Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 19, No. 3, pp. 489–505, 2023

PARETO H-EIGENVALUE INCLUSION INTERVALS FOR TENSOR EIGENVALUE COMPLEMENTARITY PROBLEMS*

Gang Wang, Chong Wang and Qiuling Hou

Abstract: In this paper, we reviIn this paper, we establish tight Pareto H-eigenvalue inclusion intervals for the tensor eigenvalue complementarity problems based on partitioning the tensor index set. To reduce computations, we propose new S-type Pareto H-eigenvalue inclusion intervals, and verify the efficiency of the obtained results by running examples. As applications, we propose some sufficient conditions for checking the strict copositivity and the strict semi-positivity of tensors.

Key words: *tensor eigenvalue complementarity problems, pareto H-eigenvalues, pareto H-eigenvalue inclusion intervals, strict copositivity*

Mathematics Subject Classification: *15A18, 15A69, 90C33*

---

1 Introduction

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an m-th order n dimensional real tensor and x be a real n-vector and $N = \{1, 2, \cdots, n\}$. $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$(\mathcal{A}x^{m-1})_i = \sum_{(i_2,\ldots,i_m)\in N} a_{ii_2\cdots i_m} x_{i_2} \ldots x_{i_m}.$$

Consider the following tensor eigenvalue complementarity problems: to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash\{0\}$ such that

$$0 \leq x \perp (\lambda \mathcal{I} x^{m-1} - \mathcal{A}x^{m-1}) \geq 0,$$

where $a \perp b$ means that the two vectors a, b are perpendicular to each other, and $\mathcal{I} \in \mathbb{R}^{[m,n]}$ is a unit tensor whose entries are

$$\mathcal{I}_{i_1 i_2 \cdots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

$(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash\{0\}$ is called a Pareto H-eigenpair. Further, if $x + \lambda \mathcal{I} x^{m-1} - \mathcal{A}x^{m-1} > 0$, then (λ, x) is called a strict Pareto H-eigenpair.

*This work is supported by the Natural Science Foundation of Shandong Province (ZR2020MA025, ZR2019PA016) and the National Natural Science Foundation of China (12071250, 11901343)

The concept of Pareto H-eigenpair of the tensor eigenvalue complementarity problems was introduced by Ling *et al.* [9] and Song *et al.* [17], which is a natural generalization of the matrix eigenvalue complementarity problem [1, 5, 14]. It is worth noting that Pareto H-eigenvalues are closely related to H-eigenvalues of $\mathcal{A}$ introduced by Lim [8] and Qi [11, 12], respectively.

Definition 1.1. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $\lambda \in \mathbb{C}, x \in \mathbb{C}^n \backslash \{0\}$. Then (λ, x) is called an eigenpair of tensor $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})^\top$. (λ, x) is called an H-eigenpair if they are both real. Further, H-eigenvalue λ of $\mathcal{A}$ is said to be H^+-eigenvalue, if its eigenvector $x \in \mathbb{R}_+^n \backslash \{0\}$.

Obviously, H^+-eigenvalues of $\mathcal{A}$ is its Pareto H-eigenvalues. However, Song *et al.* [17] pointed out that the converse results cannot hold. In order to clarify the difference between H^+-eigenvalues and Pareto H-eigenvalues, we give the following example.

Example 1.2. Let $\mathcal{A} \in \mathbb{R}^{[4,2]}$ with $a_{1112} = a_{1222} = 2, a_{2111} = a_{2122} = -2$, and other entries be all zero.

Let $(\lambda, x = (x_1, x_2)^\top)$ be a Pareto eigenpair of $(\mathcal{A}, \mathcal{I})$. Consequently,

$$\begin{cases} 2x_1(x_1^2 x_2 + x_2^3) = \lambda x_1^4; \\ 2x_2(-x_1^3 - x_1 x_2^2) = \lambda x_2^4; \\ x_1 \geq 0, x_2 \geq 0, \lambda x_1^3 - 2x_1^2 x_2 - 2x_2^3 \geq 0, \lambda x_2^3 + 2x_1^3 + 2x_1 x_2^2 \geq 0. \end{cases}$$

Thus, $(\lambda, x) = (0, (a, 0)^\top)$ and $(\lambda, x) = (0, (0, a)^\top)$ are Pareto eigenpairs of $(\mathcal{A}, \mathcal{I})$ with $a > 0$. However, $(\lambda, x) = (0, (a, 0)^\top)$ and $(\lambda, x) = (0, (0, a)^\top)$ cannot satisfy the following equations:

$$\begin{cases} 2x_1^2 x_2 + 2x_2^3 = \lambda x_1^3; \\ -2x_1^3 - 2x_1 x_2^2 = \lambda x_2^3. \end{cases}$$

As a result, Pareto H-eigenvalues of tensor eigenvalue complementarity problems make some practical problems provide more natural and exact mathematical representations for specific real difficulties. Many studies have recently been conducted on this topic [2, 3, 9, 10, 16, 24]. Ling *et al.* [9] investigated important properties of the Pareto H-eigenvalue, including the bound for the number of Pareto H-eigenvalues. Song *et al.* [15, 16] provided a large number of structured tensors to ensure the existence of solutions to tensor complementarity problems. However, finding the largest Pareto H-eigenvalue is NP-hard [9], and verifying structured tensors, such as strictly copositive tensors, is challenging [15, 16]. Thus, some researchers turned to investigating inclusion intervals to characterize the distribution of Pareto H-eigenvalues. Xu *et al.* [22] constructed S-inclusion intervals to locate Pareto H-eigenvalues, and proposed sufficient conditions to guarantee the strict copositivity of a tensor. However, choosing an inappropriate S may cause the above inclusion intervals to be inaccurate. Inspired by the articles [6, 7, 9, 10, 16, 18, 19, 20, 21], we develop inclusion intervals that do not require selecting S to locate Pareto H-eigenvalues based on dividing the tensor index set. Further, we propose some sufficient conditions to identify the strict copositivity and the strict semi-positivity by Pareto H-eigenvalue inclusion intervals under mild conditions. These constitutes the main motivation of the paper.

The remainder of this paper is organized as follows. In Section 2, important properties of the tensor eigenvalue complementarity problems are recalled. In Section 3, we propose sharp Pareto H-eigenvalue inclusion intervals for tensor the eigenvalue complementarity problems. In Section 4, we provide some sufficient conditions to check the strict copositivity and the strict semi-positivity of tensors. The given numerical experiments show their validity.

2 Preliminary

In this section, we shall begin with some definitions and important properties of Pareto H-eigenvalue [11, 15, 22].

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, we denote

$$[\mathcal{A}]_+ := ([a_{i_1 i_2 \ldots i_m}]_+) \in \mathbb{R}^{[m,n]}, [\mathcal{A}]_- := ([a_{i_1 i_2 \ldots i_m}]_-) \in \mathbb{R}^{[m,n]},$$

where $[a_{i_1 i_2 \ldots i_m}]_+ = \max\{0, a_{i_1 i_2 \ldots i_m}\}, [a_{i_1 i_2 \ldots i_m}]_- = \max\{0, -a_{i_1 i_2 \ldots i_m}\}$.

Define

$$r_i(\mathcal{A})_+ = \sum_{\delta_{i i_2 \cdots i_m}=0} [a_{i i_2 \cdots i_m}]_+, r_i(\mathcal{A})_- = \sum_{\delta_{i i_2 \cdots i_m}=0} [a_{i i_2 \cdots i_m}]_-,$$

$$r_i^j(\mathcal{A})_+ = \sum_{\substack{\delta_{i i_2 \cdots i_m} = 0, \\ \delta_{j i_2 \cdots i_m} = 0}} [a_{i i_2 \cdots i_m}]_+, r_i^j(\mathcal{A})_- = \sum_{\substack{\delta_{i i_2 \cdots i_m} = 0, \\ \delta_{j i_2 \cdots i_m} = 0}} [a_{i i_2 \cdots i_m}]_-.$$

In order to investigate the existence of solutions for the tensor eigenvalue complementarity problems, Qi [13] and Song *et al.* [15] introduced (strictly) semi-positive and (strictly) copositive tensors as follows.

Definition 2.1 (Definition 2.3 of [15]). Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is said to be

(i) semi-positive if for each $x \geq 0$ and $x \neq 0$, there exists $k \in N$ such that

$$x_k > 0 \text{ and } (\mathcal{A}x^{m-1})_k \geq 0;$$

(ii) strictly semi-positive if for each $x \geq 0$ and $x \neq 0$, there exists $k \in N$ such that

$$x_k > 0 \text{ and } (\mathcal{A}x^{m-1})_k > 0.$$

Definition 2.2. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is said to be

(i) copositive if $\mathcal{A}x^m \geq 0$ for any $x \in \mathbb{R}^n_+$;

(ii) strictly copositive if $\mathcal{A}x^m > 0$ for any $x \in \mathbb{R}^n_+ \backslash \{0\}$;

(iii) symmetric if

$$a_{i_1 \ldots i_m} = a_{i_{\pi(1)} \ldots i_{\pi(m)}}, \ \forall \ \pi \in \Gamma_m,$$

where Γ_m is the permutation group of m indices.

When $\mathcal{A}$ is symmetric, Song *et al.* [15] proposed the equivalent relation between (strictly) semi-positivity and (strictly) copositivity.

Lemma 2.3 (Theorems 3.3-3.4 of [15]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be symmetric. Then $\mathcal{A}$ is (strictly) semi-positive if and only if it is (strictly) copositive.*

Lemma 2.4 (Proposition 2.1 of [15]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ is strictly copositive, then $a_{i \cdots i} > 0$, $\forall i \in N$.*

In the following, we propose the relation between Pareto H-eigenvalues and (strictly) copositivity.

Lemma 2.5 (Corollary 3.5 of [17]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be symmetric. Then $\mathcal{A}$ is (strictly) copositive if and only if all Pareto H-eigenvalues of $\mathcal{A}$ are non-negative (positive).*

Lemma 2.6. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *and* $[\mathcal{A}]_- \in \mathbb{R}^{[m,n]}$ *be symmetric. If* $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = diag(a_{11\cdots 1}, a_{22\cdots 2}, \cdots, a_{nn\cdots n}) - [\mathcal{A}]_-$ *is strictly copositive, then* $\mathcal{A}$ *is strictly copositive.*

Proof. Since $\mathcal{A} = [\mathcal{A}]_+ - [\mathcal{A}]_-$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = diag(a_{11\cdots 1}, a_{22\cdots 2}, \cdots, a_{nn\cdots n}) - [\mathcal{A}]_-$, we define

$$\mathcal{A}^* = [\mathcal{A}]_+ - diag(a_{11\cdots 1}, a_{22\cdots 2}, \cdots, a_{nn\cdots n})$$

with $\mathcal{A} = \mathcal{A}^* + \mathcal{B}$. Consequently, $\mathcal{A}^*$ is a nonnegative tensor. Taking into account that $[\mathcal{A}]_-$ is symmetric, we obtain $\mathcal{B}$ is symmetric. Since $\mathcal{B}$ is strictly copositive, we deduce

$$\min_{\substack{x \geq 0 \\ \mathcal{I}x^m = 1}} \mathcal{A}x^m = \min_{\substack{x \geq 0 \\ \mathcal{I}x^m = 1}} (\mathcal{A}^* + \mathcal{B})x^m \geq \min_{\substack{x \geq 0 \\ \mathcal{I}x^m = 1}} \mathcal{B}x^m > 0.$$

Thus, $\mathcal{A}$ is strictly copositive. □

We end this section with the Pareto H-eigenvalue inclusion set of [22].

Lemma 2.7 (Theorem 4.5 of [22])**.** *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *and* $S \subset N$ *be a nonempty proper set. Then, we have* $\sigma(\mathcal{A}) \subseteq \Phi^S(\mathcal{A}) \bigcap \Psi(\mathcal{A})$*, where*

$$\Phi^S(\mathcal{A}) := \Bigg(\bigcup_{\substack{i \in S, \\ j \in \overline{S}}} \Phi_{i,j}(\mathcal{A}) \Bigg) \bigcup \Bigg(\bigcup_{\substack{i \in \overline{S}, \\ j \in S}} \Phi_{i,j}(\mathcal{A}) \Bigg),$$

where

$$\Phi_{i,j}(\mathcal{A}) := \Big\{ \lambda \in \mathbb{R} : (|\lambda - a_{ii\cdots i}| - r_i^j(\mathcal{A})_+)|\lambda - a_{jj\cdots j}| \leq [a_{ij\cdots j}]_+ \max\{r_j(\mathcal{A})_+, r_j(\mathcal{A})_-\} \Big\}$$
$$\bigcup \Big\{ \lambda \in \mathbb{R} : (|\lambda - a_{ii\cdots i}| - r_i^j(\mathcal{A})_-)|\lambda - a_{jj\cdots j}| \leq [a_{ij\cdots j}]_- \max\{r_j(\mathcal{A})_+, r_j(\mathcal{A})_-\} \Big\},$$

$$\Psi(\mathcal{A}) = \Big\{ \lambda \in \mathbb{R} : -n^{\frac{m-2}{2}} ||[\mathcal{A}]_-||_F \leq \lambda \leq n^{\frac{m-2}{2}} ||[\mathcal{A}]_+||_F \Big\}.$$

3 Pareto H-Eigenvalues Inclusion Intervals

As we know, choosing inappropriate an index set S might cause the above inclusion set being inaccurate in some cases. First, we propose a nonparametric Pareto H- eigenvalue inclusion interval.

Theorem 3.1. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *and* $\sigma(\mathcal{A})$ *denote the set of all Pareto* H*-eigenvalues with* $\sigma(\mathcal{A}) \neq \emptyset$*. Then,*

$$\sigma(\mathcal{A}) \subseteq Q(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{\substack{j \in N, \\ i \neq j}} \Big[Q_{i,j}(\mathcal{A}) = U_{i,j}(\mathcal{A}) \bigcup V_{i,j}(\mathcal{A}) \Big],$$

where

$$U_{i,j}(\mathcal{A}) = \Big\{ z \in \mathbb{R} : | (z - a_{i\cdots i})(z - a_{j\cdots j}) - a_{ij\cdots j}a_{ji\cdots i} |$$
$$\leq | z - a_{j\cdots j} | \max[r_i^j(\mathcal{A})_+, r_i^j(\mathcal{A})_-] + | a_{ij\cdots j} | \max[r_j^i(\mathcal{A})_+, r_j^i(\mathcal{A})_-] \Big\},$$
$$V_{i,j}(\mathcal{A}) = \Big\{ z \in \mathbb{R} : | z - a_{i\cdots i} | \leq \max[\delta_i^j(\mathcal{A})_+, \delta_i^j(\mathcal{A})_-] \Big\},$$

$$\delta_i^j(\mathcal{A})_+ = \sum_{\substack{\delta_{ii_2\cdots i_m}=0,\\ i_2,\cdots,i_m\neq j}} [a_{ii_2\cdots i_m}]_+, \delta_i^j(\mathcal{A})_- = \sum_{\substack{\delta_{ii_2\cdots i_m}=0,\\ i_2,\cdots,i_m\neq j}} [a_{ii_2\cdots i_m}]_-.$$

Proof. Let (λ, x) be a Pareto H-eigenpair. Then,

$$\lambda x_i^m = \sum_{(i_2,i_3,\cdots,i_m)\in N} a_{ii_2\cdots i_m} x_i x_{i_2}\cdots x_{i_m}, \forall i \in N. \tag{3.1}$$

Define $x_p = \max_{i\in N} x_i > 0$. For any $q \neq p$, recalling the p-th and q-th equations of (3.1), we deduce

$$\lambda x_p^m = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_p x_{i_2}\cdots x_{i_m} + a_{p\cdots p} x_p^m + a_{pq\cdots q} x_p x_q^{m-1},$$

$$\lambda x_q^m = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m} x_q x_{i_2}\cdots x_{i_m} + a_{q\cdots q} x_q^m + a_{qp\cdots p} x_q x_p^{m-1}.$$

We now break up the argument into two cases.
Case I: $x_q > 0$, for any $q \neq p$. Then,

$$(\lambda - a_{p\cdots p}) x_p^{m-1} - a_{pq\cdots q} x_q^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_{i_2}\cdots x_{i_m}, \tag{3.2}$$

$$(\lambda - a_{q\cdots q}) x_q^{m-1} - a_{qp\cdots p} x_p^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m} x_{i_2}\cdots x_{i_m}. \tag{3.3}$$

Solving (3.2) and (3.3) for x_q, we obtain

$$\begin{aligned}
&((\lambda - a_{p\cdots p})(\lambda - a_{q\cdots q}) - a_{pq\cdots q} a_{qp\cdots p}) x_p^{m-1} \\
&= (\lambda - a_{q\cdots q}) \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_{i_2}\cdots x_{i_m} + a_{pq\cdots q} \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= (\lambda - a_{q\cdots q}) \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_{i_2}\cdots x_{i_m} \\
&+ a_{pq\cdots q} \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} ([a_{qi_2\cdots i_m}]_+ - [a_{qi_2\cdots i_m}]_-) x_{i_2}\cdots x_{i_m}.
\end{aligned} \tag{3.4}$$

Taking modulus in the above equation (3.4) and using the triangle inequality yield

$$
\begin{aligned}
&| (\lambda - a_{p\cdots p})(\lambda - a_{q\cdots q}) - a_{pq\cdots q}a_{qp\cdots p} \mid x_p^{m-1} \\
&\leq | \lambda - a_{q\cdots q} || \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_+ x_{i_2}\cdots x_{i_m} - \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_- x_{i_2}\cdots x_{i_m} \mid \\
&+ | a_{pq\cdots q} || \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_+ x_{i_2}\cdots x_{i_m} - \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_- x_{i_2}\cdots x_{i_m} \mid \\
&\leq | \lambda - a_{q\cdots q} \mid \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_+ x_{i_2}\cdots x_{i_m}, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_- x_{i_2}\cdots x_{i_m}\} \\
&+ | a_{pq\cdots q} \mid \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_+ x_{i_2}\cdots x_{i_m}, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_- x_{i_2}\cdots x_{i_m}\} \\
&\leq | \lambda - a_{q\cdots q} \mid \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_+, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_-\} x_p^{m-1} \\
&+ | a_{pq\cdots q} \mid \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_+, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_-\} x_p^{m-1},
\end{aligned}
$$

where the second inequality holds from $|a - b| \leq \max\{a, b\}$, a and b are two nonnegative real numbers. That is,

$$
\begin{aligned}
&| (\lambda - a_{p\cdots p})(\lambda - a_{q\cdots q}) - a_{pq\cdots q}a_{qp\cdots p} \mid \\
&\leq | \lambda - a_{q\cdots q} \mid \max\{r_p^q(\mathcal{A})_+, r_p^q(\mathcal{A})_-\} + | a_{pq\cdots q} \mid \max\{r_q^p(\mathcal{A})_+, r_q^p(\mathcal{A})_-\}.
\end{aligned}
$$

Case II: $x_q = 0$, for any $q \neq p$, one has

$$
(\lambda - a_{p\cdots p})x_p^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_{i_2}\cdots x_{i_m}. \tag{3.5}
$$

Taking modulus in the above equation (3.5) and using the triangle inequality give

$$
\begin{aligned}
&| \lambda - a_{p\cdots p} \mid x_p^{m-1} \leq | \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_{i_2}\cdots x_{i_m} \mid \\
&\leq \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_+ x_{i_2}\cdots x_{i_m}, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_- x_{i_2}\cdots x_{i_m}\} \\
&\leq \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_+, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_-\} x_p^{m-1}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
| \lambda - a_{p\cdots p} \mid &\leq \max\{ \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_+, \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ i_2,\cdots,i_m \neq q}} [a_{pi_2\cdots i_m}]_-\} \\
&= \max\{\delta_p^q(\mathcal{A})_+, \delta_p^q(\mathcal{A})_-\}.
\end{aligned}
$$

Summarizing Cases I and II, we have $\lambda \in Q_{p,q}(\mathcal{A})$. From the arbitrariness of q, we deduce $\lambda \in \bigcap_{j\in N, j\neq p} Q_{p,j}(\mathcal{A})$. Consequently, $\lambda \in \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} Q_{i,j}(\mathcal{A})$. □

We obtain inclusion interval $Q(\mathcal{A})$ by computing $n(n-1)$ intervals $Q_{i,j}(\mathcal{A})$ by Theorem 3.1. By S-partitioning index set of $\mathcal{A}$, we propose improved S-inclusion interval $M^S(\mathcal{A})$ by computing $2|S|(n-|S|)$ intervals $M^S_{i,j}(\mathcal{A})$ and reduce the calculation cost.

Given a nonempty proper set $S \subset N$, we define

$$\triangle^N := \{(i_2, i_3, \cdots, i_m) : each\ i_j \in N\ for\ j = 2, \cdots, m\},$$

$$\triangle^S := \{(i_2, i_3, \cdots, i_m) : each\ i_j \in S\ for\ j = 2, \cdots, m\},$$

and then

$$\overline{\triangle^S} = \triangle^N \backslash \triangle^S.$$

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $i \in S$, we have

$$r_i(\mathcal{A})_+ = r_i^{\triangle^S}(\mathcal{A})_+ + r_i^{\overline{\triangle^S}}(\mathcal{A})_+, r_i(\mathcal{A})_- = r_i^{\triangle^S}(\mathcal{A})_- + r_i^{\overline{\triangle^S}}(\mathcal{A})_-,$$

where

$$r_i^{\triangle^S}(\mathcal{A})_+ = \sum_{\substack{(i_2,\cdots,i_m)\in\triangle^S,\\ \delta_{ii_2\cdots i_m}=0}} [a_{ii_2\cdots i_m}]_+, r_i^{\triangle^S}(\mathcal{A})_- = \sum_{\substack{(i_2,\cdots,i_m)\in\triangle^S\\ \delta_{ii_2\cdots i_m}=0}} [a_{ii_2\cdots i_m}]_-,$$

$$r_i^{\overline{\triangle^S}}(\mathcal{A})_+ = \sum_{(i_2,\cdots,i_m)\in\overline{\triangle^S}} [a_{ii_2\cdots i_m}]_+, r_i^{\overline{\triangle^S}}(\mathcal{A})_- = \sum_{(i_2,\cdots,i_m)\in\overline{\triangle^S}} [a_{ii_2\cdots i_m}]_-.$$

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $S \subset N$ a nonempty proper set. Then,*

$$\sigma(\mathcal{A}) \subseteq M^S(\mathcal{A}) = \bigcup_{\substack{i\in S,\\ j\in\overline{S}}} M^S_{i,j}(\mathcal{A}) \bigcup \bigcup_{\substack{i\in\overline{S},\\ j\in S}} M^{\overline{S}}_{i,j}(\mathcal{A}),$$

where

$$\begin{aligned} M^S_{i,j}(\mathcal{A}) = \Big\{ & z \in \mathbb{R} : |z - a_{i\cdots i}|(|z - a_{j\cdots j}| - \max\{r_j^{\overline{\triangle^S}}(\mathcal{A})_+, r_j^{\overline{\triangle^S}}(\mathcal{A})_-\}) \\ & \leq \max\{r_i(\mathcal{A})_+, r_i(\mathcal{A})_-\} \max\{r_j^{\triangle^S}(\mathcal{A})_+, r_j^{\triangle^S}(\mathcal{A})_-\} \Big\}, \\ M^{\overline{S}}_{i,j}(\mathcal{A}) = \Big\{ & z \in \mathbb{R} : |z - a_{i\cdots i}|(|z - a_{j\cdots j}| - \max\{r_j^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+, r_j^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_-\}) \\ & \leq \max\{r_i(\mathcal{A})_+, r_i(\mathcal{A})_-\} \max\{r_j^{\triangle^{\overline{S}}}(\mathcal{A})_+, r_j^{\triangle^{\overline{S}}}(\mathcal{A})_-\} \Big\}. \end{aligned}$$

Proof. Let (λ, x) be a Pareto H-eigenpair. Setting $x_p = \max_{i\in S} x_i$ and $x_q = \max_{i\in\overline{S}} x_i$, one has $\max\{x_p, x_q\} > 0$. We now break up the argument into three cases.

Case I: $x_p x_q > 0$ and $x_p \geq x_q$. That is, $x_p = \max_{i\in N} x_i$. Recalling the p-th equation of (3.1), we have

$$
\begin{aligned}
(\lambda - a_{p\cdots p})x_p^m &= \sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} a_{pi_2\cdots i_m} x_p x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_p x_{i_2}\cdots x_{i_m} \\
&= \sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_p x_{i_2}\cdots x_{i_m} \\
&+ \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_p x_{i_2}\cdots x_{i_m}.
\end{aligned}
$$

Taking modulus in the above equation and using the triangle inequality, one has

$$
\begin{aligned}
&|\lambda - a_{p\cdots p}|x_p^m \leq |\sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_p x_{i_2}\cdots x_{i_m}| \\
&+ |\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} ([a_{pi_2\cdots i_m}]_+ - [a_{pi_2\cdots i_m}]_-) x_p x_{i_2}\cdots x_{i_m}| \\
&\leq \max\{\sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}\} \\
&+ \max\{\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}\} \\
&\leq \max\{\sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} [a_{pi_2\cdots i_m}]_+, \sum_{(i_2,\cdots,i_m)\in\triangle^{\overline{S}}} [a_{pi_2\cdots i_m}]_-\} x_p x_q^{m-1} \\
&+ \max\{\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_+, \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^{\overline{S}}},\\ \delta_{pi_2\cdots i_m}=0}} [a_{pi_2\cdots i_m}]_-\} x_p^m \\
&= \max\{r_p^{\triangle^{\overline{S}}}(\mathcal{A})_+, r_p^{\triangle^{\overline{S}}}(\mathcal{A})_-\} x_p x_q^{m-1} + \max\{r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+, r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_-\} x_p^m,
\end{aligned}
$$

where the second inequality holds from $|a-b| \leq \max\{a,b\}$. Hence,

$$
(|\lambda - a_{p\cdots p}| - \max\{r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+, r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_-\})x_p^{m-1} \leq \max\{r_p^{\triangle^{\overline{S}}}(\mathcal{A})_+, r_p^{\triangle^{\overline{S}}}(\mathcal{A})_-\} x_q^{m-1}. \quad (3.6)
$$

Meanwhile, it follows from the q-th equation of (3.1) that

$$
\begin{aligned}
(\lambda - a_{q\cdots q})x_q^m &= \sum_{\substack{(i_2,\cdots,i_m)\in N,\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m} x_q x_{i_2}\cdots x_{i_m} \\
&= \sum_{\substack{(i_2,\cdots,i_m)\in N,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\cdots,i_m)\in N,\\ \delta_{qi_2\cdots i_m}=0}} [a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}
\end{aligned}
$$

and

$$
|\lambda - a_{q\cdots q}|x_q^{m-1} \leq \max\{r_q(\mathcal{A})_+, r_q(\mathcal{A})_-\} x_p^{m-1}. \quad (3.7)
$$

Multiplying (3.6) with (3.7) gives

$$\begin{aligned}&|\lambda - a_{q\cdots q}|(|\lambda - a_{p\cdots p}| - \max\{r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+, r_p^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_-\})\\ &\le \max\{r_q(\mathcal{A})_+, r_q(\mathcal{A})_-\}\max\{r_p^{\triangle^{\overline{S}}}(\mathcal{A})_+, r_p^{\triangle^{\overline{S}}}(\mathcal{A})_-\},\end{aligned}$$

which means that $\lambda \in M_{q,p}^{\overline{S}}(\mathcal{A}) \subseteq M^S(\mathcal{A})$.
Case II: $x_p x_q > 0$ and $x_q \ge x_p$. That is, $x_q = \max_{i\in N} x_i$. Referring to the q-th equation of (3.1), we get

$$\begin{aligned}(\lambda - a_{q\cdots q})x_q^m &= \sum_{(i_2,\cdots,i_m)\in\triangle^S} a_{qi_2\cdots i_m}x_q x_{i_2}\cdots x_{i_m} + \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m}x_q x_{i_2}\cdots x_{i_m}\\ &= \sum_{(i_2,\cdots,i_m)\in\triangle^S}([a_{qi_2\cdots i_m}]_+ - [a_{qi_2\cdots i_m}]_-)x_q x_{i_2}\cdots x_{i_m}\\ &\quad+ \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}([a_{qi_2\cdots i_m}]_+ - [a_{qi_2\cdots i_m}]_-)x_q x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Taking modulus in the equation above and using the triangle inequality, we deduce

$$\begin{aligned}&|\lambda - a_{q\cdots q}|x_q^m \le |\sum_{(i_2,\cdots,i_m)\in\triangle^S}([a_{qi_2\cdots i_m}]_+ - [a_{qi_2\cdots i_m}]_-)x_q x_{i_2}\cdots x_{i_m}|\\ &+|\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}([a_{qi_2\cdots i_m}]_+ - [a_{qi_2\cdots i_m}]_-)x_q x_{i_2}\cdots x_{i_m}|\\ &\le \max\{\sum_{(i_2,\cdots,i_m)\in\triangle^S}[a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m}, \sum_{(i_2,\cdots,i_m)\in\triangle^S}[a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}\}\\ &+\max\{\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}[a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m}, \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}[a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}\}\\ &\le \max\{\sum_{(i_2,\cdots,i_m)\in\triangle^S}[a_{qi_2\cdots i_m}]_+, \sum_{(i_2,\cdots,i_m)\in\triangle^S}[a_{qi_2\cdots i_m}]_-\}x_q x_p^{m-1}\\ &+\max\{\sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}[a_{qi_2\cdots i_m}]_+, \sum_{\substack{(i_2,\cdots,i_m)\in\overline{\triangle^S},\\ \delta_{qi_2\cdots i_m}=0}}[a_{qi_2\cdots i_m}]_-\}x_q^m\\ &= \max\{r_q^{\triangle^S}(\mathcal{A})_+, r_q^{\triangle^S}(\mathcal{A})_-\}x_q x_p^{m-1} + \max\{r_q^{\overline{\triangle^S}}(\mathcal{A})_+, r_q^{\overline{\triangle^S}}(\mathcal{A})_-\}x_q^m.\end{aligned}$$

Hence,

$$(|\lambda - a_{q\cdots q}| - \max\{r_q^{\overline{\triangle^S}}(\mathcal{A})_+, r_q^{\overline{\triangle^S}}(\mathcal{A})_-\})x_q^{m-1} \le \max\{r_q^{\triangle^S}(\mathcal{A})_+, r_q^{\triangle^S}(\mathcal{A})_-\}x_p^{m-1}. \quad (3.8)$$

Following the similar arguments to the proof of (3.7), we obtain

$$|\lambda - a_{p\cdots p}|x_p^{m-1} \le \max\{r_p(\mathcal{A})_+, r_p(\mathcal{A})_-\}x_q^{m-1}. \quad (3.9)$$

Multiplying (3.8) with (3.9) gives

$$\begin{aligned}&|\lambda - a_{p\cdots p}|(|\lambda - a_{q\cdots q}| - \max\{r_q^{\overline{\triangle^S}}(\mathcal{A})_+, r_q^{\overline{\triangle^S}}(\mathcal{A})_-\})\\ &\le \max\{r_p(\mathcal{A})_+, r_p(\mathcal{A})_-\}\max\{r_q^{\triangle^S}(\mathcal{A})_+, r_q^{\triangle^S}(\mathcal{A})_-\},\end{aligned}$$

which shows that $\lambda \in M^S_{p,q}(\mathcal{A}) \subseteq M^S(\mathcal{A})$.
Case III: $x_p x_q = 0$. Without loss of generality, let $x_p > 0$ and $x_q = 0$. Then by (3.6),

$$|\lambda - a_{p\cdots p}| - \max\{r_p^{\overline{\triangle^S}}(\mathcal{A})_+, r_p^{\overline{\triangle^S}}(\mathcal{A})_-\} \leq 0.$$

For any $q \in \bar{S}$, it holds that

$$\begin{aligned} &|\lambda - a_{q\cdots q}|(|\lambda - a_{p\cdots p}| - \max\{r_p^{\overline{\triangle^S}}(\mathcal{A})_+, r_p^{\overline{\triangle^S}}(\mathcal{A})_-\}) \\ &\leq \max\{r_q(\mathcal{A})_+, r_q(\mathcal{A})_-\} \max\{r_p^{\triangle^S}(\mathcal{A})_+, r_p^{\triangle^S}(\mathcal{A})_-\}, \end{aligned}$$

which implies that $\lambda \in M^S_{q,p}(\mathcal{A}) \subseteq M^S(\mathcal{A})$.
Combining Cases I, II and III, we conclude the desired results. □

Next, we introduce Example 4.3 of [22] to show that the results in Theorem 3.1 and Theorem 3.2 are sharper than that in Theorem 4.5 of [22] under certain cases.

Example 3.3. Let $\mathcal{A} \in \mathbb{R}^{[3,3]}$ with $a_{233} = 1, a_{322} = -1, a_{231} = -5, a_{312} = 6, a_{321} = -6$, and other entries be all zero.

Recalling Theorem 3.1 and Theorem 3.2, we obtain

$$\begin{aligned} &r_1(\mathcal{A})_+ = 0, r_1(\mathcal{A})_- = 0, r_1^2(\mathcal{A})_+ = 0, r_1^2(\mathcal{A})_- = 0, r_1^3(\mathcal{A})_+ = 0, r_1^3(\mathcal{A})_- = 0, \\ &r_2(\mathcal{A})_+ = 1, r_2(\mathcal{A})_- = 5, r_2^1(\mathcal{A})_+ = 1, r_2^1(\mathcal{A})_- = 5, r_2^3(\mathcal{A})_+ = 0, r_2^3(\mathcal{A})_- = 5, \\ &r_3(\mathcal{A})_+ = 6, r_3(\mathcal{A})_- = 7, r_3^1(\mathcal{A})_+ = 6, r_3^1(\mathcal{A})_- = 7, r_3^2(\mathcal{A})_+ = 6, r_3^2(\mathcal{A})_- = 6. \end{aligned}$$

Recalling that $S = \{1, 3\}, \overline{S} = \{2\}$ in [22], we compute $\Phi^S(\mathcal{A})$ as follows:

$$\Phi^S(\mathcal{A}) = [-3 - \sqrt{14}, 3 + \sqrt{14}] \approx [-6.7417, 6.7417].$$

Following the classification of $S = \{1, 3\}, \overline{S} = \{2\}$ of [22], we have

$$\begin{aligned} &r_1^{\triangle^{\overline{S}}}(\mathcal{A})_+ = 0, r_1^{\triangle^{\overline{S}}}(\mathcal{A})_- = 0, r_1^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+ = 0, r_1^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_- = 0, \\ &r_2^{\triangle^S}(\mathcal{A})_+ = 1, r_2^{\triangle^S}(\mathcal{A})_- = 5, r_2^{\overline{\triangle^S}}(\mathcal{A})_+ = 0, r_2^{\overline{\triangle^S}}(\mathcal{A})_- = 0, \\ &r_3^{\triangle^{\overline{S}}}(\mathcal{A})_+ = 0, r_3^{\triangle^{\overline{S}}}(\mathcal{A})_- = 1, r_3^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_+ = 6, r_3^{\overline{\triangle^{\overline{S}}}}(\mathcal{A})_- = 6, \end{aligned}$$

and

$$\delta_i^j(\mathcal{A})_+ = 0, \delta_i^j(\mathcal{A})_- = 0, \forall i, j = 1, 2, 3, i \neq j.$$

According to Theorem 3.2, we obtain

$$\begin{aligned} M^S(\mathcal{A}) &= M^S_{1,2}(\mathcal{A}) \bigcup M^S_{3,2}(\mathcal{A}) \bigcup M^{\overline{S}}_{2,1}(\mathcal{A}) \bigcup M^{\overline{S}}_{2,3}(\mathcal{A}) \\ &= [-\sqrt{35}, \sqrt{35}] \approx [-5.9161, 5.9161] \subseteq [-3 - \sqrt{14}, 3 + \sqrt{14}] = \Phi^S(\mathcal{A}), \end{aligned}$$

where $M^S_{1,2}(\mathcal{A}) = \{0\}, M^S_{3,2}(\mathcal{A}) = [-\sqrt{35}, \sqrt{35}], M^{\overline{S}}_{2,1}(\mathcal{A}) = \{0\}, M^{\overline{S}}_{2,3}(\mathcal{A}) = [-\sqrt{11}, \sqrt{11}]$.
Referring to Theorem 3.1, one has

$$\begin{aligned} Q(\mathcal{A}) &= [Q_{1,2}(\mathcal{A}) \bigcap Q_{1,3}(\mathcal{A})] \bigcup [Q_{2,1}(\mathcal{A}) \bigcap Q_{2,3}(\mathcal{A})] \bigcup [Q_{3,1}(\mathcal{A}) \bigcap Q_{3,2}(\mathcal{A})] \\ &= [-\frac{5 + \sqrt{41}}{2}, \frac{5 + \sqrt{41}}{2}] \approx [-5.7016, 5.7016] \subseteq [-3 - \sqrt{14}, 3 + \sqrt{14}] = \Phi^S(\mathcal{A}), \end{aligned}$$

where $Q_{1,2}(\mathcal{A}) \bigcap Q_{1,3}(\mathcal{A}) = \{0\}, Q_{2,1}(\mathcal{A}) \bigcap Q_{2,3}(\mathcal{A}) = [-5, 5], Q_{3,1}(\mathcal{A}) \bigcap Q_{3,2}(\mathcal{A}) = [-\frac{5+\sqrt{41}}{2}, \frac{5+\sqrt{41}}{2}]$.

4 Checking the Strict Copositivity of Tensors

In this section, we establish sharp sufficient conditions to verify the strict copositivity of real tensors based on Theorems 3.1-3.2. We begin this section with a sufficient condition for judging strict copositivity of [22].

Lemma 4.1 (Theorem 4.9 of [22])**.** *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $[\mathcal{A}]_- \in \mathbb{R}^{[m,n]}$ be symmetric, and $a_{ii\cdots i} \geq r_i(\mathcal{A})_-$ for each $i \in [n]$. If $a_{ij\cdots j} < 0$ for each $i \in [n]$ and $j \in [n]\backslash\{i\}$, and there exists $k \in [n]$ such that $a_{kk\cdots k} > r_k(\mathcal{A})_-$, then $\mathcal{A}$ is strictly copositive.*

Theorem 4.2. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $[\mathcal{A}]_- \in \mathbb{R}^{[m,n]}$ be symmetric. Let $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = diag(a_{11\cdots 1}, a_{22\cdots 2}, \cdots, a_{nn\cdots n}) - [\mathcal{A}]_-$. If $b_{i\cdots i} > 0$, for each $i, j \in N, j \neq i$ such that*

$$b_{j\cdots j}(b_{i\cdots i} - r_i^j(\mathcal{B})_-) > \mid b_{ij\cdots j} b_{ji\cdots i} \mid + \mid b_{ji\cdots i} \mid r_j^i(\mathcal{B})_- \tag{4.1}$$

and

$$b_{i\cdots i} > \delta_i^j(\mathcal{B})_-, \tag{4.2}$$

then $\mathcal{A}$ is strictly copositive and strictly semi-positive.

Proof. Let $\lambda \in Q(\mathcal{B})$ be a Pareto H-eigenvalue. By the definition of $\mathcal{B}$, we obtain $b_{i\cdots i} = a_{i\cdots i}$ for all $i \in N$. Suppose on the contrary that $\lambda \leq 0$. Now, we apply Theorem 3.1 to tensor $\mathcal{B}$ and break up the argument into two cases.
Case I. There exists $p \in N$ such that $\lambda \in U_{p,q}(\mathcal{B})$ for all $q \in N, q \neq p$. That is,

$$\begin{aligned}
&|(\lambda - b_{p\cdots p})(\lambda - b_{q\cdots q}) - b_{pq\cdots q} b_{qp\cdots p} \mid \\
&\leq \mid \lambda - b_{q\cdots q} \mid \max\{r_p^q(\mathcal{B})_+, r_p^q(\mathcal{B})_-\} + |b_{pq\cdots q}| \max\{r_q^p(\mathcal{B})_+, r_q^p(\mathcal{B})_-\} \\
&= \mid \lambda - b_{q\cdots q} \mid r_p^q(\mathcal{B})_- + |b_{pq\cdots q}| r_q^p(\mathcal{B})_-.
\end{aligned}$$

Further,

$$\begin{aligned}
&|(\lambda - b_{p\cdots p})(\lambda - b_{q\cdots q}) \mid - \mid b_{pq\cdots q} b_{qp\cdots p} \mid \leq \mid (\lambda - b_{p\cdots p})(\lambda - b_{q\cdots q}) - b_{pq\cdots q} b_{qp\cdots p} \mid \\
&\leq \mid \lambda - b_{q\cdots q} \mid r_p^q(\mathcal{B})_- + |b_{pq\cdots q}| r_q^p(\mathcal{B})_-,
\end{aligned}$$

equivalently,

$$\mid \lambda - b_{q\cdots q} \mid (\mid \lambda - b_{p\cdots p} \mid - r_p^q(\mathcal{B})_-) \leq \mid b_{pq\cdots q} b_{qp\cdots p} \mid + |b_{pq\cdots q}| r_q^p(\mathcal{B})_-. \tag{4.3}$$

It follows from (4.1), (4.3), $b_{i\cdots i} > 0$ and $\lambda \leq 0$ that

$$\begin{aligned}
0 &\leq \mid b_{pq\cdots q} b_{qp\cdots p} \mid + |b_{pq\cdots q}| r_q^p(\mathcal{B})_- < b_{q\cdots q}(b_{p\cdots p} - r_p^q(\mathcal{B})_-) \\
&\leq \mid \lambda - b_{q\cdots q} \mid (\mid \lambda - b_{p\cdots p} \mid - r_p^q(\mathcal{B})_-) \leq \mid b_{pq\cdots q} b_{qp\cdots p} \mid + |b_{pq\cdots q}| r_q^p(\mathcal{B})_-,
\end{aligned}$$

which the contradiction arises. Thus, $\lambda > 0$.
Case II. There exists $p \in N$ such that $\lambda \in V_{p,q}(\mathcal{B})$ for all $q \in N, q \neq p$. That is

$$\mid \lambda - b_{p\cdots p} \mid \leq \max\{\delta_p^q(\mathcal{B})_+, \delta_p^q(\mathcal{B})_-\} = \delta_p^q(\mathcal{B})_-.$$

Taking into account $b_{i\cdots i} > 0$ and $\lambda \leq 0$, one has

$$b_{p\cdots p} \leq \delta_p^q(\mathcal{B}),$$

which contradicts (4.2). Thus, $\lambda > 0$.

Combining Cases I and II, we obtain $\mathcal{A}$ and $\mathcal{B}$ are strictly copositive by Lemmas 2.5 and 2.6. □

Theorem 4.3. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}, [\mathcal{A}]_- \in \mathbb{R}^{[m,n]}$ *be symmetric, and* $S \subset N$ *a nonempty proper set. Let* $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = diag(a_{11\cdots 1}, a_{22\cdots 2}, \cdots, a_{nn\cdots n}) - [\mathcal{A}]_-$. *If* $b_{i\cdots i} > 0$, *for each* $i \in S$ *and each* $j \in \overline{S}$ *such that*

$$b_{i\cdots i}(b_{j\cdots j} - r_j^{\overline{\triangle S}}(\mathcal{B})_-) > r_i(\mathcal{B})_- r_j^{\triangle S}(\mathcal{B})_- \tag{4.4}$$

and

$$b_{j\cdots j}(b_{i\cdots i} - r_i^{\overline{\triangle \overline{S}}}(\mathcal{B})_-) > r_j(\mathcal{B})_- r_i^{\triangle \overline{S}}(\mathcal{B})_-, \tag{4.5}$$

then $\mathcal{A}$ *is strictly copositive and strictly semi-positive.*

Proof. Let $\lambda \in M^S(\mathcal{B})$ be a Pareto H-eigenvalue. Suppose on the contrary that $\lambda \leq 0$. Now, we apply Theorem 3.2 to tensor $\mathcal{B}$ and break up the argument into two cases.
Case I. There exists $p \in S$ and $q \in \overline{S}$ with $\lambda \in M^S_{p,q}(\mathcal{B})$, that is,

$$\begin{aligned}
&|\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - r_q^{\overline{\triangle S}}(\mathcal{B})_-) \\
=& |\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - \max\{r_q^{\overline{\triangle S}}(\mathcal{B})_+, r_q^{\overline{\triangle S}}(\mathcal{B})_-\}) \\
\leq& \max\{r_p(\mathcal{B})_+, r_p(\mathcal{B})_-\} \max\{r_q^{\triangle S}(\mathcal{B})_+, r_q^{\triangle S}(\mathcal{B})_-\} = r_p(\mathcal{B})_- r_q^{\triangle S}(\mathcal{B})_-.
\end{aligned} \tag{4.6}$$

Further, It follows from (4.4), (4.6), $b_{i\cdots i} > 0$ and $\lambda \leq 0$ that

$$\begin{aligned}
0 &\leq r_p(\mathcal{B})_- r_q^{\triangle S}(\mathcal{B})_- < b_{p\cdots p}(b_{q\cdots q} - r_q^{\overline{\triangle S}}(\mathcal{B})_-) \\
&\leq |\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - r_q^{\overline{\triangle S}}(\mathcal{B})_-) \leq r_p(\mathcal{B})_- r_q^{\triangle S}(\mathcal{B})_-,
\end{aligned}$$

which the contradiction arises. Thus, $\lambda > 0$.
Case II. There exists $p \in \overline{S}$ and $q \in S$ with $\lambda \in M^{\overline{S}}_{p,q}(\mathcal{B})$, equivalently,

$$\begin{aligned}
&|\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - r_q^{\overline{\triangle S}}(\mathcal{B})_-) \\
=& |\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - \max\{r_q^{\overline{\triangle \overline{S}}}(\mathcal{B})_+, r_q^{\overline{\triangle \overline{S}}}(\mathcal{B})_-\}) \\
\leq& \max\{r_p(\mathcal{B})_+, r_p(\mathcal{B})_-\} \max\{r_q^{\triangle \overline{S}}(\mathcal{B})_+, r_q^{\triangle \overline{S}}(\mathcal{B})_-\} = r_p(\mathcal{B})_- r_q^{\triangle \overline{S}}(\mathcal{B})_-.
\end{aligned} \tag{4.7}$$

Using (4.5), (4.7), $b_{i\cdots i} > 0$ and $\lambda \leq 0$, we deduce

$$\begin{aligned}
0 &\leq r_p(\mathcal{B})_- r_q^{\triangle \overline{S}}(\mathcal{B})_- < b_{p\cdots p}(b_{q\cdots q} - r_q^{\overline{\triangle \overline{S}}}(\mathcal{B})_-) \\
&\leq |\lambda - b_{p\cdots p}| (|\lambda - b_{q\cdots q}| - r_q^{\overline{\triangle \overline{S}}}(\mathcal{B})_-) \leq r_p(\mathcal{B})_- r_q^{\triangle \overline{S}}(\mathcal{B})_-,
\end{aligned}$$

which the contradiction arises. Thus, $\lambda > 0$.

Summing up Cases I and II, we obtain $\mathcal{A}$ and $\mathcal{B}$ are strictly copositive by Lemmas 2.5 and 2.6. □

Remark 4.4. By Lemma 2.3, we obtain that the strict copositivity is equivalent to the strict semi-positivity of a tensor under the condition that it is symmetric. Thus, Theorems 4.2 and 4.3 provide sharp conditions to verify the strict semi-positivity of $\mathcal{A}$ when $[\mathcal{A}]_-$ is symmetric. Meanwhile, Xu *et al.* [23] introduced generalized row strictly diagonally dominant tensors if and only if $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ satisfies

$$a_{i\cdots i} > r_i(\mathcal{A})_-, \forall i \in N.$$

In analogy to the generalized row strictly diagonally dominant condition, Theorems 4.2 and 4.3 can guarantee the strict semi-positivity under weak conditions. Therefore, Theorems 4.2 and 4.3 can be regarded as a generalization of the conclusion of article [23].

Remark 4.5. To identify the strict copositivity, we require $[\mathcal{A}]_-$ is symmetric in Theorems 4.2-4.3. For general tensors, symmetry is a relatively strict condition. Importantly, $\mathcal{A}x^m$ can be strictly copositive even if $[\mathcal{A}]_-$ is not symmetric. To tackle this problem, we may symmetrize the tensors $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ as follows:

$$\tilde{a}_{i_1 i_2 \cdots i_m} = \begin{cases} a_{i_1 i_2 \cdots i_m} & \text{if } i_1 = i_2 = \cdots = i_m, \\ \frac{1}{m!} \sum_{i_2 \cdots i_m \in \Gamma_m} a_{i_1 i_2 \cdots i_m} & \text{otherwise,} \end{cases}$$

where $\tilde{\mathcal{A}} = (\tilde{a}_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is the symmetrization tensor under permutation group Γ_m.

Remark 4.6. Strict semi-positivity of tensors is important to guarantee the existence of solutions for tensor eigenvalue complementarity problems. However, identifying the strict semi-positivity is not easy [16]. Based on Theorems 4.2 and 4.3, we can quickly check whether $\mathcal{A}$ is strictly semi-positive when $[\mathcal{A}]_-$ is symmetric or symmetrization tensor $[\tilde{\mathcal{A}}]_-$ is symmetric.

The following example shows that the results given in Theorem 4.2 and 4.3 can verify the strict copositivity of tensors more accurate than that of Theorem 4.9 of [22] under certain cases.

Example 4.7. Let $\mathcal{A} \in \mathbb{R}^{[3,2]}$ with

$$a_{111} = 3, a_{112} = -1, a_{121} = -1, a_{122} = 1,$$

$$a_{222} = 1, a_{212} = 0, a_{221} = 0, a_{211} = -1.$$

It is easy to see that $[\mathcal{A}]_-$ is symmetric with

$$a_{111} = 3 > 2 = r_1(\mathcal{A})_-, a_{222} = 1 \geq 1 = r_2(\mathcal{A})_-.$$

However, $a_{122} = 1 > 0$. Therefore, we cannot judge whether $\mathcal{A}$ is strictly copositive by Theorem 4.9 of [22].
According to Theorem 4.2, we have

$$b_{111} = 3, b_{112} = -1, b_{121} = -1, b_{222} = 1, b_{211} = -1,$$

$$r_1^2(\mathcal{B})_- = 2, r_2^1(\mathcal{B})_- = 0, \delta_1^2(\mathcal{B})_- = 0, \delta_2^1(\mathcal{B})_- = 0,$$

and

$$b_{222}(b_{111} - r_1^2(\mathcal{B})_-) = 1 > \mid b_{122} b_{211} \mid + \mid b_{211} \mid r_2^1(\mathcal{B})_- = 0,$$

$$b_{111} = 3 > \delta_1^2(\mathcal{B})_- = 0,$$

$$b_{111}(b_{222} - r_2^1(\mathcal{B})_-) = 3 > \mid b_{211} b_{122} \mid + \mid b_{122} \mid r_1^2(\mathcal{B})_- = 0,$$

$$b_{222} = 1 > \delta_2^1(\mathcal{B})_- = 0.$$

The conditions of Theorem 4.2 are satisfied, which show that $\mathcal{B}$ is strictly copositive. Further, $\mathcal{A}$ is strictly copositive and strictly semi-positive by Lemmas 2.3 and 2.6.
By Theorem 4.3, set $S = \{1\}$ and $\overline{S} = \{2\}$, we have

$$r_1^{\triangle^{\overline{S}}}(\mathcal{B})_- = 0, r_1^{\overline{\triangle^{\overline{S}}}}(\mathcal{B})_- = 2, r_2^{\triangle^{S}}(\mathcal{B})_- = 1, r_2^{\overline{\triangle^{S}}}(\mathcal{B})_- = 0,$$

and

$$b_{111}(b_{222} - r_2^{\overline{\triangle^S}}(\mathcal{B})_-) = 3 > r_1(\mathcal{B})_- r_2^{\triangle^S}(\mathcal{B})_- = 2,$$

$$b_{222}(b_{111} - r_1^{\overline{\triangle^{\overline{S}}}}(\mathcal{B})_-) = 1 > r_2(\mathcal{B})_- r_1^{\triangle^{\overline{S}}}(\mathcal{B})_- = 0,$$

which means that $\mathcal{B}$ is strictly copositive. Further, $\mathcal{A}$ is strictly copositive and strictly semi-positive by Lemmas 2.3 and 2.6. Indeed, we can verify

$$\mathcal{B}x^3 = 3x_1^3 - 3x_1^2x_2 + x_2^3 = (x_1 - x_2)^2(2x_1 + x_2) + x_1^3 > 0, \forall x \in \mathbb{R}_+^2 \backslash \{0\},$$

which shows that $\mathcal{A}$ and $\mathcal{B}$ are strictly copositive and strictly semi-positive.

When $[\mathcal{A}]_-$ is asymmetric, we still verify the strict copositivity of $\mathcal{A}$ by Theorems 4.2-4.3.

Example 4.8. Let $\mathcal{A} \in \mathbb{R}^{[3,2]}$ with

$$a_{111} = 3, a_{112} = -2, a_{121} = -1, a_{122} = 1,$$

$$a_{222} = 1, a_{212} = 0, a_{221} = 0, a_{211} = 0.$$

Since $[a_{112}]_- = 2, [a_{121}]_- = 1$ and $[a_{211}]_- = 0$, we know that $[\mathcal{A}]_-$ is asymmetric. Therefore, we cannot directly use Theorems 4.2-4.3 to judge whether $\mathcal{A}$ is strictly copositive. Symmetrizing $[\mathcal{A}]_-$, we obtain $[\tilde{\mathcal{A}}]_-$ with

$$\widetilde{a}_{111} = 0, \widetilde{a}_{112} = -1, \widetilde{a}_{121} = -1, \widetilde{a}_{122} = 0,$$

$$\widetilde{a}_{222} = 0, \widetilde{a}_{212} = 0, \widetilde{a}_{221} = 0, \widetilde{a}_{211} = -1.$$

It is easy to see that $[\widetilde{\mathcal{A}}]_-$ is symmetric and $\widetilde{\mathcal{B}} = \mathcal{B}$ of Example 4.1. Thus, $\mathcal{A}$ and $\mathcal{B}$ are strictly copositive by Theorems 4.2-4.3. Indeed, one has

$$\mathcal{A}x^3 = 3x_1^3 - 3x_1^2x_2 + x_1x_2^2 + x_2^3 = (x_1 - x_2)^2(2x_1 + x_2) + x_1^3 + x_1x_2^2 > 0, \forall x \in \mathbb{R}_+^2 \backslash \{0\},$$

which implies that $\mathcal{A}$ is strictly copositive and strictly semi-positive.

In the following, selecting appropriate S may affect the judgment of strict copositivity of tensors.

Example 4.9. Let $\mathcal{A} \in \mathbb{R}^{[3,3]}$ with

$$a_{111} = 9, a_{112} = -1, a_{121} = -1, a_{113} = -2, a_{131} = -2, a_{122} = -1, a_{133} = 1,$$

$$a_{222} = 12, a_{212} = -1, a_{221} = -1, a_{223} = -2, a_{232} = -2, a_{211} = -1, a_{233} = -1,$$

$$a_{333} = 5, a_{313} = 1, a_{331} = 1, a_{323} = -1, a_{332} = -1, a_{311} = -2, a_{322} = -2,$$

and other entries be all zero.

By computing, we can verify that $[\mathcal{A}]_-$ is symmetric and

$$a_{111} = 9 > 7 = r_1(\mathcal{A})_-, a_{222} = 12 > 8 = r_2(\mathcal{A})_-, a_{333} = 5 < 6 = r_3(\mathcal{A})_-,$$

which implies that Theorem 4.9 of [22] is not suitable to judge whether $\mathcal{A}$ is strictly copositive.

Recalling Theorem 4.2, we obtain

$$b_{111} = 9, b_{112} = -1, b_{121} = -1, , b_{113} = -2, b_{131} = -2, b_{122} = -1,$$

$$b_{222}=12, b_{212}=-1, b_{221}=-1, b_{223}=-2, b_{232}=-2, b_{211}=-1, b_{233}=-1,$$
$$b_{333}=5, b_{323}=-1, b_{332}=-1, b_{311}=-2, b_{322}=-2,$$
$$r_1^2(\mathcal{B})_-=6, r_1^3(\mathcal{B})_-=7, r_2^1(\mathcal{B})_-=7, r_2^3(\mathcal{B})_-=7, r_3^1(\mathcal{B})_-=4, r_3^2(\mathcal{B})_-=4,$$
$$\delta_1^2(\mathcal{B})_-=4, \delta_1^3(\mathcal{B})_-=3, \delta_2^1(\mathcal{B})_-=5, \delta_2^3(\mathcal{B})_-=3, \delta_3^1(\mathcal{B})_-=4, \delta_3^2(\mathcal{B})_-=2,$$

and

$$b_{222}(b_{111}-r_1^2(\mathcal{B})_-)=36>\mid b_{122}b_{211}\mid+\mid b_{211}\mid r_2^1(\mathcal{B})_-=8,$$
$$b_{111}=9>\delta_1^2(\mathcal{B})_-=4,$$
$$b_{333}(b_{111}-r_1^3(\mathcal{B})_-)=10>\mid b_{133}b_{311}\mid+\mid b_{311}\mid r_3^1(\mathcal{B})_-=8,$$
$$b_{111}=9>\delta_1^3(\mathcal{B})_-=3,$$
$$b_{111}(b_{222}-r_2^1(\mathcal{B})_-)=45>\mid b_{211}b_{122}\mid+\mid b_{122}\mid r_1^2(\mathcal{B})_-=7,$$
$$b_{222}=12>\delta_2^1(\mathcal{B})_-=5,$$
$$b_{333}(b_{222}-r_2^3(\mathcal{B})_-)=25>\mid b_{233}b_{322}\mid+\mid b_{322}\mid r_3^2(\mathcal{B})_-=10,$$
$$b_{222}=12>\delta_2^3(\mathcal{B})_-=3,$$
$$b_{111}(b_{333}-r_3^1(\mathcal{B})_-)=9>\mid b_{311}b_{133}\mid+\mid b_{133}\mid r_1^3(\mathcal{B})_-=0,$$
$$b_{333}=5>\delta_3^1(\mathcal{B})_-=4,$$
$$b_{222}(b_{333}-r_3^2(\mathcal{B})_-)=12>\mid b_{322}b_{233}\mid+\mid b_{233}\mid r_2^3(\mathcal{B})_-=9,$$
$$b_{333}=5>\delta_3^2(\mathcal{B})_-=2.$$

All conditions of Theorem 4.2 are satisfied. Hence, $\mathcal{B}$ is strictly copositive and strictly semi-positive. Further, $\mathcal{A}$ is strictly copositive and strictly semi-positive by Lemmas 2.3 and 2.6. According to Theorem 4.3, we compute

$$r_1(\mathcal{B})_-=7, r_2(\mathcal{B})_-=8, r_3(\mathcal{B})_-=6, r_3^{\overline{\triangle S}}(\mathcal{B})_-=4, r_3^{\triangle \overline{S}}(\mathcal{B})_-=2.$$

Setting $S=\{1,3\}$ and $\overline{S}=\{2\}$, we obtain

$$b_{222}(b_{333}-r_3^{\overline{\triangle S}}(\mathcal{B})_-)=12<r_2(\mathcal{B})_- r_3^{\triangle \overline{S}}(\mathcal{B})_-=16.$$

Thus, we cannot verify that $\mathcal{B}$ is strictly copositive.

Using Theorem 3.2 of [17], we compute Pareto H-eigenvector $x=(0.6401, 0.5891, 0.8109)^\top$ with the minimum Pareto H-eigenvalue $\lambda_{\min}$ as follows

$$\min_{\substack{x_i\geq 0\\ x_1^3+x_2^3+x_3^3=1}} \mathcal{B}x^3=\lambda_{\min}=1.2453>0,$$

which implies that $\mathcal{B}$ is strictly copositive and strictly semi-positive. Further, $\mathcal{A}$ is strictly copositive and strictly semi-positive.

5 Conclusion

In this paper, we established tight Pareto H-eigenvalue inclusion intervals based on partitioning index set of the tensors. Meanwhile, checkable sufficient conditions were proposed to verify the strict copositivity, as well as the strict semi-positivity of real tensors. Further studies can be considered to develop some algorithms by Pareto H-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems, as done in [4] for solving the matrix eigenvalue complementarity problems.

Acknowledgement

The authors are very indebted to the reviewers for their valuable comments and corrections, which improved the original manuscript of this paper.

References

[1] S. Adly and H. Rammal, A new method for solving Pareto eigenvalue complementarity problems, *Comput. Optim. Appl.* 55 (2013) 703–731.

[2] Z. Chen and L. Qi, A semismooth Newton method for tensor eigenvalue complementarity problem, *Comput. Optim. Appl.* 65 (2016) 109–126.

[3] J. Fan, J. Nie and A. Zhou, Tensor eigenvalue complementarity problems, *Math. Program.* 170 (2018) 507–539.

[4] L. Fernandes, J. Judice, H. Sherali and M. Fukushima, On the computation of all eigenvalues for the eigenvalue complementarity problem, *J. Global Optim.* 59 (2014) 307–326.

[5] J. Judice, H. Sherali and I. Ribeiro, The eigenvalue complementarity problem, *Comput. Optim. Appl.* 37 (2007) 139–156.

[6] C. Li, A. Jiao and Y. Li, An S-type eigenvalue localization set for tensors, *Linear Algebra Appl.* 493 (2016) 469–483.

[7] C. Li, Y. Li and K. Xu, New eigenvalue inclusion sets for tensors, *Numer. Linear Algebra Appl.* 21 (2014) 39–50.

[8] L.H. Lim, Singular values and eigenvalues of tensors: A variational approach, in: *CAMSAP'05: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Puerto Vallarta, 2005, pp. 129-132.

[9] C. Ling, H. He and L. Qi, On the cone eigenvalue complementarity problem for higher-order tensors, Comput. Optim. Appl. 63 (2016) 143–168.

[10] C. Ling, H. He and L. Qi, Higher-degree eigenvalue complementarity problems for tensors, Comput. Optim. Appl. 64 (2016) 149–176.

[11] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Comput. 40 (2005) 1302–1324.

[12] L. Qi and Z. Luo, *Tensor Analysis: Spectral Properties and Special Tensors*, SIAM, 2017.

[13] L. Qi, Symmetric nonnegative tensors and copositive tensors, *Linear Algebra Appl.* 439 (2013) 228–238.

[14] A. Seeger, Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions, *Linear Algebra Appl.* 292 (1999) 1–14.

[15] Y. Song and L. Qi, Properties of some classes of structured tensors, J. Optim. Theory. Appl. 165 (2015) 854–873.

[16] Y. Song and L. Qi, Tensor complementarity problem and semi-positive tensors, J. Optim. Theory. Appl. 169 (2016) 1069–1078.

[17] Y. Song and L. Qi, Eigenvalue analysis of constrained minimization problem for homogeneous polynomial, J. Global Optim. 64 (2016) 563–575.

[18] G. Wang, G. Zhou and L. Caccetta, Sharp Brauer-type eigenvalue inclusion theorems for tensors, Pac. J. Optim. 14 (2018) 227–244.

[19] G. Wang, G. Zhou and L. Caccetta, Z-eigenvalue inclusion theorems for tensors, Discrete Contin. Dyn. Syst. Ser-B. 22 (2017) 187–198.

[20] G. Wang, Y. Wang and Y. Wang, Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors, Linear Multilinear Algebra. 68 (2020) 1817–1834.

[21] K. Wang, J. Cao and H. Pei, Robust extreme learning machine in the presence of outliers by iterative reweighted algorithm, Appl. Math. Comput. 377 (2020): 125186.

[22] Y. Xu and Z. Huang, Pareto eigenvalue inclusion sets for tensors, J. Ind. Manag. Optimi. 19 (2023) 2123–2139.

[23] Y. Xu, W. Gu and Z. Huang, Estimations on upper and lower bounds of solutions to a class of tensor complementarity problems, Front. Math. China. 14(3) (2019) 661–671.

[24] L. Zhang and C. Chen, A Newton-type algorithm for the tensor eigenvalue complementarity problem and some applications, Math. Comput. 90 (2021) 215–231.

Manuscript received 17 April 2022
revised 6 August 2022
accepted for publication 20 August 2022

Gang Wang
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: wgglj1977@163.com

Chong Wang
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: 17865577207@163.com

Qiuling Hou
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: qiuilinghouhou@163.com