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# CONNECTEDNESS OF THE SOLUTION SETS FOR GENERALIZED VECTOR EQUILIBRIUM PROBLEMS*

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**Abstract:** In this paper, the connectedness and path connectedness of the sets of efficient solutions and Henig efficient solutions for generalized vector equilibrium problems, are established. By virtue of a class of strongly monotone sublinear functions, we establish the union relation between the solution sets of efficient solutions and Henig efficient solutions of generalized vector equilibrium problems and the solution sets of a series of nonlinear scalar problems. Moreover, we obtain the connectedness and path connectedness of the sets of efficient solutions and Henig efficient solutions for a class of generalized vector equilibrium problems under some mild conditions.

**Key words:** *generalized vector equilibrium problem, connectedness, nonlinear scalarization, efficient solution*

**Mathematics Subject Classification:** *49J40, 90C29, 90C31*

## 1 Introduction

The vector equilibrium problem (for short, VEP), which can be regarded as a unified framework of several problems, such as optimization problems, variational inequality problems, complementary problems, has found many important applications in mathematical physics, economics theory, operations research, management science, engineering design, and others. Recently, the vector equilibrium problem has been investigated by many researchers intensively. The existence theory related to solutions for vector equilibrium problems (for short, VEPs) has been investigated intensively by many authors (see, for example, [1, 7, 8, 11, 18]). One of the most important problems for the study of VEPs is to investigate the properties of the set of solutions. Among many desirable properties of the solution sets, the connectedness is of considerable interest, since it provides the possibility of continuously moving from one solution to any other solution (see, for example, [3, 9, 19, 20] and the references therein).

Many researchers employed the linear scalarization method and the nonlinear scalarization method to obtain the connectedness results for various kinds of VEPs. By using the linear scalarization method, Gong [10] established the connectedness of the sets of Henig efficient solutions and weakly efficient solutions for the vector-valued Hartman-Stampacchia variational inequalities in normed spaces. Gong [12] obtained the connectedness and path

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connectedness of the sets of Henig efficient solutions, globally efficient solutions, superefficient solutions and cone-Benson efficient solutions, and the connectedness of the weakly efficient solution set for monotone VEPs with a specific form. By virtue of the linear scalarization method, Gong and Yao [13] first investigated the connectedness of the efficient solution set for VEPs with monotone bifunctions in real locally convex Hausdorff topological vector spaces. Han and Huang [15] established the connectedness of the sets of globally efficient solutions, Henig efficient solutions, weakly efficient solutions, superefficient solutions and efficient solutions for the generalized vector equilibrium problem (for short, GVEP) with the help of the linear scalarization method. In [16], by virtue of the linear scalarization method, Han and Huang obtained the connectedness of the sets of weakly efficient solutions and efficient solutions and the path connectedness of the weakly efficient solution set for generalized vector quasi-equilibrium problems. By virtue of the linear scalarization method, Cui and Li [4] obtained the path connectedness of the efficient solution set for generalized vector quasi-equilibrium problems.

By using the nonlinear scalarization method, Xu and Zhang [23] investigated the connectedness and path connectedness of the set of efficient solutions for the VEP with the help of the oriented distance function with respect to natural partial ordering. Peng et al. [20] studied the connectedness and path connectedness of solution sets for weak generalized symmetric Ky Fan inequality problems with respect to addition-invariant set by nonlinear scalarization technique. Recently, Shao et al. [21] investigated the connectedness and path connectedness of the efficient solution set for the VEP via free-disposal sets with the help of the nonlinear scalarization method.

We observe that the nonlinear scalarization functions used in the previous research, are mainly the Gerstewitz function and the oriented distance function. In [23] and [21], the authors tried to separate the convex cone (or the free-disposal set) with the origin removed and a nonconvex set by nonlinear scalarization functions related to the Gerstewitz function and the oriented distance function. However, some limitations on the objective mapping must be added to build the union relation, as we can see in Theorem 3.3 in [23] and Theorem 5.3 in [21]. To avoid these additional conditions, we apply a different nonlinear scalarization function, which is introduced by using the elements of the augmented dual cone (see [17]). By virtue of the functions from this class, any closed cone with a certain separation property can be separated. Moreover, as the nonlinear scalarization function in our paper is actually sublinear, we can apply the idea of approximation to characterize the efficient solution set and the Henig efficient solution set for (GVEP). Hence, we establish the union relation and the (path) connectedness results for the efficient solution set and the Henig efficient solution set under some mild conditions, which improve the corresponding results in [4, 10, 12, 15, 21, 23].

The paper is organized as follows. In Section 2, we recall some main notions and definitions. In Section 3, we give the union relation between the sets of efficient solutions and Henig efficient solutions for the GVEP and the solution sets of a series of nonlinear scalar problems with the help of a different nonlinear scalarization function. In Section 4, the connectedness and path connectedness of the sets of efficient solutions and Henig efficient solutions for the GVEP are obtained. In Section 5, we give some conclusions of this paper.

## 2 Preliminaries

We will recall some notations and definitions, which will be used through all the paper. Let $X$ and $Z$ be two real normed vector spaces, where $Z$ is finite dimensional. The closed unit balls in $X$ and $Z$ are denoted by $B_X$ and $B_Z$, respectively. The open unit ball and the unit sphere in $Z$ are denoted by $B_Z^o$ and $U_Z$, respectively. Let $C \subseteq Z$ be a closed, convex and pointed cone with nonempty interior. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $Z^*$ be the topological dual space of $Z$ and $C^*$ be the topological dual cone of $C$, defined by

$$C^* = \{f \in Z^* : f(c) \geq 0, \forall c \in C\}.$$

Denote the quasi-interior of $C^*$ by $C^\#$, i.e.

$$C^\# = \{f \in Z^* : f(c) > 0, \forall c \in C \backslash \{0\}\}.$$

The closure, the interior and the boundary of a set $M \subseteq Z$ are denoted by cl$M$, int$M$ and bd$M$, respectively. The intersection of all convex subsets of $X$ that contain $M$ is called the convex hull of $M$, denoted by conv$(M)$. The cone generated by a set $D$ is denoted by cone$(D)$:

$$\text{cone}(D) = \{\lambda d : \lambda \geq 0 \text{ and } d \in D\}.$$

A nonempty convex subset $B$ of the convex cone $C$ is called a base of $C$, if $C = \text{cone}(B)$ and $0 \notin \text{cl}(B)$. It is easy to see that $C^\# \neq \emptyset$ if and only if $C$ has a base. Let $C_U = C \cap U_Z = \{z \in C : \|z\| = 1\}$ denote the norm-base of the cone $C$. Obviously, we have that $C = \text{cone}(C_U)$. Let $z = (z_1, z_2, \ldots, z_n) \in R^n$. $\|z\|_1 = \sum_{i=1}^n |z_i|$, $\|z\|_2 = (z_1^2 + \cdots + z_n^2)^{1/2}$, and $\|z\|_\infty = \max\{|z_1|, \ldots, |z_n|\}$ denote the $l_1$, $l_2$ and $l_\infty$ norms of the vector $z$, respectively.

If $C$ has a base $B$, we can associate $C$ with another closed convex cone $C_\varepsilon(B)$, defined by $C_\varepsilon(B) = \text{cone}(B + \varepsilon B_Z)$, where $0 < \varepsilon < \delta := \inf\{\|b\| : b \in B\}$. Clearly, we have $C \backslash \{0\} \subseteq \text{int}C_\varepsilon(B)$.

Let $A$ be a nonempty subset of $X$ and $F : X \times X \to 2^Z$ be a set-valued mapping. In this paper, we consider the following generalized vector equilibrium problem consisting of finding $x_0 \in A$ such that

$$(GVEP) \quad F(x_0, y) \cap (-\Omega) = \emptyset, \ \forall y \in A, \tag{2.1}$$

where $\Omega \cup \{0\}$ is a convex cone in $Z$.

Let $W(A, F)$ denote the set of all weakly efficient solutions of (GVEP), i.e.

$$W(A, F) = \{x \in A : F(x, y) \cap (-\text{int}C) = \emptyset, \forall y \in A\}$$

and $E(A, F)$ denote the set of all efficient solutions of (GVEP), i.e.

$$E(A, F) = \{x \in A : F(x, y) \cap (-C \backslash \{0\}) = \emptyset, \forall y \in A\}.$$

Suppose that $C$ has a base $B$. A vector $x \in A$ is called a Henig efficient solution of (GVEP) if, for some $0 < \varepsilon < \delta := \inf\{\|b\| : b \in B\}$,

$$F(x, y) \cap -\text{int}C_\varepsilon(B) = \emptyset, \forall y \in A.$$

Denote by $H(A, F)$ the set of all Henig efficient solutions of (GVEP).

**Definition 2.1** ([2]). A set-valued mapping $G : X \rightrightarrows Z$ is said to be

(i) upper semicontinuous (u.s.c.) at $u_0 \in X$ if, for any neighborhood $V$ of $G(u_0)$, there exists a neighborhood $U(u_0)$ of $u_0$ such that

$$G(u) \subseteq V, \ \forall u \in U(u_0).$$

(ii) lower semicontinuous (l.s.c.) at $u_0 \in X$ if, for any $x \in G(u_0)$ and any neighborhood $V$ of $x$, there exists a neighborhood $U(u_0)$ of $u_0$ such that

$$G(u) \cap V \neq \emptyset, \ \forall u \in U(u_0).$$

We say that $G$ is u.s.c. (resp. l.s.c.) on X if it is u.s.c. (resp. l.s.c.) at each point $u \in X$, respectively. $G$ is said to be continuous on $X$ if it is both u.s.c. and l.s.c. on $X$.

**Lemma 2.2** ([2])**.** *A set-valued mapping $\Phi : X \rightrightarrows Z$ is l.s.c. at $u_0 \in X$ if and only if for any sequence $\{u_n\} \subseteq X$ with $u_n \to u_0$ and for any $x_0 \in \Phi(u_0)$, there exists $x_n \in \Phi(u_n)$ $(n = 1, 2, \dots)$ such that $x_n \to x_0$.*

**Definition 2.3** ([17])**.** Let $C$ be a convex cone in a normed space $Z$, and let $g : M \subseteq Z \to \mathbb{R}$ be a given function on $M$.
(a) The function $g$ is called $C$-monotone on $M$ iff, for each $y_1, y_2 \in M$, one has

$$y_1 - y_2 \in C \Rightarrow g(y_1) \geq g(y_2).$$

(b) The function $g$ is called strongly $C$-monotone on $M$ iff, for each $y_1, y_2 \in M$, one has

$$y_1 - y_2 \in C \backslash \{0\} \Rightarrow g(y_1) > g(y_2).$$

(c) The function $g$ is called strictly $C$-monotone on $M$ iff, for each $y_1, y_2 \in M$, one has

$$y_1 - y_2 \in \text{int}C \Rightarrow g(y_1) > g(y_2).$$

**Definition 2.4.** Let $D$ be a nonempty convex subset of $X$. A set-valued mapping $\Phi : D \rightrightarrows Z$ is said to be

(i) [14] $C$-convex if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, one has

$$t\Phi(x_1) + (1 - t)\Phi(x_2) \subseteq \Phi(tx_1 + (1 - t)x_2) + C.$$

(ii) [6] properly quasi $C$-convex if, for any $x_1, x_2 \in D$ and for any $t \in [0, 1]$, one has

$$\text{either } \Phi(x_1) \subseteq \Phi(tx_1 + (1 - t)x_2) + C \text{ or}$$

$$\Phi(x_2) \subseteq \Phi(tx_1 + (1 - t)x_2) + C.$$

(iii) strongly proper quasi $C$-convex if for any $x_1, x_2 \in D$, and for any $t \in ]0, 1[$, one has

$$\text{either } \Phi(x_1) \subseteq \Phi(tx_1 + (1 - t)x_2) + C \backslash \{0\} \text{ or}$$

$$\Phi(x_2) \subseteq \Phi(tx_1 + (1 - t)x_2) + C \backslash \{0\}.$$

(iv) [16] naturally quasi $C$-convex if, for any $x_1, x_2 \in D$, and for any $t \in [0, 1]$, there exists $\lambda \in [0, 1]$ such that

$$\lambda\Phi(x_1) + (1 - \lambda)\Phi(x_2) \subseteq \Phi(tx_1 + (1 - t)x_2) + C.$$

**Remark 2.5.** Definition 2.4 (iii) is a generalization of the concept of strongly proper quasi $C$-convexity in [23].

**Definition 2.6** ([8])**.** Let $A$ be a nonempty subset of a linear space $X$. A set-valued mapping $T : A \rightrightarrows X$ is said to be a $KKM$ mapping iff, for any finite subset $\{y_1, \dots, y_m\}$ of $A$, we have

$$\text{conv}(y_1, \dots, y_m) \subseteq \bigcup_{i=1}^{m} T(y_i).$$

**Definition 2.7** ([24])**.** A topological space $Y$ is said to be connected iff there do not exist nonempty open subsets $V_i \subset Y,\ i = 1, 2$, such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. $Y$ is said to be path-connected (or arcwise connected) iff $\forall x, y \in Y$ there exists a continuous mapping $\gamma : [0, 1] \to Y$, such that $\gamma(0) = x, \gamma(1) = y$.

**Lemma 2.8** (see Theorem 3.1 in [22])**.** *Let $Z$ be a normed vector space. Assume that $A$ is a nonempty and connected subset of a normed vector space $X$ and $F : A \rightrightarrows Z$ is an upper semicontinuous set-valued mapping with nonempty connected values. Then $F(A)$ is connected.*

**Lemma 2.9** (see Theorem 5.2 in [9])**.** *Let $A$ be a paracompact Hausdorff path-connected space and $Z$ be a Banach space. Assume that*

(i) *$F : A \rightrightarrows Z$ is a lower semicontinuous set-valued mapping;*

(ii) *for each $x \in A$, $F(x)$ is nonempty, closed and convex.*

*Then, $F(A)$ is a path-connected set.*

**Lemma 2.10** (Fan-KKM Theorem in [5])**.** *Let $A$ be a nonempty subset of a Hausdorff topological vector space $X$ and $T : A \rightrightarrows X$ is a $KKM$ mapping with closed values. If there exists $y_0 \in A$ such that $T(y_0)$ is compact, then $\bigcap_{y \in A} T(y) \neq \emptyset$.*

## 3 Scalarization for (GVEP)

To build the scalarization of $H(A, F)$ and $E(A, F)$ for (GVEP), we first present some useful results which are introduced in [17].

**Definition 3.1** ([17])**.** Let

$$C^{a*} = \{(z^*, \alpha) \in C^{\#} \times \mathbb{R}_+ : z^*(z) - \alpha \|z\| \geqslant 0 \text{ for all } z \in C\},$$

$$C^{ao} = \{(z^*, \alpha) \in C^{\#} \times \mathbb{R}_+ : z^*(z) - \alpha \|z\| > 0 \text{ for all } z \in \text{int}C\},$$

$$C^{a\#} = \{(z^*, \alpha) \in C^{\#} \times \mathbb{R}_+ : z^*(z) - \alpha \|z\| > 0 \text{ for all } z \in C\backslash\{0\}\},$$

where the ordering cone $C$ is assumed to have a nonempty interior in the definition of $C^{ao}$.

These three cones can be regarded as augmented dual cones of $C$. The relationship between the three kinds of augmented dual cones $C^{a*}$, $C^{ao}$ and $C^{a\#}$ is straightforward from the definitions:

$$C^{a\#} \subset C^{ao} \subset C^{a*}.$$

**Lemma 3.2** ([17])**.** *Let $(Z, \|\cdot\|)$ be a real normed space partially ordered by a pointed closed convex cone $C$. Let $z^* \in Z^*\backslash\{0\}$ and $\alpha \in \mathbb{R}_+$, and let a function $g_{(z^*,\alpha)} : Z \to \mathbb{R}$ be defined as*

$$g_{(z^*,\alpha)}(z) = z^*(z) + \alpha \|z\|. \tag{3.1}$$

*Then, the function $g_{(z^*,\alpha)}$ is $C$-monotone, strictly $C$-monotone (if $\text{int}C \neq \emptyset$), and strongly $C$-monotone on $Z$ if and only if $(z^*, \alpha) \in C^{a*}$, $(z^*, \alpha) \in C^{ao}$, $(z^*, \alpha) \in C^{a\#}$, respectively.*

**Definition 3.3** ( [17])**.** Let $C$ and $K$ be closed cones of a normed space $(Z, \|\cdot\|)$ with norm-base $C_U$ and $K_U$, respectively. Let $K_U^{\partial} = K_U \cap \text{bd}(K)$, and let $\widetilde{C}$ and $\widetilde{K}^{\partial}$ be the closures of the sets $\text{conv}(C_U)$ and $\text{conv}(K_U^{\partial} \cup \{0_Z\})$, respectively. The cones $C$ and $K$ are said to have the separation property with respect to the norm $\|\cdot\|$ if

$$\widetilde{C} \cap \widetilde{K}^{\partial} = \emptyset. \tag{3.2}$$

In order to obtain the connectedness and the path connectedness of $E(A,F)$ and $H(A,F)$, we need to establish the relation between $E(A,F)$ (or $H(A,F)$) and the union of solution sets for a series of nonlinear scalar problems. We define the following set-valued mapping $G(z^*,\alpha) : C^{a\#} \rightrightarrows X$ as

$$G(z^*,\alpha) = \{\bar{x} \in A : z^*(z) + \alpha \|z\| \geq 0, \forall z \in F(\bar{x},y), \forall y \in A\}. \tag{3.3}$$

In the following, we need to assume that $G(z^*,\alpha)$ is nonempty for any $(z^*,\alpha) \in C^{a\#}$. Actually, we can prove that $G(z^*,\alpha) \neq \emptyset$ under some suitable conditions.

**Theorem 3.4.** *Assume that*

(i) *for each $x \in A$, $F(x,x) \subseteq C$;*

(ii) *$A$ is a compact and convex set;*

(iii) *for each $y \in A$, $F(\cdot,y)$ is l.s.c. on $A$;*

(iv) *for each $x \in A$, $F(x,\cdot)$ is naturally quasi $C$-convex on $A$.*

*Then, for each $(z^*,\alpha) \in C^{a\#}$, $G(z^*,\alpha)$ is nonempty.*

*Proof.* For each $(z^*,\alpha) \in C^{a\#}$, define

$$\psi(y) = \{x \in A : z^*(z) + \alpha \|z\| \geq 0, \forall z \in F(x,y)\},\ y \in A. \tag{3.4}$$

For each $z \in F(y,y) \subseteq C$ and $(z^*,\alpha) \in C^{a\#}$, we have $z^*(z) + \alpha \|z\| \geq 0$. Hence $y \in \psi(y)$ and it is nonempty for each $y \in A$.
We claim that $\psi(y)$ is closed. Let $\{x_k\} \subseteq \psi(y)$ be any sequence with $x_k \to \bar{x}$. For each $y \in A$, since $F(\cdot,y)$ is l.s.c. on $A$, it follows that for any $\bar{z} \in F(\bar{x},y)$, there exist a sequence $\{z_k\} \subseteq Z$ and $k_0 > 0$ such that

$$z_k \in F(x_k,y)(k \geq k_0), z_k \to \bar{z}.$$

Since $x_k \in \psi(y)$ and $z_k \to \bar{z}$, we have that

$$z^*(\bar{z}) + \alpha \|\bar{z}\| \geq 0,\ \forall \bar{z} \in F(\bar{x},y).$$

Hence, $\bar{x} \in \psi(y)$, $\psi(y)$ is closed.
Since $A$ is compact, we have that $\psi(y)$ is compact.
Next, we claim that $\psi : A \rightrightarrows A$ is a $KKM$ mapping. Suppose that $\psi$ is not a $KKM$ mapping on the contrary, then there exist a finite subset $\{y_1,\ldots,y_m\} \subseteq A$ and $y_0 \in \text{conv}(\{y_1,\ldots,y_m\})$, such that

$$y_0 \notin \psi(y_i), \forall i = 1,2,\ldots,m.$$

Then we have that for any $i \in \{1,2,\ldots,m\}$, there exists $\tilde{z}_i \in F(y_0,y_i)$ such that

$$z^*(\tilde{z}_i) + \alpha \|\tilde{z}_i\| < 0. \tag{3.5}$$

Since $y_0 \in \text{conv}(\{y_1,\ldots,y_m\})$, there exist $\lambda_i \geq 0 (i = 1,2,\ldots,m)$ with $\sum_{i=1}^m \lambda_i = 1$, such that $y_0 = \sum_{i=1}^m \lambda_i y_i$. Owing to the naturally quasi $C$-convexity of $F(y_0,\cdot)$, there exist $t_i \geq 0$ $(i = 1,2,\cdot,m)$ with $\sum_{i=1}^m t_i = 1$, such that

$$\sum_{i=1}^m t_i F(y_0,y_i) \subseteq F(y_0,\sum_{i=1}^m \lambda_i y_i) + C = F(y_0,y_0) + C. \tag{3.6}$$

By the $C$-monotonity of $g_{(z^*,\alpha)}$, we have that for any $z_i \in F(y_0, y_i)$, there exists $z_0 \in F(y_0, y_0)$, such that

$$z^*(\sum_{i=1}^{m} t_i z_i) + \alpha \left\| \sum_{i=1}^{m} t_i z_i \right\| \geq z^*(z_0) + \alpha \|z_0\| \geq 0.$$

Let $z_i = \tilde{z}_i$. With the help of (3.5), there exists $\tilde{z}_0 \in F(y_0, y_0)$ such that

$$z^*(\tilde{z}_0) + \alpha \|\tilde{z}_0\| \leq z^*(\sum_{i=1}^{m} t_i \tilde{z}_i) + \alpha \left\| \sum_{i=1}^{m} t_i \tilde{z}_i \right\| < 0. \tag{3.7}$$

This is a contradiction to $y_0 \in \psi(y_0)$. Therefore, $\psi$ is a $KKM$ mapping. By virtue of Lemma 2.10, we have that $G(z^*, \alpha) = \bigcap_{y \in A} \psi(y) \neq \emptyset$ for each $(z^*, \alpha) \in C^{a\#}$. □

**Remark 3.5.** The Fan-KKM theorem is applied to show that the solution set $G(z^*, \alpha)$ is nonempty, for each $(z^*, \alpha) \in C^{a\#}$. Compared with Theorem 4.1 of [21] and Theorem 3.2 of [23], the continuity (or $C$-continuity) of $F(\cdot, \cdot)$ is weakened to the lower-semicontinuity of $F(\cdot, y)$, and the properly quasi $C$-convexity of $F(x, \cdot)$ is weakened to the naturally quasi $C$-convexity of $F(x, \cdot)$.
We note that the inequality (3.7) holds as $g_{(z^*,\alpha)}$ is sublinear, which is different from the nonlinear scalarization functions in [23] and [21]. Because of this, the assumption of the properly quasi $C$-convexity can be weakened to the naturally quasi $C$-convexity in Theorem 3.4.

**Theorem 3.6.** *Let $(Z, \|\cdot\|)$ be a reflexive Banach space partially ordered by a closed convex pointed cone $C$. Suppose that $C$ has a bounded base $B$. Let $A$ be a nonempty subset of $X$. Assume that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0, 1)$. If $F(x, A) = \bigcup_{y \in A} F(x, y)$ is compact for each $x \in A$, then we have*

$$E(A, F) = \bigcup_{(z^*,\alpha) \in C^{a\#}} G(z^*, \alpha).$$

*Proof.* For any $x \in \bigcup_{(z^*,\alpha) \in C^{a\#}} G(z^*, \alpha)$, there exists $(z_0^*, \alpha_0) \in C^{a\#}$ such that $x \in G(z_0^*, \alpha_0)$. Hence,

$$z_0^*(z) + \alpha_0 \|z\| \geq 0, \forall z \in F(x, y), \forall y \in A. \tag{3.8}$$

Suppose that $x \notin E(A, F)$, then there exists $y_0 \in A$, such that

$$F(x, y_0) \cap (-C \backslash \{0\}) \neq \emptyset.$$

By the definition of $C^{a\#}$, there exists $z_0 \in F(x, y_0)$ such that $z_0^*(z_0) + \alpha_0 \|z_0\| < 0$, which contradicts (3.8). Thus, we get that $x \in E(A, F)$.
Next, we show that

$$E(A, F) \subseteq \bigcup_{(z^*,\alpha) \in C^{a\#}} G(z^*, \alpha).$$

Let $x \in E(A, F)$. We have

$$F(x, y) \cap (-C \backslash \{0\}) = \emptyset, \forall y \in A.$$

Let $T = \text{cone}(F(x, A))$. Then $T$ is a closed cone and

$$T \cap (-C \backslash \{0\}) = \emptyset.$$

Consider the norm-bases of $T$, $-C$ and $-C_\varepsilon(B)$, denoted by $T_U$, $-C_U$ and $-C_U^\varepsilon$, respectively. Since $Z$ is finite dimensional, it follows that $T_U$, $-C_U$ and $-C_U^\varepsilon$ are compact. As $T_U \cap (-C_U) = \emptyset$, there exists $\gamma > 0$ such that $d(-C_U, T_U) = \gamma > 0$, where $d(-C_U, T_U) = \inf\{\|z_1 - z_2\| : z_1 \in -C_U, z_2 \in T_U\}$.

Since $B$ is bounded, there exists a real number $m_0$ such that

$$0 < m_0 = \inf\{\|z\| : z \in B\}.$$

Let $\tilde{\varepsilon} = \dfrac{m_0\gamma}{4+\gamma}$. (If $\dfrac{m_0\gamma}{4+\gamma} \geq 1$, we can choose $\tilde{\varepsilon} = \dfrac{m_0\gamma}{k(4+\gamma)}$, where $k$ is large enough and $k > 1$.)

We claim that $d(\tilde{z}, -C_U) \leq \dfrac{\gamma}{2}$, for any $\tilde{z} \in -(C_U^{\tilde{\varepsilon}} \backslash C_U)$.

For any $\tilde{z} \in -(C_U^{\tilde{\varepsilon}} \backslash C_U)$ and $z \in -C_U$, we have

$$\tilde{z} = -\tilde{\lambda}(\tilde{b} + \tilde{\varepsilon} b_z),\ \ z = -\lambda b,$$

where $\tilde{\lambda} > 0$, $\lambda > 0$, $\tilde{b} \in B$, $b \in B$, $b_z \in B_Z \backslash \{0\}$.

Consider the case that $b = \tilde{b}$, we can get

$$d(\tilde{z}, -C_U) \leq \left\| -\tilde{\lambda}(\tilde{b} + \tilde{\varepsilon} b_z) + \lambda \tilde{b} \right\| \leq \tilde{\lambda}\tilde{\varepsilon} + \left|\lambda - \tilde{\lambda}\right| \left\|\tilde{b}\right\|. \tag{3.9}$$

As $\|\tilde{z}\| = \|z\| = 1$, we have $\tilde{\lambda} = \dfrac{1}{\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|}$ and $\lambda = \dfrac{1}{\left\|\tilde{b}\right\|}$. Then, we can get

$$\left|\lambda - \tilde{\lambda}\right| = \frac{\left|\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\| - \left\|\tilde{b}\right\|\right|}{\left\|\tilde{b}\right\| \left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|} \leq \frac{\tilde{\varepsilon}}{\left\|\tilde{b}\right\| \left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|}. \tag{3.10}$$

By (3.9) and (3.10), we have

$$d(\tilde{z}, -C_U) \leq \tilde{\varepsilon} \left( \tilde{\lambda} + \frac{1}{\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|} \right) = \frac{2\tilde{\varepsilon}}{\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|}. \tag{3.11}$$

Since $\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\| \geq m_0 - \tilde{\varepsilon} > 0$, it follows that

$$\frac{2\tilde{\varepsilon}}{\left\|\tilde{b} + \tilde{\varepsilon} b_z\right\|} \leq \frac{2\tilde{\varepsilon}}{m_0 - \tilde{\varepsilon}} \leq \frac{\gamma}{2}. \tag{3.12}$$

From (3.11) and (3.12), we have $d(\tilde{z}, -C_U) \leq \dfrac{\gamma}{2}$.

For any $\tilde{z} \in -(C_U^{\tilde{\varepsilon}} \backslash C_U)$ and $t \in T_U$, we have

$$\|\tilde{z} - t\| \geq (\|z_0 - t\| - \|\tilde{z} - z_0\|) \geq \frac{\gamma}{2} > 0,$$

where $z_0 \in -C_U$ satisfies that $\|\tilde{z} - z_0\| = d(\tilde{z}, -C_U)$.

Thus, we can deduce that for any $\tilde{z} \in -(C_U^{\tilde{\varepsilon}} \backslash C_U)$, $d(\tilde{z}, T_U) \geq \dfrac{\gamma}{2} > 0$ and

$$-C\backslash\{0\} \subseteq (-C_{\tilde{\varepsilon}}(B))\backslash\{0\} \subseteq Z\backslash T.$$

Since $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0,1)$, by virtue of Theorem 4.4 in [17], it follows that there exists $(z_0^*, \alpha_0) \in C^{a\#}$ such that

$$-C\backslash\{0\} \subseteq \{z \in Z : z_0^*(z) + \alpha_0 \|z\| < 0\} \subseteq (-C_{\bar{\varepsilon}}(B))\backslash\{0\} \subseteq Z\backslash T. \tag{3.13}$$

From (3.13), we have that there exsits $(z_0^*, \alpha_0) \in C^{a\#}$ such that

$$z_0^*(t) + \alpha_0 \|t\| \geq 0, \forall t \in T.$$

Therefore, $x \in \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha)$. □

**Example 3.7.** Let $C = \{(z_1, z_2) : z_1 \leq z_2 \leq 2z_1, z_1 \geq 0\}$. It is easy to show that $C$ and $C_\varepsilon(B)$ satisfy the separation property with respect to the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$. See Example 4.9 in [17].
Let $C = \mathbb{R}^n_+$. Then, $C$ and $C_\varepsilon(B)$ satisfy the separation property with respect to $\|\cdot\|_1$ for every $0 < \varepsilon < 1$. See Theorem 5.9 in [17].

**Proposition 3.8.** *Suppose that $C$ is a closed convex pointed cone with a bounded base $B$. Then, for each $(z^*, \alpha) \in C^{a\#}$, there exists $0 < \varepsilon < \delta := \inf\{\|b\| : b \in B\}$ such that $-\mathrm{int}C_\varepsilon(B) \subseteq \{z | g_{(z^*,\alpha)}(z) < 0\}$.*

*Proof.* (i) For each $(z^*, \alpha) \in C^{a\#}$, we can easily get that $-C\backslash\{0\} \subseteq \{z|g_{(z^*,\alpha)}(z) < 0\}$.
(ii) By the definitions of $C^{a\#}$ and $g_{(z^*,\alpha)}$, we have that there exist $M > 0$ and $\gamma < 0$ such that

$$0 < M = \max\{g_{(z^*,\alpha)}(-b_z) : b_z \in B_Z\},$$
$$0 > \gamma = \max\{g_{(z^*,\alpha)}(-b) : b \in B\}.$$

Let $\varepsilon = \dfrac{-\gamma}{nM}$, for a certain $n$ large enough (to ensure $\varepsilon < \delta$). For $z \in -(\mathrm{int}C_\varepsilon(B)\backslash C)$, there exist $\lambda > 0$, $b \in B$ and $0 \neq b_z \in B_Z$ such that

$$z = \lambda(-b - \varepsilon b_z).$$

Since

$$g_{(z^*,\alpha)}(z) \leq \lambda g_{(z^*,\alpha)}(-b) + \lambda\varepsilon g_{(z^*,\alpha)}(-b_z) < \lambda(\gamma + \varepsilon M) < 0,$$

it follows that $g_{(z^*,\alpha)}(z) < 0$ and thus proves the proposition. □

**Theorem 3.9.** *Let $(Z, \|\cdot\|)$ be a reflexive Banach space partially ordered by a closed convex pointed cone $C$. Suppose that $C$ has a bounded base $B$. Let $A$ be a nonempty subset of $X$. Assume that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0,1)$. Then,*

$$H(A, F) = \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha).$$

*Proof.* Let $x \in H(A, F)$. Then there exists $0 < \bar{\varepsilon} < \delta := \inf\{\|b\| : b \in B\}$ such that

$$F(x, y) \cap -\mathrm{int}C_{\bar{\varepsilon}}(B) = \emptyset, \text{ for all } y \in A.$$

With the help of Theorem 4.4 in [17], since $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0,1)$, it follows that for each $\varepsilon \in (0,1)$, there exists $(z_0^*, \alpha_0) \in C^{a\#}$ such that

$$-C\backslash\{0\} \subseteq \{z \in Z : z_0^*(z) + \alpha_0 \|z\| < 0\} \subseteq -C_\varepsilon(B).$$

Let $\tilde{\varepsilon} > 0$ be small enough such that $C_{\tilde{\varepsilon}}(B) \subseteq \text{int}C_{\bar{\varepsilon}}(B)$. Then,

$$-C\backslash\{0\} \subseteq \{z \in Z : z_0^*(z) + \alpha_0 \|z\| < 0\} \subseteq -C_{\tilde{\varepsilon}}(B) \subseteq -\text{int}C_{\bar{\varepsilon}}(B).$$

So, there exists $(z_0^*, \alpha_0) \in C^{a\#}$ such that $z_0^*(z) + \alpha_0 \|z\| \geq 0, \forall z \in F(x, y), \forall y \in A$.
Hence, we obtain that $H(A, F) \subseteq \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha)$.
Next, we show that

$$H(A, F) \supseteq \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha).$$

Let us suppose, on the contrary, that there exists $(\tilde{z}^*, \tilde{\alpha}) \in C^{a\#}$ such that $x \in G(\tilde{z}^*, \tilde{\alpha})$ and $x \notin H(A, F)$. Then for any $0 < \varepsilon < \delta := \inf\{\|b\| : b \in B\}$, there exist $y \in A$ and $z \in F(x, y)$ such that $z \in -\text{int}C_\varepsilon(B)$. By Proposition 3.8, for $(\tilde{z}^*, \tilde{\alpha}) \in C^{a\#}$, there exists $0 < \tilde{\varepsilon} < \delta$ such that

$$-\text{int}C_{\tilde{\varepsilon}}(B) \subseteq \{z | \tilde{z}^*(z) + \tilde{\alpha} \|z\| < 0\}.$$

Let $\varepsilon = \tilde{\varepsilon}$. Then there exists $\tilde{z} \in -\text{int}C_{\tilde{\varepsilon}}(B) \cap F(x, A)$ such that

$$\tilde{z}^*(\tilde{z}) + \tilde{\alpha} \|\tilde{z}\| < 0.$$

This is a contradiction to $x \in \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha)$. So, $H(A, F) \supseteq \bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*, \alpha)$.
□

**Remark 3.10.** The proof method of Theorem 3.6 and Theorem 3.9 is different from that in the previous literature, which is mainly reflected in the following aspects.

(i) We apply the idea of approximation to characterize the efficient solution set and the Henig efficient solution set for (GVEP). By virtue of the concept of the norm-base of the cone, we show that there exists $\tilde{\varepsilon} \in (0, 1)$ such that $-C\backslash\{0\} \subseteq (-C_{\tilde{\varepsilon}}(B))\backslash\{0\} \subseteq Z\backslash T$, when $T \cap (-C\backslash\{0\}) = \emptyset$, in Theorem 3.6. That is to say, we can find $-C_{\tilde{\varepsilon}}(B)$ which is close enough to $-C$. The approximation theorem in [17] shows that every cone $-C$, which satisfies the separation property with $-C_{\tilde{\varepsilon}}(B)$, can be approximated arbitrarily closely by the sublevel set of the strongly monotonically increasing sublinear function $g_{(z^*,\alpha)}$, where $(z^*, \alpha) \in C^{a\#}$. Then we can obtain the relation $-C\backslash\{0\} \subseteq \{z \in Z : z_0^*(z) + \alpha_0 \|z\| < 0\} \subseteq (-C_{\tilde{\varepsilon}}(B))\backslash\{0\} \subseteq Z\backslash T$, which helps us to characterize the efficient solution set for (GVEP).
In the proof of Theorem 3.9, we also apply the idea of approximation. Different from the proof of Theorem 3.6, we obtain that every sublevel set of $g_{(z^*,\alpha)}$ contains the set $-\text{int}C_{\tilde{\varepsilon}}(B)$ for some $0 < \epsilon < \delta := \inf\{\|b\| : b \in B\}$ with the help of the sublinear property of $g_{(z^*,\alpha)}$.

(ii) The scalarization function $g_{(z^*,\alpha)}$ in this paper is different from the two mainly used nonlinear scalarization functions, Gerstewitz function and oriented distance function. Actually, $g_{(z^*,\alpha)}$ is a sublinear monotonically increasing function defined by using the elements of the augmented dual cone. With the help of the sublinear property of $g_{(z^*,\alpha)}$, we can not only establish the union relation of the efficient solution set for (GVEP), but also get rid of some restrictive assumptions caused by the commonly used nonlinear scalarization functions.

**Remark 3.11.** Although the scalarization results for $E(A, F)$ and $H(A, F)$ seem the same, the conditions for $E(A, F)$ are stronger. It is necessary for $F(x, A)$ to be compact in Theorem 3.6, but not in Theorem 3.9.

## 4 Connectedness

In this section, we will establish the connectedness and the path connectedness of $E(A,F)$ and $H(A,F)$. We first give some useful lemmas.

**Lemma 4.1.** *Let $A$ be a compact set in $X$. Assume that $F(\cdot,y)$ is l.s.c. for each $y \in A$. Then, $G(\cdot,\cdot)$ is u.s.c. on $C^{a\#}$.*

*Proof.* Suppose $G(\cdot,\cdot)$ is not u.s.c. at $(z_0^*,\alpha_0) \in C^{a\#}$. Then there exist an open neighborhood $W$ of $G(z_0^*,\alpha_0)$ and a sequence $(z_n^*,\alpha_n)$ converging to $(z_0^*,\alpha_0)$ ($z_n^* \to z_0^*$ by weak* topology), such that $G(z_n^*,\alpha_n) \nsubseteq W$, $\forall n \in N$. Therefore, there exists $x_n \in G(z_n^*,\alpha_n)$ such that

$$x_n \notin W,\ \forall n \in N. \tag{4.1}$$

As the set $A$ is compact, we can suppose that there exists $x_0 \in A$ such that $x_n \to x_0$, without loss of generality. Since $x_n \in G(z_n^*,\alpha_n)$, we have

$$z_n^*(z_n) + \alpha_n \|z_n\| \geq 0,\ \forall z_n \in F(x_n,y),\ \forall y \in A.$$

For every $y \in A$, $F(\cdot,y)$ is l.s.c. at $x_0$. So, for every $y \in A$, for any sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$, and for any $z_0 \in F(x_0,y)$, there exist $k_0 > 0$ and $\tilde{z_n} \in F(x_n,y)$ $(n > k_0)$ such that $\tilde{z_n} \to z_0$. Then, $z_n^*(\tilde{z_n}) + \alpha_n \|\tilde{z_n}\| \geq 0$. As $\tilde{z_n} \to z_0$ and $(z_n^*,\alpha_n) \to (z_0^*,\alpha_0)$, we have

$$z_0^*(z_0) + \alpha_0 \|z_0\| \geq 0,\ \forall z_0 \in F(x_0,y),\ \forall y \in A.$$

Hence, $x_0 \in G(z_0^*,\alpha_0) \subseteq W$. This is a contradiction to (4.1). Thus, $G(\cdot,\cdot)$ is u.s.c. on $C^{a\#}$. □

**Lemma 4.2.** *Assume that for each $y \in A$, $F(\cdot,y)$ is properly quasi C-concave on A. Then, $G(z^*,\alpha)$ is convex for each $(z^*,\alpha) \in C^{a\#}$.*

*Proof.* Let $x_1, x_2 \in G(z^*,\alpha)$. Then

$$z^*(z_i) + \alpha \|z_i\| \geq 0,\ \forall z_i \in F(x_i,y),\ \forall y \in A,\ i = 1,2.$$

Since $F(\cdot,y)$ is properly quasi C-concave on A, it follows that for any $\lambda \in (0,1)$, we have

$$\text{either} F(\lambda x_1 + (1-\lambda)x_2, y) \subseteq F(x_1,y) + C,$$

$$\text{or } F(\lambda x_1 + (1-\lambda)x_2, y) \subseteq F(x_2,y) + C.$$

By the C-monotonicity of $g_{(z^*,\alpha)}$ for each $(z^*,\alpha) \in C^{a\#}$, we have for any $z \in F(\lambda x_1 + (1-\lambda)x_2, y)$ and for any $y \in A$,

$$z^*(z) + \alpha \|z\| \geq z^*(z_1) + \alpha \|z_1\|,$$

$$\textit{or } z^*(z) + \alpha \|z\| \geq z^*(z_2) + \alpha \|z_2\|.$$

Then,

$$z^*(z) + \alpha \|z\| \geq 0,\ \forall z \in F(\lambda x_1 + (1-\lambda)x_2, y),\ \forall y \in A.$$

So $\lambda x_1 + (1-\lambda)x_2 \in G(z^*,\alpha)$ and $G(z^*,\alpha)$ is convex for each $(z^*,\alpha) \in C^{a\#}$. □

**Lemma 4.3.** *Let $A$ be a compact and convex set. Suppose that,*

(i) *for each $y \in A$, $F(\cdot,y)$ is strongly proper quasi C-concave on A;*

(ii) *for each* $x \in A$, $F(x, A)$ *is compact;*

(iii) *for each* $y \in A$, $F(\cdot, y)$ *is l.s.c. on* $A$.

*Then,* $G(\cdot,\cdot)$ *is l.s.c. on* $C^{a\#}$.

*Proof.* Suppose that there exists $(z_0^*, \alpha_0) \in C^{a\#}$ such that $G(\cdot,\cdot)$ is not l.s.c. at $(z_0^*, \alpha_0)$. Then there exist $x_0 \in G(z_0^*, \alpha_0)$, a neighborhood $W_0$ of 0 and a sequence $\{(z_n^*, \alpha_n)\}$ with $(z_n^*, \alpha_n) \to (z_0^*, \alpha_0)$ ($\{z_n^*\}$ converging to $z_0^*$ with weak* topology), such that

$$(x_0 + W_0) \cap G(z_n^*, \alpha_n) = \emptyset,\ \forall n \in N. \tag{4.2}$$

Consider the following two cases.
**Case 1** $G(z_0^*, \alpha_0)$ is a singleton.
Let $x_n \in G(z_n^*, \alpha_n)$. As $A$ is a compact set, with loss of generality, we can suppose that $x_n \to \bar{x}$. We can obtain $\bar{x} \in G(z_0^*, \alpha_0)$, which is similar to the proof in Lemma 4.1. Since $G(z_0^*, \alpha_0)$ is a singleton, we have $\bar{x} = x_0$ and $x_n \to x_0$. We can imply that $x_n \in (x_0 + W_0) \cap G(z_n^*, \alpha_n)$ with $n$ large enough. This is a contradiction to (4.2).
**Case 2** $G(z_0^*, \alpha_0)$ is not a singleton.
Then there exists $x' \in G(z_0^*, \alpha_0)$ such that $x' \neq x_0$ and

$$z_0^*(z') + \alpha_0 \|z'\| \geq 0,\ \forall z' \in F(x', y),\ \forall y \in A, \tag{4.3}$$

$$z_0^*(z_0) + \alpha_0 \|z_0\| \geq 0,\ \forall z_0 \in F(x_0, y),\ \forall y \in A. \tag{4.4}$$

For any $\lambda \in (0, 1)$, as $F(\cdot, y)$ is strongly proper quasi C-concave on A, we have

$$\text{either } F(\lambda x' + (1-\lambda)x_0, y) \subseteq F(x', y) + C\backslash\{0\},$$

$$\text{or } F(\lambda x' + (1-\lambda)x_0, y) \subseteq F(x_0, y) + C\backslash\{0\}.$$

Then, for any $\lambda \in (0, 1)$, we have

$$z_0^*(z_\lambda) + \alpha_0 \|z_\lambda\| > 0,\ \forall z_\lambda \in F(\lambda x' + (1-\lambda)x_0, y),\ \forall y \in A. \tag{4.5}$$

By virtue of (4.2), there exists $\lambda_0 \in (0, 1)$ such that $x(\lambda_0) = \lambda_0 x' + (1-\lambda_0)x_0 \in x_0 + W_0$ and $x(\lambda_0) \notin G(z_n^*, \alpha_n)$, $\forall n \in N$. Then, there exist $y_n \in A$ and $z_n \in F(x(\lambda_0), y_n)$ such that

$$z_n^*(z_n) + \alpha_n \|z_n\| < 0,\ \forall n \in N. \tag{4.6}$$

As $F(x, A)$ is compact for each $x \in A$ and $z_n \in F(x(\lambda_0), y_n) \subseteq F(x(\lambda_0), A)$ for any $n \in N$, we can assume that there exists $z_0 \in F(x(\lambda_0), A)$ such that $z_n \to z_0$. From $(z_n^*, \alpha_n) \to (z_0^*, \alpha_0)$ and $z_n \to z_0$, we can deduce that

$$z_0^*(z_0) + \alpha_0 \|z_0\| \leq 0.$$

It is a contradiction to (4.5). Therefore, $G(\cdot,\cdot)$ is l.s.c. on $C^{a\#}$. □

**Lemma 4.4.** *Let* $A$ *be a compact set. Assume that* $F(\cdot, y)$ *is l.s.c. for each* $y \in A$. *Then,* $G(z^*, \alpha)$ *is closed for each* $(z^*, \alpha) \in C^{a\#}$.

*Proof.* Let $\{x_n\} \subseteq G(z^*, \alpha)$ with $x_n \to x_0$. Since $\{x_n\} \subseteq G(z^*, \alpha)$, we have that

$$z^*(z_n) + \alpha \|z_n\| \geq 0,\ \forall z_n \in F(x_n, y),\ \forall y \in A. \tag{4.7}$$

Since $A$ is compact, we can assume $x_0 \in A$, without loss of generality. As $F(\cdot, y)$ is l.s.c., for any $y \in A$ and for any $z_0 \in F(x_0, y)$, we can obtain that there exists $\{z_n\} \subseteq Z$ with $z_n \to z_0$ and $z_n \in F(x_n, y)$ for $n \in N$ large enough. Together with (4.7), we can get

$$z^*(z_0) + \alpha \|z_0\| \geq 0, \ \forall z_0 \in F(x_0, y), \ \forall y \in A.$$

Therefore, $x_0 \in G(z^*, \alpha)$. □

Next, we give the connectedness and the path connectedness of $E(A, F)$ and $H(A, F)$ for (GVEP).

**Theorem 4.5.** *Let $C$ be a closed, pointed and convex cone of a reflexive Banach space $(Z, \|\cdot\|)$. Suppose that $C$ has a bounded base $B$. Let $A$ be a compact and convex set. Suppose that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0, 1)$. Assume that*

(i) *for each $x \in A$, $F(x, x) \subseteq C$;*

(ii) *for each $x \in A$, $F(x, A)$ is compact;*

(iii) *for each $x \in A$, $F(x, \cdot)$ is naturally quasi $C$-convex on $A$, and for each $y \in A$, $F(\cdot, y)$ is properly quasi $C$-concave on $A$;*

(iv) *for each $y \in A$, $F(\cdot, y)$ is l.s.c. on $A$.*

*Then, $E(A, F)$ is a connected set.*

*Proof.* From Theorem 3.4 and Lemma 4.2, we have that $G(z^*, \alpha)$ is nonempty and connected for each $(z^*, \alpha) \in C^{a\#}$. By virtue of Lemma 4.1, we obtain that $G(\cdot, \cdot)$ is u.s.c. on $C^{a\#}$. It can be deduced from Theorem 3.6 that

$$E(A, F) = \bigcup_{(z^*, \alpha) \in C^{a\#}} G(z^*, \alpha).$$

Hence, by means of Lemma 2.9, we get that $E(A, F)$ is a connected set. □

**Theorem 4.6.** *Let $C$ be a closed, pointed and convex cone of a reflexive Banach space $(Z, \|\cdot\|)$. Suppose that $C$ has a bounded base $B$. Let $A$ be a compact and convex set. Suppose that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0, 1)$. Assume that*

(i) *for each $x \in A$, $F(x, x) \subseteq C$;*

(ii) *for each $x \in A$, $F(x, A)$ is compact;*

(iii) *for each $x \in A$, $F(x, \cdot)$ is naturally quasi $C$-convex on $A$, and for each $y \in A$, $F(\cdot, y)$ is strongly proper quasi $C$-concave on $A$;*

(iv) *for each $y \in A$, $F(\cdot, y)$ is l.s.c. on $A$.*

*Then, $E(A, F)$ is a path-connected set.*

*Proof.* Since $C^{a\#}$ is a metric space with the distance defined by

$$d((z_1^*, \alpha_1), (z_2^*, \alpha_2)) = \|z_1^* - z_2^*\| + |\alpha_1 - \alpha_2|,$$

it follows that $Z^* \times \mathbb{R}_+$ is a paracompact Hausdorff space. It is clearly that $C^{a\#}$ is convex. Hence, we obtain that $C^{a\#}$ is path-connected.

By virtue of Theorem 3.4, Lemma 4.2 and Lemma 4.4, we have that $G(z^*,\alpha)$ is nonempty, closed and convex for each $(z^*,\alpha)\in C^{a\#}$. With the help of Lemma 4.3, we have that $G(\cdot,\cdot)$ is l.s.c. on $C^{a\#}$. From Theorem 3.6, we can obtain that

$$E(A,F)=\bigcup_{(z^*,\alpha)\in C^{a\#}} G(z^*,\alpha).$$

Hence, $E(A,F)$ is a path-connected set. □

Now, we give the following example to illustrate Theorem 4.5 and 4.6.

**Example 4.7.** Let $Z=\mathbb{R}^2$ and $A=[-1,1]$. Let $C=\mathbb{R}^2_+$ and $F(x,y)=(f_1(x)+f_2(y))E_1+E_2$, $x,y\in A$, where $E_1=[-1,0]$, $E_2=[1,2]$ and

$$f_1(x)=\begin{cases}1, & x\neq 0,\\ 0, & x=0.\end{cases}$$

$$f_2(y)=\begin{cases}0, & y\neq 0,\\ 1, & y=0.\end{cases}$$

It is easy to check that $F(x,x)\subseteq C$ for each $x\in A$. For each $x\in A$ and each $y\in A$, $F(x,y)$ is compact. For each $x\in A$, $F(x,A)$ is closed. The closed cones $C$ and $C_\varepsilon$ satisfy the separation property for all $\varepsilon\in(0,1)$. Through a simple calculation, we can get that $F(x,\cdot)$ is naturally quasi C-convex on $A$ for each $x\in A$. And $F(\cdot,y)$ is strongly proper quasi C-concave and l.s.c. on $A$ for each $y\in A$. Hence, the conditions (i)-(iv) of Theorem 4.6 are satisfied. By virtue of Theorem 4.6, $E(A,F)$ is a path-connected set. Actually, we can compute that $E(A,F)=\{0\}$, which is obviously a path-connected set.

**Remark 4.8.** Theorem 4.5 and 4.6 improve the results in [23], [21], [15] and [4] in the following aspects.

(i) Compared with Theorem 4.1 and 4.2 in [23], Theorem 5.4 and 5.5 in [21] and Corollary 3.1 in [4], the continuity of $F(\cdot,\cdot)$ (or the $C$-continuity of $F(\cdot,\cdot)$) is weakened to the lower semicontinuity of $F(\cdot,y)$ in Theorem 4.5 and 4.6 in this paper. The continuity (or semicontinuity) of $F(x,\cdot)$ is not necessary.

(ii) The properly quasi $C$-convexity of $F(x,\cdot)$ in [23] and [21] is weakened to naturally quasi $C$-convexity. The condition that $F(x,\cdot)$ is $C$-convexlike in [4] is removed.

(iii) The condition (v) of Theorem 4.1 and 4.2 in [23] and the condition (vi) of Theorem 5.4 and 5.5 in [21] are removed in our paper. For the set $F(x,A)$, we only need that it is a compact for each $x\in A$, without other additional conditions in [23] and [21].

(iv) Compared with Theorem 4.2 in [15], the assumption that there exists $v_0\in -\mathrm{int}C$ such that $S(v_0)\neq\emptyset$ is removed. We think that the assumption in [15] excludes the case that $F(x,x)=0$, which means it can not be applied to vector optimization problems. In addition, the method used in [15] is only applicable to the derivation of the results of connectedness, not to the results of the path connectedness.

Next, we give the following examples to show that our results hold when the results in [15], [23], [21] and [4] are not applicable.

**Example 4.9.** Let $X = \mathbb{R}$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $A = [0, 2]$. Define $F(x, y) = (f_1(x, y), f_2(x, y))$, $x, y \in A$, where

$$f_1(x,y) = \begin{cases} -x^2 + y^2, & y \geq 1, \\ -x^2 + (y-1)^2 + 2, & y < 1, \end{cases} \quad and \quad f_2(x,y) = \begin{cases} sinx - siny, & y < 1, \\ sinx - sin1, & y \geq 1. \end{cases}$$

(i) It is easy to check that $F(x, y) \in C$ for each $x \in A$. For each $x \in A$, $F(x, A)$ is compact. We can easily check that $F(x, \cdot)$ is naturally quasi C-convex on $A$ for each $x \in A$ and $F(\cdot, y)$ is strongly proper quasi C-concave on $A$ for each $y \in A$. We can also get that $F(\cdot, y)$ is continuous on $A$ for each $y \in A$. Hence, all the conditions of Theorem 4.6 hold. Through a direct calculation, we obtain that $E(A, F) = A = [0, 2]$. Hence, $E(A, K)$ is a path-connected set and Theorem 4.6 is valid.

(ii) As $f_1(x, \cdot)$ is neither continuous nor $C$-continuous on $A$, Theorem 5.5 in [21], Theorem 4.2 in [23] and Corollary 3.1 in [4] do not hold.

(iii) We can calculate that $F(x, x) \notin \text{int}C$ for each $x \in A$. The condition (i) of Theorem 4.2 in [15] does not hold. Hence, Theorem 4.2 in [15] is not applicable.

**Example 4.10.** Let $X = \mathbb{R}^2$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $A = [-1, 1] \times [-1, 1]$. Define $F(x, y) = (f_1(x, y), f_2(x, y))$, $x, y \in A$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ where

$$f_1(x, y) = -y_1 + x_1, \ f_2(x, y) = y_2^2 - x_2^2 + \frac{1}{4}.$$

(i) For each $x \in A$, we have $F(x, x) = (0, \frac{1}{4}) \in C \backslash \text{int}C$. Then, the condition (i) of Theorem 4.6 holds and the condition (i) of Theorem 4.2 in [15] does not hold.

(ii) As $F(\cdot, y)$ is continuous on $A$ and $A$ is compact, we have that $F(x, A)$ is compact for each $x \in A$. It is easily to check that the conditions (iii) and (iv) are satisfied. Then, all conditions of Theorem 4.6 hold. By a direct computation, we obtain $E(A, F) = \{x \in A | x_1 \in [-1, 1], x_2 \in (-\frac{1}{2}, \frac{1}{2})\} \bigcup \{(1, \frac{1}{2})\}$, which is a path-connected set.

(iii) Let $x_0 = (0, \frac{1}{2})$. We can calculate that $-F(x_0, A) = [-1, 1] \times [-1, 0]$. It not hard to check that the condition (v) of Theorem 4.2 in [23] and the condition (vi) of Theorem 5.5 in [21] do not hold.

**Theorem 4.11.** *Let $C$ be a closed, pointed and convex cone of a reflexive Banach space $(Z, \|\cdot\|)$. Suppose that $C$ has a bounded base $B$. Suppose that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0, 1)$. Let $A$ be a compact and convex set. Assume that*

(i) *for each $x \in A$, $F(x, x) \subseteq C$;*

(ii) *for each $x \in A$, $F(x, \cdot)$ is naturally quasi C-convex on $A$, and for each $y \in A$, $F(\cdot, y)$ is properly quasi C-concave on $A$;*

(iii) *for each $y \in A$, $F(\cdot, y)$ is l.s.c. on $A$.*

*Then, $H(A, F)$ is a connected set.*

**Theorem 4.12.** *Let $C$ be a closed, pointed and convex cone of a reflexive Banach space $(Z, \|\cdot\|)$. Suppose that $C$ has a bounded base $B$. Suppose that $C$ and $C_\varepsilon(B)$ satisfy the separation property for all $\varepsilon \in (0, 1)$. Let $A$ be a compact and convex set. Assume that*

(i) *for each* $x \in A$, $F(x,x) \subseteq C$;

(ii) *for each* $x \in A$, $F(x,A)$ *is compact;*

(iii) *for each* $x \in A$, $F(x,\cdot)$ *is naturally quasi* $C$*-convex on* $A$, *and for each* $y \in A$, $F(\cdot,y)$ *is strongly proper quasi* $C$*-concave on* $A$;

(iv) *for each* $y \in A$, $F(\cdot,y)$ *is l.s.c. on* $A$.

*Then,* $H(A,F)$ *is a path-connected set.*

The proofs of Theorem 4.11 and 4.12 are similar to the proofs of Theorem 4.5 and 4.6.

**Remark 4.13.** Theorem 4.11 and 4.12 improve the results in [10], [12] and [15] in the following aspects.

(i) Compared with Theorem 4.1 in [15], the assumption $S(0) \neq \emptyset$ is removed, which means that we do not need the assumption that there exists $x_0 \in A$ such that $F(x_0,K) \subseteq C$. And the $C$-concavity of $F(\cdot,y)$ is weakened to the properly quasi $C$-concavity. In addition, as we mentioned in Remark 4.8, the method in [15] is only suitable for the discussion of the connectedness, not for the path connectedness.

(ii) Compared with Theorem 4.2 in [12] and Theorem 4.2 in [10], the (weakly) $C$-lower semicontinuity of $F(x,\cdot)$ is removed, and the $C$-convexity of $F(x,\cdot)$ is weakened to naturally quasi $C$-convexity. Theorem 4.2 in [12] and Theorem 4.2 in [10] are only applicable to the monotone VEP with a specific form, where $F(x,y) = \psi(y)+\varphi(x,y)-\psi(x)$ in [12]. In Theorems 4.11 and 4.12, the assumptions of monotonicity is removed. And Theorems 4.11 and 4.12 can be applied to a more general case of $F(x,y)$.

Finally, we give an application of the main results to vector optimization problems. Let $f : X \to Z$ and $A \subseteq X$. Consider the following vector optimization problem (for short, VOP):

$$\text{Min } f(x) \ s.t. \ x \in A.$$

Let $E(A,f)$ denote the efficient solution set of (VOP), i.e.,

$$E(A,f) = \{x \in A : f(y) - f(x) \notin -\mathbb{R}^n_+ \backslash \{0\}, \ \forall y \in A\}.$$

Then, we have the following result.

**Corollary 4.14.** *Assume that*

(i) $A$ *is a nonempty, compact and convex set;*

(ii) $f$ *is continuous on* $A$;

(iii) $f$ *is strongly proper quasi* $C$*-convex on* $A$.

*Then,* $E(A,f)$ *is connected and path-connected.*

## 5 Conclusions

In this paper, we considered (GVEP) in the reflexive Banach space $(Z, \|\cdot\|)$. By virtue of a special class of strongly monotone sublinear functions, we constructed the union relation between the sets of efficient solutions and Henig efficient solutions and the solution sets of a series of nonlinear scalar problems. We have obtained some results on the connectedness and path connectedness of the sets of efficient solutions and Henig efficient solutions for (GVEP) under some mild conditions.

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