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A SIMPLE INTRODUCTION TO HIGHER ORDER LIFTINGS FOR BINARY PROBLEMS

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Abstract: A short, simple, and self-contained proof is presented showing that n-th lifting for the max-cut-polytope is exact. The proof re-derives the known observations that the max-cut-polytope is the projection of a higher-dimensional regular simplex and that this simplex coincides with the n-th semidefinite lifting. An extension to reduce the dimension of higher order liftings and to include linear equality and inequality constraints concludes this paper.

Key words: *binary problems, higher order liftings*

Mathematics Subject Classification: *90-01, 05C50, 05C70, 52B11*

1 Introduction

Starting with the work of Gomory [5] an elegant general cutting plane procedure became available for solving integer linear programs. Unfortunately, even in two dimensions there is no universal bound on the number of cutting plane steps needed for solving an integer linear program. Also, as the cutting planes at each step remove the leading digits of selected inequality constraints, rounding errors systematically accumulate and thus necessitate an increasingly higher precision for the numerical computations. To overcome these shortcomings Sherali and Adams, [14], Lovász and Schrijver, [12], Balas, Ceria, and Cornuéjols[2], Lasserre, [8], and others proposed convex exact higher order liftings for hard combinatorial problems where the number of lifting steps is bounded in terms of the dimension of the problem and for which no systematic issues concerning numerical accuracy have been reported to date. From a theoretical point of view some of the shortcomings of Gomory-cuts could thus be eliminated. But the shortcomings are not eliminated in full as the dimensions of the liftings are large and refer to binary variables for which also the number of Gomory-cuts can be bounded in terms of the dimension of the problem.

The aim of the present paper is to start with a particularly simple example of binary problem (without any constraints), namely the max-cut problem in order to derive a very simple analysis of higher order liftings (Sections 2 and 3). In a second step a reduction of the dimension of the liftings is considered, and finally, linear constraints are added to this formulation. All the results presented here are well known, and some relations to earlier work are detailed in the final section.

$\boxed{1.1}$ Notation

Given $n \in \mathbb{N}$, let $N := \{1, \dots, n\}$. For $I, J \subseteq N$ let $I \triangle J := (I \cup J) \backslash (I \cap J)$ be the symmetric difference of I and J. For $x \in \{\pm 1\}^n$ let $\vec{x} \in \mathbb{R}^{2^n}$ denote the augmented vector with components $\vec{x}_I := \prod_{i \in I} x_i$ for $I \subseteq N$.

By convention, when I is the empty set, $\vec{x}_\emptyset := 1$. Also note that for $I, J \subseteq N$, the product $\vec{x}_I \vec{x}_J$ is given by $\vec{x}_I \vec{x}_J = \vec{x}_{I \triangle J}$.

The space of real symmetric $k \times k$ matrices is denoted by $\mathcal{S}^k$ and the cone of real symmetric positive semidefinite $k \times k$ matrices is denoted by $\mathcal{S}^k_+$. The trace inner product of $X, Y \in \mathcal{S}^k$ inducing the Frobenius norm $\| \, . \, \|_F$ is denoted by $\langle X, Y \rangle$. The all-one-vector is denoted by e, its dimension being evident from the context. The max-cut-polytope is denoted by $\mathbf{MC} := \operatorname{conv}(\{xx^T \mid x \in \{\pm 1\}^n\})$, see e.g. [3, 4, 13].

$\boxed{2}$ The Max-Cut-Polytope

Being able to minimize a linear function over the max-cut-polytope,

$$\min \ \langle X, Q \rangle \quad \text{where} \quad X \in \mathbf{MC}$$

(for a given matrix $Q \in \mathcal{S}^n$) is equivalent to being able to solve the max-cut-problem, see, for example [3]. This is a difficult NP-complete problem. However, the semidefinite approximation or "spectrahedron" $\mathbf{SH} := \{X \in \mathcal{S}^n_+ \mid \operatorname{Diag}(X) = e\}$ is an "easily computable" (see e.g. [13]) outer approximation of $\mathbf{MC}$ with an excellent (see e.g. [4]) approximation guarantee that is in a certain sense best possible (see e.g. [6]).
For illustration the case $n = 3$ is considered: The vector $x = (1, 1, -1)^T$, for example, defines the vertex

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (1, 1, -1) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

of $\mathbf{MC}$. As x and $-x$ generate the same vertex, the set $\mathbf{MC}$ has 4 vertices for $n = 3$ and matrices X in $\mathbf{MC}$ only have three degrees of freedom namely the lower diagonal part (by symmetry, and since the diagonal is fixed to all one). The free entries of all matrices in $\mathbf{MC}$ form a 3-dimensional simplex with the vertices $(1, 1, 1)^T$, $(1, -1, -1)^T$, $(-1, 1, -1)^T$, $(-1, -1, 1)^T$. The arithmetic mean of the last three vertices leads to the matrix $X^{(2)}$ below. The set $\mathbf{SH}$ contains $\mathbf{MC}$, and edges and vertices of $\mathbf{MC}$ also lie at the boundary of $\mathbf{SH}$, but the surface of $\mathbf{SH}$ bulges out over the 2-dimensional faces of $\mathbf{MC}$, and the worst approximation of $\mathbf{MC}$ by $\mathbf{SH}$ is attained at

$$X^{(1)} := \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \in \mathbf{SH} \qquad \text{with} \qquad X^{(2)} := \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$$

being the nearest matrix in $\mathbf{MC}$. To improve the approximation quality, rather than considering the semidefinite approximation $\mathbf{SH}$ to the convex hull of all "± 1"-matrices xx^T, one can form the augmented "± 1"-vector

$$\vec{x} := (\vec{x}_\emptyset, \ \vec{x}_{\{1\}}, \ \vec{x}_{\{2\}}, \ \vec{x}_{\{3\}}, \ \vec{x}_{\{1,2\}}, \ \vec{x}_{\{1,3\}}, \ \vec{x}_{\{2,3\}}, \ \vec{x}_{\{1,2,3\}})^T \in \mathbb{R}^8$$

where $\vec{x}_\emptyset = 1$ and $\vec{x}_{\{1,2\}}$ for example stands for $x_1 \cdot x_2$. Then consider a semidefinite approximation to the convex hull of all "± 1"-matrices $\breve{X} = \vec{x}\vec{x}^T \in \mathcal{S}^8_+$ with entries $\breve{X}_{I,J} =$

$\vec{x}_I \cdot \vec{x}_J = \vec{x}_{I \vartriangle J}$ for $I, J \subset \{1,2,3\}$. Here, the symmetric differences represent equations such as

$$\vec{x}_{\{1,2\}} \cdot \vec{x}_{\{1,3\}} = x_1 x_2 \cdot x_1 x_3 = x_2 x_3 = \vec{x}_{\{2,3\}} = \vec{x}_{\{1,2\} \vartriangle \{1,3\}}$$

using $x_1^2 = 1$ for the equation in the middle. Considering all equations of this form leads to the matrix $\breve{X} = \vec{x}\vec{x}^T$ being given by

$$\breve{X} = \left(\begin{array}{c|ccc|cccc}
1 & \vec{x}_{\{1\}} & \vec{x}_{\{2\}} & \vec{x}_{\{3\}} & \vec{x}_{\{1,2\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{2,3\}} & \vec{x}_{\{1,2,3\}} \\ \hline
\vec{x}_{\{1\}} & 1 & \vec{x}_{\{1,2\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{2\}} & \vec{x}_{\{3\}} & \vec{x}_{\{1,2,3\}} & \vec{x}_{\{2,3\}} \\
\vec{x}_{\{2\}} & \vec{x}_{\{1,2\}} & 1 & \vec{x}_{\{2,3\}} & \vec{x}_{\{1\}} & \vec{x}_{\{1,2,3\}} & \vec{x}_{\{3\}} & \vec{x}_{\{1,3\}} \\
\vec{x}_{\{3\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{2,3\}} & 1 & \vec{x}_{\{1,2,3\}} & \vec{x}_{\{1\}} & \vec{x}_{\{2\}} & \vec{x}_{\{1,2\}} \\ \hline
\vec{x}_{\{1,2\}} & \vec{x}_{\{2\}} & \vec{x}_{\{1\}} & \vec{x}_{\{1,2,3\}} & 1 & \vec{x}_{\{2,3\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{3\}} \\
\vec{x}_{\{1,3\}} & \vec{x}_{\{3\}} & \vec{x}_{\{1,2,3\}} & \vec{x}_{\{1\}} & \vec{x}_{\{2,3\}} & 1 & \vec{x}_{\{1,2\}} & \vec{x}_{\{2\}} \\
\vec{x}_{\{2,3\}} & \vec{x}_{\{1,2,3\}} & \vec{x}_{\{3\}} & \vec{x}_{\{2\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{1,2\}} & 1 & \vec{x}_{\{1\}} \\
\vec{x}_{\{1,2,3\}} & \vec{x}_{\{2,3\}} & \vec{x}_{\{1,3\}} & \vec{x}_{\{1,2\}} & \vec{x}_{\{3\}} & \vec{x}_{\{2\}} & \vec{x}_{\{1\}} & 1
\end{array} \right).$$

The 3×3-submatrix of $\breve{X}$ in the upper left box (rows and columns starting with $\vec{x}_{\{1\}}$, $\vec{x}_{\{2\}}$, $\vec{x}_{\{3\}}$) is contained in **SH**, and if this submatrix is set to the matrix $X^{(1)}$ from above and the remaining variables are set to $\alpha = \vec{x}_{\{1\}}$, $\beta = \vec{x}_{\{2\}}$, $\gamma = \vec{x}_{\{3\}}$, and $\delta = \vec{x}_{\{1,2,3\}}$, then one obtains

$$\breve{X}^{(1)} := \left(\begin{array}{c|ccc|cccc}
1 & \alpha & \beta & \gamma & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \delta \\ \hline
\alpha & 1 & -\frac{1}{2} & -\frac{1}{2} & \beta & \gamma & \delta & -\frac{1}{2} \\
\beta & -\frac{1}{2} & 1 & -\frac{1}{2} & \alpha & \delta & \gamma & -\frac{1}{2} \\
\gamma & -\frac{1}{2} & -\frac{1}{2} & 1 & \delta & \alpha & \beta & -\frac{1}{2} \\ \hline
-\frac{1}{2} & \beta & \alpha & \delta & 1 & -\frac{1}{2} & -\frac{1}{2} & \gamma \\
-\frac{1}{2} & \gamma & \delta & \alpha & -\frac{1}{2} & 1 & -\frac{1}{2} & \beta \\
-\frac{1}{2} & \delta & \gamma & \beta & -\frac{1}{2} & -\frac{1}{2} & 1 & \alpha \\
\delta & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \gamma & \beta & \alpha & 1
\end{array} \right).$$

The argument to show that $\breve{X}^{(1)}$ cannot be positive semidefinite is a bit technical: By considering symmetric permutations of rows and columns one can assume without loss of generality that $\alpha = \beta = \gamma$ if the leading 4×4 principal submatrix of $\breve{X}^{(1)}$ is to be positive semidefinite. And by taking the Schur complement with respect to the $(1,1)$-element it follows that $\alpha = \beta = \gamma = 0$. By interchanging the rows and columns starting with γ and δ, the same argument leads to $\delta = 0$. But for $\alpha = \beta = \gamma = \delta = 0$ the overall matrix $\breve{X}^{(1)}$ has an eigenvalue $-\frac{1}{2}$ with eigenvector e.

A more careful analysis shows that the numbers "$-\frac{1}{2}$" in $\breve{X}^{(1)}$ would need to be replaced by "$-\frac{1}{3}$" (corresponding to the matrix $X^{(2)} \in$ **MC** above) in order to allow for a positive semidefinite completion. Semidefiniteness of $\breve{X}$ seems to imply that the 3×3-submatrix of $\breve{X}$ in the upper left box is indeed contained in **MC** (and by convexity actually coincides with **MC**). Instead of showing this in detail, a generalization of this result will be discussed in the next subsection with a new and simple general proof.

$\boxed{2.1}$ A High-Dimensional Simplex

Consider the convex hull of all $\vec{x}\vec{x}^T$ where $\vec{x}$ is an augmented $\{\pm 1\}$-vector, i.e. consider the polytope

$$\mathbf{S} := \text{conv}(\ \{\ \vec{x}\vec{x}^T \mid x \in \{\pm 1\}^n\ \}\ ) \subset \mathcal{S}^{2^n}.$$

Proposition 2.1. *For $x, y \in \{\pm 1\}^n$ with $x \neq y$ it always follows $\vec{x}^T\vec{y} = 0$ and $\|\vec{x} - \vec{y}\|_2 = 2^{(n+1)/2}$. And for the associated vertices $\vec{x}\vec{x}^T$ and $\vec{y}\vec{y}^T$ of $\mathbf{S}$ it follows that $\langle \vec{x}\vec{x}^T, \vec{y}\vec{y}^T\rangle = 0$ and $\|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F = 2^{(2n+1)/2}$.*

Proof. Consider the case that y differs from x exactly in the components $1, \ldots, k$, i.e. $x_1y_1 = \ldots = x_ky_k = -1$. As is well known and easy to verify the number of subsets of $\{1, \ldots, k\}$ with an even number of elements is 2^{k-1} and the number of subsets of $\{1, \ldots, k\}$ with an odd number of elements also is 2^{k-1}. If an even number of elements from $\{1, \ldots, k\}$ is contained in $I \subseteq N$, then $\vec{x}_I = \vec{y}_I$, else $\vec{x}_I = -\vec{y}_I$. Similarly when some subset $\{i_1, \ldots, i_k\}$ of N is considered in place of $\{1, \ldots, k\}$. Hence exactly half the entries of $\vec{x}$ and $\vec{y}$ differ, so that $\vec{x}^T\vec{y} = 0$ and also $\langle \vec{x}\vec{x}^T, \vec{y}\vec{y}^T\rangle = \text{trace}(\vec{x}\vec{x}^T\vec{y}\vec{y}^T) = 0$. This implies $\|\vec{x} - \vec{y}\|_2^2 = \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 = 2 \cdot 2^n = 2^{n+1}$ and likewise $\|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F^2 = 2^{2n+1}$. ∎

Remark 2.2. There are 2^n subsets $I \subseteq N$, and in the following, rows and columns of $X \in \mathcal{S}^{2^n}$ will be indexed by such subsets I. Let

$$\mathcal{A} := \{X \in \mathcal{S}^{2^n} \mid X_{\emptyset,\emptyset} = 1,\ \ X_{I,J} = X_{K,L} \ \text{ for any } I, J, K, L \subseteq N \text{ with } I \triangle J = K \triangle L\}. \tag{2.1}$$

As in the 3-dimensional example of **MC** above, the linear equations in (2.1) relating $X_{I,J}$ and $X_{K,L}$ represent simple equalities that are satisfied for all vertices of $\mathbf{S}$ such as

$$\vec{x}_{\{i,j\}}\vec{x}_{\{i,k\}} = \vec{x}_{\{j\}}\vec{x}_{\{k\}} = \vec{x}_{\emptyset}\vec{x}_{\{j,k\}}.$$

As each vertex of $\mathbf{S}$ satisfies the linear equations in (2.1) this implies that $\mathbf{S} \subset \mathcal{A}$. Since $X_{\emptyset,\emptyset} = 1$, it follows in particular that $X_{I,I} = 1$ for all $I \subseteq N$ so that there are only $2^n - 1$ "free" matrix entries $X_{I,J}$ of X in $\mathcal{A}$ depending on $I \triangle J \neq \emptyset$. Thus, the dimension of $\mathcal{A}$ is $2^n - 1$.

Remark 2.3. Within $\mathcal{A}$ the set $\mathbf{S}$ is a regular simplex with 2^n vertices. It is well known and easy to see that $\mathbf{S}$ is centered about the identity matrix $I \in \mathcal{S}^{2^n}$. In particular, $\mathbf{S}$ is full-dimensional within $\mathcal{A}$. As shown in Proposition 2.1 the vertices are perpendicular to each other which seems to contradict the fact that the vertices of a full-dimensional regular simplex centered about the origin of some Euclidean space share an angle of more than 90 degrees to each other. However, $\mathcal{A}$ is embedded in a higher-dimensional space and does not contain the origin; instead all vertices $\vec{x}\vec{x}^T$ of $\mathbf{S}$ are centered about the identity matrix in $\mathcal{A}$ reducing the pairwise angle to 90 degrees.

The first central observation used in this note is:

> The projection of $\mathbf{S}$ onto the rows and columns associated with $\vec{x}_{\{1\}}, \ldots, \vec{x}_{\{n\}}$ is the max-cut-polytope $\mathbf{MC} = \text{conv}(\{xx^T \mid x \in \{\pm 1\}^n\})$ in $\mathcal{S}^n$.

Indeed for $n = 1$ the two subsets of N are $\emptyset$ and N. ✓

Now let $\hat{n} := n - 1 \geq 1$ and $\hat{N} := \{1, \ldots, \hat{n}\}$. The subsets of N are given by I and $I \cup \{n\}$ where $I \subseteq \hat{N}$. By induction hypothesis, $2^{\hat{n}-1}$ of the sets I and also $2^{\hat{n}-1}$ of the sets $I \cup \{n\}$ have even cardinality. Thus, the claim follows from $2^{\hat{n}-1} + 2^{\hat{n}-1} = 2^{n-1}$. ✓ ∎

Each vertex $X \in \mathbf{S}$ has an all-one-diagonal, and so does the average of all vertices. The off-diagonal elements of X are given by $X_{I,K} = \prod_{i \in I \triangle K} x_i$ so that among all $x \in \{\pm 1\}^n$, half of the entries $X_{I,K}$ are 1, and the average is zero.

The max-cut-polytope has 2^{n-1} vertices as x and $z := -x$ generate the same vertex, i.e. $xx^T = zz^T$, but $\vec{x}$ and $\vec{z}$ do not, i.e. $\vec{x}\vec{x}^T \neq \vec{z}\vec{z}^T$.

The second central observation used in this note (with a new short proof below) is that the semidefinite relaxation $\tilde{\mathbf{S}}$ of $\mathbf{S}$ coincides with the n-th lifting for the max-cut-polytope and also coincides with $\mathbf{S}$. This observation implies the known fact (established for example in [8, 12]) that the semidefinite liftings of sufficiently high order do represent the exact convex hull, and thus also the max-cut-polytope. The n-th lifting in [8, 12] can also be seen as a high dimensional linear extension of $\mathbf{MC}$ as introduced in Theorem 3 of [15]; the above representation of $\mathbf{S}$ via its 2^n facets is a very simple form of such linear extension.

2.2 Semidefinite Representation of the Simplex

Note that $\mathbf{S} \subseteq \tilde{\mathbf{S}}$, where the semidefinite relaxation $\tilde{\mathbf{S}}$ is given by

$$\tilde{\mathbf{S}} := \mathcal{S}_+^{2^n} \cap \mathcal{A}. \tag{2.2}$$

with $\mathcal{A}$ defined in (2.1). The set $\tilde{\mathbf{S}}$ is essentially the same as the n-th lifting for the max-cut-polytope defined in [1]. Same as $\mathbf{S}$, also $\tilde{\mathbf{S}}$ is contained in the (2^n-1)-dimensional affine space $\mathcal{A}$. In fact, as shown next, both sets coincide.

Lemma 2.4. *The sets* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *coincide.*

Proof. Both, $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are full-dimensional, bounded, closed, convex subsets of the (2^n-1) dimensional affine subspace $\mathcal{A}$ of (2.1) containing the identity matrix I in $\mathcal{S}^{2^n}$ in their relative interior.

Next, it is shown that all relative boundary points $X \in \partial\mathbf{S}$ are also at the relative boundary of $\tilde{\mathbf{S}}$,

$$\partial\mathbf{S} \subset \partial\tilde{\mathbf{S}},$$

i.e. that all relative boundary points $X \in \partial\mathbf{S}$ have rank at most $2^n - 1$.
Indeed, let X be a boundary point of the simplex $\mathbf{S}$. Then, X is a convex combination of vertices of $\mathbf{S}$ with the exception of at least one vertex $\vec{y}\vec{y}^T$ of $\mathbf{S}$. Let $\vec{x}\vec{x}^T$ be some vertex different form $\vec{y}\vec{y}^T$. By Proposition 2.1,

$$2^{2n+1} = \|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F^2 = \|\vec{x}\vec{x}^T\|_F^2 + \|\vec{y}\vec{y}^T\|_F^2 - 2(\vec{x}^T\vec{y})^2 = 2^{2n+1} - 2(\vec{x}^T\vec{y})^2$$

so that $(\vec{x}^T\vec{y})^2 = \vec{y}^T(\vec{x}\vec{x}^T)\vec{y} = 0$. As this is true for all other vertices $\vec{x}\vec{x}^T$ it follows that $\vec{y}^T X\vec{y} = 0$, i.e. X has rank at most $2^n - 1$.
Now, let $I \neq X \in \mathbf{S}$ be given. Since $\mathbf{S}$ is bounded the line starting at I and passing through X will cross the relative boundary of $\mathbf{S}$ at some point $\bar{X} \in \partial\mathbf{S} \subset \partial\tilde{\mathbf{S}}$. Since $\tilde{\mathbf{S}}$ is closed it follows $\bar{X} \in \tilde{\mathbf{S}}$ and by convexity also $X \in \tilde{\mathbf{S}}$. This shows $\mathbf{S} \subset \tilde{\mathbf{S}}$.
Conversely, let $\tilde{X} \in \tilde{\mathbf{S}}$ be given. If $\tilde{X} \notin \mathbf{S}$ the line segment from I to $\tilde{X}$ intersects the boundary of $\mathbf{S}$ at some point $\hat{X} \in \partial\mathbf{S} \subset \partial\tilde{\mathbf{S}}$. But $\hat{X}$ being in the open segment between I in the relative interior of $\tilde{\mathbf{S}}$ and $\tilde{X} \in \tilde{\mathbf{S}}$ cannot be at the boundary of $\tilde{\mathbf{S}}$. This contradiction shows that also $\tilde{\mathbf{S}} \subset \mathbf{S}$. □

Remark 2.5. Some definitions of higher order liftings contain redundancies such as identical rows and columns. The set $\tilde{\mathbf{S}}$ is the n-th lifting after eliminating identical rows and columns. For $1 \leq k < n$, liftings of order k can be defined in a similar way by considering augmented vectors $\vec{x}$ with components $\vec{x}_I$ where $I \subseteq N$ has cardinality at most k. The corresponding semidefinite approximation of the max-cut-polytope is defined in an analogous way as the projection of the semidefinite relaxation for $\vec{x}\vec{x}^T$ onto rows and columns associated with

$\vec{x}_{\{1\}},\dots,\vec{x}_{\{n\}}$. The previous lemma implies for any subset $M\subset N$ of cardinality at most k that the restriction of the k-th lifting to the matrix with entries $X_{I,J}$ for $I,J\subseteq M$ is exact, indicating that the accuracy of the lifting is improving when increasing k.

This concludes the main part of this note.

3 Reduced Representations

Following an observation in [10], in this section the size of the representation is reduced by eliminating half of the subsets of $\vec{x}$. A reduced representation of the max-cut-polytope is obtained, for example, when considering vectors $\vec{x}^{\,o}\in\{\pm1\}^{2^{n-1}}$ with components $\vec{x}^{\,o}_I := \prod_{i\in I}x_i$ where $x\in\{\pm1\}^n$ is as before and $I\subseteq N$ has *odd cardinality only.* For sets $I,J\subseteq N$ with odd cardinality it follows that

$$|I\triangle J| = |I|+|J|-2|I\cap J|$$

is even so that the rank-1-matrix $Y:=\vec{x}^{\,o}\vec{x}^{\,oT}\in\mathcal{S}^{2^{n-1}}$ only has entries $Y_{I,J}=\vec{x}_{I\triangle J}$ with subsets $I\triangle J$ of even cardinality $|I\triangle J|$. For a matrix $Y\in\mathcal{S}_+^{2^{n-1}}$ with rows and columns indexed by odd-cardinality subsets linear equality constraints as in (2.2) also define a semidefinite relaxation $\tilde{\mathbf{S}}_1$ of the max-cut-polytope.

Similarly, one can define a reduced representation by considering *only even-cardinality* subsets I (including the empty set) for the definition of a second vector $\vec{x}^{\,e}$ with entries $\vec{x}^{\,e}_I := \prod_{i\in I}x_i$. The resulting matrix $Z:=\vec{x}^{\,e}\vec{x}^{\,eT}$ has entries $Z_{I,J}=\vec{x}_{I\triangle J}$ with subsets $I\triangle J$ that are also of even cardinality $|I\triangle J|$. Let the associated semidefinite relaxation be denoted by $\tilde{\mathbf{S}}_2$. It turns out that the above liftings of order $k=n$ are also exact:

To see this, let $\mathbf{S}_1 := \mathrm{conv}(\ \{\ \vec{x}^{\,o}\vec{x}^{\,oT}\mid x\in\{\pm1\}^n\ \}\ )$ and $\mathbf{S}_2 := \mathrm{conv}(\ \{\ \vec{x}^{\,e}\vec{x}^{\,eT}\mid x\in\{\pm1\}^n\ \}\ )$ where $\vec{x}^{\,o}$ and $\vec{x}^{\,e}$ are defined as above.

Lemma 3.1. *The set $\mathbf{S}_1$ coincides with its semidefinite relaxation $\tilde{\mathbf{S}}_1$, and $\mathbf{S}_2$ coincides with its semidefinite relaxation $\tilde{\mathbf{S}}_2$.*

Proof. Let $x\in\{\pm1\}^n$ and $\vec{x}^{\,o}$ with components $\vec{x}^{\,o}_J := \prod_{j\in J}x_j$ for odd-cardinality J be given. Likewise let $y\in\{\pm1\}^n$ and $\vec{y}^{\,o}$ be given and assume that x and y differ in the components $1\le k\le n-1$. (Since $\vec{x}^{\,o}\vec{x}^{\,oT}$ only has entries with even-cardinality indices, the case $k=n$ generates the same matrix $\vec{x}^{\,o}\vec{x}^{\,oT}=\vec{y}^{\,o}\vec{y}^{\,oT}$.) As in the proof of Proposition 2.1, if an even number of elements from $\{1,\dots,k\}$ is contained in $I\subseteq N$, then $\vec{x}^{\,o}_I=\vec{y}^{\,o}_I$, else $\vec{x}^{\,o}_I=-\vec{y}^{\,o}_I$, and thus, again as in the proof of Proposition 2.1,

$$\|\vec{x}^{\,o}-\vec{y}^{\,o}\|_2 = \|\vec{x}^{\,o}+\vec{y}^{\,o}\|_2 = 2\sqrt{2^{n-2}} = 2^{n/2}.$$

Since $\vec{x}^{\,o}$ and $\vec{x}^{\,e}$ are based on a disjoint union of all indices $I\subseteq N$ it follows from the above and from Proposition 2.1 that also $\|\vec{x}^{\,e}-\vec{y}^{\,e}\|_2=2^{n/2}$ for different $\vec{x}^{\,e},\vec{y}^{\,e}\in\{\pm1\}^{2^{n-1}}$. Thus, in both cases Proposition 2.1 is valid just with a different constant distance. Since there are only 2^{n-1} vertices in $\mathbf{S}_1$ or $\mathbf{S}_2$, an argument as for the proof of Lemma 2.4 is applicable as well: A boundary point of $\mathbf{S}_1$ is an affine combination of all vertices except one, i.e. it is an affine combination of $2^{n-1}-1$ vertices, each of which has rank one, so that a boundary point of $\mathbf{S}_1$ has rank at most $2^{n-1}-1$. (In difference to the proof of Lemma 2.4, an explicit element of the null space is not stated with this argument.) □

Observe that both relaxations can be combined by relating equivalent entries of both reduced representations $Y=\vec{x}^{\,o}\vec{x}^{\,oT}$ and $Z=\vec{x}^{\,e}\vec{x}^{\,eT}$ with equality constraints. A larger matrix inequality is thus replaced with two smaller matrix inequalities (namely Y and Z being positive semidefinite). This will be referred to as mixed reduced lifting below.

Proposition 3.2. *For even numbers n the above mixed reduced lifting of order $n/2$ is exact.*

Proof. First, for even numbers n and $x \in \{\pm 1\}^n$ let $\vec{\hat{x}}$ have components $\vec{\hat{x}}_I := \prod_{i\in I} x_i$ where $I \subseteq N$ only has at most $|I| \leq n/2$ entries. Then observe that the proof of Lemma 2.4 also applies in the lower-dimensional setting $\hat{X} = \vec{\hat{x}}\vec{\hat{x}}^T \in \mathcal{S}^{2^{n-1}}$ with 2^{n-1} equidistant extreme points. This implies a slight strengthening of Lemma 2.4, namely for even n the lifting of order $n/2$ is exact.

Now let $\vec{\hat{x}}^o$ have components $\vec{\hat{x}}_I^o := \prod_{i\in I} x_i$ where $x \in \{\pm 1\}^n$ is as before and $I \subseteq N$ has odd cardinality only and $|I| \leq n/2$. Likewise assume that $\vec{\hat{x}}^e$ has components $\vec{\hat{x}}_I^e := \prod_{i\in I} x_i$ where $x \in \{\pm 1\}^n$ and $I \subseteq N$ has even cardinality only and $|I| \leq n/2$. Thus,

$$\vec{\hat{x}} = \Pi \begin{pmatrix} \vec{\hat{x}}^o \\ \vec{\hat{x}}^e \end{pmatrix}$$

for some permutation matrix Π.

Consider the block-structured simplex

$$\mathbf{BS} := \operatorname{conv}\left(\left\{ \begin{pmatrix} \vec{\hat{x}}^o \vec{\hat{x}}^{o\,T} & 0 \\ 0 & \vec{\hat{x}}^e \vec{\hat{x}}^{e\,T} \end{pmatrix} \mid x \in \{\pm 1\}^n \right\} \right)$$

where the dimensions of the matrix blocks follow from the context. As the non-zero entries of a matrix in $\mathbf{BS}$ are associated with even cardinality subsets only, the vectors x and $-x$ generate the same matrix. Hence, $\mathbf{BS}$ is the convex hull of 2^{n-1} points contained in an $(2^{n-1}-1)$-dimensional affine subspace of $\mathcal{S}^{2^{n-1}}$. Up to a permutation, the mixed reduced lifting of $\mathbf{BS}$ coincides with the semidefinite approximation of $\vec{\hat{x}}\vec{\hat{x}}^T$ projected to a block diagonal format. This projection is consistent with respect to the semidefinite ordering and with respect to the equality constraints "$\hat{X}_{I,J} = \hat{X}_{K,L}$ for $I \triangle J = K \triangle L$", so that the arguments in the proof of Lemma 2.4 are applicable again. □

4 Including Linear Constraints

A key observation used in the approach of Lovász and Schrijver [12] concerns the inclusion of inequalities: Let a feasible set $\mathbf{IP}$ be given by the convex hull of ± 1-vectors satisfying linear inequalities $(a^{(j)})^T x + \alpha_j \geq 0$ for $j \in M$ with some finite set M,

$$\mathbf{IP} = \operatorname{conv}(\{x \mid x \in \{\pm 1\}^n, \ \ (a^{(j)})^T x + \alpha_j \geq 0 \text{ for } j \in M\}).$$

Then, since squared variables are identical to one, any finite product of these inequalities can be expressed as linear inequalities in terms of elements of the augmented vectors $\vec{x}$, the constant terms α_j being represented via $\vec{x}_\emptyset = 1$. Analogously, products of inequalities of the form $(\pm x_i + 1) \geq 0$ for $1 \leq i \leq n$ can be represented as follows: For a vector $p \in \{0,1\}^n$ let the vector $\bar{\mathbf{p}} \in \mathbb{R}^{2^n}$ be defined by the identity

$$\bar{\mathbf{p}}^T \vec{x} \equiv \prod_{i=1}^n ((-1)^{p_i} x_i + \vec{x}_\emptyset) \quad \text{for any } x \in \{\pm 1\}^n \text{ and the associated } \vec{x} \in \{\pm 1\}^{2^n}.$$

In the next lemma it is observed that an exact relaxation of $\mathbf{IP}$ is obtained when augmenting the components of the vectors $a^{(j)} \in \mathbb{R}^n$ for $j \in M$ to vectors

$$\bar{a}^{(j)} := (\alpha_j, (a^{(j)})^T, 0, \ldots, 0) \in \mathbb{R}^{2^n}$$

and when forming the semidefinite relaxation of order n with the same constraints as in (2.2) and with $2^n|M|$ additional constraints

$$\langle \bar{\mathbf{p}}(\bar{a}^{(j)})^T + \bar{a}^{(j)}(\bar{\mathbf{p}})^T, X\rangle \geq 0 \quad \text{for } j \in M \text{ and all } p \in \{0,1\}^n \text{ and their associated } \bar{\mathbf{p}}. \tag{4.1}$$

Lemma 4.1. *The first components $X_{\emptyset,\{1\}}, \ldots, X_{\emptyset,\{n\}}$ of the above semidefinite relaxation represent the exact convex hull $\mathbb{P}$ of all vectors $x \in \{\pm 1\}^n$ satisfying the linear inequalities $(a^{(j)})^T x + \alpha_j \geq 0$ for $j \in M$.*

Proof. For $M = \emptyset$ the all-one-diagonal of X and semidefiniteness of X imply the trivial fact that the vector $(X_{\emptyset,\{1\}}, \ldots, X_{\emptyset,\{n\}})^T$ is contained in the convex hull $[-1,1]^n$ of all $\{\pm 1\}$-vectors in $\mathbb{R}^n$. When $M \neq \emptyset$ assume that $\bar{x} \in \{\pm 1\}^n$ is a vector violating the constraint $(a^{(j)})^T \bar{x} + \alpha_j \geq 0$ for some $j \in M$. Selecting $p = (1 - \bar{x})/2$ in (4.1) and using

$$\bar{\mathbf{p}}^T \vec{\bar{x}} = \prod_{i=1}^{n} ((-1)^{p_i} \bar{x}_i + \vec{\bar{x}}_\emptyset) = 2^n > 0$$

it follows that (4.1) is violated by $X = \vec{\bar{x}}\vec{\bar{x}}^T$. On the other hand $\bar{\mathbf{p}}^T \vec{x} = 0$ for any other $x \in \{\pm 1\}^n$, since at least one of the factors $((-1)^{p_i} x_i + \vec{x}_\emptyset)$ is zero. Thus, (4.1) is satisfied with equality by any other vertex $\vec{x}\vec{x}^T$. Due to Lemma 2.4 any point in the semidefinite relaxation of order n is a convex combination of vertices $\vec{x}\vec{x}^T$ and by the above observation only vertices $\vec{x}\vec{x}^T$ satisfying all constraints occur in the convex combination. □

Based on ideas from [12] a simple proof of Lemma 4.1 is also given in [9]. Note that the approach in [12] is slightly different: The $\{\pm 1\}$-formulation is replaced with the equivalent $\{0,1\}$-formulation, and the constraints $\pm x_i + 1 \geq 0$ are added to the constraints in M forming a set $\tilde{M}$ (with $|\tilde{M}| = 2n + |M|$). Then the $2^n M$ constraints in (4.1) are replaced with $|\tilde{M}|^n$ constraints

$$\prod_{\ell=1}^{n} ((a^{(j_\ell)})^T x + \alpha_{j_\ell}) \geq 0 \tag{4.2}$$

for all choices of $j_\ell \in \tilde{M}$. Again (4.2) can be expressed as linear constraints of the entries of X forming an extended lifting that includes all the constraints of the form (4.1). (Some of these constraints are redundant.)

When reducing the extended lifting of order n to an order less than n, the relaxation based on (4.2) is an improvement compared to (4.1), but Lemma 4.1 indicates that (4.1) is sufficient for the semidefinite lifting of order n.

Unfortunately, for all approaches, the dimensions of the liftings of order higher than one generally are too large to be computationally competitive, and therefore so far they are mostly of theoretical interest.

To conclude note that for linear equality constraints $(a^{(j)})^T x + \alpha_j = 0$ on the binary variable $x \in \{\pm 1\}^n$ the constraints (4.1) can be simplified (substantially) to just $|M|$ constraints

$$\langle \bar{a}^{(j)}(\bar{a}^{(j)})^T, X\rangle = 0 \quad \text{for } j \in M. \tag{4.3}$$

Indeed if (4.3) is satisfied, then by semidefiniteness of X the vectors $\bar{a}^{(j)}$ lie in the null space of X and then all constraints (4.1) are satisfied so that the argument in the proof of Lemma 4.1 remains valid.

$\boxed{5}$ Relations to Earlier Results

There is vast literature on liftings for hard combinatorial problems. Some selected relations are detailed next.

The observation of Proposition 2.1 can be found in [10], for example, where it is also noted that any d-dimensional 0-1 polytope can be realized as the projection of a (2^d-1)-dimensional simplex. For the max-cut-polytope, $d = \frac{n(n-1)}{2}$, but due to the regular structure of the max-cut-polytope, the dimension of the associated simplex only is 2^n-1.

Lemma 2.4 has been derived in several earlier works based on an operator M which is defined below.

Observe that for $X \in \mathcal{A}$ the first column $\mathbf{x}$ of X (with components $\mathbf{x}_I = X_{I,\emptyset}$ for $I \subseteq N$ and $\mathbf{x}_\emptyset = 1$) uniquely determines the remaining entries $X_{I,J} = X_{I\triangle J,\emptyset}$ of X (see Definition (2.1) of $\mathcal{A}$). In [8, 10] and others, the notation $X = M(\mathbf{x})$ is used in this situation, and M is denoted a *moment matrix*.

Let the 2^n vertices of $\mathbf{S}$ be denoted by $\vec{x}^{(i)}(\vec{x}^{(i)})^T$ for $1 \le i \le 2^n$. Proposition 2.1 implies that the vertices are pairwise orthogonal to each other and that $Z := [\vec{x}^{(1)}, \dots, \vec{x}^{(2^n)}]$ satisfies $Z^T Z = 2^n I$ i.e. $2^{-n/2}Z$ is an orthogonal matrix. (Here, I denotes the identity matrix in $\mathcal{S}^{2^n}$.)

The linear identity

$$M(\mathbf{x}) = 2^{-n} Z \mathrm{Diag}(Z^T \mathbf{x}) Z^T \tag{5.1}$$

can be verified for any vector $\mathbf{x} = \vec{x}^{(i)}$ with $1 \le i \le 2^n$ since $2^{-n} Z^T \vec{x}^{(i)}$ is the i-th canonical unit vector in $\mathbb{R}^{2^n}$ and the first column of $M(\vec{x}^{(i)}) = \vec{x}^{(i)}(\vec{x}^{(i)})^T$ is $\vec{x}^{(i)}$ (using $(\vec{x}^{(i)})_\emptyset = 1$). Since the $\vec{x}^{(i)}$ form a basis of $\mathbb{R}^{2^n}$, (5.1) is indeed valid for all $\mathbf{x} \in \mathbb{R}^{2^n}$, in particular for all $\mathbf{x}$ with $\mathbf{x}_\emptyset = 1$, i.e. for all $\mathbf{x}$ with $M(\mathbf{x}) \in \mathcal{A}$.

Since $2^{-n/2}Z$ is an orthogonal matrix, relation (5.1) actually is an eigenvalue decomposition. The eigenvectors $\vec{x}^{(i)}$ are independent of the particular choice of $\mathbf{x}$ i.e. of $X = M(\mathbf{x}) \in \mathcal{A}$, and the vector of eigenvalues is given by $Z^T\mathbf{x}$. Thus, $X = M(\mathbf{x})$ satisfies

$$X \in \tilde{\mathbf{S}} \Longleftrightarrow Z^T \mathbf{x} \ge 0 \Longleftrightarrow X \in \mathbf{S},$$

where the second equivalence uses the fact that $2^n = \mathrm{trace}(X) = e^T Z^T \mathbf{x}$ so that $Z^T\mathbf{x} \ge 0$ is equivalent to $Z^T\mathbf{x}$ being a convex combination of canonical unit vectors multiplied by 2^n.

This is a second proof of Lemma 2.4 that is closer to the proofs in the earlier papers [8, 10] where relation (5.1) is derived directly by algebraic arguments and then used to establish Lemma 2.4.

A benefit of the introduction of the operator M lies in the observation that M applied to a convolution of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{2^n}$ can be easily analyzed and exploited, see e.g. Lemma 2 and Lemma 3 in [10].

Another derivation of Lemma 2.4 can be found in Lemma 8.15 in [11], and a simple proof of a result equivalent to Lemma 2.4 is also given in [2].

$\boxed{6}$ Conclusion

This paper is intended as a reference providing a self-contained and simple proof of the known fact that the n-th semidefinite lifting of binary problems is exact. The unconstrained case of the max-cut-polytope is considered first and the results are then extended to reduced liftings with just rows corresponding to odd/even subsets of variables, a mixed reduced setting, and finally, the constrained case with both inequalities and equalities. A brief discussion of some related work concludes this paper.

While the quality of the liftings improves with the order of the liftings, the dimension of the liftings grows dramatically during the first lifting steps, so that even low orders of the lifting are computationally very expensive. It is still an open problem to modify the liftings such that they can be solved efficiently. For a survey exploiting algebraic symmetry properties it is referred to [7].

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