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D-GAP FUNCTIONS AND ERROR BOUNDS FOR VECTOR EQUILIBRIUM PROBLEMS

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Abstract: In this article, we generalize the notion of D-gap function for a *weak* vector equilibrium problem $weakVEP$, that constitutes an unconstrained optimization reformulation for $weakVEP$. The construction of the D-gap function is based on the scalarization and the regularization of a gap function in the sense of Fukushima [12]. We give conditions under which the D-gap functions provide a global error bound for $weakVEP$.

Key words: *vector equilibrium problem, unconstrained optimization reformulations, gap function, global error bound, strongly monotone*

Mathematics Subject Classification: *49J40, 47J20, 90C33*

1 Introduction

Equilibrium problems provide a general setting for analyzing several other problems such as optimization problems, variational inequalities, complementarity problems, fixed point problems, etc. A fundamental generalization of the equilibrium problem is the vector equilibrium problem which includes vector optimization problem, vector variational inequality, vector saddle point problem and vector complementarity problem as special cases. Following the developments in the field of variational inequalities, various classes of equilibrium problems have been investigated in the literature; see [5, 16, 13, 26, 34, 19, 9, 3, 1, 11] and the references therein. A weak vector equilibrium problem (for short, $weakVEP$) is given by:

$$weakVEP: \qquad \text{Find an} \quad x \in K \quad \text{such that} \quad f(x,y) \notin -int\, \mathbb{R}^m_+, \quad \forall y \in K, \tag{1.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(x,y) = (f_1(x,y), f_2(x,y), ..., f_m(x,y))$ with $f(x,x) = 0$, $\forall x \in \mathbb{R}^n$ and K is a nonempty closed convex subset of $\mathbb{R}^n$. Let $sol(weakVEP)$ denote the solution set of $weakVEP$. For $m = 1$, $weakVEP$ collapses to the classic scalar equilibrium problem EP which consists of finding an $x \in K$ such that

$$f(x,y) \geq 0 \quad \forall y \in K.$$

The weak *Stampacchia* vector variational inequality $(SVVI)^w$ which consists of finding an $x \in K$ such that

$$(\langle F_1(x), y-x\rangle, \langle F_2(x), y-x\rangle, ..., \langle F_m(x), y-x\rangle) \notin -int\, \mathbb{R}^m_+, \quad \forall y \in K, \tag{1.2}$$

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where $F_i : \mathbb{R}^n \to \mathbb{R}^n$, is an instance of a $weakVEP$ with $f_i(x, y) = \langle F_i(x), y - x\rangle$.

It is well known that gap functions play a vital role in transforming a variational inequality into an equivalent optimization problem and are crucial in devising error bounds. These error bounds provide effective estimates for the distance between any arbitrary point and the solution set of a variational inequality and they are key in giving a stopping criterion for iterative algorithms. There has been a ton of research effort on the gap functions for variational inequalities, see [10, 23, 27, 6, 4, 20, 32]. Fukushima's regularization [12] of a gap function gave a well-defined constrained optimization reformulation for a variational inequality. Unconstrained optimization reformulations for a variational inequality was proposed in [33] through introducing a generalized D-gap function (or the 'difference' gap function), which essentially is the difference of two regularized gap functions, the generalized versions of those in [29]. Since the D-gap function provides an unconstrained reformulation, a global error bound for a variational Inequality was constructed in [33] using the D-gap function. This construction of the D-gap function has been generalized to a vector variational inequality in [4].

Turning to the case of equilibrium problems, the concept of D-gap function is extended to a scalar equilibrium problem by Zhang and Han [34]. Gap functions for vector equilibrium problems have also gained a keen interest in the recent past. Li et al. [22] extended the notion of gap a function to a vector equilibrium problem based on nonlinear scalarization. Gap functions for a nonsmooth convex vector optimization problem, a special case of a vector equilibrium problems have been investigated in [8]. Gap functions for a system of vector equilibrium problems have been investigated in [15, 21]. Set-valued gap functions for a vector equilibrium problem have been constructed in [24, 31] through conjugate duality in vector optimization. Khan et al. [16] and Zhang et al. [35] have independently proposed Fukushima's regularization of a gap function for vector equilibrium problems and developed error bounds. These gap functions give a constrained minimization reformulation for a vector equilibrium problem. To the best of our knowledge, the construction of D-gap function for a vector equilibrium problem has not been attempted so far. Motivated by the above studies, we extend the notion of D-gap function for a $weakVEP$ and derive error bounds in terms of the D-gap function. Our results generalize the existing theory of gap functions and error bounds in [36, 33, 34, 4].

This paper is structured as follows. In Section 2, we propose a regularized version of the gap function for the $weakVEP$ based on scalarization and study some properties of the same. In Section 3, we construct D-gap functions for a $weakVEP$ which lead to an unconstrained optimization reformulation of $weakVEP$. Section 4 deals with the construction error bounds for $weakVEP$ in terms of the regularized gap function and the D-gap function. In section 5, we give some applications and numerical examples to support our results.

2 Regularized Gap Function

In this section, we propose a gap function for $weakVEP$ based on the scalarization link in [13]. Further, we introduce a regularized version of this gap function in the spirit of Fukushima [12]. For any $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, consider the following linear scalarization EP_ξ of the $weakVEP$:

$$EP_\xi : \quad \text{Find } x \in K \text{ such that } \langle \xi, f(x, y)\rangle \geq 0, \quad \forall y \in K. \tag{2.1}$$

For any $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $sol(EP_\xi)$ denote the set of solutions of EP_ξ. It is easy to check that any solution x^* of EP_ξ solves $weakVEP$. Under the assumption of convexity on each $f_i(x,.)$ for any $x \in K$, Lemma 2.1 in [13] establishes that, for any $x^* \in weakVEP$, there exists a $\xi^* \in \mathbb{R}^m_+ \setminus \{0\}$ such that x^* solves EP_{ξ^*}. We restate the following theorem from [13].

Theorem 2.1. *For any $x \in K$, assume that $f_i(x, \cdot)$ is convex for each i, $1 \leq i \leq m$. Then the following relation holds:*

$$sol(weakVEP) \;=\; \bigcup_{\xi \in \mathbb{R}^m_+ \setminus \{0\}} sol(EP_\xi).$$

Definition 2.2. Let $K \subseteq \mathbb{R}^n$. A function $p : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be a gap function for $weakVEP$ if it satisfies the following properties:
i) $p(x) \geq 0, \quad \forall x \in K$;
ii) $p(x^*) = 0$ if and only if x^* solves $weakVEP$.

In the sequel, we will always consider the following assumption, that will not be explicitly mentioned.

Assumption A. For every $x \in K$ and $1 \leq i \leq m$, $f_i(x,.)$ is a convex function on K.

Proposition 2.3. *If Assumption A holds, then the following function $\phi(x)$ is a gap function for $weakVEP$:*

$$\phi(x) := \min_{\xi \in S^m} \max_{y \in K} \langle \xi, -f(x,y) \rangle, \tag{2.2}$$

where S^m denotes the unit simplex in $\mathbb{R}^m_+$, i.e., $S^m := \{x \in \mathbb{R}^m_+ : \sum_{i=1}^m x_i = 1\}$.

Proof. Proof goes along the lines of the proof of Theorem 2.2 in [4]. □

Remark 2.4. The gap function $\phi(x)$ given by (2.2) is clearly an extension of the gap function proposed in [26] for a scalar EP. When $weakVEP$ represents the weak Stampacchia vector variational inequality $(SVVI)^w$, i.e., when $f_i(x,y) = \langle F_i(x), y - x \rangle$ where $F_i : \mathbb{R}^n \to \mathbb{R}^n$, we recover the following gap function θ introduced in [4] for $(SVVI)^w$:

$$\theta(x) := \min_{\xi \in S^m} \max_{y \in K} \left\langle \sum_{i=1}^m \xi_i F_i(x), x - y \right\rangle.$$

Note that the gap function θ is not finite valued in general, unless K is assumed to be compact. To overcome this, a suitable regularization of θ in line with Fukushima [12] is considered in [4]. We attempt such a regularization for the gap function ϕ in (2.2), by adding a function that is strongly concave in y, to the term $\langle \xi, -f(x,y) \rangle$ in ϕ so that the maximum is attained.

Consider a function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the following conditions B1) -B5):

B1) H is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$;

B2) $H(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$;

B3) $H(x,y) = 0$ if and only if $x = y$;

B4) $H(x,.)$ is strongly convex uniformly in x; i.e., there exists a $\lambda > 0$ such that, for any $x \in \mathbb{R}^n$,

$$H(x, y_1) - H(x, y_2) \geq \langle \nabla_y H(x, y_2), y_1 - y_2 \rangle + \lambda \|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n;$$

B5) $\nabla_y H(x,.)$ is uniformly Lipschitz continuous; i.e., there exists a constant $L \geq 0$ such that for any $x \in \mathbb{R}^n$

$$\|\nabla_y H(x, y_1) - \nabla_y H(x, y_2)\| \leq L\|y_1 - y_2\|, \quad \forall\, y_1, y_2 \in \mathbb{R}^n\,.$$

For example, for any $k > 0$, the function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined below satisfies B1) -B5):

$$H(x, y) = k\|x - y\|^2.$$

Since $H(x, y)|_{x=y} = 0$ and $H(x,.)$ is strongly convex, $y := x$ is a global minimum of the function $H(x,.)$ on K and hence the following holds true:

$$x = y \quad \Longleftrightarrow \quad \nabla_y H(x, y) = 0. \tag{2.3}$$

Now, for any $x, y \in \mathbb{R}^n$, using B3), B5) and (2.3)

$$\begin{aligned} H(x,y) = H(x,y) - H(x,x) &= \int_0^1 \langle \nabla_y H(x, x + t(y-x)), y - x \rangle dt \\ &= \int_0^1 \langle \nabla_y H(x, x + t(y-x)) - \nabla_y H(x,x), y - x \rangle dt \\ &\leq \int_0^1 L\|y - x\|^2 t dt, \quad \text{where } L \text{ is the constant in B5)} \end{aligned}$$

Hence, if H satisfies B1)-B5) then for any $x, y \in \mathbb{R}^n$

$$H(x, y) \leq \frac{L}{2}\|y - x\|^2 \quad \text{where } L \text{ is the Lipschitz constant in B5)} \tag{2.4}$$

We now define a regularized version ϕ_α of the gap function ϕ as

$$\phi_\alpha(x) := \min_{\xi \in S^m} \max_{y \in K} \{\langle \xi, -f(x, y) \rangle - \alpha H(x, y)\}, \; \alpha > 0, \tag{2.5}$$

where S^m denotes the unit simplex in $\mathbb{R}^m_+$. For a given $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let us denote

$$\psi_\alpha(x, \xi) := \max_{y \in K} [\langle \xi, -f(x, y) \rangle - \alpha H(x, y)]. \tag{2.6}$$

Equivalently,

$$\psi_\alpha(x, \xi) := -\min_{y \in K} [\langle \xi, f(x, y) \rangle + \alpha H(x, y)]. \tag{2.7}$$

Remark 2.5. The function ϕ_α generalizes the gap function for a scalar equilibrium problem proposed in [26] and the gap function constructed in [4] for a vector variational inequality. We observe that for any $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m_+ \setminus \{0\}$ the function $\gamma^\alpha_{(x,\xi)}(y) := \langle \xi, f(x, y) \rangle + \alpha H(x, y)$ is strongly convex on K so that the minimum in (2.7) exists and is unique.

We recall the following that will be used in the sequel.

Definition 2.6 ([7])**.** The generalized directional derivative of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, at x in the direction $d \in \mathbb{R}^n$ is defined as

$$f^{\circ}(x; d) = \overline{\lim_{y \to x, t \downarrow 0}} \frac{f(y + td) - f(y)}{t}. \tag{2.8}$$

Definition 2.7. A function $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strongly monotone on $K \subset \mathbb{R}^n$ with modulus $\mu > 0$ if

$$f(x, y) + f(y, x) \leq \; -\mu \|y - x\|^2, \quad \forall x, y \in K.$$

Definition 2.8 ([7])**.** A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be regular at $x \in \mathbb{R}^n$ if

1. for all $d \in \mathbb{R}^n$, the one sided directional derivative $f'(x; d)$ exists.
2. for all $d \in \mathbb{R}^n$, $f'(x; d) = f^{\circ}(x; d)$.

Lemma 2.9. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be continuous in (x, y) and let C be a compact set in $\mathbb{R}^m$. Then the function*

$$g(x) = \min_{y \in C} f(x, y)$$

is a continuous function in x.

Theorem 2.10 (Sion's minimax theorem, [17])**.** *Let $A \subset \mathbb{R}^n$ be compact and convex and Let $B \subset \mathbb{R}^m$ be convex. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfy the following conditions:*

(i) *$f(x, .)$ is upper-semicontinuous and concave on B for each $x \in A$.*

(ii) *$f(., y)$ is lower-semicontinuous and convex on A for each $y \in B$.*

Then

$$\min_{x \in A} \sup_{y \in B} f(x, y) = \sup_{y \in B} \min_{x \in A} f(x, y).$$

Theorem 2.11. *For any $\alpha > 0$ function ϕ_α is finite valued. If $f_i, i = 1, ..., m$ is continuous in (x, y), then ϕ_α is a continuous function.*

Proof. Let $x \in \mathbb{R}^n$. Since $f_i(x, y), i = 1, ..., m$ is convex in y, for a given $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m_+ \setminus \{0\}$ the function $y \mapsto \langle \xi, f(x, y) \rangle$ is convex on K. Since $H(x, .)$ is strongly convex, the function

$$\gamma^{\alpha}_{(x,\xi)}(y) := \langle \xi, f(x, y) \rangle + \alpha H(x, y)$$

is strongly convex on K. Hence $\psi_\alpha(x, \xi)$ given by (2.7) is finite valued. Let us denote the unique minimum in (2.7) by $y_\alpha(x, \xi)$, that is,

$$y_\alpha(x, \xi) := argmin_{y \in K} [\langle \xi, f(x, y) \rangle + \alpha H(x, y)] \tag{2.9}$$

which is the unique maximum in (2.6). Further, we note that $\psi_\alpha(x, \cdot)$ is lower semi continuous being maximum of a family of linear functions. Therefore, using compactness of S^m we can conclude that the function $\phi_\alpha(x) = \min_{\xi \in S^m} \; \psi_\alpha(x, \xi)$ is finite valued.

Let us now assume that $f_i, i = 1, ..., m$ is continuous in (x, y) and let $y_\alpha(x, \xi)$ be given by (2.9). Applying Theorem 4.3.3 of [2], we obtain that the map $y_\alpha : \mathbb{R}^n \times S^m \to K$ is upper semi continuous on $\mathbb{R}^n \times S^m$ according to Berge. Being single-valued, y_α is continuous on $\mathbb{R}^n \times S^m$. Now, $\psi_\alpha(x, \xi)$ in (2.6) can be written as $\psi_\alpha(x, \xi) := \langle \xi, -f(x, y_\alpha(x, \xi)) \rangle - \alpha H(x, y_\alpha(x, \xi))$. Since each f_i and H are continuous on $\mathbb{R}^n \times R^n$, $\psi_\alpha(x, \xi)$ is continuous on $\mathbb{R}^n \times S^m$. Since S^m is compact, continuity of ϕ_α follows from Lemma 2.9. $\square$

The proof of the following theorem goes along the lines of the proof of Theorem 2.7 in [4]. We provide the proof for the sake of completeness.

Theorem 2.12. *Let $\alpha > 0$ and let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B4). Then $\phi_\alpha(x) \geq 0$ for all $x \in K$. Further, $\phi_\alpha(\bar{x}) = 0,\ \bar{x} \in K$ if and only if $\bar{x}$ solves weakVEP.*

Proof. For a fixed $(x, \xi) \in K \times S^m$, consider $\psi_\alpha(x, \xi)$ defined by (2.6). Since $f(x, x) = H(x, x) = 0$, we have $\psi_\alpha(x, \xi) \geq 0, \forall (x, \xi) \in K \times S^m$. This implies $\phi_\alpha(x) \geq 0, \forall x \in K$. Assume that there exists an $\bar{x} \in K$ such that $\phi_\alpha(\bar{x}) = 0$. Then there exists a $\xi' \in S^m$ such that

$$0 = \psi_\alpha(\bar{x}, \xi') = \max_{y \in K} [\langle \xi', -f(\bar{x}, y)\rangle - \alpha H(\bar{x}, y)] = -\min_{y \in K} \gamma^\alpha_{(\bar{x}, \xi')}(y), \tag{2.10}$$

where $\gamma^\alpha_{(\bar{x}, \xi')}(y) = \langle \xi', f(\bar{x}, y)\rangle + \alpha H(\bar{x}, y)$. Since $\gamma^\alpha_{(\bar{x}, \xi')}(\bar{x}) = 0$, $\bar{x}$ is a global minimum of the strongly convex map $y \mapsto \gamma^\alpha_{(\bar{x}, \xi')}(y)$ over the convex set K. It is well-known [30] that this is equivalent to

$$-\nabla_y \gamma^\alpha_{(\bar{x}, \xi')}(\bar{x}) \in N_K(\bar{x}), \tag{2.11}$$

where $N_K(\bar{x})$ is the normal cone to the set K at $\bar{x}$. Hence we have

$$\left\langle \sum_{i=1}^m \xi'_i \nabla_y f_i(\bar{x}, \bar{x}) + \alpha \nabla_y H(\bar{x}, \bar{x}), y - \bar{x} \right\rangle \geq 0, \quad \forall y \in K. \tag{2.12}$$

Since (2.3) holds for H, (2.12) collapses to the following:

$$\left\langle \sum_{i=1}^m \xi'_i \nabla_y f_i(\bar{x}, \bar{x}), y - \bar{x} \right\rangle \geq 0, \quad \forall y \in K.$$

Denoting $f_\xi(x, y) = \langle \xi, f(x, y)\rangle$, the above inequality means $-\nabla_y f_{\xi'}(\bar{x}, y)|_{y=\bar{x}} \in N_K(\bar{x})$, which is equivalent to saying that $\bar{x}$ is a global minimum of the function $f_{\xi'}(\bar{x}, .)$ on K, i.e.,

$$\langle \xi', f(\bar{x}, y)\rangle \geq 0, \quad \forall y \in K.$$

This implies $\bar{x}$ solves $EP_{\xi'}$. Applying Theorem 2.1, we obtain that $\bar{x}$ solves $weakVEP$.

Conversely, assume that $\bar{x}$ is a solution to $weakVEP$. By Theorem 2.1, there exists a $\bar{\xi} \in \mathbb{R}^m_+ \setminus \{0\}$ such that

$$\langle \bar{\xi}, f(\bar{x}, y\rangle \geq 0, \quad \forall y \in K. \tag{2.13}$$

Setting $\xi' := \bar{\xi} / \sum_{i=1}^m \bar{\xi}_i$, we have $\xi' \in S^m$ for which (2.13) holds. Since $f(\bar{x}, \cdot)$ is convex, taking $\nabla_y H(\bar{x}, \bar{x}) = 0$ into account, it is clear that (2.13) implies (2.12) for any $\alpha > 0$. Using the equivalence of (2.12), (2.11) and (2.10), we obtain $\psi_\alpha(\bar{x}, \xi') = 0$. Since $\psi_\alpha(\bar{x}, \xi) \geq 0$, $\forall \xi \in S^m$, we conclude that

$$\phi_\alpha(\bar{x}) = \min_{\xi \in S^m} \psi_\alpha(\bar{x}, \xi) = 0.$$

Hence the proof. □

Remark 2.13. The above theorem shows that ϕ_α is a gap function for $weakVEP$ and hence $weakVEP$ is equivalent to the constrained minimization problem $\min_{x \in K} \phi_\alpha(x)$. Though we do not need to use gap functions for solving $weakVEP$, they help in devising error bounds for $weakVEP$ which are crucial in measuring how close is a feasible point from the solution set of the problem. In view of this, one would look for gap functions that are well-behaved, in the sense that the value of the gap function should decrease as we approach the solution set. This is achieved if the gap function is continuous. However, we saw in Theorem 2.11 that ϕ_α is continuous only if each f_i is continuous in (x, y). The following result states a property of the gap function ϕ_α under a weakened assumption on f_i, that is, when f_i's are assumed to be upper-semi continuous in (x, y).

Theorem 2.14. *Let $f_i(x, y)$ be upper semi-continuous in (x, y), for each $i, 1 \leq i \leq m$. Let $x^* \in \mathbb{R}^n$ and let $(x_n) \longrightarrow x^*$. Then, there exist a sub-sequence (x_{n_j}) of (x_n) such that*

$$\liminf_{j \to \infty} \phi_\alpha(x_{n_j}) \geq \phi_\alpha(x^*).$$

Proof. Let us denote

$$g_\alpha(\xi, x, y) = -\sum_{i=1}^{m} \xi_i f_i(x, y) - \alpha H(x, y).$$

Then $\psi_\alpha(x, \xi)$ in (2.6) is given by

$$\psi_\alpha(x, \xi) = \max_{y \in K} g_\alpha(\xi, x, y).$$

Since each f_i is upper semi continuous in (x, y) and H is continuous in (x, y), $g_\alpha(\xi, x, y)$ is lower semi continuous in (ξ, x, y). This implies that $\psi_\alpha(x, \xi)$ is lower semi continuous in (x, y) being supremum of a family of lower semi continuous functions [using Theorem 7.1, [30]]. We have

$$\phi_\alpha(x) = \inf_{\xi \in S^m} \psi_\alpha(x, \xi).$$

For any fixed x, note that $\psi_\alpha(x, \xi) \geq 0$ for all $\xi \in S^m$ and is finite valued. Hence $\psi_\alpha(x, \xi)$ is a proper lower semi-continuous function. Since S^m is compact, for each $n \in \mathbb{N}$, there exists a $\xi_n \in S^m$ such that

$$\phi_\alpha(x_n) = \psi_\alpha(x_n, \xi_n),$$

and there exists a sub-sequence (ξ_{n_j}) of (ξ_n) such that $(\xi_{n_j}) \longrightarrow \xi^*$ as $j \longrightarrow \infty$. Hence

$$\liminf_{j \longrightarrow \infty} \phi_\alpha(x_{n_j}) = \liminf_{j \longrightarrow \infty} \psi_\alpha(x_{n_j}, \xi_{n_j})$$

Since $\psi_\alpha(x, \xi)$ is lower semi-continuous in (x, ξ),

$$\phi_\alpha(x^*) \leq \psi_\alpha(x^*, \xi^*) \leq \liminf_{j \longrightarrow \infty} \psi_\alpha(x_{n_j}, \xi_{n_j}),$$

which implies

$$\phi_\alpha(x^*) \leq \liminf_{j \longrightarrow \infty} \phi_\alpha(x_{n_j}).$$

□

Proposition 2.15. *The gap function ϕ_α can be equivalently expressed as*

$$\phi_\alpha(x) = \max_{y \in K} \left\{ \min_{1 \leq i \leq m} -f_i(x, y) - \alpha H(x, y) \right\} \tag{2.14}$$

Further, ϕ_α is a lower-semicontinuous function.

Proof. The function ϕ_α in (2.5) is given by

$$\phi_\alpha(x) = \min_{\xi \in S^m} \max_{y \in K} \{\langle \xi, -f(x,y)\rangle - \alpha H(x,y)\}, \ \alpha > 0.$$

Let $x \in \mathbb{R}^n$. Note that for each i, the function $-f_i(x,y)$ is concave in y and is finite valued. Hence $-f_i(x,y)$ is continuous in y. Since H is continuous, for any $\xi \in \mathbb{R}^m$, the map $y \mapsto -\langle \xi, f(x,y)\rangle - \alpha H(x,y)$ is concave and continuous in y.

On the other hand, for each fixed $y \in K$, the map $\xi \mapsto -\langle \xi, f(x,y)\rangle - \alpha H(x,y)$ is linear and continuous in ξ. Hence using Theorem 2.10 we conclude that

$$\phi_\alpha(x) = \max_{y \in K} \min_{\xi \in S^m} \{\langle \xi, -f(x,y)\rangle - \alpha H(x,y)\} = \max_{y \in K} \{ \min_{\xi \in S^m} \langle \xi, -f(x,y)\rangle - \alpha H(x,y)\}.$$

Now $S^m = Co\{e_1, e_2, .., e_m\}$ where each e_i is a unit vector in $\mathbb{R}^m$ with 1 in the i'th place. Hence we have

$$\min_{\xi \in S^m} \sum_{i=1}^m \xi_i f_i(x,y) = \min_{\xi \in \{e_1, e_2, .., e_m\}} \sum_{i=1}^m \xi_i f_i(x,y) = \min_{1 \le i \le m} f_i(x,y).$$

Hence

$$\phi_\alpha(x) = \max_{y \in K} \{ \min_{1 \le i \le m} -f_i(x,y) - \alpha H(x,y)\}$$

Since each $y \mapsto f_i(x,.)$ is continuous, the function $y \mapsto \min_{1 \le i \le m} -f_i(x,y)$ is continuous. Using continuity of H, the function

$$v_\alpha(x,y) = \left\{ \min_{1 \le i \le m} -f_i(x,y) \right\} - \alpha H(x,y)$$

is continuous in y. Hence $\phi_\alpha(x) = \max_{y \in K} v_\alpha(x,y)$ given by (2.14) is lower-semi continuous, being maximum of a family of continuous functions. □

Though we started with scalarization approach to define ϕ_α, in view of the construction of the D-gap function, we observe that the alternative expression of ϕ_α in the Proposition 2.15 is advantageous for computation, as it doesn't involve a scalarization parameter. In fact we will use this form to plot the graph of ϕ_α for some sample problems in the later sections. The other form of ϕ_α (with scalarization parameter) is useful in the study of error bounds involving the D-gap function. We notice that both the forms of ϕ_α have an edge in the study of some important properties.

2.1 A special case: the regularized gap function for VVI

In this section we consider a special case of $weak(VEP)$ called the weak *Stampacchia* vector variational inequality $(SVVI)^w$ which consists of finding an $x \in K$ such that

$$(\langle F_1(x), y - x\rangle, \langle F_2(x), y - x\rangle, ..., \langle F_m(x), y - x\rangle) \notin -int\, \mathbb{R}^m_+, \quad \forall y \in K,$$

where $F_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, ..., m$. This is an instance of $weakVEP$ with $f_i(x,y) = \langle F_i(x), y - x\rangle$ for each $i = 1, ..., m$. We analyse some properties of regularized gap function ϕ_α for this case under the necessary assumptions on the functions involved. We consider the following assumption B6) on H:

B6) H is locally Lipschitz in x, uniformly for y in K; that is, for each $x \in \mathbb{R}^n$, there exists a $\delta_x > 0$ and $G_x > 0$ such that, $H(.,y)$ is Lipschitz on $B(x,\delta_x)$ with Lipschitz constant G_x, for all $y \in K$.

The following result is a restated version of Proposition 2.3 [28] for the case of the scalar variational inequality problem $VI(F_\xi, K)$ where $\xi \in S^m$.

Theorem 2.16. *Let $\alpha > 0$ and let $\xi \in S^m$. Then, $\|y_\alpha(x,\xi) - y_\alpha(x',\xi)\| \leq \frac{1}{2\lambda} L\|x - x'\| + \frac{1}{2\lambda\alpha}\|F_\xi(x) - F_\xi(x')\|$ for all $x, x' \in \mathbb{R}^n$, where λ and L are the constants in $B4$) and $B5$) respectively.*

Theorem 2.17. *Let F_i, $i = 1, ..., m$ be continuously differentiable on $\mathbb{R}^n$ and let H satisfy the assumption $B6$). Then ϕ_α is locally Lipschitz on $\mathbb{R}^n$.*

Proof. Let $\xi \in S^m$ and $x \in \mathbb{R}^n$ be fixed. Since each F_i is continuously differentiable on $\mathbb{R}^n$, F_i locally Lipschitz on $\mathbb{R}^n$(Corollary to Proposition 2.2.1 in [7]). It is easy to see that $F_\xi = \sum_{i=1}^m \xi_i F_i$ is locally Lipschitz $\mathbb{R}^n$. That is, there exists a $\delta_x^\xi > 0$ such that F_ξ is Lipschitz on $B(x,\delta_x^\xi)$. Let L_x^ξ denote the Lipschitz constant of F_ξ on $B(x,\delta_x^\xi)$.

Now, for a fixed $y \in K$ and $x_1, x_2 \in B(x,\delta_x^\xi)$,

$$\begin{aligned}
&|\langle F_\xi(x_1), x_1 - y\rangle - \langle F_\xi(x_2), x_2 - y\rangle| \\
&\quad = |\langle F_\xi(x_1), x_1\rangle + \langle F_\xi(x_1), x_2\rangle - \langle F_\xi(x_1), x_2\rangle - \langle F_\xi(x_2), x_2\rangle + \langle F_\xi(x_2) - F_\xi(x_1), y\rangle| \\
&\quad \leq \|F_\xi(x_1)\|\|x_1 - x_2\| + \|F_\xi(x_1) - F_\xi(x_2)\|\|x_2\| + \|F_\xi(x_2) - F_\xi(x_1)\|\|y\| \\
&\quad \leq \|F_\xi(x_1)\|\|x_1 - x_2\| + L_x^\xi\|x_1 - x_2\|\|x_2\| + L_x^\xi\|x_1 - x_2\|\|y\|
\end{aligned}$$

Let C be a compact set in $\mathbb{R}^n$ such that $B(x,\delta_x^\xi) \subset C$. Since F_ξ is continuous, there exist constants $M > 0$ and $N > 0$ such that $\|F_\xi(x_1)\| \leq M$ and $\|x_1\|, \|x_2\| \leq N$. Hence

$$|\langle F_\xi(x_1), x_1 - y\rangle - \langle F_\xi(x_2), x_2 - y\rangle| \leq (M + NL_x^\xi + L_x^\xi\|y\|)\|x_1 - x_2\| \tag{2.15}$$

Hence the map $x \mapsto \langle F_\xi(x), x - y\rangle$ is Lipschitz on $B(x,\delta_x^\xi)$.

Since H is locally Lipschitz in x, uniformly for y in K, there exists a $\delta_x > 0$ such that for all $y \in K$, $H(.,y)$ is Lipschitz on $B(x,\delta_x)$ (say with a Lipschitz constant $G_x > 0$). Setting $\delta_\xi^*(x) = \min\{\delta_x^\xi, \delta_x\}$, we conclude from (2.15) that the function

$$f_{\alpha,\xi}(x,y) = \langle F_\xi(x), x - y\rangle - \alpha H(x,y) \tag{2.16}$$

is Lipschitz on $B(x,\delta_\xi^*(x))$ with Lipschitz constant $M + NL_x^\xi + L_x^\xi\|y\| + \alpha G_x$.

Now

$$\psi_\alpha(x,\xi) = \max_{y\in K}\langle F_\xi(x), x - y\rangle - \alpha H(x,y) = \max_{y\in K} f_{\alpha,\xi}(x,y) \tag{2.17}$$

Since the maximum in $\psi_\alpha(x,\xi)$ is attained at $y_\alpha(x,\xi)$,

$$\psi_\alpha(x,\xi) = f_{\alpha,\xi}(x, y_\alpha(x,\xi)) = \langle F_\xi(x), x - y_\alpha(x,\xi)\rangle - \alpha H(x, y_\alpha(x,\xi))$$

To show that $x \mapsto \psi_\alpha(x,\xi)$ is locally Lipschitz, we need to prove the following claim.

Claim: The map $x \mapsto f_{\alpha,\xi}(x, y_\alpha(x,\xi))$ is locally Lipschitz on $\mathbb{R}^n$.

The above claim can be established if we prove that, for any $z \in B(x, \delta_\xi^*(x))$, the map $x \mapsto f_{\alpha,\xi}(x, y_\alpha(z,\xi))$ is Lipschitz on $B(x, \delta_\xi^*(x))$.

Let $z \in B(x, \delta_\xi^*(x))$. Clearly, $y_\alpha(z,\xi) \in K$. Since $z \in B(x, \delta_\xi^*(x))$, using Theorem 2.16 for the points $z, x \in \mathbb{R}^n$, there exists a constant $D_x^\xi > 0$ such that $\|y_\alpha(z,\xi)\| \leq D_x^\xi$. Noting that $f_{\alpha,\xi}$ is given by (2.16), from the discussion above, the function $x \mapsto f_{\alpha,\xi}(x, y_\alpha(z,\xi))$ is Lipschitz on $B(x, \delta_\xi^*(x))$ with Lipschitz constant $M + NL_x^\xi + L_x^\xi D_x^\xi + \alpha G_x$ for any $z \in B(x, \delta_\xi^*(x))$ (refer to the Lipschitz constant $M + NL_x^\xi + L_x^\xi \|y_\alpha(z,\xi)\| + \alpha G_x$ below the equation (2.16)).

Since $x \in B(x, \delta_\xi^*(x))$, we thus conclude that the map $x \mapsto f_{\alpha,\xi}(x, y_\alpha(x,\xi))$, that is, the map $x \mapsto \psi_\alpha(x,\xi)$ is Lipschitz on $B(x, \delta_x^*(x))$ with Lipschitz constant $M + NL_x^\xi + L_x^\xi D_x^\xi + \alpha G_x$.

Now, the function $\phi_\alpha(x)$ given by

$$\phi_\alpha(x) = \min_{\xi \in S^m} \psi_\alpha(x,\xi)$$

is finite valued, it is Lipschitz on $B(x, \delta_x^*(x))$. Therefore ϕ_α is locally Lipschitz. $\square$

Our next step is to find an expression for the generalized sub-differential of the function ϕ_α. We need the following Lemma.

Lemma 2.18. *For a fixed $\alpha > 0$ and $\xi \in \mathbb{R}_+^m$ let $f_{\alpha,\xi}(.,y)$ be given by (2.17). If F_i continuously differentiable for each $i \in \{1, ..., m\}$ then $f_{\alpha,\xi}(.,y)$ defined by (2.16) is regular. That is,*

$$f^\circ_{\alpha,\xi}(x, y; .) = f'_{\alpha,\xi}(x, y; .),$$

where f° is the generalized derivative given by (2.8) and f' is the one-sided directional derivative, the derivatives being with respect to x.

Proof. Let $d \in \mathbb{R}^n$. Then

$$f^\circ_{\alpha,\xi}(x, y; d) = \overline{\lim_{y' \to x, t \downarrow 0}} \frac{f_{\alpha,\xi}(y' + td, y) - f_{\alpha,\xi}(y', y)}{t} \tag{2.18}$$

$$\begin{aligned}
&\frac{f_{\alpha,\xi}(y' + td, y) - f_{\alpha,\xi}(y', y)}{t} \\
= \quad & \frac{\langle F_\xi(y' + td), y' + td - y\rangle - \alpha H(y' + td, y) - \langle F_\xi(y'), y' - y\rangle + \alpha H(y', y)}{t} \\
= \quad & \frac{\langle F_\xi(y' + td) - F_\xi(y'), y' - y\rangle}{t} + \frac{\langle F_\xi(y' + td), td\rangle}{t} - \frac{\alpha[H(y' + td, y) - H(y', y)]}{t}
\end{aligned}$$

Since limits exist for the terms in the above expression, $f^\circ_{\alpha,\xi}(x, y; d)$ in (2.18) is given by

$$\overline{\lim_{y' \to x, t \downarrow 0}} \frac{\langle F_\xi(y' + td) - F_\xi(y'), y' - y\rangle}{t} + \overline{\lim_{y' \to x, t \downarrow 0}} \frac{\langle F_\xi(y' + td), td\rangle}{t} \\
- \overline{\lim_{y' \to x, t \downarrow 0}} \frac{\alpha[H(y' + td, y) - H(y', y)]}{t}$$

Plugging in $F_\xi = \sum_{i=1}^m \xi_i F_i$,

$$\overline{\lim_{y'\to x, t\downarrow 0}} \frac{\langle F_\xi(y'+td) - F_\xi(y'), y'-y\rangle}{t} = \langle \sum_{i=1}^m \xi_i F_i'(x;d), x-y\rangle;$$

$$\overline{\lim_{y'\to x, t\downarrow 0}} \frac{\langle F_\xi(y'+td), td\rangle}{t} = \langle F_\xi(x), d\rangle$$

and

$$\overline{\lim_{y'\to x, t\downarrow 0}} \frac{\alpha[H(y'+td, y) - H(y', y)]}{t} = H'(x, y; d).$$

Hence

$$f^\circ_{\alpha,\xi}(x, y; d) = \left\langle \sum_{i=1}^m \xi_i F_i'(x;d), x-y \right\rangle + \langle F_\xi(x), d\rangle + H'(x, y; d) = f'_{\alpha,\xi}(x, y; d). \tag{2.19}$$

Hence the proof.

□

Theorem 2.19. *Let $x \in \mathbb{R}^n$. Assume that each F_i be continuously differentiable on $\mathbb{R}^n$. If each F_i is regular at $x \in \mathbb{R}^n$, then the Clarke's sub-differential of ϕ_α is given by*

$$\partial^\circ \phi_\alpha(x) = Co\{F_\xi(x) + \langle \nabla F_\xi(x), x - y_\alpha(x,\xi)\rangle - \alpha(x - y_\alpha(x,\xi)) : \xi \in \Omega(x)\},$$

where $\Omega(x) = \{\xi \in S^m : \phi_\alpha(x) = \psi_\alpha(x,\xi)\}$.

Proof. Since each F_i is continuously differentiable, it follows from Theorem 3.2 in [33] that $\psi_\alpha(.,\xi)$ is continuously differentiable and the gradient of $\psi_\alpha(.,\xi)$ is given by

$$\nabla_x \psi_\alpha(x,\xi) = F_\xi(x) + \langle \nabla F_\xi(x), x - y_\alpha(x,\xi)\rangle - \alpha(x - y_\alpha(x,\xi)). \tag{2.20}$$

In view of Theorem 4.2 in [14], we will need the following conditions to hold for ψ_α:

(i) The map $\xi \mapsto \psi_\alpha(x,\xi)$ is continuous.

(ii) The map $(\xi, x) \mapsto -\nabla\psi_\alpha(x,\xi)$ is upper semi continuous at (ξ, x) for all $\xi \in \Omega(x)$.

(iii) The function $-\psi_\alpha(.,\xi)$ is regular at x.

Since each F_i is continuous, it follows from Theorem 2.11 that the the map $(x,\xi) \mapsto \psi_\alpha(x,\xi)$ is continuous. Hence (i) holds true.

Let the sequence (ξ_n, x_n) converge to (ξ, x). Using (2.20), for any $n \in \mathbb{N}$

$$\nabla_x \psi_\alpha(x_n,\xi_n) = F_{\xi_n}(x_n) + \langle \nabla F_{\xi_n}(x_n), x_n - y_\alpha(x_n,\xi_n)\rangle - \alpha(x_n - y_\alpha(x_n,\xi_n)). \tag{2.21}$$

From Lemma 2.4 in [4], the map $(\xi, x) \mapsto y_\alpha(x,\xi)$ is continuous. It follows from the proof of Theorem 2.11 that the map $(\xi, x) \mapsto \psi_\alpha(x,\xi)$ is continuous. Further, since each F_i is continuous, $\nabla_x\psi_\alpha(x_n,\xi_n)$ converges to $\nabla_x\psi_\alpha(x,\xi)$. Hence (ii) holds.

It remains to show that the function $-\psi_\alpha(x,\xi)$ is regular at x. From (2.17)

$$\psi_\alpha(x,\xi) = \max_{y\in K} \langle F_\xi(x), x-y\rangle - \alpha H(x,y) = \max_{y\in K} f_{\alpha,\xi}(x,y)$$

where $f_{\alpha,\xi}(x,y) = \langle F_\xi(x), x-y\rangle - \alpha H(x,y)$. Observe that the mapping $y \mapsto f_{\alpha,\xi}(x,y)$ is continuous. Further, the map $(x,y) \mapsto \nabla_x g_{\alpha,\xi}(x,y)$ is continuous at x. Since F_i is regular, from Lemma 2.18, the function $f_{\alpha,\xi}(.,y)$ is also regular. Theorem 4.1 in [14] implies that the function $\psi_\alpha(.,\xi)$ is regular at x. Hence (iii) is verified.

Therefore using Theorem 4.2 in [14],

$$\partial\phi_\alpha(x) = Co\{\nabla\psi_\alpha(x,\xi)\} : \xi \in \Omega(x)\}.$$

□

Remark 2.20. With reference to the study on the regularized gap function ϕ_α in [4], the results obtained in this section deepen the analysis on properties of ϕ_α, such as, lower semi-continuity, Lipschitz continuity of ϕ_α and an expression for the Clarke's sub-differential of ϕ_α. We further note that there have been studies on 'minimizing sequences' and 'stationary sequences' of gap functions for nonlinear complementarity problems (NCP) and scalar variational inequalities (VI) which involve gradients of the gap functions. The Clarke's sub-differential of ϕ_α in Theorem 2.19 can be an important tool to investigate the connection between minimizing sequences and stationary sequences of ϕ_α at least for the case of a vector variational inequality involving functions that are continuously differentiable or Lipschitz.

3 D-Gap Functions

In this section, we construct D-gap functions for $weakVEP$ along the lines of the constructions in [4, 33], using the structure $\psi_\alpha(x,\xi)$ that appears in the regularized gap function ϕ_α. These D-gap functions give an unconstrained minimization reformulation for $weakVEP$ and also provide a global error bounds for the distance between any given point and the set $sol(weakVEP)$.

Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy the conditions B1)-B4). We define the D-gap function $\phi_{\alpha\beta}$ as

$$\phi_{\alpha\beta}(x) := \min_{\xi\in S^m}\{\psi_\alpha(x,\xi) - \psi_\beta(x,\xi)\}, 0 < \alpha < \beta, \tag{3.1}$$

where

$$\psi_\alpha(x,\xi) := \max_{y\in K}[\langle\xi, -f(x,y)\rangle - \alpha H(x,y)].$$

We recall that for any $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m_+ \setminus \{0\}$, the maximization problem $\max_{y\in K}[\langle\xi, -f(x,y)\rangle - \alpha H(x,y)]$ has a unique maximizer denoted by $y_\alpha(x,\xi)$. The following result is the analogue of Theorem 2.11 for $\phi_{\alpha\beta}$.

Theorem 3.1. *The function $\phi_{\alpha\beta}$ is a finite valued function. If each f_i is continuous in (x,y), then $\phi_{\alpha\beta}$ is continuous on $\mathbb{R}^n$.*

Proof. For a given $(x,\xi) \in \mathbb{R}^n \times S^m$ and $\alpha > 0$, we know that the function $\psi_\alpha(x,\xi)$ is finite valued (proof of Theorem 2.11). Hence for any $\beta > \alpha$, the difference function $\psi_\alpha(x,\xi) - \psi_\beta(x,\xi)$ is finite valued. Thus $\phi_{\alpha\beta}$ is finite valued.
The proof of continuity of $\phi_{\alpha\beta}$ is similar to the proof of Theorem 2.11, working with the difference function $\psi_\alpha(x,\xi) - \psi_\beta(x,\xi)$ instead of $\psi_\alpha(x,\xi)$. □

Theorem 3.2. *Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B4). For a fixed $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $y_\alpha(x,\xi)$ be the unique maximizer for the maximization problem in (2.6). Then $x = y_\alpha(x,\xi)$ if and only if x solves EP_ξ.*

Proof. Fix $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ and assume that x solves EP_ξ. The necessary and sufficient condition for $y_\alpha(x,\xi)$ to solve the concave maximization problem (2.6) is

$$\left\langle -\sum_{i=1}^{m} \xi_i \nabla_y f_i(x, y_\alpha(x,\xi)) - \alpha \nabla_y H(x, y_\alpha(x,\xi)),\ z - y_\alpha(x,\xi)) \right\rangle \leq 0, \quad \forall z \in K. \tag{3.2}$$

In particular for $z = x$

$$\alpha \left\langle \nabla_y H(x,\ y_\alpha(x,\xi)),\ x - y_\alpha(x,\xi) \right\rangle \geq - \left\langle \sum_{i=1}^{m} \xi_i \nabla_y f_i(x, y_\alpha(x,\xi)),\ x - y_\alpha(x,\xi) \right\rangle. \tag{3.3}$$

Since $f_i(x,.)$ is convex for each i,

$$\langle \nabla_y f_i(x, y_\alpha(x,\xi)),\ x - y_\alpha(x,\xi) \rangle \leq f_i(x,x) - f_i(x, y_\alpha(x,\xi)).$$

This implies

$$- \left\langle \sum_{i=1}^{m} \xi_i \nabla_y f_i(x, y_\alpha(x,\xi)),\ x - y_\alpha(x,\xi) \right\rangle \geq \sum_{i=1}^{m} \xi_i f_i(x, y_\alpha(x,\xi)) - \sum_{i=1}^{m} \xi_i f_i(x,x).$$

Hence from (3.3)

$$\alpha \left\langle \nabla_y H(x,\ y_\alpha(x,\xi)), x - y_\alpha(x,\xi) \right\rangle \geq \sum_{i=1}^{m} \xi_i f_i(x, y_\alpha(x,\xi)) - \sum_{i=1}^{m} \xi_i f_i(x,x).$$

Since $f(x,x) = 0$ and x solves EP_ξ, we get

$$\langle \nabla_y H(x, y_\alpha(x,\xi)), x - y_\alpha(x,\xi) \rangle \geq 0. \tag{3.4}$$

On the other hand, $B1$) and $B4$) imply

$$\langle \nabla_y H(x, y_\alpha(x,\xi)),\ x - y_\alpha(x,\xi) \rangle + \lambda \|x - y_\alpha(x,\xi)\|^2 \leq H(x,x) - H(x, y_\alpha(x,\xi)) \leq 0. \tag{3.5}$$

From (3.4) and (3.5)

$$\|x - y_\alpha(x,\xi)\|^2 \leq 0.$$

Hence $x = y_\alpha(x,\xi)$.

Conversely, let us assume that $x = y_\alpha(x,\xi)$. In view of (2.3), the inequality (3.2) reduces to

$$- \left\langle \sum_{i=1}^{m} \xi_i \nabla_y f_i(x,x), z - x \right\rangle \leq 0, \quad \forall z \in K.$$

Using the convexity of $f_i(x,.)$, the above inequality gives

$$\sum_{i=1}^{m} \xi_i f_i(x,x) - \sum_{i=1}^{m} \xi_i f_i(x,z) \leq 0, \quad \forall z \in K.$$

This implies

$$\sum_{i=1}^{m} \xi_i f_i(x,z) \geq 0, \quad \forall z \in K.$$

Hence the proof. $\square$

Corollary 3.3. *For a fixed $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $y_\alpha(x,\xi)$ be the unique maximizer in (2.6). Then x solves weakVEP if and only if there exists a $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ such that $x = y_\alpha(x,\xi)$.*

The following proposition can be derived from Proposition 3.1 in [18].

Proposition 3.4. *Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B4). For any $x \in \mathbb{R}^n$ and $\xi \in S^m$*

$$(\beta - \alpha)H(x, y_\beta(x,\xi)) \le \psi_\alpha(x,\xi) - \psi_\beta(x,\xi) \le (\beta - \alpha)H(x, y_\alpha(x,\xi)),$$

where $y_\alpha(x,\xi)$ is given by (2.9) and $0 < \alpha < \beta$.

The proof of the following theorem is a direct implication of Proposition 3.4. We however provide proof for the sake of completeness.

Theorem 3.5. *Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B4). Then $\phi_{\alpha\beta}$ is non-negative on $\mathbb{R}^n$. Further, x^* solves the weakVEP if and only if $\phi_{\alpha\beta}(x^*) = 0$.*

Proof. For a given $x \in \mathbb{R}^n$, $\xi \in S^m$ and $\beta > \alpha$, it follows from Proposition 3.4 that

$$\psi_\alpha(x,\xi) - \psi_\beta(x,\xi) \ge 0.$$

Since $x \in \mathbb{R}^n$ is arbitrary, we conclude that $\phi_{\alpha\beta}(x) \ge 0$ for all $x \in \mathbb{R}^n$.

Let us assume that x^* is a solution of $weakVEP$. Then by Theorem 2.1, there exists a ξ^* in $\mathbb{R}^m_+ \setminus \{0\}$ and hence in S^m (by normalizing) such that x^* solves EP_{ξ^*}. Hence from Theorem 3.2, $x^* = y_\alpha(x^*, \xi^*)$. From B3) and from Proposition 3.4

$$\phi_{\alpha\beta}(x^*) \le (\beta - \alpha)H(x^*, y_\alpha(x^*, \xi^*)) = 0.$$

Therefore, $\phi_{\alpha\beta}(x^*) = 0$.

Conversely, assume that $\phi_{\alpha\beta}(x^*) = 0$. Since S^m is compact, there exists a $\xi^* \in S^m$ such that $\psi_\alpha(x^*, \xi^*) - \psi_\beta(x^*, \xi^*) = 0$. Using Proposition 3.4,

$$(\beta - \alpha)H(x^*, y_\beta(x^*, \xi^*)) \le 0.$$

Since $\beta > \alpha$, we have $H(x^*, y_\beta(x^*, \xi^*)) = 0$ which implies $x^* = y_\beta(x^*, \xi^*)$. Again, using Theorem 2.1, x^* solves EP_{ξ^*} and hence solves $weakVEP$. □

3.1 D-gap function without scalarization

In view of the other gap functions available for $weakVEP$ which are free of scalarization, it is interesting to ask whether one can construct the D-gap function for $weakVEP$ without scalarization parameter. Going along the lines of the technique in [33] for any $0 < \alpha < \beta$, we define the function $\zeta_{\alpha\beta}(x) : \mathbb{R}^n \mapsto \mathbb{R}$ as :

$$\zeta_{\alpha\beta}(x) = h_\alpha(x) - h_\beta(x) \tag{3.6}$$

where h_α and h_β are given by

$$h_\alpha(x) = \max_{y \in K} \left\{ \left(\min_{1 \le i \le m} -f_i(x,y) \right) - \alpha H(x,y) \right\}; h_\beta(x) = \max_{y \in K} \left\{ \left(\min_{1 \le i \le m} -f_i(x,y) \right) - \beta H(x,y) \right\} \tag{3.7}$$

and H satisfies B1) -B4). The function h_α can be viewed as a regularization of the scalarization function considered in [25].

We further note that h_α can be equivalently expressed as the regularized gap function ϕ_α given by (2.5) (shown in Proposition 2.15). That is, the D-gap function $\zeta_{\alpha\beta}$ is nothing but the difference of the two regularized gap functions ϕ_α and ϕ_β for $0 < \alpha < \beta$.

From (3.6) and (3.7),

$$\zeta_{\alpha\beta}(x) = h_\alpha(x) - h_\beta(x) = \max_{y \in K} v_\alpha(x, y) - \max_{y \in K} v_\beta(x, y)$$

where

$$v_\alpha(x, y) = \left\{ \min_{1 \le i \le m} -f_i(x, y) \right\} - \alpha H(x, y). \tag{3.8}$$

The functions h_α and h_β are not continuous in general but only lower semi-continuous. We also note that h_α and h_β are continuous only when f_i's are continuous in (x, y) (Theorem 2.11 and Proposition 2.15). Hence the continuity of $\zeta(x)$ is not guaranteed unless all f_i's are continuous in (x, y). Since each $f_i(x, y)$ is convex in y, using the assumption of strong convexity of the function $H(x, .)$, the map $v_\alpha(x, .)$ is strongly concave on the set K and hence has a unique maximizer on K. Hence h_α and h_β are finite valued which implies that $\zeta_{\alpha\beta}$ is finite valued.

If $y_\alpha(x)$ denotes the unique maximizer of $v_\alpha(x, .)$ on K, $\zeta_{\alpha\beta}(x)$ is given by

$$\zeta_{\alpha\beta}(x) = v_\alpha(x, y_\alpha(x)) - v_\beta(x, y_\beta(x))$$

That is

$$\zeta_{\alpha\beta}(x) = \left\{ \min_{1 \le i \le m} -f_i(x, y_\alpha(x)) \right\} - \alpha H(x, y_\alpha(x)) - \left\{ \min_{1 \le i \le m} -f_i(x, y_\beta(x)) \right\} + \beta H(x, y_\beta(x)) \tag{3.9}$$

Lemma 3.6. *Let $x \in \mathbb{R}^n$ and for any $\alpha < 0$, let $v_\alpha(x, y)$ be defined by* (3.8). *Assume that for each $i \in \{1, ..., m\}$, $f_i(x, .)$ is differentiable. If $x = y_\alpha(x)$, then x solves the weakVEP.*

Proof. From definition,

$$h_\alpha(x) = \max_{y \in K} v_\alpha(x, y) = -\min_{y \in K} -v_\alpha(x, y) = -\min_{y \in K} \{ \max_{1 \le i \le m} f_i(x, y) + \beta H(x, y) \}.$$

Since $y_\alpha(x)$ maximizes $v_\alpha(x, .)$ on K, it minimizes $u_\alpha(x, .)$ on K where

$$u_\alpha(x, y) = \max_{1 \le i \le m} f_i(x, y) + \alpha H(x, y).$$

Since each $f_i(x, .)$ is convex, it is equivalent to

$$0 \in \partial_y u_\alpha(x, y_\alpha(x)) + N_K(y_\alpha(x)) \tag{3.10}$$

where $N_K(y_\alpha(x))$ is the normal cone to the set K at $y_\alpha(x)$. Since each $f_i(x, .)$ is differentiable, we have

$$0 \in Co\{\nabla_y f_i(x, y_\alpha(x)) : i \in I(x, y)\} + \alpha \nabla_y H(x, y_\alpha(x)) + N_K(y_\alpha(x)) \tag{3.11}$$

where $I(x,y) = \{i \in \{1,...,m\} : f_i(x,y) = \max_{1\leq i\leq m} f_i(x,y)\}$. Therefore there exits a $z \in Co\{\partial_y f_i(x, y_\alpha(x)) : i \in I(x,y)\} + \alpha\nabla_y H(x, y_\alpha(x))$ such that for all $y \in K$

$$\langle z, y - y_\alpha(x)\rangle \geq 0. \tag{3.12}$$

This implies that there exists an $r \in \{1,...,m\}$ and $\lambda_i \geq 0, 1 \leq i \leq r, \sum_{i=1}^{r} \lambda_i = 1$ such that

$$z = \sum_{i=1}^{r} \lambda_i \nabla_y f_i(x, y_\alpha(x)) + \alpha\nabla_y H(x, y_\alpha(x)).$$

From (3.12),

$$\left\langle \sum_{i=1}^{r} \lambda_i \nabla_y f_i(x, y_\alpha(x)), y - y_\alpha(x) \right\rangle + \langle \alpha\nabla_y H(x, y_\alpha(x)), y - y_\alpha(x)\rangle \geq 0. \tag{3.13}$$

Since $x = y_\alpha(x)$, $\nabla_y H(x, y_\alpha(x)) = 0$ and hence there exists an $i \in \{1,...,r\}$ such that

$$\langle \nabla_y f_i(x, y_\alpha(x)), y - y_\alpha(x)\rangle \geq 0.$$

Using convexity of $f_i(x,.)$ we have

$$f_i(x,y) - f_i(x, y_\alpha(x)) \geq \langle \nabla_y f_i(x, y_\alpha(x)), y - y_\alpha(x)\rangle \geq 0.$$

Since $x = y_\alpha(x)$, we have $f_i(x, y_\alpha(x)) = 0$ and hence we conclude that there exists an $i \in \{1,...,r\}$ satisfying

$$f_i(x,y) \geq 0, \quad \forall y \in K.$$

Hence x solves $weakVEP$. □

Theorem 3.7. *Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B4) and let $0 < \alpha < \beta$. Then $\zeta_{\alpha\beta}$ is non-negative on $\mathbb{R}^n$. Further, x^* solves the $weakVEP$ if and only if $\zeta_{\alpha\beta}(x^*) = 0$.*

Proof. From (3.9) we have

$$\zeta_{\alpha\beta}(x) = \left\{\min_{1\leq i\leq m} -f_i(x, y_\alpha(x))\right\} - \alpha H(x, y_\alpha(x)) - \left\{\min_{1\leq i\leq m} -f_i(x, y_\beta(x))\right\} + \beta H(x, y_\beta(x))$$

Since $y_\alpha(x)$ maximizes $v_\alpha(x,.) = \{\min_{1\leq i\leq m} -f_i(x,.)\} - \alpha H(x,.)$,

$$\begin{aligned}\zeta_{\alpha\beta}(x) &\geq \left\{\min_{1\leq i\leq m} -f_i(x, y_\beta(x))\right\} - \alpha H(x, y_\beta(x)) - \left\{\min_{1\leq i\leq m} -f_i(x, y_\beta(x))\right\} + \beta H(x, y_\beta(x)) \\ &\geq (\beta - \alpha) H(x, y_\beta(x)).\end{aligned}$$

Hence $\zeta_{\alpha\beta}(x) \geq 0$.

Similarly, we can show that $\zeta_{\alpha\beta}(x) \leq (\beta - \alpha)H(x, y_\alpha(x))$. Hence we have the following inequality for any $x \in \mathbb{R}^n$:

$$(\beta - \alpha)H(x, y_\beta(x) \leq \zeta_{\alpha\beta}(x) \leq (\beta - \alpha)H(x, y_\alpha(x)). \tag{3.14}$$

Let x^* solve the $weakVEP$. From Proposition 2.15 and Theorem 3.5 the functions h_α and h_β are also gap functions for the $weakVEP$. Hence, $h_\alpha(x^*) = 0 = h_\beta(x^*)$ which implies that $\zeta_{\alpha\beta}(x^*) = 0$. Conversely, let $\zeta_{\alpha\beta}(x^*) = 0$. From (3.14) we have $(\beta-\alpha)H(x^*, y_\beta(x^*) \leq 0$ which implies that $x^* = y_\beta(x^*)$. Using Lemma 3.6, x^* solves $weakVEP$.

□

4 Error Bounds

This section deals with the construction of error bounds for $weakVEP$ in terms of the gap functions ϕ_α, $\phi_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ under some additional assumptions on the functions f_i and H. In what follows, $d(x,C)$ denotes the distance between a point x and the set C. We need the following additional assumptions.

C1) For each fixed y, $\nabla_y f_i(.,y)$ is Lipschitz continuous, i.e., there exists a constant $L_i \geq 0$ such that

$$\|\nabla_y f_i(x_1,y) - \nabla_y f_i(x_2,y)\| \leq L_i\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n .$$

C2) For each fixed y, $\nabla_y f_i(.,y)$ is strongly monotone, i.e., there exists a constant $a_i \geq 0$ such that

$$\|\nabla_y f_i(x_1,y) - \nabla_y f_i(x_2,y)\| \geq a_i\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in \mathbb{R}^n .$$

Theorem 4.1. *Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B5). Let each f_i is strongly monotone on K with modulus $\mu_i, i = 1, ..., m$ and satisfy the assumptions C1) and C2). Then, for any $x \in K$, there exist constants $\mu > 0$ and $M > 0$ such that*

$$d(x, sol(weakVEP)) \leq \frac{1}{\sqrt{\mu - \alpha M}}\sqrt{\phi_\alpha(x)}.$$

Proof. For a given $x \in K$, $\phi_\alpha(x)$ can be written as

$$\phi_\alpha(x) = \min_{\xi\in S^m} \psi_\alpha(x,\xi),$$

where $\psi_\alpha(x,\xi)$ is given by (2.6). Since $\psi_\alpha(x,.)$ is continuous and S^m is compact, there exists a $\xi^* \in S^m$ such that $\phi_\alpha(x) = \psi_\alpha(x,\xi^*)$. Note that ξ^* depends on the choice of x. Set $f_{\xi^*}(x,y) = \langle\xi^*, f(x,y)\rangle$. Since $f(\bar{x},\bar{x}) = 0$, any $\bar{x} \in K$ that solves EP_{ξ^*} is a solution to the problem $\min_{y\in K} f_{\xi^*}(\bar{x},y)$. This is equivalent to finding an $\bar{x} \in K$ such that

$$\langle\nabla_y f_{\xi^*}(\bar{x},y)|_{y=\bar{x}}, y - x\rangle \geq 0 \quad \forall y \in K. \tag{4.1}$$

The above variational inequality (4.1) has a unique solution provided the map $x \mapsto \nabla_y f_{\xi^*}(.,y)$ is strongly monotone and Lipschitz, both of which are guaranteed under the assumptions C1) and C2). Hence EP_{ξ^*} has a unique solution which we shall denote by x^*. By Theorem 2.1, x^* solves $weakVEP$.

From the definition of $\psi(x,\xi^*)$ in (2.6), we have

$$\phi_\alpha(x) \geq \langle\xi^*, -f(x,y)\rangle - \alpha H(x,y) \ \ \forall y \in K.$$

Setting $y = x^*$,

$$\phi_\alpha(x) \geq \langle\xi^*, -f(x,x^*)\rangle - \alpha H(x,x^*). \tag{4.2}$$

Since each f_i is strongly monotone on K with modulus μ_i, we have

$$\langle\xi^*, f(x,x^*)\rangle + \langle\xi^*, f(x^*,x)\rangle \leq -\sum_{i=1}^{m} \xi_i^*\mu_i\|x - x^*\|^2.$$

Since x^* solves EP_{ξ^*}, $\langle \xi^*, f(x^*, x)) \rangle \geq 0$ and therefore

$$\langle \xi^*, -f(x, x^*) \rangle \geq \sum_{i=1}^{m} \xi_i^* \mu_i \|x - x^*\|^2. \tag{4.3}$$

Using (2.4) and the inequalities (4.2) and (4.3)

$$\phi_\alpha(x) \geq (\mu - \frac{\alpha L}{2})\|x - x^*\|^2, \text{ where } \mu = \sum_{i=1}^{m} \xi_i^* \mu_i.$$

Therefore

$$d(x, sol(weakVEP)) \leq \|x - x^*\| \leq \frac{1}{\sqrt{\mu - \frac{\alpha L}{2}}} \sqrt{\phi_\alpha(x)}.$$

Hence the proof. □

Remark 4.2. We note that the above result can also be derived from Proposition 4.2 in [26] using the scalarization link in Theorem 2.1, by considering the scalar equilibrium problem EP_ξ. Further, there are error bounds for $weakVEP$ in terms of regularized gap functions that are free of scalarization; see for example [31, 35]. We note that the assumptions used in [31, 35] for establishing error bounds require the calculation of solution sets of all the individual scalar equilibrium problems EP given by f_i's, where as, we do not need such an assumption to devise an error bound using ϕ_α.

We now give sufficient conditions under which $\phi_{\alpha\beta}$ provides a global error bound for $weakVEP$. The proof of the result below can alternatively be derived from Theorem 2.1 using Theorem 3.1 of [5].

Theorem 4.3. *Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B5). Let C1) and C2) hold for the functions f_i, $1 \leq i \leq m$. If x^* solves EP_ξ for some $\xi \in S^m$, then there exists a $c > 0$ such that*

$$\|x - x^*\| \leq c\|y_\beta(x, \xi) - x\|, \quad \forall x \in \mathbb{R}^n,$$

where $\beta > 0$ and $y_\beta(x, \xi)$ is given by (2.9).

Proof. Let $x \in \mathbb{R}^n$ and $\xi \in S^m$. Since $y_\beta(x, \xi)$ is the unique solution of (2.6), the following necessary and sufficient condition holds:

$$\left\langle -\sum_{i=1}^{m} \xi_i \nabla_y f_i(x, y_\beta(x, \xi)) - \beta \nabla_y H(x, y_\beta(x, \xi)),\ z - y_\beta(x, \xi) \right\rangle \leq 0, \quad \forall z \in K.$$

In particular for $z = x^*$

$$\left\langle -\sum_{i=1}^{m} \xi_i \nabla_y f_i(x, y_\beta(x, \xi) - \beta \nabla_y H(x, y_\beta(x, \xi)),\ x^* - y_\beta(x, \xi) \right\rangle \leq 0. \tag{4.4}$$

Since x^* solves EP_ξ, we have

$$\sum_{i=1}^{m} \xi_i f_i(x^*, y_\beta(x, \xi)) \geq 0. \tag{4.5}$$

Convexity of each $f_i(x^*,.)$ gives

$$\left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)),\ x^* - y_\beta(x,\xi) \right\rangle \le \sum_{i=1}^m \xi_i f_i(x^*, x^*) - \sum_{i=1}^m \xi_i f_i(x^*, y_\beta(x,\xi)).$$

Since $f(x^*, x^*) = 0$, using (4.5) we get

$$\left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)),\ x^* - y_\beta(x,\xi) \right\rangle \le 0. \tag{4.6}$$

Adding (4.4) and (4.6)

$$\left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x,\xi)) - \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)), y_\beta(x,\xi) - x^* \right\rangle \le$$
$$-\beta \left\langle \nabla_y H(x, y_\beta(x,\xi),\ y_\beta(x,\xi) - x^* \right\rangle.$$

Rearranging the above inequality

$$\left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x,\xi)) - \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)), x - x^* \right\rangle \le$$
$$- \left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x,\xi)) - \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)), y_\beta(x,\xi) - x \right\rangle$$
$$-\beta \left\langle \nabla_y H(x, y_\beta(x,\xi)),\ x - x^* \right\rangle - \beta \left\langle \nabla_y H(x, y_\beta(x,\xi)),\ y_\beta(x,\xi) - x \right\rangle. \tag{4.7}$$

From the assumption B4) on H, there exists a $\lambda > 0$ such that

$$\langle \nabla_y H(x, y_\beta(x,\xi)),\ x - y_\beta(x,\xi) \rangle \le H(x,x) - H(x, y_\beta(x,\xi)) - \lambda \|x - y_\beta(x,\xi)\|^2 \le 0. \tag{4.8}$$

From C1) and C2) the following two inequalities hold:

$$\left\| \left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x,\xi)) - \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)), x - x^* \right\rangle \right\| \ge \mu^* \|x - x^*\|^2, \tag{4.9}$$

where $\mu^* = \sum_{i=1}^m \xi_i \mu_i$, μ_i is the modulus of strong monotonicity of $\nabla_y f_i(., y_\beta(x,\xi))$;

$$\left\| \left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x,\xi)) - \sum_{i=1}^m \xi_i \nabla_y f_i(x^*, y_\beta(x,\xi)), y_\beta(x,\xi) - x \right\rangle \right\| \le$$
$$L^* \|x - x^*\| \|y_\beta(x,\xi) - x\|, \tag{4.10}$$

where $L^* = \sum_{i=1}^m \xi_i L_i$, Ł$_i$ is the Lipschitz constant for $\nabla_y f_i(., y_\beta(x,\xi))$ on K.

Assumptions B5) and (2.3) imply that

$$\|\langle \nabla_y H(x, y_\beta(x,\xi)) - \nabla_y H(x,x),\ x - x^* \rangle\| \le L \|x - x^*\| \|y_\beta(x,\xi) - x\|. \tag{4.11}$$

Applying Cauchy-Schwarz inequality for (4.7) and substituting (4.9), (4.10) and (4.11) in (4.7), we get

$$\mu^* \|x - x^*\|^2 \le L^* \|x - x^*\| \|y_\beta(x,\xi) - x\| + \beta L \|x - x^*\| \|y_\beta(x,\xi) - x\|.$$

Therefore

$$\|x - x^*\| \leq \frac{L^* + \beta L}{\mu^*} \|y_\beta(x, \xi) - x\|.$$

Hence the proof. □

Theorem 4.4. *Let f_i, $i = 1, ..., m$ satisfy C1) and C2) and let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B5). Then there exist constants $c > 0$ and $\lambda > 0$ such that for any $x \in \mathbb{R}^n$*

$$d(x, sol(weakVEP)) \leq \frac{c}{\sqrt{\lambda(\beta - \alpha)}} \sqrt{\phi_{\alpha\beta}(x)}.$$

i.e., $\sqrt{\phi_{\alpha\beta}}$ provides a global error bound for the $weakVEP$.

Proof. For a given $x \in \mathbb{R}^n$ and any $0 < \alpha < \beta$, the function $g_{\alpha\beta}(x, .) := \psi_\alpha(x, .) - \psi_\beta(x, .)$ is continuous over S^m. Since S^m is compact, there exists a $\xi^* \in S^m$ such that $\phi_{\alpha\beta}(x) = g_{\alpha\beta}(x, \xi^*)$. As in the proof of Theorem 4.1, under the assumptions C1) and C2), the problem EP_{ξ^*} has a unique solution which we shall denote by x^*. Now using Theorem 4.3, there exists a $c > 0$ such that

$$\|x - x^*\| \leq c\|y_\beta(x, \xi^*) - x\|. \tag{4.12}$$

From B3) and B4), we have

$$\begin{aligned} H(x, y_\beta(x, \xi^*)) &= H(x, y_\beta(x, \xi^*)) - H(x, x) \\ &\geq \langle \nabla_y H(x, x), y_\beta(x, \xi^*) - x\rangle + \lambda\|x - y_\beta(x, \xi^*)\|^2, \end{aligned}$$

where λ is the modulus of strong convexity of H. Hence by using (2.3)

$$H(x, y_\beta(x, \xi^*)) \geq \lambda\|x - y_\beta(x, \xi^*)\|^2.$$

The above inequality and Proposition 3.4 give

$$\|y_\beta(x, \xi^*) - x\| \leq \sqrt{\frac{1}{\lambda(\beta - \alpha)} g_{\alpha\beta}(x, \xi^*)}. \tag{4.13}$$

Noting that $\phi_{\alpha\beta}(x) = g_{\alpha\beta}(x, \xi^*)$, from (4.12) and (4.13)

$$d(x, sol(weakVEP)) \leq \frac{c}{\sqrt{\lambda(\beta - \alpha)}} \sqrt{\phi_{\alpha\beta}(x)}.$$

Hence the proof.

□

The following result gives an estimate for the the distance between any point x to the solution set of EP_ξ, a linear scalarization of the $weakVEP$. This is useful for formulating an error bound for the $weakVEP$ interms of $\zeta_{\alpha\beta}$.

Theorem 4.5. *Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy B1)-B5). Let C1) and C2) hold for the functions f_i, $1 \leq i \leq m$. Then there exists a $\xi \in S^m$ such that*

$$d(x, sol(EP_\xi)) \leq c^*\|y_\beta(x) - x\|, \quad \forall x \in \mathbb{R}^n,$$

where $\beta > 0$, $c > 0$ and $y_\beta(x)$ is the unique maximizer of $v_\beta(x, .)$ given by (3.8).

Proof. Let $x \in \mathbb{R}^n$ and let $y_\beta(x)$ is the unique maximizer of $v_\beta(x,.)$ on K. From (3.13), there exists an $r \in \{1,...,m\}$ and $\xi_i \geq 0, 1 \leq \xi_i \leq r, \sum_{i=1}^r \xi_i = 1$ such that

$$\left\langle \sum_{i=1}^r \xi_i \nabla_y f_i(x, y_\beta(x)), y - y_\beta(x) \right\rangle + \langle \beta \nabla_y H(x, y_\beta(x)), y - y_\beta(x) \rangle \geq 0.$$

Setting $\xi_i = 0$ for $r < i \leq m$, we can conclude the existence of an element $\xi \in S^m$ such that

$$\left\langle \sum_{i=1}^m \xi_i \nabla_y f_i(x, y_\beta(x)), y - y_\beta(x) \right\rangle + \langle \beta \nabla_y H(x, y_\alpha(x)), y - y_\beta(x) \rangle \geq 0, \tag{4.14}$$

for all $y \in K$.

Now, let x^* be a solution of the problem EP_ξ, for this particular ξ. Proceeding exactly as in the proof of Theorem 4.3 (from (4.4) on wards), the following hold true:

that is, there exists a $c^* > 0$ satisfying

$$\|x - x^*\| \leq c^* \|y_\beta(x) - x\|.$$

Hence the proof. □

Theorem 4.6. *Let H satisfy $B1) - B5)$. Let C1) and C2) hold for the functions f_i, $1 \leq i \leq m$. Then there exist constants $c^* > 0$ and $\lambda > 0$ such that for any $x \in \mathbb{R}^n$*

$$d(x, sol(weakVEP)) \leq \frac{c^*}{\sqrt{\lambda(\beta - \alpha)}} \sqrt{\zeta_{\alpha\beta}(x)}.$$

where λ is the parameter in $B4)$.

Proof. From B3) and B4)

$$\begin{aligned} H(x, y_\beta(x)) &= H(x, y_\beta(x)) - H(x, x) \\ &\geq \langle \nabla_y H(x, x), y_\beta(x) - x \rangle + \lambda \|x - y_\beta(x)\|^2, \end{aligned}$$

where λ is the parameter of strong convexity of H. Since $\nabla_y H(x, x) = 0$,

$$H(x, y_\beta(x)) \geq \lambda \|x - y_\beta(x)\|^2$$

Using (3.14) the above inequality gives

$$\lambda(\beta - \alpha)\|x - y_\beta(x)\|^2 \leq \zeta_{\alpha\beta(x)}.$$

Hence from Theorem 4.5, there exists a $\xi \in S^m$ and $c^* > 0$ such that

$$d(x, sol(EP_\xi)) \leq \frac{c^*}{\sqrt{\lambda(\beta - \alpha)}} \sqrt{\zeta_{\alpha\beta}(x)}, \quad \forall x \in \mathbb{R}^n,$$

Now, using Theorem 2.1, since $sol(EP_\xi) \subseteq sol(weakVEP)$ for any $\xi \in S^m$, proof follows. □

5 Applications

Let us consider the vector optimization problem VOP with convex objective functions. It is given as

$$\min_{x\in K} g(x) = (g_1(x), ..., g_m(x)) \tag{5.1}$$

where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a convex function for each $i = 1, ..., m$ and K is a closed convex set in $\mathbb{R}^n$. We recall the notion of weak pareto minimum. A vector $x^* \in K$ is a weak pareto minimizer of (VOP) if

$$g(x) - g(\overline{x}) \notin -int\,\mathbb{R}^m_+, \forall x \in K.$$

If $\overline{x}$ is a pareto minimizer of (VOP) with f_is are not necessarily differentiable, the following optimality condition holds:

$$(g_1'(\overline{x}, x - \overline{x}), ..., g_m'(\overline{x}, x - \overline{x})) \notin -int\,\mathbb{R}^m_+, \forall x \in K. \tag{5.2}$$

This is a $weakVEP$ with $f_i(x, y) = g_i'(\bar{x}, x - \bar{x})$.

Since g_i is convex for each i, $y \mapsto g_i'(x, y - x)$ is convex for each fixed $x \in \mathbb{R}^n$. Since it is finite valued, it is continuous y. Hence the assumptions on $f_i(x, y)$ including Assumption A are satisfied for the case $f_i(x, y) = g_i'(\bar{x}, x - \bar{x})$.

For the case when each g_i is continuously differentiable, $\overline{x}$ is a weak pareto minimum of (VOP) if and only if the following inequality holds true for $\overline{x}$:

$$(\langle \nabla g_1(\bar{x}), y - \bar{x}\rangle, \langle \nabla g_2(\bar{x}), y - \bar{x}\rangle, ..., \langle \nabla g_m(\bar{x}), y - \bar{x}\rangle) \notin -int\,\mathbb{R}^m_+, \quad \forall y \in K. \tag{5.3}$$

The above problem is a weak Stampacchia vector variational inequality defined by (1.2).

Example 5.1. Let $g_1.g_2 : \mathbb{R} \to \mathbb{R}$ be defined as

$$g_1(x) = x^2 - x, \; g_2(x) = x^2 + x$$

and $K = [-1, 1]$. We consider the vector optimization problem

$$\min_{x\in K} g(x) = (g_1(x), g_2(x)). \tag{5.4}$$

This is equivalent to solving the $weakVEP$ with $f_1(x, y) = g_1'(x, y - x)$ and $f_2(x, y) = g_2'(x, y - x)$. The regularized gap function ϕ_α for this case

$$\phi_\alpha(x) = \max_{y\in K} \{\min \{-g_1'(x, y - x), \;\; - g_2'(x, y - x)\} - \alpha H(x, y)\}.$$

Fugure 1 shows that graph of the regularized gap function $\phi_\alpha(x)$ when $\alpha = 0.1$ and $H(x, y) = |y - x|^2$. We see that $\phi_\alpha(x) = 0$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, showing that $[-\frac{1}{2}, \frac{1}{2}]$ is the set of weak Pareto solutions.

Example 5.2. Let $g_1.g_2 : \mathbb{R} \to \mathbb{R}$ be defined as

$$g_1(x) = x^2, \; g_2(x) = (x - 2)^2$$

and $K = [-3, 3]$. We consider the vector optimization problem

$$\min_{x\in K} g(x) = (g_1(x), g_2(x)). \tag{5.5}$$

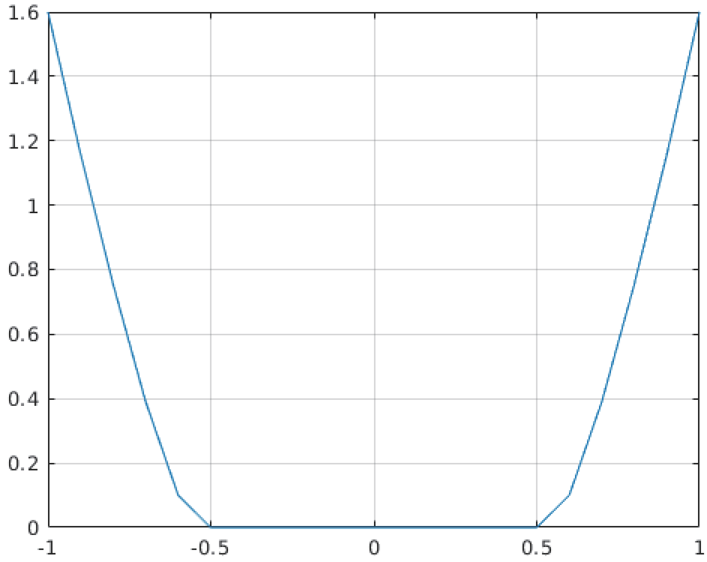

Figure 1: The Graph of ϕ_α for Example 5.1

This is equivalent to solving the $weakVEP$ with $f_1(x,y) = g_1'(x, y-x)$ and $f_2(x,y) = g_2'(x, y-x)$. The D-gap function $\zeta_{\alpha\beta}$ for this case

$$\zeta_{\alpha\beta}(x) = h_\alpha(x) - h_\alpha(y),$$

where h_α and h_β are given by

$$h_\alpha(x) = \max_{y\in K}\ \{\min\ \{-g_1'(x, y-x),\ \ -g_2'(x, y-x)\} - \alpha H(x,y)\};$$

$$h_\beta(x) = \max_{y\in K}\ \{\min\ \{-g_1'(x, y-x),\ \ -g_2'(x, y-x)\} - \beta H(x,y)\}.$$

Figure 2 is the graph of D-gap function $\zeta_{\alpha\beta}(x)$ for $\alpha = 0.1$, $\beta = 0.2$ and $H(x,y) = |y-x|^2$. We see that $\phi_\alpha(x) = 0$ for all $x \in [0,2]$, showing that $[0,2]$ is the set of weak Pareto solutions.

Example 5.3. Let $g_1.g_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$g_1(x_1,x_2) = |x_1| + x_2,\ \ g_2(x) = x_1^2 + x_2^2$$

and $K = [-2,2]\times[-2,2]$. Consider the following nonsmooth vector optimization problem in two dimensions

$$\min_{x\in K} g(x) = (g_1(x), g_2(x)). \tag{5.6}$$

The regularized gap function ϕ_α for this case

$$\phi_\alpha(x) = \max_{y\in K}\ \{\min\ \{-g_1'(x, y-x),\ \ -g_2'(x, y-x)\} - \alpha H(x,y)\}.$$

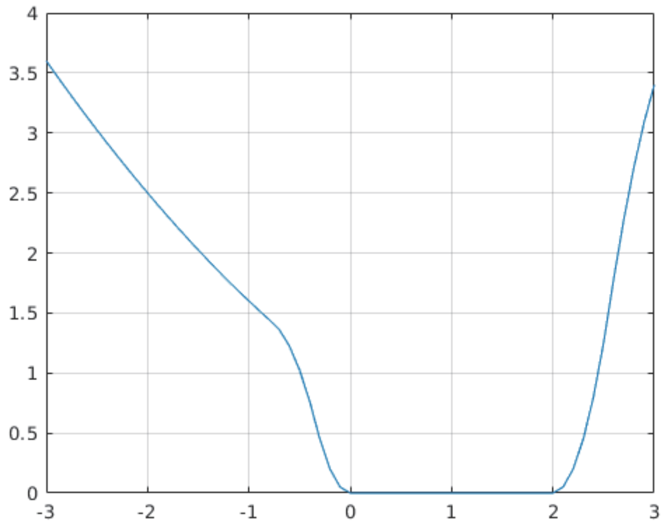

Figure 2: The Graph of $\zeta_{\alpha\beta}$ for Example 5.2

Note that the function g_1 is not differentiable on the line $\{(0, x_2) \in \mathbb{R}^2 : -2 \leq x_2 \leq 2\}$. We plotted $\phi_\alpha(x)$ in three dimensions using MATLAB for the case $\alpha = 0.5$ and $H(x, y) = \|y - x\|^2$. We see that $\phi_\alpha(x) = 0$ for all $x \in \{(0, x_2) : -2 \leq x_2 \leq 0\}$, which is the set of weak Pareto solutions.

The following examples support the assumptions C1) and C2).

Example 5.4. Let A_1 and A_2 be symmetric positive definite matrices of order $n \times n$. Let $f_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_1(x, y) = \frac{1}{2}\langle x, A_1 y\rangle + \frac{1}{2}\|y\|^2_{A_1} - \|x\|^2_{A_1} \text{ and } f_2(x, y) = \frac{1}{2}\langle x, A_2 y\rangle + \frac{1}{2}\|y\|^2_{A_2} - \|x\|^2_{A_2},$$

where $\|.\|_A$ denotes the norm in $\mathbb{R}^n$ defined by $\|x\|_A = \langle x, Ax\rangle^{\frac{1}{2}}$. Clearly, f_1 and f_2 satisfy all the basic assumptions we need. Further,

$$\nabla_y f_1(x, y) = \frac{1}{2}A_1 x + A_1 y \quad \text{and} \quad \nabla_y f_2(x, y) = \frac{1}{2}A_2 x + A_2 y.$$

Then,

$$\|\nabla_y f_1(x_1, y) - \nabla_y f_1(x_2, y)\| = \left\|\frac{1}{2}A_1 x_1 + A_1 y - \frac{1}{2}A_1 x_2 - A_1 y\right\| \leq \frac{1}{2}\|A_1\|\|x_1 - x_2\|,$$

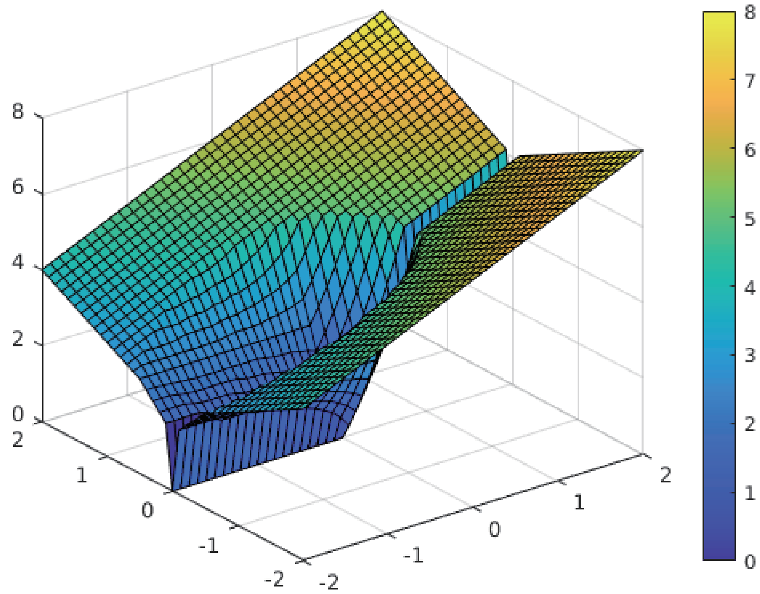

Figure 3: The Graph of ϕ_α for Example 5.3

where $\|A_1\|$ is the 2-norm(operator norm) of the matrix A_1, and

$$\begin{aligned}\langle \nabla_y f_1(x_1, y) - \nabla_y f_1(x_2, y), x_1 - x_2\rangle &= \left\langle \frac{1}{2}A_1x_1 + A_1y - \frac{1}{2}A_1x_2 - A_1y, x_1 - x_2 \right\rangle \\ &= \left\langle \frac{1}{2}A_1(x_1 - x_2), x_1 - x_2 \right\rangle \\ &= \left\langle \frac{1}{2}x_1 - x_2, A_1(x_1 - x_2) \right\rangle \\ &\geq \lambda_{min}(A_1)\|x_1 - x_2\|^2,\end{aligned}$$

where $\lambda_{min}(A_1)$ is the minimum eigenvalue of A_1. Hence, the map $x \mapsto \nabla_y f_1(., y)$ (similarly, $x \mapsto \nabla_y f_2(., y)$) is strongly monotone and Lipschitz continuous for each fixed y.

Example 5.5. Let M_1 and M_2 be symmetric $n \times n$ positive definite matrices and let $q_1, q_2 \in \mathbb{R}^n$. Let $f_1(x, y) = \langle M_1x + q_1, y - x\rangle$ and let $f_2(x, y) = \langle M_2x + q_2, y - x\rangle$. Clearly, f_1 and f_2 satisfy the assumptions we need. We have

$$\begin{aligned}\langle \nabla_y f_1(x_1, y) - \nabla_y f_1(x_2, y), x_1 - x_2\rangle &= \langle M_1x_1 + q_1 - M_1x_2 - q_1, x_1 - x_2\rangle \\ &= \langle x_1 - x_2, M_1(x_1 - x_2)\rangle \\ &\geq \lambda_{min}(M_1)\|x_1 - x_2\|^2,\end{aligned}$$

where $\lambda_{min}(M_1)$ is the minimum eigenvalue of M_1, and

$$\|\nabla_y f_1(x_1, y) - \nabla_y f_1(x_2, y)\| = \|M_1x_1 + q_1 - M_1x_2 - q_1\| \leq \|M_1\|\|x_1 - x_2\|.$$

Hence, the map $x \mapsto \nabla_y f_1(., y)$ (similarly, $x \mapsto \nabla_y f_2(., y)$) is strongly monotone and Lipschitz continuous for each fixed y.

6 Conclusions

In this paper, we generalized some gap functions for the case of a $weakVEP$ along the lines of Fukushima's regularization [12]. Under suitable assumptions on the functions involved, we constructed error bounds for a $weakVEP$ in terms of the gap functions ϕ_α, $\phi_{\alpha\beta}$ and $\zeta_{\alpha\beta}$. We noticed that it is hard to prove the continuity of the gap functions when one of the functions f_i, $1 \le i \le m$ fail to be continuous in (x, y). However, the alternative structure of the gap function ϕ_α shows that it is lower semi-continuous. The D-gap function $\zeta_{\alpha\beta}$ doesn't contain a scalarization parameter and is computationally advantageous. In view of this, the error bound in terms of $\zeta_{\alpha\beta}$ can be more useful. The gap functions and the error bounds developed in this paper can be exploited to devise stopping criterion for algorithms for solving $weakVEP$. In fact, if we generate approximate solutions for a convex multi-objective optimization problem using any known methods, such as Genetic algorithms, these gap functions can be helpful to compare solutions from any two methods and give us a better approximation.

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