors would like to thank the anonymous referees and the editor for their helpful comments and constructive suggestions toward the improvement of this paper. Kai Zhang's research was supported by National Natural Science Foundation of China (Project 11871347). Yang's work was partially supported by the Research Grants Council of Hong Kong (PolyU 152182/19E). This work is also supported by Shenzhen Humanities & Social Sciences key research bases.

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Manuscript received 22 July 2022
revised 8 November 2022
accepted for publication 15 January 2023

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