Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 18, No. 4, pp. 867–881, 2022

PESSIMISTIC REFERENTIAL-UNCOOPERATIVE LINEAR BILEVEL MULTI-FOLLOWERS DECISION MAKING WITH APPLICATION TO WATER RESOURCES OPTIMAL ALLOCATION*

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Abstract: In this paper, a pessimistic linear bilevel multi-followers programming problem in a referential-uncooperative situation (for short, PLBMF-RU) is presented where the followers are uncooperative while having cross reference to decision making information of the other followers. The model and the solution definitions of PLBMF-RU is first presented. Then, a simple example is developed to illustrate it. Furthermore, the penalized problems are proposed based on the penalty function method to solve PLBMF-RU. Finally, a case study of water resources optimal allocation illustrates the feasibility of the proposed model. This may provide a new way to discuss the pessimistic linear bilevel multi-followers programming problem in a referential-uncooperative situation.

Key words: *bilevel programming, pessimistic formulation, multiple followers, penalty function*

Mathematics Subject Classification: *90C26, 90C30*

1 Introduction

Bilevel programming plays an exceedingly important role in different application fields, such as transportation, economics, ecology, engineering and others [12]. It has been developed and researched by many authors, e.g., see the recent monographs [6, 11, 29]. When the set of solutions of the lower level problem is not a singleton, it is difficult for the leader to optimize his choice unless he knows the follower's reaction to his choice. In this situation, at least two approaches have been suggested: optimistic (or strong) formulation and pessimistic (or weak) formulation [7, 22]. However, most research on bilevel programming relates to the optimistic formulation. Interested readers can refer to the surveys [9, 12, 25, 30, 31, 36] and the references therein.

However, the pessimistic formulation is very popular in practice. There are at least three reasons for this. First, for the actual bilevel programming problems, a non-cooperative behavior of the followers can frequently be observed in applications. Second, in fact, the

*This work was partially supported by the National Natural Science Foundation of China (Nos. 12071379, 11871383, 12126412), the Humanities and Social Sciences Research Project of Anhui Province University (SK2021ZD0049), the Natural Science Foundation of Chongqing (cstc2021jcyj-msxmX0925, cstc2022ycjh-bgzxm0097), the Youth Top Talent Program of Chongqing Talents (CQYC2021059311) and the Fundamental Research Funds for the Central Universities (XDJK2020B048).

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leader in general has incomplete information about the followers, i.e., he may not know the values of all parameters in the lower level problem. The leader would like to create a safety margin to bound the damage resulting from undesirable selections of the followers. Finally, if the leader can obtain a pessimistic solution, this result will provide him with a policy-making reference that prompts him to make a rational decision to effectively control risk. Note that, several studies has been contributed to pessimistic (weak) bilevel programming problems [18] from different subjects: existing results of solutions [1, 2, 3, 4, 16, 20, 21, 26, 39], optimality conditions [10, 13, 14], approximation algorithm [8, 15, 32], penalty method [2, 37], reduction method [34, 40] and so on.

The above stated pessimistic bilevel programming problems are limited to a specific situation with one-leader-and-one-follower. However, for the actual bilevel programming problems, the lower level problem often involves multiple decision makers. Moreover, the most basic situation is that these followers are uncooperative and they do cross reference information by considering other followers' decision results in each of their own decision objectives and constraints. It is worthwhile noting that, Lu et al. [23] call this case as a referential-uncooperative situation, and this paper will study the pessimistic linear bilevel multi-follower programming problem on this situation. Note that several papers have been devoted to bilevel multi-follower decision making problems dealing with different situations, we cite, for example, Refs. [24, 38]. The reader is also referred to the book on bilevel multi-follower decision making [35].

The remainder of the paper is organized as follows. The pessimistic linear bi-level multi-followers programming problem is presented in a referential-uncooperative situation (PLBMF-RU) in Section 2. Under some assumptions, the penalized problems of PLBMF-RU are proposed in Section 3. A case study is given to illustrate the PLBMF-RU model and its feasibility in Section 4. Finally, the conclusions are given.

2 PLBMF-RU

Under the bilevel optimizaiton framework, if the followers do not have any shared control decision variable, it is called an uncooperative relationship. However, if either of them has a reference or consideration of other followers' decision information in their objective or constraint functions, the followers are defined having a referential-uncooperative relationship. When there is a referential-uncooperative relationship in a bilevel optimization model, the leader in general has incomplete information about the followers, e.g., he may not know the values of all parameters of the followers or the relations among the followers. Then the leader may be risk-averse, and would need a safety margin to bound the damage resulting from the undesirable selections of the followers. This situation is called a pessimistic linear bilevel multi-followers programming problem in a referential-uncooperative situation.

Consider the following pessimistic linear bilevel programming problem (P) where k followers are involved in a referential-uncooperative situation:

$$\begin{aligned} \min_{x,y_1,\ldots,y_k} \quad & c^T x + \sum_{i=1}^{k} \sup_{y_i \in \Psi_i(x,y_{-i})} d_i^T y_i \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k)$ and $\Psi_i(x, y_{-i})$ is the set of solutions of the ith follower's problem

$$\min_{y_i \geq 0} \; u_i^T y_i$$
$$\text{s.t.} \;\; A_i x + \sum_{j=1}^{k} B_{ij} y_j \leq b_i.$$

Here, $x, c \in \mathbb{R}^n$, $y_i, d_i, u_i \in \mathbb{R}^{m_i}$, $A_i \in \mathbb{R}^{q_i \times n}$, $B_{ij} \in \mathbb{R}^{q_i \times m_j}$, $b_i \in \mathbb{R}^{q_i}$, $i, j = 1, 2, \ldots, k$, and X is a closed subset of $\mathbb{R}^n$.

Problem (P) can be equivalently transformed into the following problem :

$$\min_{x, y_1, \ldots, y_k} \; c^T x + \sum_{i=1}^{k} \; d_i^T y_i$$
$$\text{s.t.} \quad x \in X,$$

where y_i is a solution of the following problem $(P(x, y_{-i}))$,

$$\max_{y_i \in \Psi_i(x, y_{-i})} \; d_i^T y_i.$$

In order to assure that the problem is well-posed, it will now be assumed that a solution to problem (P) exists.

Next, we give some definitions of problem (P).

Definition 2.1. (a) Constraint region of problem (P):

$$S = \{(x, y_1, \ldots, y_k) : x \in X, A_i x + \sum_{j=1}^{k} B_{ij} y_j \leq b_i, y_i \geq 0, i = 1, 2, \ldots, k\}.$$

(b) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : (x, y_1, \ldots, y_k) \in S\}.$$

(c) Feasible set for the ith follower:

$$S_i(x, y_{-i}) = \{y_i : A_i x + \sum_{j=1}^{k} B_{ij} y_j \leq b_i, y_i \geq 0\}.$$

(d) The ith follower's rational reaction set:

$$\Psi_i(x, y_{-i}) = \{y_i : y_i \in \operatorname{Arg\,min}[u_i^T y_i : y_i \in S_i(x, y_{-i})]\}.$$

(e) Inducible region or feasible region of the leader:

$$IR = \{(x, y_1, \ldots, y_k) : (x, y_1, \ldots, y_k) \in S, y_i \in \Psi_i(x, y_{-i}), i = 1, 2, \ldots, k\}.$$

Note that, $(x^*, y_1^*, \ldots, y_k^*) \in IR$ means that, $\forall i = 1, 2, \ldots, k$,

$$u_i^T y_i \geq u_i^T y_i^*$$

for any $y_i \in S_i(x^*, y_{-i}^*)$ where $y_{-i}^* = (y_1^*, \ldots, y_{i-1}^*, y_{i+1}^*, \ldots, y_k^*)$. Then, a popular solution concept is the so-called Nash equilibrium defined as $(y_1^*, \ldots, y_k^*)$ with respect to x^*. The interested reader can refer to Refs. [5, 17].

Definition 2.2. A point $(x^*, y_1^*, \ldots, y_k^*) \in IR$ is called a pessimistic solution to problem (P), if

$$c^T x^* + \sum_{i=1}^{k} f_i(x^*, y_{-i}^*) \le c^T x + \sum_{i=1}^{k} f_i(x, y_{-i}),\ \forall\,(x, y_1, \ldots, y_k) \in IR,$$

where $f_i(x, y_{-i}) = \sup_{y_i \in \Psi_i(x, y_{-i})} d_i^T y_i$ for all $i = 1, 2, \ldots, k$.

To illustrate these concepts, assume that $k = 2$ followers exist, and that for each $i = 1, 2$, the ith follower controls over the vector $y_i = (y_i^1, y_i^2) \in \mathbb{R}^2$. Consider the following example which is a pessimistic linear bilevel programming problem where two followers are involved in a referential-uncooperative situation.

Example 2.3.

$$\begin{aligned} \min_{x, y_1, y_2}\ & \{-5x + \sup_{y_1 \in \Psi_1(x, y_{-1})} (y_1^1 + 2y_1^2) + \sup_{y_2 \in \Psi_2(x, y_{-2})} (y_2^1 + 2y_2^2)\} \\ \text{s.t.}\ & \quad 0 \le x \le 1, \end{aligned}$$

where $\Psi_1(x, y_{-1})$ and $\Psi_2(x, y_{-2})$ are the sets of solutions of the followers' problem respectively,

$$\begin{aligned} \min_{y_1 \ge 0}\ & -y_1^1 - y_1^2 \\ \text{s.t.}\ & y_1^1 + y_1^2 \le x, \\ & y_1^1 + y_1^2 \le x + y_2^1 + y_2^2 - 0.4, \\ \min_{y_2 \ge 0}\ & -y_2^1 - y_2^2 \\ \text{s.t.}\ & y_2^1 + y_2^2 \le x, \\ & y_2^1 + y_2^2 \le x + y_1^1 + y_1^2 - 0.6. \end{aligned}$$

Here, the constraint region is given by

$$\begin{aligned} S = \big\{(x, y_1, y_2) : 0 \le x \le 1,\ & y_1^1 + y_1^2 \le x,\ y_1^1 + y_1^2 \le x + y_2^1 + y_2^2 - 0.4, \\ & y_2^1 + y_2^2 \le x,\ y_2^1 + y_2^2 \le x + y_1^1 + y_1^2 - 0.6,\ y_1, y_2 \ge 0\big\}. \end{aligned}$$

The vector $(0.6, 0, 0, 0, 0) \in S$, but it is not a feasible point since, given $x = 0.6$ and $y_2 = (0, 0)$, $y_1 = (0.2, 0)$ (rather than $y_1 = (0, 0)$) minimizes the first follower's problem. The vector $(0.5, 0.1, 0, 0, 0)$ is feasible, since $(0.1, 0, 0, 0)$ is a Nash equilibrium point with respect to $x = 0.5$. It is not difficult to show that the inducible region IR for this example is given by

$$IR = \left\{ (x, y_1, y_2) : \begin{aligned} & y_1^1 + y_1^2 = \begin{cases} x, & y_2^1 + y_2^2 \ge 0.4; \\ x + y_2^1 + y_2^2 - 0.4, & y_2^1 + y_2^2 < 0.4; \end{cases} \\ & y_2^1 + y_2^2 = \begin{cases} x, & y_1^1 + y_1^2 \ge 0.6; \\ x + y_1^1 + y_1^2 - 0.6, & y_1^1 + y_1^2 < 0.6; \end{cases} \\ & (x, y_1, y_2) \in S \end{aligned} \right\}.$$

Given IR, it is easy to check that the feasible point $(0.5, 0.1, 0, 0, 0)$ of S is the pessimistic solution for this example and that the leader's objective value is -2.3.

3 Penalty Problems

In this section, the penalized problems of PLBMF-RU are proposed under some suitable conditions. To establish theoretical results, we need the following assumptions throughout the paper.

Assumption A:

(A1) For any $x \in X$, the sets $S_i(x, y_{-i})$ are non-empty, and there exists compact subsets W_i such that $S_i(x, y_{-i}) \subset W_i$ for all $x \in X$, $i = 1, 2, \ldots, k$.

(A2) The set X is a bounded non-empty polyhedron.

Then we have the following lemma which provides the existence of solutions to problem (P).

Lemma 3.1. *If assumptions* (A1) *and* (A2) *are satisfied, then problem* (P) *has at least one solution.*

Proof. Under assumptions (A1) and (A2), for each $i = 1, 2, \ldots, k$, it follows from [11, Theorem 4.3] that $f_i(x, y_{-i})$ is continuous. Using the results of [28, Lemma 1], we have that IR is a non-empty compact set. Therefore, the proof follows from the Weierstrass's Theorem. □

The dual problem of the lower level problem in (P) can be written as:

$$\begin{aligned} \max_{z_i \geq 0} \; & -(b_i - A_i x - \sum_{j=1, j\neq i}^{k} B_{ij} y_j)^T z_i \\ \text{s.t.} \; & -B_{ii}^T z_i \leq u_i. \end{aligned}$$

Let $Z_i := \{z_i : -B_{ii}^T z_i \leq u_i, z_i \geq 0\}$, and denote the ith follower's duality gap by $\pi_i(x, y_1, \ldots, y_k, z_i) := u_i^T y_i + (b_i - A_i x - \sum_{j=1, j\neq i}^{k} B_{ij} y_j)^T z_i$.

It follows from the dual theory that problem $(P(x, y_{-i}))$ is written as follows:

$$\begin{aligned} \max_{y_i, z_i} \; & d_i^T y_i \\ \text{s.t.} \; & \pi_i(x, y_1, \ldots, y_k, z_i) = 0, \\ & y_i \in S_i(x, y_{-i}), \\ & z_i \in Z_i. \end{aligned}$$

For $\rho_i > 0 \, (i = 1, 2, \ldots, k)$, we consider the following penalized problem $(P_{\rho_i}(x, y_{-i}))$:

$$\begin{aligned} \max_{y_i, z_i} \; & d_i^T y_i - \rho_i \pi_i(x, y_1, \ldots, y_k, z_i) \\ \text{s.t.} \; & y_i \in S_i(x, y_{-i}), \\ & z_i \in Z_i. \end{aligned}$$

Let $V(\cdot)$ represent the set of vertices of the set to be concerned.

Lemma 3.2. *Under assumption* (A1)*, for* $\rho_i > 0$*, problem* $(P_{\rho_i}(x, y_{-i}))$ *has at least one solution in* $V(S_i(x, y_{-i}) \times V(Z_i)$.

Proof. For $\rho_i > 0$, we have

$$\sup_{\substack{y_i \in S_i(x, y_{-i}) \\ z_i \in Z_i}} [d_i^T y_i - \rho_i \pi_i(x, y_1, \ldots, y_k, z_i)] \le \sup_{y_i \in S_i(x, y_{-i})} d_i^T y_i$$
$$= \max_{y_i \in S_i(x, y_{-i})} d_i^T y_i.$$

Then, it follows from (A1) that the objective function of the linear programming problem $(P_{\rho_i}(x, y_{-i}))$ is bounded from above. Hence, the latter problem has at least one solution in $V(S_i(x, y_{-i})) \times V(Z_i)$. □

The dual of problem $(P_{\rho_i}(x, y_{-i}))$ is written as:

$$\begin{aligned}
\min_{t_i, v_i \ge 0} \ & (b_i - A_i x - \sum_{j=1, j\ne i}^{k} B_{ij} y_j)^T t_i + u_i^T v_i \\
\text{s.t.} \ & - B_{ii}^T t_i \le -d_i + \rho_i u_i, \\
& B_{ii} v_i \le \rho_i (b_i - A_i x - \sum_{j=1, j\ne i}^{k} B_{ij} y_j).
\end{aligned}$$

Then, we can get the following intermediate penalized problem $(\bar{P})$:

$$\begin{aligned}
\min_{x, y_i, z_i, t_i, v_i, i=1,2,\ldots,k} \ & c^T x + \sum_{i=1}^{k} [(b_i - A_i x - \sum_{j=1, j\ne i}^{k} B_{ij} y_j)^T t_i + u_i^T v_i] \\
\text{s.t.} \ & - B_{ii}^T t_i \le -d_i + \rho_i u_i, \\
& B_{ii} v_i \le \rho_i (b_i - A_i x - \sum_{j=1, j\ne i}^{k} B_{ij} y_j), \\
& (x, y_1, \ldots, y_k) \in S, \ z_i \in Z_i, \\
& t_i, v_i \ge 0, i = 1, 2, \ldots, k.
\end{aligned}$$

For simplicity, let

$$\begin{aligned}
& Z_{k+1}(\rho_1, \ldots, \rho_k) := \big\{(x, y_1, \ldots, y_k, v_1, \ldots, v_k) : (x, y_1, \ldots, y_k) \in S, \\
& B_{ii} v_i \le \rho_i (b_i - A_i x - \sum_{j=1, j\ne i}^{k} B_{ij} y_j), v_i \ge 0, i = 1, 2, \ldots, k\big\}, \\
& Z_{k+2}(\rho_1, \ldots, \rho_k) := \big\{(z_1, \ldots, z_k, t_1, \ldots, t_k) : z_i \in Z_i, -B_{ii}^T t_i \le -d_i + \rho_i u_i, \\
& \qquad t_i \ge 0, i = 1, \ldots, k\big\}.
\end{aligned}$$

Then, we have the following results which provide the existence of solutions to problem $(\bar{P})$.

Theorem 3.3. *Under Assumption A, for fixed values of $\rho_i > 0$ $(i = 1, 2, \ldots, k)$, problem $(\bar{P})$ has at least one solution in $V(Z_{k+1}(\rho_1, \ldots, \rho_k)) \times V(Z_{k+2}(\rho_1, \ldots, \rho_k))$.*

Proof. Note that problem $(\bar{P})$ is a disjoint bilinear programming problem for fixed values of $\rho_i > 0$ $(i = 1, 2, \ldots, k)$. Similar the proof of Theorem 3.2 in [2], we can conclude that problem $(\bar{P})$ has at least one solution in $V(Z_{k+1}(\rho_1, \ldots, \rho_k)) \times V(Z_{k+2}(\rho_1, \ldots, \rho_k))$. □

For any $\eta > 0$, let $\mathcal{W}_\eta = \{(\alpha, \beta) : B\beta \le \eta(b - A\alpha)\}$ and $\mathcal{W} = \{(\alpha, \xi) : B\xi \le b - A\alpha\}$.

Lemma 3.4. *If $(\alpha_\eta^*, \beta_\eta^*) \in V(\mathcal{W}_\eta)$, then there exists $(\alpha^*, \xi^*) \in V(\mathcal{W})$ such that $\alpha_\eta^* = \alpha^*$.*

Proof. By the definition of $\mathcal{W}$, we have $\left(\alpha_\eta^*, \frac{\beta_\eta^*}{\eta}\right) \in \mathcal{W}$. Let $(\alpha^1, \xi^1), \ldots, (\alpha^r, \xi^r)$ be the distinct vertices of $\mathcal{W}$. Since any point in $\mathcal{W}$ can be written as a convex combination of these vertices, let $\left(\alpha_\eta^*, \frac{\beta_\eta^*}{\eta}\right) = \sum_{i=1}^{\hat{r}} \kappa^i(\alpha^i, \xi^i)$, where $\sum_{i=1}^{\hat{r}} \kappa^i = 1$, $\kappa^i > 0$, $i = 1, \ldots, \hat{r}$, and $\hat{r} \le r$. Then, it follows that

$$(\alpha_\eta^*, \beta_\eta^*) = \sum_{i=1}^{\hat{r}} \kappa^i(\alpha^i, \eta\xi^i).$$

Note that $(\alpha^i, \eta\xi^i) \in \mathcal{W}_\eta$. Hence, the last equality implies that $\hat{r} = 1$. Because $(\alpha_\eta^*, \beta_\eta^*)$ is a vertex of $\mathcal{W}_\eta$, a contradiction results unless $\hat{r} = 1$. Thus, there exists $(\alpha^*, \xi^*) \in V(\mathcal{W})$ such that $\alpha^* = \alpha_\eta^*$ and $\beta_\eta^* = \eta\xi^*$. □

Let

$$\begin{aligned} Z_{k+3} \quad := \quad & \Big\{(x, y_1, \ldots, y_k, v_1, \ldots, v_k) : (x, y_1, \ldots, y_k) \in S, v_i \ge 0, \\ & B_{ii}v_i \le b_i - A_i x - \sum_{j=1, j\neq i}^{k} B_{ij}y_j, \ i = 1, 2, \ldots, k\Big\}. \end{aligned}$$

The following result follows from Theorem 3.3 and Lemma 3.4 directly.

Corollary 3.5. *Under Assumption A, for fixed values of $\rho_i > 0$ $(i = 1, 2, \ldots, k)$, problem $(\bar{P})$ has at least one solution in $V(Z_{k+3}) \times V(Z_{k+2}(\rho_1, \ldots, \rho_k))$.*

Obviously, if a solution of problem $(\bar{P})$ satisfies $\pi_i(x, y_1, \ldots, y_k, z_i) = 0$ $(i = 1, 2, \ldots, k)$, it is also a solution of problem $(\hat{P})$. We may add the duality gap $\pi_i(x, y_1, \ldots, y_k, z_i) = 0$ $(i = 1, 2, \ldots, k)$ in problem $(\bar{P})$ by considering the following problem $(\hat{P})$:

$$\begin{aligned} \min_{x, y_i, z_i, t_i, v_i, i=1,2,\ldots,k} \quad & c^T x + \sum_{i=1}^{k} [(b_i - A_i x - \sum_{j=1, j\neq i}^{k} B_{ij}y_j)^T t_i + u_i^T v_i] \\ \text{s.t.} \quad & -B_{ii}^T t_i \le -d_i + \rho_i u_i, \\ & B_{ii}v_i \le \rho_i(b_i - A_i x - \sum_{j=1, j\neq i}^{k} B_{ij}y_j), \\ & (x, y_1, \ldots, y_k) \in S, \ z_i \in Z_i, \\ & \pi_i(x, y_1, \ldots, y_k, z_i) = 0, \\ & t_i, v_i \ge 0, i = 1, 2, \ldots, k, \end{aligned}$$

The constraints $\pi_i(x, y_1, \ldots, y_k, z_i) = 0, i = 1, 2, \ldots, k$, ensure that, whatever the values ρ_i selected, we have $(x, y_1, \ldots, y_k) \in IR$.

Now, for $\gamma_i > 0$ $(i = 1, 2, \ldots, k)$, we consider the following problem $(\tilde{P})$ which is a penalized problem of $(\hat{P})$:

$$\begin{aligned}\min_{x,y_i,z_i,t_i,v_i,i=1,2,\ldots,k} \quad & c^T x + \sum_{i=1}^{k} [(b_i - A_i x - \sum_{j=1,j\neq i}^{k} B_{ij} y_j)^T t_i + u_i^T v_i] \\ & + \sum_{i=1}^{k} \gamma_i \pi_i(x, y_1, \ldots, y_k, z_i) \\ \text{s.t.} \quad & - B_{ii}^T t_i \leq -d_i + \rho_i u_i, \\ & B_{ii} v_i \leq \rho_i (b_i - A_i x - \sum_{j=1,j\neq i}^{k} B_{ij} y_j), \\ & (x, y_1, \ldots, y_k) \in S, \ z_i \in Z_i, \\ & t_i, v_i \geq 0, i = 1, 2, \ldots, k.\end{aligned}$$

We can characterize the relations between problems $(\tilde{P})$ and $(\hat{P})$.

Theorem 3.6. *Let Assumption A hold, $\rho_i > 0$ and $\gamma_i > 0$ $(i = 1, 2, \ldots, k)$. Assume that $(x, y_1, \ldots, y_k,\ z_1, \ldots, z_k,\ t_1, \ldots, t_k, v_1, \ldots, v_k)$ be a solution of problem $(\tilde{P})$. Then, there exist finite values $\gamma_i^* > 0$ such that it is also a solution of problem $(\hat{P})$ for all $\gamma_i \geq \gamma_i^*$ $(i = 1, 2, \ldots, k)$.*

Proof. Let $(x^*, y_1^*, \ldots, y_k^*, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*, v_1^*, \ldots, v_k^*)$ solve problem $(\hat{P})$. Then,

$$(x^*, y_1^*, \ldots, y_k^*, v_1^*, \ldots, v_k^*) = \sum_{m=1}^{M} l_m (x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m)$$

for $l_m > 0$ with $\sum_{m=1}^{M} l_m = 1$ where $\{(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m)\}$ are vertices of Z_{k+3}.

Note that, for $i = 1, 2, \ldots, k$,

$$0 = \pi_i(x^*, y_1^*, \ldots, y_k^*, z_i^*) = \sum_{m=1}^{M} l_m \pi_i(x^m, y_1^m, \ldots, y_k^m, z_i^*).$$

Since $(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*) \in Z_{k+3}$ and $\pi_i(x^m, y_1^m, \ldots, y_k^m, z_i^*) \geq 0$, we have

$$\pi_i(x^m, y_1^m, \ldots, y_k^m, z_i^*) = 0, 1 \leq m \leq M.$$

Let $Q(x, y_1, \ldots, y_k, v_1, \ldots, v_k, z_1, \ldots, z_k, t_1, \ldots, t_k)$ be the objective function of problem $(\hat{P})$. Then, we have

$$0 = \sum_{m=1}^{M} l_m \Big[Q(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*) - Q(x^*, y_1^*, \ldots, y_k^*, v_1^*, \ldots, v_k^*, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*) \Big] \geq 0$$

which implies that

$$Q(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*) = Q(x^*, y_1^*, \ldots, y_k^*, v_1^*, \ldots, v_k^*, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*).$$

That is, $\{(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*)\}$ solve problem $(\hat{P})$.

Now, we may fix $(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m)$ and similarly show that $\{(z_1^q, \ldots, z_k^q, t_1^q, \ldots, t_k^q)\}$, $1 \leq q \leq \bar{q}$, of $Z_{k+2}(\rho_1, \ldots, \rho_k)$ such that $(x^m, y_1^m, \ldots, y_k^m, v_1^m, \ldots, v_k^m, z_1^q, \ldots, z_k^q, t_1^q, \ldots, t_k^q)$ solves problem $(\hat{P})$ for all $1 \leq m \leq M, 1 \leq q \leq \bar{q}$. Therefore, following a similar line of analysis to that in Zangwill [33], restricting the optimization problem to the finite set of vertices only, the result follows immediately. □

Theorem 3.7. *Let Assumption A hold. For the fixed values of* $\rho_i > 0$ $(i = 1, 2, \ldots, k)$, *let* $(x^*, y_1^*, \ldots, y_k^*, z_1^*, \ldots, z_k^*, t_1^*, \ldots, t_k^*, v_1^*, \ldots, v_k^*)$ *be a solution of problem* $(\hat{P})$. *Then,* $(x^*, y_1^*, \ldots, y_k^*)$ *is a solution of problem* (P).

Proof. It follows from [2, Theorem 3.3] that if a solution of problem $(\bar{P})$ satisfies $\pi_i(x, y_1, \ldots, y_k, z_i) = 0$ $(i = 1, 2, \ldots, k)$, it is also a solution of problem (P). Therefore, it is easy to conclude that $(x^*, y_1^*, \ldots, y_k^*)$ is a solution of problem (P) . □

To summarise, there are two ways to solve problem (P). One of them is to solve problem $(\tilde{P})$, increasing γ_i until a solution is obtained for which these duality gap are equal to zero. The other one is to solve problem $(\hat{P})$ via solvers (e.g. BARON, MSNLP) which are popularly used for mathematical programming models. Clearly, these allow us to use the constrained nonlinear programming tools to solve PLBMF-RU. Therefore, these may provide us with a new way to discuss the pessimistic linear bilevel multi-follower programming problem in a referential-uncooperative situation.

4 PLBMF-RU for Water Resources Optimal Allocation

In this section, a water resources optimal allocation problem will be solved by constructing a pessimistic referential-uncooperative linear bilevel multi-follower decision making model.

It is well-known that fresh water is very crucial and precious resources of our living and production. With the accelerating promotion of economic globalization, one of the biggest problems is that fresh water is in increasingly high demand and is in a growingly short supply. The conflict between increasing demand and the relative shortage of fresh water becomes current difficulties faced by China and the world. Particularly, in China, the limited available water is unable to meet the growing demand for such water. The contradiction between supply and demand of water resources has become an important factor restricting China's booming economy. Therefore, the higher requirements to the water resources optimal allocation are put forward.

Note that the water resources are public welfare. In addition to meeting the water demand of individual user, the water management department also need to allocate a certain public water rights to ensure the public interest (Ecological protection, environmental purification and other social welfare). Without loss of generality, suppose that the total amount of water allocated to the water management department is denoted by Q. The public water right is w, and the water resources rate is κ. The initial water rights for the ith water user is q_i $(i = 1, 2, \ldots, n)$. Clearly, the total amount of public water rights and all water users is not more than Q. That is to say,

$$w + \sum_{i=1}^{n} q_i \leqslant Q.$$

Furthermore, suppose that the revenue function of ith water user is $f_i(q_i)$ where $f_i(0) = 0$ and $f_i^{'}(q_i) > 0$, and the ith user has a minimum water demand δ_i respectively. Because an individual water user is rational, the purpose of user is to maximize his own interests. Then, maximizing the ith water user's revenue function is deemed to be the ith follower problem of pessimistic bilevel problem, i.e.,

$$\begin{aligned}\max_{q_i \geq 0}\ & g_i(q_i) = f_i(q_i) - \kappa q_i \\ \text{s.t.}\ & w + \sum_{i=1}^{n} q_i \leqslant Q, \\ & q_i \geqslant \delta_i.\end{aligned}$$

On the other hand, the total social benefits contain the gains from the public water consumption and the interest from all water users. Note that, the water management department regulates the water market by distributing the initial water rights, and then maximizes the total social benefits. The mathematical formulation of the upper level model is given by

$$\max_{w \geq 0}\ h(w) + \sum_{i=1}^{n} g_i(q_i)$$

where $h(0) = 0$ and $h^{'}(w) > 0$.

Stated thus, based on problem (P), a pessimistic referential-uncooperative bilevel multi-follower water resources optimal allocation is constructed as follows:

$$\begin{aligned}\max_{w, q_1, \ldots, q_n}\ & h(w) + \sum_{i=1}^{n} \min_{q_i}\ g_i(q_i) \\ \text{s.t.}\ & w \geq 0,\end{aligned} \tag{4.1}$$

where q_i is a solution of the ith follower's problem,

$$\begin{aligned}\max_{q_i \geq 0}\ & g_i(q_i) = f_i(q_i) - \kappa q_i \\ \text{s.t.}\ & w + \sum_{i=1}^{n} q_i \leqslant Q, \\ & q_i \geqslant \delta_i.\end{aligned}$$

In order to easily show the application for the proposed PLBMF-RU model, we can assign a value to a variable respectively. The experimental data employed for the model (4.1) is provided as follows: $n = 2$, $h(w) = 1.5w$, $\kappa = 0.2$, $Q = 1$, $f_1(q_1) = 0.6q_1$, $f_2(q_2) = 0.4q_2$, $\delta_1 = 0.25$, and $\delta_2 = 0.15$. Therefore, a water resources optimal allocation problem is

Table 4.1: Numerical results of problem (4.3) via Algorithm 1

ρ_1	ρ_2	γ_1	γ_2	x	y_1	y_2	f	π_1	π_2
1	1	1	1	0.6	0.25	0.15	-1.03	0	0
10	10	10	10	0.6	0.25	0.15	-1.03	0	0
100	100	100	100	0.6	0.25	0.15	-1.03	0	0

established by simplifying it into the following linear PLBMF-RU decision making model:

$$\begin{aligned}\max_{w,q_1,q_2}\ & 1.5w+\sum_{i=1}^{2}\min_{q_i}\ g_i(q_i)\\ \text{s.t.}\ & w\geq 0,\\ & \max_{q_1\geq 0}\ g_1(q_1)=0.4q_1\\ & \quad \text{s.t.}\ w+q_1+q_2\leqslant 1,\\ & \qquad\ q_1\geqslant 0.25,\\ & \max_{q_2\geq 0}\ g_2(q_2)=0.2q_2\\ & \quad \text{s.t.}\ w+q_1+q_2\leqslant 1,\\ & \qquad\ q_2\geqslant 0.15.\end{aligned}\tag{4.2}$$

To simply solve problem (4.2), we transform it into the following problem:

$$\begin{aligned}\min_{x,y_1,y_2}\ & -1.5x+\sum_{i=1}^{2}\max_{y_i}\ g_i(y_i)\\ \text{s.t.}\ & x\geq 0,\\ & \min_{y_1\geq 0}\ g_1(y_1)=-0.4y_1\\ & \quad \text{s.t.}\ x+y_1+y_2\leqslant 1,\\ & \qquad\ y_1\geqslant 0.25,\\ & \min_{y_2\geq 0}\ g_2(y_2)=-0.2y_2\\ & \quad \text{s.t.}\ x+y_1+y_2\leqslant 1,\\ & \qquad\ y_2\geqslant 0.15.\end{aligned}\tag{4.3}$$

The numerical results of problem (4.3) are listed as in Table 4.1 where $f=-1.5x+\sum_{i=1}^{2}g_i(y_i)$. It follows from Table 4.1 that a solution of problem (4.2) occurs at the point $(w,q_1,q_2)=(0.6,0.25,0.15)$. The result shows that an optimal solution for the water management department, who is risk-averse, is to take the public water right variable as 0.6 through anticipating all possible responses of the water users from the worst-case point of view. Each user is assumed to execute simultaneously his individually optimal choice after decisions of the water management department. That is, the two users will take values of their decision making variables 0.25 and 0.15 respectively as their response for the water management department.

5 Conclusions

A pessimistic linear bilevel multi-follower programming problem in a referential-uncooperative situation (PLBMF-RU) occurs commonly in management and planning of many organizations. In this paper, the model and the solution definitions of PLBMF-RU decision problem are proposed. Moreover, for solving such a problem, the penalized problems are presented which extends the general penalty function method from dealing with optimistic one-leader-and-one-follower situation to complex pessimistic referential-uncooperative multiple followers situation. This paper further illustrates an application example of water resources optimal allocation. For the future research, some meta-heuristic search methods to solve the large-scale pessimistic referential-uncooperative bilevel multi-follower problems will be considered. Furthermore, it will be interesting to use the objective penalty method [19] or smoothing method [27] to solve PLBMF-RU.

Acknowledgements

The authors are grateful to the anonymous referees and the associated editor for their valuable comments and suggestions, which help to improve the paper greatly.

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Manuscript received 5 July 2021
revised 27 January 2022, 26 March 2022
accepted for publication 31 March 2022

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