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THE HYPERTREE WITH THE SECOND LARGEST SPECTRAL RADIUS AMONG UNIFORM HYPERTREES WITH GIVEN SIZE AND STABILITY NUMBER*

Li Su and Honghai Li†

Abstract: In this paper, we determine the set of candidates for the hypertree which attain the second largest spectral radius among all uniform hypertrees with given size and stability number mainly by means of matching polynomial method.

Key words: *hypertree, adjacency tensor, spectral radius, matching polynomial*

Mathematics Subject Classification: *15A18, 05C65, 05C31*

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1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair (V, E), where V is a finite set and $E \subseteq \mathcal{P}(V)$ and $\mathcal{P}(V)$ stands for the power set of V. The elements of V are referred to as *vertices* and the elements of E are called *edges*. A hypergraph $\mathcal{H}$ is r*-uniform* if every edge $e \in E(\mathcal{H})$ contains precisely r vertices, which is denoted by $V(e)$ to be distinguished from the edge e. For a vertex $v \in V$, we denote by $E_{\mathcal{H}}(v)$ (or simply $E(v)$) the set of edges containing v. The cardinality $|E_v|$ is the *degree* of v, denoted by $d_{\mathcal{H}}(v)$ (or simply $d(v)$). A vertex of $\mathcal{H}$ is called an *isolated vertex* if its degree is zero, and is a *core vertex* if its degree is equal to one and an *intersection vertex* otherwise. If any two edges in $\mathcal{H}$ share at most one vertex, then $\mathcal{H}$ is said to be a *linear hypergraph*.

In a hypergraph $\mathcal{H}$, two vertices are *adjacent* if there is an edge containing both of them and two edges are called *adjacent* if their intersection is not empty. A vertex v is said to be *incident* to an edge e if $v \in e$. A *walk* of hypergraph $\mathcal{H}$ is defined to be an alternating sequence of vertices and edges $v_1e_1v_2e_2 \cdots v_\ell e_\ell v_{\ell+1}$ satisfying that both v_i and v_{i+1} are incident to e_i for $1 \leqslant i \leqslant \ell$. This walk is called a *path (of length l)* if all edges are distinct and all vertices are distinct except the possible case of $v_1 = v_{\ell+1}$, and when this happens, i.e., $v_1 = v_{\ell+1}$, it is called *closed*. A closed path of length at least two is called a *cycle*. A hypergraph $\mathcal{H}$ is called *connected* if for any vertices u, v, there is a walk connecting u and v. A hypergraph $\mathcal{H}$ is called *acyclic* if it contains no cycle. A connected and acyclic hypergraph is called a *hypertree*. By definition, every hypertree is linear.

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†Corresponding author

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Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph of order n and size m. A subset $S \subseteq V$ is *stable* if for any $e \in E$, there exists at least one $v \in V(e)$ such that $v \notin S$. A stable set in $\mathcal{H}$ is a *maximum stable set* if there is no stable set in $\mathcal{H}$ of greater cardinality. The *stability number*, denoted by $\alpha(\mathcal{H})$, is the cardinality of a maximum stable set in $\mathcal{H}$. A *matching* of the hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$ and the *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the maximum cardinality of a matching. For the hypergraph $\mathcal{H}$, we use $m(\mathcal{H}, k)$ to denote the number of matchings consisting of k edges. See [1] for more details.

Zhang et al. [18] introduced the polynomial $\varphi(\mathcal{H}, x)$ of a hypertree $\mathcal{H}$ and obtained some properties between the eigenvalues of $\mathcal{H}$ and this polynomial, where $\varphi(\mathcal{H}, x) = \sum_{k=0}^{\nu(\mathcal{H})}(-1)^k m(\mathcal{H}, k)x^{(\nu(\mathcal{H})-k)r}$.

Su et al. [11] redefined the *matching polynomial* of $\mathcal{H}$ as

$$\varphi(\mathcal{H}, x) = \sum_{k\geq 0}(-1)^k m(\mathcal{H}, k)x^{n-kr}.$$

This definition seems more appropriate as it guarantees that matching polynomials of hypergraphs of the same order have the same degree and spectral radius of $\mathcal{H}$ is still the maximum real root of $\varphi(\mathcal{H}, x)$ with algebraic multiplicity one.

In [5], some transformations on hypergraphs such as "edge-moving" and "edge-releasing" were introduced and the first two largest spectral radii of hypertrees on n vertices were characterized. Yuan et al. [17] further determined the first eight uniform hypertrees on n vertices with the largest spectral radii. Xiao et al. [15] characterized the unique uniform hypertree with the maximum spectral radius among all uniform hypertrees with a given degree sequence. The first two largest spectral radii of uniform hypertrees with given diameter were characterized in [16]. The matching polynomial method was introduced in [11] to characterize the first $\lfloor \frac{d}{2} \rfloor + 1$ hypertrees among all r-uniform hypertrees with given size m and diameter d, and the first two minimal hypertrees among all r-uniform hypertrees with given size. Further, the first two hypertrees were characterized among all r-uniform hypertrees with given size and strong stability number in [13]. The authors [14] determined the hypertree with the largest spectral radius among all hypertrees with given size and stability number.

In this paper, as a continuation of work in [14], we determine the set of candidates for the second largest hypertrees among all r-uniform hypertrees with given size and stability number. The structure of the remaining part of the paper is as follows: In Section 2, some definitions and results related to the spectral radius of hypergraph are presented. In Section 3, some properties on stability number of hypertrees are obtained. In the last section, we find out the set of candidates for the second largest hypertrees among all r-uniform hypertrees with given size and stability number.

2 Preliminaries

Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph on n vertices. A *subgraph* $\mathcal{H}' = (V', E')$ of $\mathcal{H}$ is a hypergraph with $V' \subseteq V$ and $E' \subseteq E$. A *proper subgraph* $\mathcal{H}'$ of $\mathcal{H}$ is subgraph of $\mathcal{H}$ with $\mathcal{H}' \neq \mathcal{H}$. For a vertex subset $S \subset V$, let $\mathcal{H} - S$ be the subgraph of $\mathcal{H}$ by deleting all the vertices in S and their incident edges. When $S = \{v\}$, $\mathcal{H} - S$ is simply written as $\mathcal{H} - v$. For an edge e with $V(e) = \{v_1, \ldots, v_r\} \in E(\mathcal{H})$, let $\mathcal{H} \setminus e$ stand for the subgraph of $\mathcal{H}$ obtained by deletion of the edge e from $\mathcal{H}$, i.e. $\mathcal{H} \setminus e = (V, E \setminus \{e\})$, and $\mathcal{H} - V(e)$ stand for the subgraph $\mathcal{H} - \{v_1, \ldots, v_r\}$. For two r-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$ with $V(\mathcal{G}) \cap V(\mathcal{H}) = \emptyset$, we use $\mathcal{G}\dot{\cup}\mathcal{H}$ to denote the disjoint union of $\mathcal{G}$ and $\mathcal{H}$. Use $t\mathcal{G}$ to denote the disjoint union of t copies of $\mathcal{G}$, where t is a positive integer.

An edge e of $\mathcal{H}$ is called a *pendent edge* if e contains exactly $r-1$ core vertices. If e is not a pendent edge, it is called a *non-pendent edge.* The intersection vertex of a pendent edge is called *support vertex.* An r-uniform hypergraph $\mathcal{H}$ is called a *hyperstar*, denoted by S_m^r, if there is a partition of the vertex set V as $V = \{v\} \cup V_1 \cup \cdots \cup V_m$ such that $|V_1| = \cdots = |V_m| = r-1$, and $E = \{\{v\} \cup V_i \mid i = 1, \ldots, m\}$, and v is the *center* of S_m^r. Use P_m^r to denote the r-uniform hypertree with m edges which is a path.

For positive integers r and n, a real *tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_r})$ of order r and dimension n refers to a multidimensional array (also called *hypermatrix*) with entries $a_{i_1 i_2 \cdots i_r}$ such that $a_{i_1 i_2 \cdots i_r} \in \mathbb{R}$ for all $i_1, i_2, \ldots, i_r \in [n]$, where $[n] = \{1, 2, \ldots, n\}$.

Qi [8] and Lim [6] independently introduced the concepts of tensor eigenvalues. Let $\mathcal{A}$ be an order r dimension n tensor. If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x = \lambda x^{[r-1]},$$

where $x^{[r-1]}$ is a vector with i-th entry x_i^{r-1}, then λ is called an *eigenvalue* of $\mathcal{A}$, x is called an *eigenvector* of $\mathcal{A}$ corresponding to the eigenvalue λ. The *spectral radius* of $\mathcal{A}$ is the maximum modulus of the eigenvalues of $\mathcal{A}$.

Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph on n vertices. The adjacency tensor of $\mathcal{H}$ is defined as the order r and dimension n tensor $\mathcal{A}(\mathcal{H}) = (a_{i_1 i_2 \cdots i_r})$, whose $(i_1 i_2 \cdots i_r)$-entry is

$$a_{i_1 i_2 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \ldots, i_r\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The *spectral radius of hypergraph* $\mathcal{H}$ is defined as spectral radius of its adjacency tensor, denoted by $\rho(\mathcal{H})$. In [3] the weak irreducibility of nonnegative tensors was defined. It was proved that an r-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathcal{A}(\mathcal{H})$ is weakly irreducible (see [3]). Part of the Perron-Frobenius theorem for nonnegative tensors is stated in the following.

Theorem 2.1 ([9])**.** *Let $\mathcal{A}$ be a nonnegative tensor of order r and dimension n, where $r, n \geq 2$. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector corresponding to it. If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is a positive eigenvalue of $\mathcal{A}$ with a positve eigenvector x. Furthermore, $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with a positive eigenvector, and x is the unique positive eigenvector associated with $\rho(\mathcal{A})$, up to a multiplicative constant.*

The unique positive eigenvector x with $\sum_{i=1}^n x_i^r = 1$ corresponding to $\rho(\mathcal{H})$ is called the *principal eigenvector* of $\mathcal{H}$.

An *edge-moving operation* on hypergraphs was introduced in [5]. Let $\mathcal{H} = (V, E)$ be a hypergraph with $u \in V$ and $e_1, \cdots, e_k \in E$, such that $u \notin e_i$ for $i = 1, \cdots, k$. Suppose that $v_i \in e_i$ and write $e_i' = (e_i \setminus \{v_i\}) \cup \{u\}$ $(i = 1, \cdots, k)$. Let $\mathcal{H}' = (V, E')$ be the hypergraph with $E' = (E \setminus \{e_1, \cdots, e_k\}) \cup \{e_1', \cdots, e_k'\}$. Note that $v_1, \cdots, v_k$ need not be distinct. Assume that $v_1, \ldots, v_r$ are all distinct vertices of them. Then we say that $\mathcal{H}'$ is obtained from $\mathcal{H}$ by *moving edges* $(e_1, \cdots, e_k)$ from $(v_1, \cdots, v_k)$ (or from $v_1, \cdots, v_r$) to u.

Lemma 2.2 ([5])**.** *Let $\mathcal{H}$ be a connected and uniform hypergraph, $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by moving edges $(e_1, \cdots, e_k)$ from $(v_1, \cdots, v_k)$ to u, and $\mathcal{H}'$ contains no multiple edges. If x is the principal eigenvector of $\mathcal{H}$ corresponding to $\rho(\mathcal{H})$, and suppose that $x_u \geq \max_{1 \leq i \leq k}\{x_{v_i}\}$, then $\rho(\mathcal{H}') > \rho(\mathcal{H})$.*

The matching polynomial is fundamental in determining extremal spectral radius of hypertrees, and so some basic properties of matching polynomial of hypergraph are listed as follows.

Lemma 2.3 ([11]). *Let $\mathcal{G}$ and $\mathcal{H}$ be two r-uniform hypergraphs. Then the following statements hold.*

1. $\varphi(\mathcal{G}\dot{\cup}\mathcal{H},x)=\varphi(\mathcal{G},x)\varphi(\mathcal{H},x)$.
2. $\varphi(\mathcal{G},x)=\varphi(\mathcal{G}\setminus e,x)-\varphi(\mathcal{G}-V(e),x)$ *if e is an edge of $\mathcal{G}$.*
3. *If $u\in V(\mathcal{G})$ and $I=\{i|e_i\in E_u\}$, for any $J\subseteq I$, we have*
$$\varphi(\mathcal{G},x)=\varphi(\mathcal{G}\setminus\{e_i|i\in J\},x)-\sum_{i\in J}\varphi(\mathcal{G}-V(e_i),x)$$
and
$$\varphi(\mathcal{G},x)=x\varphi(\mathcal{G}-u,x)-\sum_{e\in E_u}\varphi(\mathcal{G}-V(e),x).$$

Let $\mathcal{T}$ and $\mathcal{T}'$ be acyclic uniform hypergraphs of n vertices. We define $\mathcal{T}'\preceq\mathcal{T}$ iff $\varphi(\mathcal{T}',x)\geq\varphi(\mathcal{T},x)$ for every $x\geq\rho(\mathcal{T})$; let $\mathcal{T}'\prec\mathcal{T}$ iff $\mathcal{T}'\preceq\mathcal{T}$ and $\varphi(\mathcal{T}',x)-\varphi(\mathcal{T},x)\neq 0$ at the point $x=\rho(\mathcal{T})$, which implies that $\varphi(\mathcal{T}',x)-\varphi(\mathcal{T},x)>0$ for any $x\geq\rho(\mathcal{T})$. Note that $\mathcal{T}'\prec\mathcal{T}$ ($\mathcal{T}'\preceq\mathcal{T}$, resp.) implies $\rho(\mathcal{T}')<\rho(\mathcal{T})$ ($\rho(\mathcal{T}')\leq\rho(\mathcal{T})$, resp.).

Let $\mathcal{T}$ be an r-uniform hypertree, and u be a vertex of $\mathcal{T}$. We consider such edge e incident to u satisfying that edges incident to vertices in $V(e)\setminus\{u\}$ are only pendent edges. Let a be the number of support vertices of $V(e)\setminus\{u\}$. We define the edge e to be

- L-type if $a=0$, equivalently, e is a pendent edge.
- F-type if $a>0$, equivalently, e is not L-type.
- $\bar{F}$-type if e is F-type and the degree of each of the a support vertices is two.
- $\hat{F}$-type if e is F-type and only one of the a support vertices has degree greater than two.
- W-type if e is F-type and $a=r-1$.
- $\bar{W}$-type if e is $\bar{F}$-type and $a=r-1$.
- $\hat{W}$-type if e is $\hat{F}$-type and $a=r-1$.
- $\bar{K}$-type if e is $\bar{F}$-type but not $\bar{W}$-type.
- $\hat{K}$-type if e is $\hat{F}$-type but not $\hat{W}$-type.

Let $\mathcal{T}_j$ be an r-uniform hypertree with v_j as a vertex, for $j=1,\ldots,l$ with $l\geq 2$. We denote by $R(\mathcal{T}_1,\ldots,\mathcal{T}_l)$ the r-uniform hypertree obtained by identifying vertices $v_1,\ldots,v_l$ into a single vertex which is called *center vertex*, see (a) of Figure 1. $R(\mathcal{T}_1,\mathcal{T}_2)$ may be written as $\mathcal{T}_1(v_1,v_2)\mathcal{T}_2$, known as the *coalescence* of $\mathcal{T}_1$ and $\mathcal{T}_2$ at v_1,v_2. If some $\mathcal{T}_j$ consists of only one $\bar{F}$-type with b support vertices, then $\mathcal{T}_j$ is simply written as the number b in $R(\mathcal{T}_1,\ldots,\mathcal{T}_l)$. For convenience, b^x in $R(\mathcal{T}_1,\ldots,b^x,\ldots,\mathcal{T}_l)$ means the hypertree represented by the number b appears x times.

Let e be an edge with a ($a\leq r-2$) disjoint pendent edges attached, and u, v be its two core vertices. Denote by $R(\mathcal{T}_1,\ldots,\mathcal{T}_h;a;\mathcal{T}_{h+1},\ldots,\mathcal{T}_l)$ an r-uniform hypertree obtained by identifying the center vertex of $R(\mathcal{T}_1,\ldots,\mathcal{T}_h)$ and that of $R(\mathcal{T}_{h+1},\mathcal{T}_{h+2},\ldots,\mathcal{T}_l)$ with u and v, respectively. See (b) of Figure 1. Vertices u and v are called *center vertices* and edge e is called *center edge* of $R(\mathcal{T}_1,\ldots,\mathcal{T}_h;a;\mathcal{T}_{h+1},\ldots,\mathcal{T}_l)$. When $\mathcal{T}_{h+1},\ldots,\mathcal{T}_l$ are all trivial, $R(\mathcal{T}_1,\ldots,\mathcal{T}_h;a;\mathcal{T}_{h+1},\ldots,\mathcal{T}_l)$ is just $R(\mathcal{T}_1,\ldots,\mathcal{T}_h,a)$.

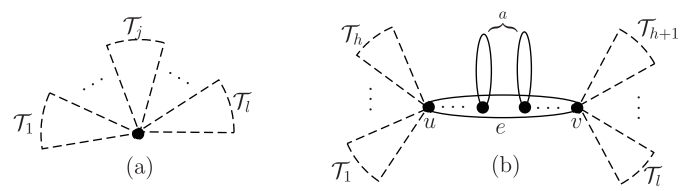

Figure 1: (a) $R(\mathcal{T}_1, \ldots, \mathcal{T}_l)$; (b) $R(\mathcal{T}_1, \ldots, \mathcal{T}_h; a; \mathcal{T}_{h+1}, \ldots, \mathcal{T}_l)$.

Lemma 2.4 ([13]). *Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$ be nontrivial r-uniform hypertrees, $w_i \in V(\mathcal{G}_i)$ for $i = 1, 2$ and $w_0 \in V(\mathcal{H})$. For $i = 1, 2$, let $\mathcal{G}_i(w_i, w_0)\mathcal{H}$ be the coalescence of $\mathcal{G}_i$ and $\mathcal{H}$. If $\mathcal{G}_1 \succeq \mathcal{G}_2$ and $\mathcal{G}_1 - w_1 \preceq \mathcal{G}_2 - w_2$, then $\mathcal{G}_1(w_1, w_0)\mathcal{H} \succeq \mathcal{G}_2(w_2, w_0)\mathcal{H}$. Furthermore, if at least one of $\mathcal{G}_1 \succ \mathcal{G}_2$ and $\mathcal{G}_1 - w_1 \prec \mathcal{G}_2 - w_2$ holds, then $\mathcal{G}_1(w_1, w_0)\mathcal{H} \succ \mathcal{G}_2(w_2, w_0)\mathcal{H}$.*

Lemma 2.5 ([11]). *Let H and G be arbitrary r-uniform hypertrees with H nontrivial. Let a, b, c be integers.*

(a) *If $0 \le b < a \le r - 1$, then*

$$R(H, 0; a-1; G, b) \succ R(H; b; G, a) \tag{2.1}$$

and

$$R(H, 0; a-1; G, b) \succ R(H, 0; b-1; G, a). \tag{2.2}$$

(b) *If $a > c$, then*

$$R(H, a, 0; b; 0, c) \succ R(H, c, 0; b; 0, a). \tag{2.3}$$

Lemma 2.6 ([13]). *For any r-uniform hypertree H and nonnegative integers $s \ge 3$ and $t \le r - 2$, we have*

$$R(H, r-1, 0^{s-1}; t; 0^2) \succ R(H, r-1, 0; t; 0^s).$$

Lemma 2.7 ([13]). *For any r-uniform hypertree H and integers a and b with $1 \le b \le a \le r - 2$, we have*

$$R(H, a+1, b-1) \succ R(H, a, b).$$

3 Stability Number

The following two results in [14] will be used in the sequel.

Lemma 3.1 ([14]). *Let $\mathcal{T}$ be a hypertree with $|E(\mathcal{T})| \ge 2$, and $C(\mathcal{T})$ be the set of all core vertices of $\mathcal{T}$. Then there exists a maximum stable set $S(\mathcal{T})$ of $\mathcal{T}$ such that $C(\mathcal{T}) \subseteq S(\mathcal{T})$.*

Henceforth, we assume that maximum stable set under consideration always contains all of core vertices and so no support vertex.

Let $\mathcal{H}$ be an r-uniform hypertree with a F-type edge e attached at vertex u. Suppose that the component containing u of $\mathcal{H} \setminus e$ is nontrivial. Let $C(e)$ denote the vertex subset obtained by taking all core vertices from e and from all pendent edges attached at support vertices of $V(e) \setminus \{u\}$. Obviously, $u \notin C(e)$.

Lemma 3.2 ([14])**.** *Let $\mathcal{H}$ be an r-uniform hypergraph with $u \in V(\mathcal{H})$ and $e \in E_{\mathcal{H}}(u)$, and $\mathcal{H}'$ be the component containing u of $\mathcal{H} \setminus e$, which is nontrivial.*

(1) *If e is an F-type edge incident to u, then $\alpha(\mathcal{H}) = \alpha(\mathcal{H}') + |C(e)|$.*

(2) *If e is an L-type edge incident to u, and there exists a maximum stable set $S(\mathcal{H}')$ of $\mathcal{H}'$ such that $u \notin S(\mathcal{H}')$, then $\alpha(\mathcal{H}) = \alpha(\mathcal{H}') + r - 1$.*

From Lemma 3.2, the stability number of hypergraph keeps unchanged after moving an F-type edge, or moving a pendent edge from a vertex which is incident to at least two pendent edges to a vertex which is incident to at least one pendent edge. Combining with Lemma 2.2 yields the following result which will be used repeatedly for characterizing the structure of extremal hypertrees.

Lemma 3.3. *Let $\mathcal{H}$ be an r-uniform hypertree and x be the principal eigenvector of $\mathcal{H}$.*

(1) *After moving some F-type edges from some vertices, say $v_1, \ldots, v_k$, to a vertex, say u, for which $x_u \geq \max\{x_{v_1}, \ldots, x_{v_k}\}$, the resulting hypertree has the same stability number as $\mathcal{H}$ and larger spectral radius than $\mathcal{H}$.*

(2) *For some vertices $u_1, \ldots, u_l, u_{l+1}$ of $\mathcal{H}$, if $|L_i| \geq 2$ for all $i = 1, \ldots, l$ and $|L_{l+1}| \geq 1$, where L_i is the set of L-type edges incident to vertex u_i for $i = 1, \ldots, l+1$, then the hypertree obtained from $\mathcal{H}$ by moving at most $|L_i| - 1$ L-type edges incident to u_i for each $i = 1, \ldots, l$ to u_{l+1} has the same stability number as $\mathcal{H}$ and larger spectral radius than $\mathcal{H}$.*

For a hypertree $\mathcal{T}$, let $N(\mathcal{T})$ denote the maximum distance among support vertices of $\mathcal{T}$, and $CO(\mathcal{T})$ denote the nontrivial component of the hypertree obtained from $\mathcal{T}$ by deleting all pendent edges from $\mathcal{T}$. Denote by $\mathcal{T}(m, r, \alpha)$ the set of all r-uniform hypertrees with m edges and strong stability number α. For any nonnegative integer d, let $\mathcal{T}_d(m, r, \alpha) = \{\mathcal{T} \in \mathcal{T}(m, r, \alpha) | N(CO(\mathcal{T})) = d\}$.

Lemma 3.4. *For any $\mathcal{T} \in \mathcal{T}_d(m, r, \alpha)$ with $d \geq 2$, there exists a hypertree in $\mathcal{T}_j(m, r, \alpha)$ such that it has larger spectral radius than $\mathcal{T}$ and $j \in \{0, 1\}$.*

Proof. For $\mathcal{T} \in \mathcal{T}_d(m, r, \alpha)$ with $d \geq 2$, choose two vertices v_1 and v_2 of $\mathcal{T}$ such that they are support vertices of $CO(\mathcal{T})$ and the distance between them in $CO(\mathcal{T})$ is equal to d. Note that both v_1 and v_2 are incident to F-type edges. Let x be the principal eigenvector of $\mathcal{T}$. Without loss of generality, assume that $x_{v_1} \geq x_{v_2}$. Denote by $\mathcal{T}_1$ the hypertree obtained from $\mathcal{T}$ by moving all the F-type edges from v_2 to v_1, which have the following properties:

(1) $\alpha(\mathcal{T}_1) = \alpha(\mathcal{T})$.

(2) $\rho(\mathcal{T}_1) > \rho(\mathcal{T})$.

(3) $N(CO(\mathcal{T}_1)) \leq N(CO(\mathcal{T}))$.

The statements (1) and (2) hold by Lemma 3.3. Obviously $\mathcal{T}_1$ has no F-type edges incident to v_2 and at least two F-type edges incident to v_1. Thus the statement (3) holds as well.

If $N(CO(\mathcal{T}_1)) \geq 2$, then continue the process and obtain the sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ satisfying that

(1) $\alpha(\mathcal{T}) = \alpha(\mathcal{T}_1) = \alpha(\mathcal{T}_2) = \cdots$

(2) $\rho(\mathcal{T}) < \rho(\mathcal{T}_1) < \rho(\mathcal{T}_2) < \cdots$

(3) $N(CO(\mathcal{T})) \geq N(CO(\mathcal{T}_1)) \geq N(CO(\mathcal{T}_2)) \geq \cdots$

Since each of $\rho(\mathcal{T}_i)$ is a finite real number, the process will certainly terminate after a finite number of steps, say at some $\mathcal{T}_j$. This means that $N(CO(\mathcal{T}_j)) < 2$ and we are done. □

4 Main Results

We have shown in [14] that the hypertree $A(m,r,\alpha) = R((r-1)^g, 0^l, s)$ ($g \geq 0$, $l \geq 1$) has the largest spectral radius in $\mathcal{T}(m,r,\alpha)$. Now we further consider the problem of characterizing the hypertree in $\mathcal{T}(m,r,\alpha)$ with the second largest spectral radius. Note that S_m^r is the only hypertree with $\alpha = n-1$, where $n = m(r-1)+1$. For $\alpha = n-2$, it is easy to see that $R(0^2; 0; 0^{m-3})$ uniquely has the second largest spectral radius in $\mathcal{T}(m,r,n-2)$ ($m \geq 5$) by Lemma 2.2.

From now on we assume that $m \geq 6$ and $\alpha \leq n-3$. For given α and $r-1$, there exist two (unique) integers q and a such that

$$\alpha = q(r-1) + a, \quad 0 \leq a < r-1. \tag{4.1}$$

Let $s = r-1-a$. Consider linear equations as follows

$$\begin{cases} gr + l = m - s - 1, \\ g(r-1) + l = q - s, \end{cases} \tag{4.2}$$

which obviously has the unique integral solutions for g, l.

Lemma 4.1. *The largest hypertree in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$ ($\alpha \leq n-3$, $m \geq 6$) must be one of the following*

$$\begin{cases} R((r-1)^g, 0^{l-1}; s-1; 0^2), & \text{if } l \geq 3,\ g = 0, \text{ and } 2 \leq s \leq r-1. \\ R((r-1)^{g-1}, 0^{l-1}, s; r-2; 0^2), & \text{if } l \geq 2,\ g \geq 1, \text{ and } 1 \leq s \leq r-1. \\ R((r-1)^g, 0^{l-1}, s+1), & \text{if } l = 1,\ g \geq 1, \text{ and } s = r-2. \\ R((r-1)^g, 0^{l-1}, s-1, 1), & \text{if } l \geq 2,\ g \geq 0, \text{ and } 2 \leq s \leq r-1. \\ R((r-1)^{g-1}, 0^l, r-2, s+1), & \text{if } l \geq 1,\ g \geq 1, \text{ and } 1 \leq s \leq r-3. \\ R((r-1)^g, 0^{l-1}; 0; s), & \text{if } l = 1,\ g = 1, \text{ and } 1 \leq s \leq r-1. \\ R((r-1)^{g-1}, 0^l; r-2; s, 0), & \text{if } l \geq 1,\ g \geq 2, \text{ and } 1 \leq s \leq r-1. \end{cases}$$

Proof. Suppose that $\tilde{\mathcal{T}} \in \mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$ has the largest spectral radius. We consider four cases according to the number of support vertices in $CO(\tilde{\mathcal{T}})$ and the value of $N(CO(\tilde{\mathcal{T}}))$.

Case 1. $CO(\tilde{\mathcal{T}})$ has no support vertices. In this case, $CO(\tilde{\mathcal{T}})$ must be a single edge because $\tilde{\mathcal{T}}$ is not a hyperstar, and so $\tilde{\mathcal{T}}$ is the r-uniform hypertree obtained from an edge e, say $V(e) = \{v_1, \ldots, v_r\}$, by attaching a_i pendent edges at vertex v_i in e for $i = 1, \ldots, r$. Because $\tilde{\mathcal{T}}$ is not $A(m,r,\alpha)$, we can assume that for some positive integer s, $a_1 \geq a_2 \cdots \geq a_{s+1} > 0 = a_{s+2} = \cdots = a_r$, $2 \leq s \leq r-1$ and $a_1 \geq a_2 \geq 2$. By the maximality of $\tilde{\mathcal{T}}$ and Lemma 2.2, $a_1 = m-s-2$, $a_2 = 2$, $a_3 = \ldots = a_{s+1} = 1$, and so $\tilde{\mathcal{T}}$ can only be $R((r-1)^g, 0^{l-1}; s-1; 0^2)$ with $g = 0$, $l \geq 3$, and $2 \leq s \leq r-1$.

Case 2. $N(CO(\tilde{\mathcal{T}})) = 0$. Then $CO(\tilde{\mathcal{T}})$ has just one support vertex, say u. Let F_u and L_u denote the set of all F-type edges and all L-type edges incident to u in $\tilde{\mathcal{T}}$, respectively. Note that $E_{\tilde{\mathcal{T}}}(u) = F_u \cup L_u$ and $|F_u| \geq 2$. We discuss two cases according to whether there exists non-$\bar{F}$-type edge in F_u or not.

Case 2.1. There exists a non-$\bar{F}$-type edge in F_u, say e. Then e must be $\hat{F}$-type and is the only one by (2) of Lemma 3.3. In this case, we have the following claims.

Claim 1. $L_u \neq \emptyset$.

Proof. Suppose, by contradiction, that $L_u = \emptyset$. Since $\alpha \leq n-3$ and $m > 5$, $\tilde{\mathcal{T}}$ has at least three support vertices. We can choose a support vertex v in an edge f of F_u such that the hypertree $\mathcal{T}_1$ obtained from $\tilde{\mathcal{T}}$ by moving all pendent edges incident to v from v to u is still in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$. Moreover, $\mathcal{T}_1$ is isomorphic to the hypertre obtained from $\tilde{\mathcal{T}}$ by moving all edges incident to u but f from u to v. However, $\rho(\mathcal{T}_1) > \rho(\tilde{\mathcal{T}})$ by Lemma 2.2, which is a contradiction with the maximality of $\tilde{\mathcal{T}}$. □

Claim 2. There is at most one $\bar{K}$-type edge in F_u.

Proof. Otherwise, F_u contains two $\bar{K}$-type edges, a contradiction to the maximality of $\tilde{\mathcal{T}}$ by Lemma 2.7. □

Claim 3. The unique $\hat{F}$-type edge e must be $\hat{W}$-type. More precisely, there exists a vertex $v \in V(e) \setminus \{u\}$ such that $d(v) = 3$ and $d(w) = 2$ for all $w \in V(e) \setminus \{u, v\}$.

Proof. Because e is $\hat{F}$-type, there exists a unique vertex, say v, in $V(e) \setminus \{u\}$ such that $d(v) \geq 3$. Suppose, by contradiction, that e is a $\hat{K}$-type edge. Choose another edge e_1 in F_u as $|F_u| \geq 2$. Then e_1 must be either $\bar{K}$-type or $\bar{W}$-type. We consider two cases as follows.

- e_1 is a $\bar{K}$-type edge. In this case, we move $d(v)-2$ pendent edges from v to u, and the resulting hypertree is denoted by $\mathcal{T}_1$, which is isomorphic to the one obtained from $\tilde{\mathcal{T}}$ by moving all edges incident with u but e and a pendent edge from u to v. Note that $\mathcal{T}_1 \neq A(m,r,\alpha)$ because it has at least two $\bar{K}$-type edges, but has larger spectral radius than $\tilde{\mathcal{T}}$. This is a contradiction with the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$.
- e_1 is a $\bar{W}$-type edge. In this case, $\tilde{\mathcal{T}}$ may be viewed as $R(H,0;a;G,r-1)$, where $H = R(0^{d(v)-2})$, G is the component of $\tilde{\mathcal{T}} \setminus \{e, e_1\}$ containing the vertex u. Since e is a $\hat{K}$-type edge, $a+1 < r-1$ and then $\rho(R(H,0;a;G,r-1)) < \rho(R(H,0;r-2;G,a+1))$ by (2) of Lemma 2.5, and $\alpha(R(H,0;a;G,r-1)) = \alpha(R(H,0;r-2;G,a+1))$ by (2) of Lemma 3.2, a contradiction.

Suppose, by contradiction, that $d(v) \geq 4$. From the claims established above, we assume that $\tilde{\mathcal{T}} = R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^k)$ with $\tau \geq 1$, and $k \geq 3$. We consider three cases according to the values of ϵ and τ.

- $\epsilon \geq 1$. In this case, let $H = R((r-1)^{\epsilon-1}, 0^{\tau-1}, \mu)$, then $\tilde{\mathcal{T}}$ may be viewed as $R(H, r-1, 0; r-2; 0^k)$. Thus $\rho(R(H, r-1, 0; r-2; 0^k)) < \rho(R(H, r-1, 0^{k-1}; r-2; 0^2))$ by Lemma 2.6, but $\alpha(R(H, r-1, 0; r-2; 0^k)) = \alpha(R(H, r-1, 0^{k-1}; r-2; 0^2))$ by (2) of Lemma 3.2, a contradiction.
- $\epsilon = 0$ but $\tau \geq 2$. In this case, we can move $d(v)-3$ pendent edges from v to u and the resulting hypertree is denoted by $\mathcal{T}_2$, which is isomorphic to the one obtained from $\tilde{\mathcal{T}}$ by moving all edges incident to u but e and two pendent edges from u to v. Thus $\rho(\mathcal{T}_2) > \rho(\tilde{\mathcal{T}})$ by Lemma 2.2, but $\alpha(\mathcal{T}_2) = \alpha(\tilde{\mathcal{T}})$ by (2) of Lemma 3.3, a contradiction.
- $\epsilon = 0$ and $\tau = 1$. In this case, let $H = R(0^{k-2})$, then $\tilde{\mathcal{T}}$ may be viewed as $R(H, 0, 0; r-2; 0, \mu)$. From (3) of Lemma 2.5, $\rho(R(H,0,0;r-2;0,\mu)) < \rho(R(H,\mu,0;r-2;0,0))$, but $\alpha(R(H,0,0;r-2;0,\mu)) = \alpha(R(H,\mu,0;r-2;0,0))$ by (2) of Lemma 3.2, a contradiction.

Consequently, $\tilde{\mathcal{T}}$ can only be $R((r-1)^{g-1}, 0^{l-1}, s; r-2; 0^2)$ with $l \geq 2$, $g \geq 1$, and $1 \leq s \leq r-1$. □

Case 2.2. F_u consists of $\bar{F}$- type edges. We consider two cases according to $L_u = \varnothing$ or not.

- $L_u = \varnothing$. Suppose that there exists a $\bar{K}$-type edge in F_u. Since $|F_u| \geq 2$, we can choose another $\bar{F}$-type edge and a support vertex, say v, on it. Moving the pendent edge from v to u, the resulting hypertree is in $\mathcal{T}(m,r,\alpha)\backslash\{A(m,r,\alpha)\}$, but has larger spectral radius, a contradiction. Thus F_u consists of $\bar{W}$-type edges, and then $\tilde{\mathcal{T}} = R((r-1)^g, 0^{l-1}, s+1)$ with $l = 1$, $g \geq 1$, and $s = r-2$.
- $L_u \neq \varnothing$. In this case, from the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m,r,\alpha)\backslash\{A(m,r,\alpha)\}$, F_u has exactly two $\bar{K}$-type edges. Thus, either $\tilde{\mathcal{T}} = R((r-)^{g-1}, 0^l, r-2, s+1)$ with $l \geq 1$, $g \geq 1$, and $1 \leq s \leq r-3$ or $\tilde{\mathcal{T}} = R((r-1)^g, 0^{l-1}, s-1, 1)$ with $l \geq 2$, $g \geq 0$, and $2 \leq s \leq r-1$.

Case 3. $N(CO(\tilde{\mathcal{T}})) = 1$. In this case, all support vertices of $CO(\tilde{\mathcal{T}})$ lie in one edge, say e_0. Assume that $e_0 = \{v_1, \ldots, v_r\}$. Note that each $E_{\tilde{\mathcal{T}}}(v_i) \setminus \{e_0\}$ consists of F-type and L-type edges, say $E_{\tilde{\mathcal{T}}}(v_i) \setminus \{e_0\} = F_i \cup L_i$, where F_i and L_i stand for the set of all F-type and L-type edges incident to v_i, respectively, for all $i = 1, \ldots, r$. Because $N(CO(\tilde{\mathcal{T}})) = 1$, there are at least two of $F_1, \ldots, F_r$ are nonempty. Let x be the principal eigenvector of $\tilde{\mathcal{T}}$. Without loss of generality, suppose that x_{v_1} takes the maximum value of all x_{v_i} for which $F_i \neq \emptyset$ $(1 \leq i \leq r)$, and $F_2 \neq \emptyset$. By Lemma 3.3, we know that

- all F-type edges must be $\bar{F}$-type and $|L_i| \leq 1$ $(i = 2, \ldots, r)$.
- $|F_2| = 1$ and $F_3 = F_4 = \cdots = F_r = \emptyset$.
- $F_1 \cup F_2$ contains at most one $\bar{K}$-type edge.

Thus, $\tilde{\mathcal{T}}$ must be the form of $R((r-1)^\epsilon, d, 0^\tau; f; h, 0^\delta)$, where $d = r-1$ or $h = r-1$, $0 \leq f \leq r-2$, and $\delta \leq 1$. We now consider two cases according to the value of $f + \delta$.

Case 3.1. $f + \delta = 0$. In this case, $\tilde{\mathcal{T}} = R((r-1)^\epsilon, d, 0^\tau; 0; h)$. We claim that $\epsilon + \tau = 0$. Suppose, by contradiction, that $\epsilon + \tau \geq 1$. Let $G = R((r-1)^\epsilon, 0^\tau)$, $H = R(h)$, then $\tilde{\mathcal{T}}$ may be viewed as $R(H; 0; G, d)$. Deduce from (1) of Lemma 2.5 and (1) of Lemma 3.2 that $\rho(R(H; 0; G, d)) < \rho(R(H, 0; d-1; G, 0))$ and $\alpha(R(H; 0; G, d)) = \alpha(R(H, 0; d-1; G, 0))$. Because $R(H, 0; d-1; G, 0) \neq A(m, r, \alpha)$, a contradiction to the maximality of $\tilde{\mathcal{T}}$.

Consequently, $\tilde{\mathcal{T}}$ can only be $R((r-1)^g, 0^{l-1}; 0; s)$ with $g = 1$, $l = 1$, and $1 \leq s \leq r-1$.

Case 3.2. $f + \delta \neq 0$. We have claims as follows.

Claim 1. $\tau \neq 0$.

Proof. Otherwise, $\tau = 0$, it is easy to deduce a contradiction to the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$ from Lemma 3.3(1). □

Claim 2. at most one of $f + \delta, d, h$ is less than $r-1$.

Proof. Otherwise, the hypertree obtained from $\tilde{\mathcal{T}}$ by moving F-type edges from v_2 to v_1 belong to $\mathcal{T}(m,r,\alpha) \setminus \{A(m,r,\alpha)\}$, then deduce a contradiction to the maximality of $\tilde{\mathcal{T}}$ from Lemma 3.3(1). □

Claim 3. $f + \delta = d = r - 1$.

Proof. Let $H = R(h)$ and $G = R((r-1)^{\epsilon}, 0^{\tau})$, then $\tilde{\mathcal{T}}$ may be viewed as $R(H, 0^{\delta}; f; G, d)$. If $d > f + \delta$, then $R(H, 0; d-1; G, f+\delta) \in \mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$ and has larger spectral radius than $\tilde{\mathcal{T}}$ by Lemma 2.5. This is a contradiction. Thus $f + \delta \geq d$, and this, together with claim 2, implies that $f + \delta = r - 1$, and so $f = r - 2$ and $\delta = 1$.

Let $T = R((r-1)^{\epsilon}, 0^{\tau-1})$, then $\tilde{\mathcal{T}}$ may be viewed as $R(H, d, 0; r-2; 0, h)$. If $h > d$, then $\rho(R(H, d, 0; r-2; 0, h)) < \rho(R(H, h, 0; r-2; 0, d))$ by Lemma 2.5(3), but $\alpha(R(H, d, 0; r-2; 0, h)) = \alpha(R(H, h, 0; r-2; 0, d))$ from Lemma 3.2, a contradiction. Thus $h \leq d$ and this, together with claim 2, implies that $d = r - 1$. □

Consequently, $\tilde{\mathcal{T}}$ can only be $R((r-1)^{g-1}, 0^{l}; r-2; s, 0)$ with $l \geq 1$, $g \geq 2$ and $1 \leq s \leq r - 1$.

Case 4. $N(CO(\tilde{\mathcal{T}})) \geq 2$.

By Lemma 3.4, there is a $\mathcal{T}' \in \mathcal{T}_j(m, r, \alpha)$, where $j \in \{0, 1\}$, such that $\rho(\tilde{\mathcal{T}}) < \rho(\mathcal{T}')$.

If either $N(CO(\mathcal{T}')) = 1$ or $N(CO(\mathcal{T}')) = 0$ and $\mathcal{T}' \neq A(m, r, \alpha)$, then $\mathcal{T}' \in \mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$. However, this is a contradiction with the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$.

Now it remains to consider the case that $\mathcal{T}' = A(m, r, \alpha)$. Note that $\mathcal{T}'$ is obtained from $\tilde{\mathcal{T}}$ by moving F-type edge from v_2 to v_1. Due to the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$, $\tilde{\mathcal{T}}$ must be the form in Figure 2.

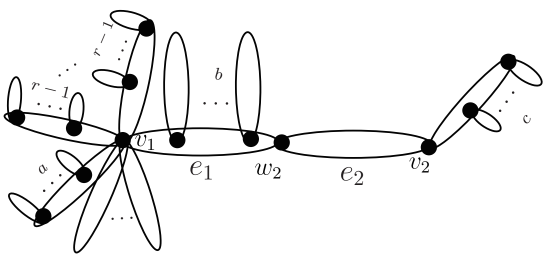

Figure 2: Possible form of $\tilde{\mathcal{T}}$, where $1 \leq a, c \leq r - 1$ and $0 \leq b \leq r - 2$

After moving the unique F-type edge from v_2 to w_2 in $\tilde{\mathcal{T}}$, the resulting hypertree is denoted by $\mathcal{T}_2$. Note that $\alpha(\mathcal{T}_2) = \alpha(\tilde{\mathcal{T}})$ by (1) of Lemma 3.2 and $\rho(\mathcal{T}_2) > \rho(\tilde{\mathcal{T}})$ because it is isomorphic to the one obtained from $\tilde{\mathcal{T}}$ by moving edge e_1 from w_2 to v_2. This is a contradiction with the maximality of $\tilde{\mathcal{T}}$ in $\mathcal{T}(m, r, \alpha) \setminus \{A(m, r, \alpha)\}$. The above statement means that the case of $N(CO(\tilde{\mathcal{T}})) \geq 2$ cannot occur, and we are done. □

Lemma 4.2. *The following orderings on hypertrees hold.*

(1) *For $\epsilon \geq 0$, $\tau \geq 0$ and $\mu \geq 1$,*

$$R((r-1)^{\epsilon+1}, 0^{\tau}, \mu - 1, 1) \succ R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^2).$$

(2) *For $\epsilon \geq 1$, $\tau \geq 0$, and $\mu \geq 0$,*

$$R((r-1)^{\epsilon}, 0^{\tau+1}, \mu + 1, r-2) \succ R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^2).$$

(3) *For* $\epsilon + \tau \geq 1$ *and* $\mu \geq 1$,

$$R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^2) \succ R((r-1)^{\epsilon}, 0^{\tau+1}; r-2; \mu, 0).$$

(4) *For* $1 \leq \mu \leq r-2$,

$$R(r-1, \mu+1) \succ R(r-1; 0; \mu).$$

Proof. (1) Let $\mathcal{G}_1 = R(r-1, \mu-1, 1)$, $\mathcal{G}_2 = R(\mu; r-2; 0^2)$, and $\mathcal{H} = R((r-1)^{\epsilon}, 0^{\tau})$, $w_i \in V(\mathcal{G}_i)$ for $i = 1, 2$ and $w_0 \in V(\mathcal{H})$. For $i = 1, 2$, let $\mathcal{G}_i(w_i, w_0)\mathcal{H}$ denote hypertrees $R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^2)$ and $R((r-1)^{\epsilon}, 0^{\tau+1}; r-2; \mu, 0)$ respectively.

By (b) of Lemma 2.3, we have

$$\begin{aligned}
\varphi(\mathcal{G}_1, x) =& x^{r-1}\varphi(R(r-1, \mu-1, 0), x) - x^{r-2}\varphi(R(r-1, \mu-1), x),\\
=& x^{2r-2}\varphi(R(r-1, \mu-1), x) - x^{2r-\mu-1}(x^r-1)^{r+\mu-2} - x^{r-2}\varphi(R(r-1, \mu-1), x),\\
=& x^{2r-2}\varphi(R(r-1, \mu-1), x) - x^{2r-\mu-1}(x^r-1)^{r+\mu-2}\\
& - x^{2r-\mu-2}(x^r-1)^{\mu-1}\varphi(R(r-1), x) + x^{(r-1)\mu-1}(x^r-1)^{r-1},\\
\varphi(\mathcal{G}_2, x) =& x^{r-1}\varphi(R(r-1, \mu), x) - x^{r-1}(x^r-1)^{r-2}\varphi(R(\mu), x),\\
=& x^{2r-2}\varphi(R(r-1, \mu-1), x) - x^{2r-\mu-2}(x^r-1)^{\mu-1}\varphi(R(r-1), x)\\
& - x^{r-1}(x^r-1)^{r-2}\varphi(R(\mu), x).
\end{aligned}$$

Subtracting the above two equations, we have

$$\begin{aligned}
&\varphi(\mathcal{G}_1, x) - \varphi(\mathcal{G}_2, x) \qquad (4.3)\\
&= x^{(r-1)\mu-1}(x^r-1)^{r-1} - x^{r-1}(x^r-1)^{r-2}\varphi(R(\mu), x) - x^{2r-\mu-1}(x^r-1)^{r+\mu-2}\\
&= x^{(r-1)\mu-1}(x^r-1)^{r-1} - x^{(r-1)(\mu+1)}(x^r-1)^{r-2}\\
&= x^{(r-1)\mu-1}(x^r-1)^{r-2}((x^r-1) - x^r)\\
&= -x^{(r-1)\mu-1}(x^r-1)^{r-2}
\end{aligned}$$

and obviously $\varphi(\mathcal{G}_1, x) < \varphi(\mathcal{G}_2, x)$ for any $x \geq \rho(\mathcal{G}_1)$. Thus $G_1 \succ G_2$.

$$\begin{aligned}
&\varphi(\mathcal{G}_1 - w_1, x) = x^{2r-\mu-2}(x^r-1)^{r+\mu-1}),\\
&\varphi(\mathcal{G}_2 - w_2, x) = x^{r-\mu-1}(x^r-1)^{r+\mu-2}\varphi(S_2), x) = x^{2r-\mu-2}(x^r-1)^{r+\mu-2}(x^r-2).
\end{aligned}$$

Subtracting the above two equations, we have

$$\varphi(\mathcal{G}_1 - w_1, x) - \varphi(\mathcal{G}_2 - w_2, x) = x^{2r-\mu-2}(x^r-1)^{r+\mu-2}$$

and obviously $\varphi(\mathcal{G}_1 - w_1, x) > \varphi(\mathcal{G}_2 - w_2, x)$ for any $x \geq \rho(\mathcal{G}_2 - w_2)$. Thus $G_1 - w_1 \prec G_2 - w_2$. The statement follows from Lemma 2.4.

(2) For convenience, denote $R((r-1)^{\epsilon}, 0^{\tau+1}, \mu+1, r-2)$ by H_1 and $R((r-1)^{\epsilon}, 0^{\tau}, \mu; r-2; 0^2)$ by H_2. By (b) of Lemma 2.3, we have

$$\begin{aligned}
\varphi(H_1, x) =& x^{2r-2}\varphi(R((r-1)^{\epsilon}, 0^{\tau}, r-2, \mu), x) - x^{2r-\mu-3}(x^r-1)^{\mu}\varphi(R((r-1)^{\epsilon}, 0^{\tau}, r-2), x)\\
& - x^{(r-1)(\tau+1)-\mu}(x^r-1)^{(r-1)(\epsilon+1)+\mu},\\
\varphi(H_2, x) =& x^{2r-2}\varphi(R((r-1)^{\epsilon}, 0^{\tau}, r-2, \mu), x) - 2x^{r-1}(x^r-1)^{r-2}\varphi(R((r-1)^{\epsilon}, 0^{\tau}, \mu), x)
\end{aligned}$$

Subtracting the above two equations and using Lemma 2.3 two more times, we have

$$\begin{aligned}&\varphi(H_1,x)-\varphi(H_2,x)\\ =&2x^{r-1}(x^r-1)^{r-2}\varphi(R((r-1)^\epsilon,0^\tau,\mu),x)-x^{2r-\mu-3}(x^r-1)^\mu\varphi(R((r-1)^\epsilon,0^\tau,r-2),x)\\ &-x^{(r-1)(\tau+1)-\mu}(x^r-1)^{(r-1)(\epsilon+1)+\mu}\\ =&x^{2r-\mu-2}(x^r-1)^{r+\mu-2}\varphi(R((r-1)^\epsilon,0^\tau),x)-2x^{(r-1)(\tau+\mu+1)}(x^r-1)^{(r-1)(\epsilon+1)-1}\\ &+x^{(r-1)(\tau+r)-\mu-1}(x^r-1)^{(r-1)\epsilon+\mu}-x^{(r-1)(\tau+1)-\mu}(x^r-1)^{(r-1)(\epsilon+1)+\mu}\\ =&-h(x)[(x^r-1)^{r-\mu-1}(2x^{r\mu}+(\tau-1)(x^r-1)^\mu)+x^{r(r-2)}((\epsilon-1)x^r+1)],\end{aligned}$$

where $h(x)=x^{(r-1)(\tau+1)-\mu}(x^r-1)^{(r-1)\epsilon+\mu-1}$. Since $\epsilon\geq 1$ and $\tau\geq 0$, we have $\varphi(H_1,x)-\varphi(H_2,x)<0$ for any $x\geq\rho(H_1)$. Thus $H_1\succ H_2$.

(3) We consider two cases according to $\epsilon\geq 1$ or $\tau\geq 1$.

Case 1. $\tau\geq 1$.

In this case, if we let $G=R((r-1)^\epsilon,0^{\tau-1})$, then $R((r-1)^\epsilon,0^\tau,\mu;r-2;0^2)=R(G,0,\mu;r-2;0,0)$ and $R((r-1)^\epsilon,0^{\tau+1};r-2;\mu,0)=R(G,0,0;r-2;\mu,0)$. Because $\mu>0$, by (3) of Lemma 2.5, we have

$$R(G,0,\mu;r-2;0,0)\succeq R(G,0,0;r-2;\mu,0),$$

with equality only when $\tau=1$ and $\epsilon=0$, in which these two hypertrees are isomorphic to each other.

Case 2. $\epsilon\geq 1$.

Let $G=R((r-1)^\epsilon,0^\tau)$, and w be its center vertex. Then $R((r-1)^\epsilon,0^\tau,\mu;r-2;0^2)$ and $R((r-1)^\epsilon,0^{\tau+1};r-2;\mu,0)$ may be viewed as $R(G,\mu;r-2;0^2)$ and $R(G,0;r-2;\mu,0)$, respectively. For convenience, denote $R((r-1)^\epsilon,0^\tau,\mu;r-2;0^2)$ and $R((r-1)^\epsilon,0^{\tau+1};r-2;\mu,0)$ by H_1 and H_2, respectively. By (b) of Lemma 2.3, we have

$$\begin{aligned}\varphi(H_1,x)=&x^{2r-\mu-2}(x^r-1)^\mu\varphi(R(G,r-1),x)-x^{2r-\mu-2}(x^r-1)^{r+\mu-2}\varphi(G,x)\\ &-x^{(r-1)\mu}(x^r-1)^{r-2}\varphi(G-w,x)\varphi(S_2,x)\\ \varphi(H_2,x)=&x^{2r-\mu-2}(x^r-1)^\mu\varphi(R(G,r-1),x)-x^{(r-1)(\mu+2)}(x^r-1)^{r-2}\varphi(G,x)\\ &-(x^r-1)^{r-2}\varphi(G-w,x)\varphi(R(\mu+1),x).\end{aligned}$$

Subtracting the above two equations, By (c) of Lemma 2.3, we have

$$\begin{aligned}&\varphi(H_1,x)-\varphi(H_2,x)\\ &=x^{r-\mu-1}(x^r-1)^{r-2}(x^{r\mu}-(x^r-1)^\mu)(x^{r-1}\varphi(G,x)-(x^r-1)\varphi(G-w,x))\\ &=x^{(r-2)(\mu+1)+1}(x^r-1)^{(r-1)(\epsilon-1)-1}(x^{r\mu}-(x^r-1)^\mu)(-\epsilon x^{r(r-1)}-(\tau-1)(x^r-1)^{r-1}).\end{aligned}$$

Because $\epsilon\geq 1$ and $\tau\geq 0$, we have $\varphi(H_1,x)-\varphi(H_2,x)<0$ for any $x\geq\rho(H_1)$. Thus $H_1\succ H_2$.

(4) Let $H=R((r-1)^\epsilon,0^{\tau-1})$. Then $R((r-1)^\epsilon,0^\tau,\mu;r-2;0^2)$ and $R((r-1)^\epsilon,0^{\tau+1};r-2;\mu,0)$ may be viewed as $R(H,\mu,0;r-2;0,0)$ and $R(H,0,0;r-2;0,\mu)$ respectively. Therefore $R(H,\mu,0;r-2;0,0)\succ R(H,0,0;r-2;0,\mu)$ by (3) of Lemma 2.5. □

We have shown in [14] that the largest spectral radius of hypertrees in $\mathcal{T}(m,r,\alpha)$ is uniquely attained by $A(m,r,\alpha)$. Further problem of characterizing which hypertrees attain the second spectral radius in $\mathcal{T}(m,r,\alpha)$ is more complicated and difficult. Although we cannot determine exactly the second hypertree in $\mathcal{T}(m,r,\alpha)$, we limit the search scope mainly to two candidates together with three sporadic cases.

Theorem 4.3. *The second largest spectral radius of hypertrees in* $\mathcal{T}(m,r,\alpha)$ ($\alpha \leq n-3$, $m \geq 6$) *must be attained by one of the following*

$$\begin{cases} R((r-1)^g, 0^{l-1}, s-1, 1), & \text{if } l \geq 2,\ g \geq 0, \text{ and } 2 \leq s \leq r-1. \\ R((r-1)^{g-1}, 0^l, r-2, s+1), & \text{if } l \geq 1,\ g \geq 1, \text{ and } 1 \leq s \leq r-3. \\ R((r-1)^g, 0^{l-1}; s-1; 0^2), & \text{if } l \geq 3,\ g = 0, \text{ and } 2 \leq s \leq r-1. \\ R((r-1)^g, 0^{l-1}, s+1), & \text{if } l = 1,\ g \geq 1, \text{ and } s = r-2. \\ R((r-1)^g, 0^{l-1}; 0; s), & \text{if } l = 1,\ g = 1, \text{ and } 1 \leq s \leq r-1. \end{cases}$$

Proof. This follows immediately from Lemma 4.1 and Lemma 4.2. □

Further, we can say something about the ordering between the first two candidates in Theorem 4.3.

Lemma 4.4. *Let* $B_1 = R((r-1)^g, 0^{l-1}, s-1, 1)$ *and* $B_2 = R((r-1)^{g-1}, 0^l, r-2, s+1)$, *and* $\rho_i = \rho(B_i)$ *for* $i = 1, 2$. *Let* $g(x) = \frac{x^r}{x^r-1}$ *and* $f(x) = 1 + g(x)^{r-2} - g(x)^\mu - g(x)^{\mu-1}$. *Then we have*

$B_1 \succeq B_2$ ($B_1 \succ B_2$, *resp.*) *if* $f(x) \geq 0$ ($f(x) > 0$, *resp.*) *for any* $x \geq \rho_1$;

$B_2 \succeq B_1$ ($B_2 \succ B_1$, *resp.*) *if* $f(x) \leq 0$ ($f(x) < 0$, *resp.*) *for any* $x \geq \rho_2$.

Particularly, $B_1 \succ B_2$ *whenever* $\mu \leq \frac{r-1}{2}$.

Proof. Let $\mathcal{G}_1 = R(\mu-1, r-1, 1)$, $\mathcal{G}_2 = R(\mu+1, r-2, 0)$ and $\mathcal{H} = R((r-1)^\epsilon, 0^\tau)$. Choose $w_i \in V(\mathcal{G}_i)$ for $i = 1, 2$ and $w_0 \in V(\mathcal{H})$ such that $B_i = \mathcal{G}_i(w_i, w_0)\mathcal{H}$ for $i = 1, 2$.

By using (c) of Lemma 2.3 to the centre vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$, we have

$$\begin{aligned} \varphi(\mathcal{G}_1, x) =& x^{2r-\mu-1}(x^r-1)^{r+\mu-1} - x^{2r-\mu-1}(x^r-1)^{r+\mu-2} \\ & - x^{(r+1)(r-1)-\mu}(x^r-1)^\mu - x^{(r-1)\mu-1}(x^r-1)^r, \\ \varphi(\mathcal{G}_2, x) =& x^{2r-\mu-1}(x^r-1)^{r+\mu-1} - x^{r-\mu-1}(x^r-1)^{r+\mu-1} \\ & - x^{r(r-1)-\mu-1}(x^r-1)^{\mu+1} - x^{(r-1)(\mu+2)+1}(x^r-1)^{r-2}. \end{aligned}$$

Subtracting the above two equations, we have

$$\varphi(\mathcal{G}_1, x) - \varphi(\mathcal{G}_2, x) = -x^{r-\mu-1}(x^r-1)^{r+\mu-2} f(x).$$

It is easy to verify that $\varphi(\mathcal{G}_1 - w_1, x) = \varphi(\mathcal{G}_2 - w_2, x)$. Thus the statements follows from Lemma 2.4.

Observe that $f(x) = g(x)^{\mu-1}(g(x)^{r-\mu-1} - 1) - (g(x)^\mu - 1)$ and $g(x) > 1$. If $\mu \leq \frac{r-1}{2}$, then $f(x) > 0$ for any $x \geq \rho_1$. We are done. □

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Li Su
School of Mathematics and Statistics, Jiangxi Normal University
Nanchang, Jiangxi 330022, China
E-mail address: lhh@jxnu.edu.cn

Honghai Li
School of Mathematics and Statistics, Jiangxi Normal University
Nanchang, Jiangxi 330022, China
E-mail address: suli@jxnu.edu.cn