Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 20, No. 1, pp. 119–133, 2024

AN AUGMENTED LAGRANGE METHOD FOR THE SECOND-ORDER CONE COUPLED CONSTRAINED VARIATIONAL INEQUALITY PROBLEM

Bin Wang, Li Wang, Juhe Sun, Yining Sun and Yanhong Yuan

Abstract: We introduce a new class of the second-order cone coupled constrained variational inequality (SOCCCVI) problem in this study. Based on the coupled constraints, this problem can be considered as a special minimization problem. The augmented Lagrange method is initially established to solve the SOCCCVI problem, and its global convergence theorem is shown by employing the saddle point inequality of the Lagrange function for the minimization problem. Moreover, we use the semi-smooth Newton method to solve the inner issue contained in the augmented Lagrange method. The numerical results are reported to verify the efficiency of the augmented Lagrange method for solving three SOCCCVI problems with 2000 variables.

Key words: *second-order cone coupled constrained variational inequality, augmented Lagrange method, Newton method, projection operator*

Mathematics Subject Classification: *47J20, 65K15*

1 Introduction

In this paper, we are interested in solving the second-order cone coupled constrained variational inequality, which is to find $x^* \in \Omega$ satisfying

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

where the set C is finitely representable as

$$C = \{y \in \Omega | - g(x^*, y) \in \mathcal{K}^m\},$$

and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $F : \Re^n \to \Re^n$, $g : \Re^n \times \Re^n \to \Re^m$, $\Omega \subseteq \Re^n$ is a convex closed set and

$$\mathcal{K}^m = \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \cdots \times \mathcal{K}^{m_p},$$

where $m_1, m_2, \ldots, m_p \geq 1$, $\sum_{i=1}^{p} m_i = m$, and each $\mathcal{K}^{m_i}$ is a second-order cone whose dimension is m_i. It is noted that the constraints of the form $-g(x, y) \in \mathcal{K}^m$ relate both the parameters and the variables, which are called coupled constraints. We will refer to (1.1) as the second-order cone coupled constrained variational inequality (SOCCCVI) problem.

 DOI: https://doi.org/10.61208/pjo-2023-015

The presence of coupled constraints makes these problems more difficult to solve, but they arise in many fields of mathematics. Among these are economic equilibrium models (see for instances [17, 23]), n-person games (see for instances [12, 20, 27]), the multicriteria decision making models (see for instances [14, 18]), quasi-variational inequalities (see for instance [5]), two-level programming (see for instance [1]) and so on. This has attracted the attention of specialists in researching those issues. Yet, there are few papers on approaches for solving variational inequalities with coupled constraints. One of the most well-known sources for a comprehensive numerical examination of variational inequality and complementarity problems is the book by Facchinei and Pang [10]. However, the variational inequality problems with coupled constraints were not addressed in this book.

Let us recall the definition of the second-order cone as follows.

The second-order cone (SOC) in $\Re^n$ $(n \geq 1)$, also called the Lorentz cone or the icecream cone, is defined as

$$\mathcal{K}^n = \{(x_1; \bar{x}) | x_1 \in \Re, \bar{x} \in \Re^{n-1}, x_1 \geq \|\bar{x}\|\}. \tag{1.2}$$

If $n = 1$, $\mathcal{K}^n$ is the set of nonnegative reals $\Re_+$. Hence the problem (1.1) in Antipin [2] is a special case of our second-order cone coupled constrained variational inequality (SOCCCVI) problem.

Several academics have been interested in the second-order cone constrained variational inequality in recent years, see for examples [21, 22, 28–30]. Neural network methods based on the Karush-Kuhn-Tucker (KKT) conditions of second-order cone constrained variational inequalities or programming issues are primarily deployed in [21, 22, 28, 29] to solve the relevant problems. In Sun and Zhang [30], a modified Newton method with Armijo line search is shown to achieve global convergence with a local super linear rate of convergence on the second-order cone constrained variational inequality problem under specific assumptions.

Arrow and Solow [4] used the augmented Lagrange function for the first time in their analysis of the differential equation method for equality constrained optimization problems. Hestenes [13] and Powell [24] pioneered the augmented Lagrange method for equality constrained optimization problems, which was later extended to nonlinear programming with equality constraints by Buys [8] and Rockafellar [26]. Bertsekas [6] provided further development of the augmented Lagrange method for dealing nonlinear optimization issues. The idea of the augmented Lagrange method was applied by Antipin [2] to solve variational inequalities problems with coupled constraints.

In this paper, we use the idea of augmented Lagrange method to solve the SOCCCVI problem (1.1) and report numerical results of the proposed method for solving three SOCCCVI problems with 2000 variables. Actually, the SOCCCVI problem can always be considered as a special minimization problem due to the coupled constraints, so we can convert the SOCCCVI problem (1.1) into several different equivalent formulas by using the saddle point inequality of the Lagrange function and the characteristics of projection operators. To build the augmented Lagrange method and demonstrate its convergence for solving the SOCCCVI problem, we need to use fundamental notions and characteristics of the second-order cone $\mathcal{K}^m$, which can be found in Bonnans and Shapiro [7]. We apply the semi-smooth Newton method to provide an approximate solution to the inner problem contained in the augmented Lagrange method, making the proposed method implementable and quick.

The paper is organized as follows. In Section 2, we present some preliminaries pertaining to the second-order cone and symmetric functions. Due to the coupled constraints, the SOCCCVI problem can be viewed as a special minimization problem, so we convert it into a saddle point problem and obtain the equivalent forms based on the characteristics of projection operators in Section 3. In Section 4, we establish the augmented Lagrange method and prove its global convergence theorem for solving the SOCCCVI problem (1.1). Finally, in Section 5, the numerical results are reported to confirm the efficiency and quickness of the augmented Lagrange function method for handling several SOCCCVI problems.

2 Preliminaries

The projection operator to a convex set is quite useful in reformulating the SOCCCVI problem (1.1) as an equation. Let C be a convex closed set, for every $x \in \Re^n$, there is a unique $\hat{x}$ in C such that

$$\|x - \hat{x}\| = \min\{\|x - y\| \mid y \in C\}.$$

The point $\hat{x}$ is the projection of x onto C, denoted by $\Pi_C(x)$. The projection operator $\Pi_C : \Re^n \to C$ is well defined over $\Re^n$ and it is a nonexpensive mapping.

Lemma 2.1 ([19]). *Let H be a real Hilbert space and $C \subset H$ be a closed convex set. For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u - \Pi_C(z) = 0$.

Next, we introduce the projection operator $\Pi_{\mathcal{K}^n} : \Re^n \to \mathcal{K}^n$ and $\mathcal{K}^n$ is the second-order cone (SOC) defined (1.2) in $\Re^n$.

In this paper, $\|\cdot\|$ is the l_2 norm. For any $x = (x_1; \bar{x}), y = (y_1; \bar{y}) \in \Re \times \Re^{n-1}$, we define their Jordan products

$$x \circ y = (x^T y; y_1\bar{x} + x_1\bar{y}). \tag{2.1}$$

Denote $x^2 = x \circ x$ and $\|x\| = \sqrt{x^2}$, where for any $y \in \mathcal{K}^n$, $\sqrt{y}$ is the unique vector in $\mathcal{K}^n$ such that $y = \sqrt{y} \circ \sqrt{y}$.

It follows from [11] that each $x = (x_1; \bar{x}) \in \Re \times \Re^{n-1}$ admits a spectral factorization, associated with $\mathcal{K}^n$, of the form

$$x = \rho_1\mu^{(1)} + \rho_2\mu^{(2)},$$

where ρ_1, ρ_2 and $\mu^{(1)}, \mu^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\rho_i = x_1 + (-1)^i\|\bar{x}\|,$$

$$u^{(i)} = \begin{cases} \frac{1}{2}\left(1; (-1)^i \frac{\bar{x}}{\|\bar{x}\|}\right), & \text{if} \quad \bar{x} \neq 0, \\ \frac{1}{2}(1; (-1)^i\omega), & \text{if} \quad \bar{x} = 0, \end{cases}$$

for $i = 1, 2$, with ω being any vector in $\Re^{n-1}$ satisfying $\|\omega\| = 1$. If $\bar{x} \neq 0$, the factorization is unique.

For $x \in \Re^n$, its factorization is as follows

$$x = \lambda_1(x)c_1(x) + \lambda_2(x)c_2(x),$$

then the projection of x onto $\mathcal{K}^n$, denoted by $\Pi_{\mathcal{K}^n}$, can be represented in the following.

$$\Pi_{\mathcal{K}^n} = [\lambda_1(x)]_+c_1(x) + [\lambda_2(x)]_+c_2(x),$$

where $[\lambda_i]_+ = \max\{0, \lambda_i\}, i = 1, 2$. We can calculate it as follows.

$$\Pi_{\mathcal{K}^n}(x) = \begin{cases} \frac{1}{2}\left(1 + \frac{x_1}{\|\bar{x}\|}\right)(\|\bar{x}\|; \bar{x}), & \text{if} \quad |x_1| < \|\bar{x}\| \\ x, & \text{if} \quad \|\bar{x}\| \leq x_1 \\ 0, & \text{if} \quad \|\bar{x}\| \leq -x_1 \end{cases}$$

Importantly, $\Pi_{\mathcal{K}^n}(\cdot)$ is semi-smooth over $\Re^n$, see for instance [9]. In the following, we denote $\Pi_{\mathcal{K}^m}(\cdot) = \Pi_{\mathcal{K}^{m_1}}(\cdot) \times \Pi_{\mathcal{K}^{m_2}}(\cdot) \times \cdots \times \Pi_{\mathcal{K}^{m_p}}(\cdot)$ in solving the SOCCCVI problem (1.1).

Now we recall the results about symmetric and skew-symmetric functions from Antipin [3].

Definition 2.2 ([3])**.** A function g from $\Re^n \times \Re^n$ into $\Re^m$ is said to be symmetric on $\Re^n \times \Re^n$ if it satisfies the condition

$$g(x, y) = g(y, x), \quad \forall x \in \Re^n, \quad \forall y \in \Re^n. \tag{2.2}$$

Properties of symmetric functions are investigated in [3], we list two of them in the following.

Property 2.1 ([3])**.** The matrices of gradient-restrictions of vector symmetric function g with respect to variable x and y onto the diagonal of the square $\Omega \times \Omega$ are identical. That is

$$\nabla_y^\top g(x, x) = \nabla_x^\top g(x, x), \quad \forall x \in \Omega. \tag{2.3}$$

Property 2.2 ([3])**.** The operator $2\nabla_y g(x, y)|_{x=y}$ is potential operator and coincides with the gradient of the restriction of the symmetric function g to the diagonal of the square $\Omega \times \Omega$, that is,

$$2\nabla_y^\top g(x, y)|_{x=y} = \nabla^\top g(x, x), \quad \forall x \in \Omega, \quad \forall y \in \Omega. \tag{2.4}$$

At the end of this section, we recall the definition of semi-smooth mappings. For a mapping $G : \Re^n \to \Re^m$, we denote by $\mathcal{J}G(x)$ its Fréchet-derivative at $x \in \Re^n$. Let D_G be the set of Fréchet-differentiable points of G in $\Re^n$. The Bouligand-Subdifferential of G at $x \in \Re^n$, denoted $\partial_B G(x)$, is

$$\partial_B G(x) := \{H \in \Re^{m \times n} : H = \lim_{k \to \infty} \mathcal{J}G(x^k) \,|\, x^k \in D_G, x^k \to x\}.$$

And the Clarke's generalized Jacobian of G at x is the convex hull of $\partial_B G(x)$, that is, $\partial G(x) = \text{conv}\{\partial_B G(x)\}$.

The semi-smooth property of mappings was introduced by Qi and Sun [25] to develop nonsmooth Newton method. Moreover, Kummer [15, 16] made a great contribution to the study of semi-smooth functions, respectively. We adopt the definition given by [10], which is equivalent to the original definition in Qi and Sun [25].

Definition 2.3 ([10])**.** Let $G : \Re^n \longrightarrow \Re^m$ be a locally Lipschitz continuous mapping. We say that G is semi-smooth at a point $x \in \Re^n$ if

(i) G is directionally differentiable at x; and

(ii) for any $\Delta x \in \Re^n$ and $H \in \partial G(x + \Delta x)$ with $\Delta x \longrightarrow 0$,

$$G(x + \Delta x) - G(x) - H(\Delta x) = \text{o}(\|\Delta x\|).$$

3 Transformations of the SOCCCVI Problem

In this section, the second-order cone coupled constrained variational inequality problem (1.1) can always be viewed as the minimization of the function $f(y) = \langle F(x^*), y - x^*\rangle$ on the set $C = \{y \in \Omega | - g(x^*, y) \in \mathcal{K}^m\}$ and $f(y) \geq 0$. Define the Lagrange function

$$\mathcal{L}(x^*, y, p) = \langle F(x^*), y - x^*\rangle + \langle p, g(x^*, y)\rangle, \quad \forall y \in C, \quad \forall p \in \mathcal{K}^m,$$

where x^* is the solution to the SOCCCVI problem (1.1). And y, p are the primal and dual variables. Since x^* is the minimum of $f(y)$ on Ω, the pair (x^*, p^*) is a saddle point of $\mathcal{L}(x^*, y, p)$, i.e., according to the saddle point theorem, satisfies the system of inequalities

$$\mathcal{L}(x^*, x^*, p) \leq \mathcal{L}(x^*, x^*, p^*) \leq \mathcal{L}(x^*, y, p^*), \quad \forall y \in \Omega, \quad \forall p \in \mathcal{K}^m. \tag{3.1}$$

This system can be represented in somewhat different manner as

$$\begin{aligned} &x^* \in \arg\min\{\langle F(x^*), y - x^*\rangle + \langle p^*, g(x^*, y)\rangle |\, y \in \Omega\}, \\ &p^* \in \arg\max\{\langle p, g(x^*, x^*)\rangle |\, p \in \mathcal{K}^m\}. \end{aligned} \tag{3.2}$$

There are other equivalent representations of system (3.2). Assuming that $g(x, y)$ is differentiable with respect to y for any x, we rewrite system (3.2) as

$$\begin{aligned} &\langle\ F(x^*) + \nabla_y g(x^*, x^*)p^*, y - x^*\rangle \geq 0, \quad \forall y \in \Omega, \\ &\langle p - p^*, -g(x^*, x^*)\rangle \geq 0, \quad \forall p \in \mathcal{K}^m. \end{aligned} \tag{3.3}$$

By using Lemma 2.1, the above system of variational inequalities is represented in the form of operator equations as

$$\begin{aligned} x^* \quad &= \Pi_\Omega(x^* - \alpha(F(x^*) + \nabla_y g(x^*, x^*)p^*)), \\ p^* \quad &= \Pi_{\mathcal{K}^m}(p^* + \alpha g(x^*, x^*)), \end{aligned} \tag{3.4}$$

where $\Pi_\Omega(\cdot)$ and $\Pi_{\mathcal{K}^m}(\cdot)$ are the projection operators of any vector onto the set Ω and the second-order cone $\mathcal{K}^m$ defined in (1.1), respectively, $\alpha > 0$ is step-length parameter.

System (3.3) can be transformed as follows. The first inequality in this system can be represented as

$$\langle\ F(x^*), y - x^*\rangle\ \ + \langle\ p^*, \nabla_y^\top g(x^*, x^*)(y - x^*)\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.5}$$

Next, taking into account the key property (2.4) of symmetric functions and convexity of vector-valued function g componentwise, we have following expression of the second term in the first inequality of (3.3) as

$$\langle p^*, \nabla_y^\top g(x^*, x^*)(y - x^*)\rangle = \frac{1}{2}\langle p^*, \nabla^\top g(x^*, x^*)(y - x^*)\rangle \leq \frac{1}{2}\langle p^*, g(y, y) - g(x^*, x^*)\rangle. \tag{3.6}$$

Finally, we can rewrite the first inequality of (3.3) in the form

$$\begin{aligned} &\langle\ F(x^*), y - x^*\rangle + \tfrac{1}{2}\langle p^*, g(y, y) - g(x^*, x^*)\rangle \geq 0, \quad \forall y \in \Omega, \\ &\langle p - p^*, -g(x^*, x^*)\rangle \geq 0, \quad \forall p \in \mathcal{K}^m. \end{aligned} \tag{3.7}$$

Thus, the second-order cone coupled constrained variational inequality problem (1.1) reduces to the system of variational inequality (3.7).

Remark 3.1. When the vector-function g is symmetric and differentiable with respect to y for any x, the transformations (3.2)-(3.4) are equivalent from each other by using the properties of the projection operators. Moreover its restriction $g(x, y)|_{x=y}$ on the diagonal of the square $\Omega \times \Omega$ is a convex function, (3.2)-(3.4) can be changed into (3.7).

4 The Augmented Lagrange Method

Now we construct the augmented Lagrange method to solve the SOCCCVI problem (1.1), it can be stated as follows.

Let $x^1 \in \Omega, p^1 \in \mathcal{K}^m$ be the initial estimated solution and Lagrange multiplier. Assume that the $k-$th iteration approximation (x^k, p^k) are known, then the next approximation (x^{k+1}, p^{k+1}) can be determined by

$$\begin{aligned} & x^{k+1} \in \arg\min\{\tfrac{1}{2}\|y - x^k\|^2 + \alpha\mathcal{M}(x^{k+1}, y, p^k) \mid y \in \Omega\}, \\ & p^{k+1} = \Pi_{\mathcal{K}^m}(p^k + \alpha g(x^{k+1}, x^{k+1})), \alpha > 0, \end{aligned} \tag{4.1}$$

where

$$\mathcal{M}(x, y, p) = \langle F(x), y - x\rangle + \frac{1}{2\alpha}\|\Pi_{\mathcal{K}^m}(p + \alpha g(x, y))\|^2 - \frac{1}{2\alpha}\|p\|^2 \tag{4.2}$$

is the augmented Lagrangian function for problem (3.1).

Note that x^{k+1} appears in both the left side and the right side of (4.1), thus (4.1) are implicit equations. It is an important issue for solving such implicit equations.

From (4.1), x^{k+1} is the minimum point. We can calculate system (4.1) as the following equivalent variational inequalities:

$$\langle x^{k+1} - x^k + \alpha(F(x^{k+1}) + \nabla_y g(u^{k+1})\Pi_{\mathcal{K}^m}(p^k + \alpha g(u^{k+1})), y - x^{k+1}\rangle \geq 0, \quad \forall y \in \Omega, \tag{4.3}$$

and

$$\langle p^{k+1} - p^k - \alpha g(u^{k+1}), p - p^{k+1}\rangle \geq 0, \quad \forall p \in \mathcal{K}^m, \tag{4.4}$$

where u^{k+1} represents (x^{k+1}, x^{k+1}).

Next we demonstrate the convergence of the augmented Lagrange method for solving the SOCCCVI problem (1.1).

Theorem 4.1. *Let the solution set of the second-order cone coupled constrained variational inequality problem (1.1) be nonempty, the function F be a monotone operator, the vector-function g be symmetric and differentiable with respect to y for any x, moreover its restriction $g(x, y)|_{x=y}$ on the diagonal of the square $\Omega \times \Omega$ be a convex function, $\Omega \subseteq \Re^n$ be a convex closed set and $\alpha > 0$. Then, the sequence $\{x^k\}$ constructed by the augmented Lagrange method (4.1) converges to a solution of the second-order cone coupled constrained variational inequality problem (1.1).*

Proof. Setting $y = x^*$ in (4.3) and taking into account the second equality of (4.1), we obtain

$$\langle x^{k+1} - x^k + \alpha(F(x^{k+1}) + \nabla_y g(u^{k+1})p^{k+1}), x^* - x^{k+1}\rangle \geq 0.$$

From the above inequality, we have

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \alpha\langle F(x^{k+1}), x^* - x^{k+1}\rangle + \alpha\langle \nabla_y g(u^{k+1})p^{k+1}, x^* - x^{k+1}\rangle \geq 0. \tag{4.5}$$

Using (3.6) and the convexity of $g(x, x)$, the last term in (4.5) can be expressed as

$$\begin{aligned} \langle p^{k+1}, \nabla_y^\top g(u^{k+1})(x^* - x^{k+1})\rangle &= \frac{1}{2}\langle p^{k+1}, \nabla^\top g(u^{k+1})(x^* - x^{k+1})\rangle \\ &\leq \frac{1}{2}\langle p^{k+1}, g(x^*, x^*) - g(u^{k+1})\rangle. \end{aligned} \tag{4.6}$$

(4.6) together with (4.5) deduce that

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \alpha\langle\, F(x^{k+1}), x^* - x^{k+1}\rangle + \frac{\alpha}{2}\langle p^{k+1}, g(x^*, x^*) - g(u^{k+1})\rangle \geq 0. \tag{4.7}$$

Setting $y = x^{k+1}$ in the first inequality in (3.7) yields that

$$\langle F(x^*), x^{k+1} - x^*\rangle + \frac{1}{2}\langle p^*, g(x^{k+1}, x^{k+1}) - g(x^*, x^*)\rangle \geq 0. \tag{4.8}$$

Summing (4.7) and (4.8), we deduce that

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \alpha\langle\, F(x^{k+1}) - F(x^*), x^* - x^{k+1}\rangle + \frac{\alpha}{2}\langle p^{k+1} - p^*, g(x^*, x^*) - g(u^{k+1})\rangle \geq 0. \tag{4.9}$$

Setting $p = p^*$ in (4.4), in view of $\langle p^{k+1}, g(x^*, x^*)\rangle \leq 0$ and $\langle p^*, g(x^*, x^*)\rangle = 0$, we have

$$\frac{1}{2}\langle p^{k+1} - p^k, p^* - p^{k+1}\rangle - \frac{\alpha}{2}\langle g(u^{k+1}) - g(x^*, x^*), p^* - p^{k+1}\rangle \geq 0. \tag{4.10}$$

Noting that $F(x)$ is a monotone operator, we summing (4.9) and (4.10). Thus we obtain

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \frac{1}{2}\langle p^{k+1} - p^k, p^* - p^{k+1}\rangle \geq 0.$$

By using the identity for arbitrary x_1, x_2 and x_3

$$\|x_1 - x_3\|^2 = \|x_1 - x_2\|^2 + 2\langle x_1 - x_2, x_2 - x_3\rangle + \|x_2 - x_3\|^2,$$

which implies that

$$\langle x_1 - x_2, x_2 - x_3\rangle = \frac{1}{2}\|x_1 - x_3\|^2 - \frac{1}{2}\big[\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2\big]. \tag{4.11}$$

By using (4.11), we get that

$$\|x^{k+1} - x^k\|^2 + \frac{1}{2}\|p^{k+1} - p^k\|^2 + \|x^{k+1} - x^*\|^2 + \frac{1}{2}\|p^{k+1} - p^*\|^2 \leq \|x^k - x^*\|^2 + \frac{1}{2}\|p^k - p^*\|^2. \tag{4.12}$$

Adding the left and right sides of equation (4.12) from $k = 0$ to $k = N$ yields that

$$\sum_{k=0}^{N}\|x^{k+1} - x^k\|^2 + \frac{1}{2}\sum_{k=0}^{N}\|p^{k+1} - p^k\|^2 + \|x^{N+1} - x^*\| + \frac{1}{2}\|p^{N+1} - p^*\|^2 \leq \|x^0 - x^*\|^2 + \frac{1}{2}\|p^0 - p^*\|^2. \tag{4.13}$$

This inequality implies the boundedness of the trajectory $\{(x^i, p^i) : i = 1, 2, \ldots, \}$, namely

$$\|x^{N+1} - x^*\|^2 + \frac{1}{2}\|p^{N+1} - p^*\|^2 \leq \|x^0 - x^*\|^2 + \frac{1}{2}\|p^0 - p^*\|^2,$$

and also the convergence of the series

$$\sum_{k=0}^{\infty}\|x^{k+1} - x^k\|^2 < \infty, \quad \sum_{k=0}^{\infty}\|p^{k+1} - p^k\|^2 < \infty.$$

Therefore, $\|x^{k+1} - x^k\|^2 \to 0$, $\|p^{k+1} - p^k\|^2 \to 0$ as $k \to \infty$. Since the sequence (x^k, p^k) is bounded, there exists an element (x', p') such that $x^{k_i} \to x'$ and $p^{k_i} \to p'$, as $i \to \infty$. Moreover

$$\|x^{k_i+1} - x^{k_i}\|^2 \to 0, \quad \|p^{k_i+1} - p^{k_i}\|^2 \to 0.$$

Considering (4.3) and (4.4) with $k = k_i$ for all $i \to \infty$ and passing to the limit as produces

$$\langle F(x') + \nabla_y g(x', x')p', y - x'\rangle \geq 0, p' = \Pi_{\mathcal{K}^m}(p' + \alpha g(x', x')),$$

$$\langle -g(x', x'), p - p'\rangle \geq 0, \quad \forall p \in \mathcal{K}^m.$$

Since these relations coincide with (3.3), we have $x' \in \Omega$ and $p' \in \mathcal{K}^m$ satisfies

$$\begin{aligned} &x' \in \arg\min\{\langle F(x'), y - x'\rangle + \langle p', g(x', y)\rangle \,|\, y \in \Omega\}, \\ &p' \in \arg\max\{\langle p, g(x', x')\rangle \,|\, p \in \mathcal{K}^m\}. \end{aligned}$$

Therefore, the above expressions coincide with (3.2) and (3.3). It follows from Remark 3.1 that any accumulation point x' of the sequence $\{x^k\}$ is a solution to the second-order cone coupled constrained variational inequality problem (1.1). Thus the sequence $\{x^k\}$ constructed by the augmented Lagrange method (4.1) converges to a solution of the second-order cone coupled constrained variational inequality problem (1.1). This completes the proof. □

5 Numerical Experiments

In this section, we discuss the case when $\Omega = \Re^n$ in the second-order cone coupled constrained variational inequality (SOCCCVI) problem (1.1). The augmented Lagrange method (4.1) for this case can be reduced to

$$\begin{aligned} &G^k(x^{k+1}) = 0, \\ &p^{k+1} = \Pi_{\mathcal{K}^m}(p^k + \alpha g(u^{k+1})), \quad \alpha > 0, \end{aligned} \tag{5.1}$$

where

$$G^k(x) = x - x^k + \alpha F(x) + \alpha \nabla_y g(x, x)\Pi_{\mathcal{K}^m}(p^k + \alpha g(x, x)).$$

Since the projection mapping $\Pi_{\mathcal{K}^m}(\cdot)$ is semi-smooth, $G^k(\cdot)$ is also semi-smooth. Thus the system (5.1) is semi-smooth. Suppose that any element in $\partial G^k(x^{k+1})$ is nonsingular and x^k is quite close to x^{k+1}, then the semi-smooth Newton method can be employed to solve the first equality in (5.1).

Newton Method ([25])

Step 1: Set $\xi^0 = x^k$ and $j = 0$.

Step 2: If $G^k(\xi^j) = 0$, stop at $x^{k+1} = \xi^j$.

Step 3: Select an element $H^j \in \partial G^k(\xi^j)$. Solve a solution $d^j \in \Re^n$ to the equation

$$G^k(\xi^j) + H^j d^j = 0.$$

Step 4: Set $\xi^{j+1} = \xi^j + d^j$ and $j = j + 1$ and go to Step 2.

Remark 5.1. In practice, the stopping criterion $G^k(\xi^j) = 0$ in Step 2 is usually replaced by $\|G^k(\xi^j)\| \leq \epsilon_0$ for some accuracy $\epsilon_0 > 0$.

In this case, we use Newton method for finding an approximate solution to the $k-$th inner problem by setting $x^{k+1} = \xi^j$ when the following condition is satisfied

$$\|G^k(\xi^j)\| \leq \epsilon_0.$$

We adopt the following condition as a stopping criterion for the augmented Lagrange method

$$r_k := \|F(x^k) + \nabla_y g(x^k, x^k)p^k\| \leq \epsilon_1.$$

Our numerical experiments are carried out in Matlab R2019a running on a PC Intel Pentium IV of 3.10 GHz CPU and with 8G RAM. Numerical results of three examples are summarized in Table 5.1, Table 5.2 and Table 5.3, where n denotes the dimension of the test problem, $\mathcal{K}$ denotes the second-order cone which is defined in (1.1), k denotes the number of outer iterations for solving the test problem, "Time" represents the CPU-time in second to reach the termination condition.

Example 5.2. Consider the SOCCCVI problem

$$\langle (N+M)x^*, y - x^* \rangle \geq 0, \quad \forall\, y \in \Re^n, \quad -g(x^*, y) \in \mathcal{K}^n, \tag{5.2}$$

where $g(x,y) = x + y \in \mathcal{K}^n \subseteq \Re^n$, N and M are positive semidefinite matrices.

In this example, $\alpha = 0.5$ is used and at the $j-$th Newton iteration H^j has the form

$$H^j = I_n + \alpha(N+M) + \alpha B^j.$$

In this case, we have $u = p + 2\alpha x = (u_1, \bar{u}) \in \mathcal{K}^n$. That is, $u_1 \in \Re^1$ and $\bar{u} = (u_2, u_3, \ldots, u_n) \in \Re^{n-1}$. Then we can compute B^j in the following

$$B^j_{11} = \begin{cases} \alpha, & |u_1| < \|\bar{u}\|, \\ 2\alpha, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The first column B^j_{i1} of B^j for $i = 2, 3, \ldots, n$ is calculated by

$$B^j_{i1} = \begin{cases} \alpha \frac{u_i}{\|\bar{u}\|}, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The first row B^j_{1k} of B^j for $k = 2, 3, \ldots, n$ is as follows

$$B^j_{1k} = \begin{cases} \alpha \frac{u_k}{\|\bar{u}\|}, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The diagonal elements B^j_{ii} of B^j for $i = 2, 3, \ldots, n$ can be computed by

$$B^j_{ii} = \begin{cases} \alpha(1 + \frac{u_1}{\|\bar{u}\|} - \frac{u_1 u_i^2}{\|\bar{u}\|^3}), & |u_1| < \|\bar{u}\|, \\ 2\alpha, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The other elements B^j_{ik} of B^j for $i, k = 2, 3, \dots, n$ and $i \neq k$ is calculated by

$$B^j_{ik} = \begin{cases} -\alpha \frac{u_1 u_i u_k}{\|\bar{u}\|^3}, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

Table 5.1 demonstrates the numerical results of the augmented Lagrange method (5.1) for solving the SOCCCVI problem (5.2).

Table 5.1. Numerical Results of the SOCCCVI Problem (5.2)

n	$\mathcal{K}$	k	$Time$	r_k	ϵ_0	ϵ_1
200	$\mathcal{K}^{100} \times \mathcal{K}^{100}$	19	6.421875e+00	9.864113e-03	10^{-6}	10^{-2}
400	$\mathcal{K}^{200} \times \mathcal{K}^{200}$	20	2.955313e+01	6.802632e-03	10^{-6}	10^{-2}
800	$\mathcal{K}^{400} \times \mathcal{K}^{400}$	20	1.489063e+02	9.987015e-03	10^{-6}	10^{-2}
1200	$\mathcal{K}^{600} \times \mathcal{K}^{600}$	21	7.424063e+02	5.123815e-03	10^{-6}	10^{-2}
2000	$\mathcal{K}^{1000} \times \mathcal{K}^{1000}$	21	2.009469e+03	6.645142e-03	10^{-6}	10^{-2}

Example 5.3. Consider the SOCCCVI problem

$$\langle P(P+Q)x^*, y - x^* \rangle \geq 0, \quad \forall\, y \in \Re^n, \quad -g(x^*, y) \in \mathcal{K}^n, \tag{5.3}$$

here $g(x, y) = x \circ y$, which is the Jordan product of x and y defined in (2.1). P and Q are positive semidefinite matrices.

In the numerical implementation of this example, $\alpha = 0.5$ is used and at the $j-$th Newton iteration H^j has the form as follows

$$H^j = I_n + \alpha P(P+Q) + \alpha B^j.$$

In this case, we have $u = p + \alpha x \circ x = (u_1, \bar{u}) \in \mathcal{K}^n$. That is, $u_1 \in \Re^1$ and $\bar{u} = (u_2, u_3, \dots, u_n) \in \Re^{n-1}$. B^j can be calculated in the following

$$B^j_{11} = \begin{cases} \frac{1}{2}(\|\bar{u}\| + u_1) + \alpha \sum\limits_{i=1}^{n} \frac{u_i}{\|\bar{u}\|} x_1 x_i, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(\|\bar{u}\| + u_1) + 2\alpha x_1^2, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The first column B^j_{i1} of B^j for $i = 2, 3, \dots, n$ is as follows

$$B^j_{i1} = \begin{cases} \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_i + \alpha \frac{u_i}{\|\bar{u}\|} x_1 x_i + \alpha(1 + \frac{u_1}{\|\bar{u}\|} - \frac{u_1 u_i^2}{\|\bar{u}\|^3}) x_i^2 - \alpha \sum\limits_{l=3}^{n} \frac{u_1 u_i u_l}{\|\bar{u}\|^3} x_i x_l, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_i + 2\alpha u_i, & \|\bar{u}\| \leq u_1, \\ 0, & \|\bar{u}\| \leq -u_1. \end{cases}$$

The first row B^j_{1k} of B^j for $k = 2, 3, \ldots, n$ can be evaluated by

$$B^j_{1k} = \begin{cases} \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_k + \alpha x_1 x_k + \alpha \frac{u_k}{\|\bar{u}\|} x_1^2, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_k + 2\alpha x_1 x_k, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

We can calculate the diagonal elements B^j_{ii} of B^j for $i = 2, 3, \ldots, n$ as follows

$$B^j_{ii} = \begin{cases} \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|}) + \alpha \frac{u_i}{\|\bar{u}\|} x_i^2 + \alpha(1 + \frac{u_1}{\|\bar{u}\|} - \frac{u_1 u_i^2}{\|\bar{u}\|^3}) x_1 x_i, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(\|\bar{u}\| + u_1) + 2\alpha x_1 x_i, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

The other elements of B^j_{ik} of B^j for $i, k = 2, 3, \ldots, n$ and $i \neq k$ can be computed by

$$B^j_{ik} = \begin{cases} \alpha \frac{u_i}{\|\bar{u}\|} x_i x_k - \alpha \frac{u_1 u_i u_k}{\|\bar{u}\|^3} x_1 x_i, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

Table 5.2 shows the numerical results of the augmented Lagrange method (5.1) for solving the SOCCCVI problem (5.3).

Table 5.2. Numerical Results of the SOCCCVI Problem (5.3)

n	$\mathcal{K}$	k	$Time$	r_k	ϵ_0	ϵ_1
200	$\mathcal{K}^{100} \times \mathcal{K}^{100}$	21	2.575625e+00	9.680187e-07	10^{-9}	10^{-6}
400	$\mathcal{K}^{200} \times \mathcal{K}^{200}$	24	1.859375e+01	6.663238e-07	10^{-9}	10^{-6}
800	$\mathcal{K}^{400} \times \mathcal{K}^{400}$	30	1.293750e+02	7.049868e-07	10^{-9}	10^{-6}
1200	$\mathcal{K}^{600} \times \mathcal{K}^{600}$	36	5.042344e+02	8.376350e-07	10^{-9}	10^{-6}
2000	$\mathcal{K}^{1000} \times \mathcal{K}^{1000}$	56	3.395703e+03	9.972165e-07	10^{-9}	10^{-6}

Example 5.4. Now we consider the last SOCCCVI problem

$$\langle x^*, y - x^* \rangle \ge 0, \quad \forall\, y \in \Re^n, \quad -g(x^*, y) \in \mathcal{K}^n, \tag{5.4}$$

where $g(x, y)$ is defined by the following

$$g(x, y) = \begin{pmatrix} y_1 e^{x_1} + x_1 e^{y_1} \\ \vdots \\ y_n e^{x_n} + x_n e^{y_n} \end{pmatrix},$$

In this experiment, $\alpha = 0.5$ is used and at the $j-$th Newton iteration H^j has the form as follows

$$H^j = (1 + \alpha) I_n + \alpha B^j$$

In this case, we have $u = (u_1, \bar{u}) \in \mathcal{K}^n$. That is, $u_1 \in \Re^1$ and $\bar{u} \in \Re^{n-1}$. We calculate that $u_i = p_i + 2\alpha x_i e^{x_i}$ and $c_i = (1 + x_i)e^{x_i}$ for $i = 1, 2, \ldots, n$. We get that

$$B^j_{11} = \begin{cases} \frac{1}{2}(\|\bar{u}\| + u_1)c_1 e^{x_1} + \alpha c_1^2, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(\|\bar{u}\| + u_1)c_1 e^{x_1} + 2\alpha c_1^2, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

We can compute the first column B^j_{i1} of B^j for $i = 2, 3, \ldots, n$ in the following

$$B^j_{i1} = \begin{cases} \alpha \frac{u_i}{\|\bar{u}\|} c_i c_1, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

The first row B^j_{1k} of B^j for $k = 2, 3, \ldots, n$ can be calculated by

$$B^j_{1k} = \begin{cases} \alpha \frac{u_k}{\|\bar{u}\|} c_k c_1, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

The diagonal elements B^j_{ii} of B^j for $i = 2, 3, \ldots, n$ as follows

$$B^j_{ii} = \begin{cases} \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_i c_i e^{x_i} + \alpha(1 + \frac{u_1}{\|\bar{u}\|} - \frac{u_1 u_i^2}{\|\bar{u}\|^3})c_i^2, & |u_1| < \|\bar{u}\|, \\ \frac{1}{2}(1 + \frac{u_1}{\|\bar{u}\|})u_i c_i e^{x_i} + 2\alpha c_i^2, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

We can compute the other elements of B^j_{ik} of B^j for $i, k = 2, 3, \ldots, n$ and $i \ne k$ in the following

$$B^j_{ik} = \begin{cases} -\alpha \frac{u_1 u_i u_k}{\|\bar{u}\|^3} c_i c_k, & |u_1| < \|\bar{u}\|, \\ 0, & \|\bar{u}\| \le u_1, \\ 0, & \|\bar{u}\| \le -u_1. \end{cases}$$

Table 5.3 expresses the numerical results of the augmented Lagrange method (5.1) for solving the SOCCCVI problem (5.4).

Table 5.3. Numerical Results of the SOCCCVI Problem (5.4)

n	$\mathcal{K}$	k	$Time$	r_k	ϵ_0	ϵ_1
200	$\mathcal{K}^{100} \times \mathcal{K}^{100}$	23	7.328125e+00	6.701724e-04	10^{-6}	10^{-3}
400	$\mathcal{K}^{200} \times \mathcal{K}^{200}$	24	2.343750e+01	6.677153e-04	10^{-6}	10^{-3}
800	$\mathcal{K}^{400} \times \mathcal{K}^{400}$	25	1.191719e+02	7.934865e-04	10^{-6}	10^{-3}
1200	$\mathcal{K}^{600} \times \mathcal{K}^{600}$	26	3.642750e+02	8.302493e-04	10^{-6}	10^{-3}
2000	$\mathcal{K}^{1000} \times \mathcal{K}^{1000}$	28	1.715453e+03	7.134481e-04	10^{-6}	10^{-3}

The above-mentioned numerical experiments demonstrate the practicality and effectiveness of the augmented Lagrange method for solving second-order cone coupled constrained variational inequality (SOCCCVI) problem.

6 Conclusions

In this paper, we initially propose a new class of the second-order cone coupled constrained variational inequality problem, which is an extension of the problem (1.1) in Antipin [2]. Unlike previous neural network methods for addressing the second-order cone variational inequality, the augmented Lagrange method is constructed for the first time to handle the SOCCCVI problem. Based on the characteristics of the coupled constraints, this problem can be transformed into a saddle point problem. Then several equivalent transformations of the primal SOCCCVI problem can be carried out by using the properties of the projection operators. By using the above transformation, the augmented Lagrange method for the SOCCCVI problem is constructed. Furthermore, we obtain the global convergence theorem for the augmented Lagrange method to solve the SOCCCVI problem. At last, we give three numerical examples which show the effectiveness and speediness of the augmented Lagrange method for solving SOCCCVI problems.

References

[1] A.S. Antipin, The convergence of proximal methods to fixed points of extremal mappings and estimates of their convergence rate, *Zh. Vychisl. Mat. Mat. Fiz.* 35 (1995), 688–704.

[2] A.S. Antipin, Solution methods for variational inequalities with coupled constraints, *Comput. Math. Math. Phys.* 40 (2000), 1239–1254.

[3] A.S. Antipin, Differential equations for equilibrium problems with coupled constraints, *Nonlinear Anal.* 47 (2001), 1833–1844.

[4] K.J. Arrow and R.M. Solow, *Gradient Methods for Constrained Maxima, with Weakened Assumptions*, Stanford University Press, Stanford, 1958.

[5] C. Baiocchi and A. Capelo, *Variational and Quasi-variational Inequalities: Applications to Free Boundary Problems*, Wiley, Chichester, 1984.

[6] D.P. Bertsekas, *Optimization and Multiplier Methods*, Academic Press, New York, 1982.

[7] J.F . Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Spring-Verlag, New York, 2000.

[8] J.D. Buys, Dual Algorithms for Constrained Optimization Problems, PhD thesis, University of Leiden, 1972.

[9] X.D. Chen, D. Sun and J. Sun, Complementarity functions and numerical experiments for second-order-cone complementarity problems, Comput. Optim. Appl. **25** (2003), 39–56.

[10] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, volume II, Springer-Verlag Inc, New York, 2003.

[11] M. Fukushima, Z.L. Luo and P. Tseng, Smoothing functions for second-order-cone complimentarity problems, *SIAM J. Optim.* 12(2002), 436–460.

[12] C.B. Garcia and W.I. Zangwill, *Pathways to Solutions, Fixed Points, and Equilibria*, Prentice Hall, New York, 1981.

[13] M.R. Hestenes, Multiplier and gradient methods, *J. Optim. Theory Appl.* 4 (1969), 303–320.

[14] V.G. Karmanov and V.V. Fedorov, *Modeling in Operations Research*, Tvema, Moscow, 1996.

[15] B. Kummer, *Newton's Method for Non-Differentiable Functions*, Advances in Mathematical Optimization , Akademie Verlag, Berlin, 1988.

[16] B. Kummer, Lipschitzian inverse functions, directional derivatives, and applications in C-1,1 optimization , *J. Optim. Theory Appl.* 70 (1991), 561–582.

[17] M.I. Levin, V.L. Makarov and A.M. Rubinov, *Mathematical Models of Economic Interaction*, Fizmatgiz, Moscow, 1993.

[18] N.N. Moiseev, *Mathematical Problems in System Analysis*, Nauka, Moscow, 1981.

[19] U. Mosco, *Implicit Variational Problems and Quasi-Variational Inequalities*, Lect. Notes Math. 543, 1976.

[20] J.F. Jr. Nash, Equilibrium points in n-person games, *Proceedings of the National Academy of Science* 36 (1950), 48–49.

[21] A. Nazemi and A. Sabeghi, A novel gradient-bsed neural network for solving convex second-order cone constrained variational inequality problems, *J. Comput. Appl. Math.* 347 (2019), 343–356.

[22] A. Nazemi and A. Sabeghi, A new neural network framework for solving convex second-order cone constrained variational inequality problems with an application in multi-finger robot hands, *J. Exp. Theor. Artif. In.* 32 (2020), 181–203.

[23] V.M. Polterovich, *Economic Equilibrium and Economic Mechanism*, Nauka, Moscow, 1990.

[24] M.J.D. Powell, *A Method for Nonlinear Constrains in Minimization Problems*, Optimization, Academic Press, New York, 1969.

[25] L. Qi and J. Sun, A nonsmooth version of Newton's method, *Math. Program.* 58 (1993), 353–367.

[26] R.T. Rockafellar, The multiplier method of hestenes and powell applied to convex programming, *J. Optim. Theory Appl.* 12 (1973), 555–562.

[27] J.B. Rosen, Existence and uniqueness of equilibrium points for concave n-person games, *Econometrica* 33 (1965), 520–534.

[28] J. Sun, J.S. Chen and C.H. Ko, Neural networks for solving second-order cone constrained variational inequality problem, *Comput. Optim. Appl.* 51 (2012), 623–648.

[29] J. Sun, W. Fu, J.H. Alcantara and J.S. Chen, A neural network based on the metric projector for solving SOCCVI problem, *IEEE Trans. Neural Netw. Learn. Syst.* 32 (2021), 2886–2900.

[30] J. Sun and L.W. Zhang, A globally convergent method based on Fischer-Burmeister operators for solving second-order cone constrained variational inequality problems, *Comput. Math. Appl.* 58 (2009), 1936–1946.

Manuscript received 22 March 2022
revised 27 August 2022
accepted for publication 24 April 2023

BIN WANG
School of Science, Shenyang Aerospace University
Shenyang 110136,
E-mail address: Chinawbin326@163.com

LI WANG
School of Science, Shenyang Aerospace University
Shenyang 110136, China
E-mail address: liwang211@163.com

JUHE SUN
School of Science, Shenyang Aerospace University
Shenyang 110136, China
E-mail address: juhesun@163.com

YINING SUN
School of Science, Shenyang Aerospace University
Shenyang 110136, China
E-mail address: yiningsun527@163.com

YANHONG YUAN
College of Economics and Management
Taiyuan University of Technology
Taiyuan, 030024, China
E-mail address: yuanyanhong36@163.com