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A REPUTATION MODEL WITH ONE VENTURE CAPITALIST AND TWO COMPETITIVE ENTREPRENEURS VIA STOCHASTIC DIFFERENTIAL GAMES*

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Abstract: This paper focuses on the study of a reputation model with one venture capitalist and two competitive entrepreneurs by applying stochastic differential games. Optimal efforts rate strategies of venture capitalist and entrepreneurs as well as mean and variance functions for the entrepreneurship chain reputation level are obtained in cases of Stakelberg game and cooperative game, respectively. Moreover, optimal incentive factors of venture capitalist on entrepreneurs are also obtained in the Stakelberg game. Finally, some discussions are given to show the difference of optimal strategies, entrepreneurship chain profits and entrepreneurship chain reputation levels between two cases, and several numerical experiments are provided to illustrate the effect of competitive coefficient on profits for the entrepreneurship chain, venture capitalist's incentive factors for entrepreneurs as well as mean and variance functions for entrepreneurship chain reputation level.

Key words: *entrepreneurship chain, reputation, entrepreneurs competition, Stakelberg game, cooperative game, stochastic optimal control*

Mathematics Subject Classification: *49L20, 90B50, 91A12, 91A23, 93E20*

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1 Introduction

As an important financing channel to support entrepreneurial activities, venture capitalist has been widely recognized for its positive role in promoting the growth of entrepreneurs [3, 4, 19, 24]. Early venture investment models, studied by many authors, usually assume that the relationship of venture capitalists and entrepreneurs is one kind of unilateral principal-agent relationship. These models discuss how venture capitalists can motivate and supervise entrepreneurs to work hard [1, 13, 18, 43] and how entrepreneurs attract and encourage more investors to provide more resources into enterprises through designing investment deals [2, 5, 7, 8, 20]. In recent years, the game process between venture capitalists and entrepreneurs has been studied extensively by many scholars after reaching investment deals in the literature [4, 12, 46]. Meanwhile, from the perspective of incentive mechanisms, many authors discussed optimal game strategies and optimal profits of venture capitalists and entrepreneurs by applying the principal-agent models and game theoretic models [16, 22, 37, 45, 47].

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A far-reaching reputation model, proposed by Nerlove and Arrow [31] in 1962, has been improved and extended to various aspects of supply chain research [14, 15, 17, 21, 23, 25, 26, 36, 44]. For instance, Jorgensen [23] applied the reputation model to supply chain with a single manufacturer and a single retailer and studied optimal control strategies in cases of Nash game and Stackelberg game. Hong [21] further applied the reputation model to areas of quality management of supply chain and analyzed profits of supplier and manufacturer in different games. Guan [15] assumed that manufacturers make efforts to improve product quality and retailers make efforts to increase the market share which both increase the reputation of products. By applying differential games, Guan [15] also obtained optimal strategies in cases of centralized, fairness-neutral decentralized, and fairness-concerned decentralized channels. On the other hand, it is well known that the entrepreneurship chain reputation level is crucial to venture capitalists and entrepreneurs. Although venture capitalists and entrepreneurs seek to maximize their own profits respectively, they will establish good reputation level of their entrepreneurship chain for long-term profits and put an end to opportunism because that improving the entrepreneurship chain reputation level can enhance the trust in their entrepreneurial chain for external investors [28, 37, 46]. In order to get an optimal contract design between companies, Fama [9] applied reputation level as an excitation mechanism in the principal-agent model. Recently, Zhao et al. [46] applied the entrepreneurial chain reputation level to the areas of venture capital, constructed a dynamic reputation model with bilateral efforts of one venture capitalist and one entrepreneur, and compared the optimal efforts strategies and the optimal return of projects under three differential games. Very recently, Luo [28] obtained the optimal incentive contracts of entrepreneur towards different kinds of venture capitalists by constructing continuous time principal-agent models with dynamic reputation level.

It is worth mentioning that all the works above mainly discuss the principal-agent models with a venture capitalist and an entrepreneur. However, in some real situations, venture capitalist would like to invest more than one entrepreneurs. In the modern economy, when a large venture capital institution is particularly optimistic about the return on investment of an industry and can not see which projects in this industry will succeed, it will invest two similar enterprises at the same time. For example, Shanda capital, for strategic investment reasons, invests two mobile application markets: Anzhi market and N-duo market ; In order to monopolize group buying and independent brand of cosmetics markets, Sequoia capital invests two cosmetics e-commerce companies: Jumei and Lefeng ; Tiger fund, because very optimistic about the development of education and training market and e-commerce platform, invests two training organizations and two e-commerce platforms: New Oriental English School and Xueersi, Jingdong and Dangdang . One venture capitalist and two competitive entrepreneurs play games with each other to achieve the goal of improving their respective profits and the entrepreneurship chain reputation level. The entrepreneurship chain reputation level which is evaluated by external venture capitalist investments depends on the efforts rates of three players, the competitive strength of two entrepreneurs and the influence of some stochastic interference factors in the entrepreneurial process. Venture capitalists want to sell their shares to the outside investors at ideal prices when they withdraw from the entrepreneurship chain. Entrepreneurs also want to show their own values and attract the external venture capital investments. As a result, three players will choose their optimal strategies to improve the entrepreneurship chain reputation level.

Based on the above realistic background, the main purpose of this paper is to study the stochastic reputation model which contains one venture capitalist and two competitive entrepreneurs in one market. We assume that competitive entrepreneurs get investment from the same venture capitalist and the profit of entrepreneur is effected by its own efforts level as well as the competitive entrepreneur's efforts level. We propose a new reputation model with a venture capitalist and two competitive entrepreneurs via stochastic differential games. We also analyze and compare optimal strategies and profits of the entrepreneurship chain as well as mean and variance functions for entrepreneurship chain reputation levels in cases of Stackelberg game and cooperative game.

The main contributions of this paper can be summarized as follows: (i) proposes a stochastic reputation model with one venture capital and two competitive entrepreneurs; (ii) derives the optimal efforts rate strategies of venture capital and entrepreneurs in both cases of the Stakelberg game and cooperative game; (iii) obtains characteristic functions of entrepreneurship chain reputation levels in the two cases; (iv) shows the influence of competitive coefficient on profits of venture capital and two competitive entrepreneurs, venture capitalist's incentive factors for entrepreneurs as well as mean and variance functions of the entrepreneurship chain reputation level.

The rest of the paper is organized as follows. The next section presents some necessary presuppositions. After that in Sections 3 and 4, we obtain optimal strategies of venture capitalist and entrepreneurs as well as mean and variance functions for the entrepreneurship chain reputation level in the cases of Stakelberg game and cooperative game, respectively. In Section 5, we compare the main results obtained in Sections 3 and 4 and provide several numerical experiments to show the impacts of competitive coefficient on the entrepreneurship chain. Finally, some conclusion remarks are given in Section 6.

2 Model Presuppositions

This paper focuses on the study of an entrepreneurship chain consisting of one venture capitalist and two competitive entrepreneurs. Both venture capitalist and entrepreneurs invest visible and invisible efforts for improving their own profits and the entrepreneurship chain reputation level. Venture capitalist provides financial support for entrepreneurs and improves their management system, while entrepreneurs provide professionals, advanced technology, equipment and sensitive market olfaction.

Presupposition 2.1. Venture capitalist and the ith entrepreneurs ($i = 1, 2$) invest efforts rate for improving their own profits and maintaining the entrepreneurship chain reputation level, respectively. Let $R(t)$ be the entrepreneurship chain reputation level. The efforts rate invested venture capitalist and entrepreneurs is an important factor in establishing $R(t)$. In order to conform to the realistic environment, the entrepreneurship chain reputation level in our model is a stochastic process which is affected by some random factors such as uncertainty of the different understanding of product by potential consumers, industry background, political environment, humanistic factors and so on [29, 38]. Thus, we consider the extended Nerlove-Arrow model described by the following stochastic differential equation (SDE):

$$\begin{cases} dR(t) = [\varphi_{vc} d_{vc}(t) + \varphi_1 d_1(t) + \varphi_2 d_2(t) - \varepsilon R(t)]\, dt + \sigma(R(t)) dB(t), \quad t \geq 0; \\ R(0) = R_0, \end{cases} \tag{2.1}$$

in the complete probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ satisfying the usual hypothesis, where φ_{vc} and φ_i represent the marginal contribution of the efforts rate of venture capital and the ith entrepreneur on the entrepreneurial chain reputation level, $d_{vc}(t)$ and $d_i(t)$ denote the efforts rate of venture capitalist and the ith entrepreneurs ($i = 1, 2$), respectively, ε is the decay rate of the reputation level, $B(t)$ is a standard one-dimensional Brownian motion, the

σ-algebra $\mathfrak{F} = (\mathfrak{F}_t)_{t\geq 0}$ generated by $B(t)$ is right-continuous and increasing, and $\sigma(G(t))$ is a function with different forms depending on uncertainty circumstances.

Some special cases of SDE (2.1) can be listed as follows.

(i) If $\varphi_2 = 0$, then SDE (2.1) reduces to

$$\begin{cases} dR(t) = [\varphi_{vc} d_{vc}(t) + \varphi_1 d_1(t) - \varepsilon R(t)]\, dt + \sigma(R(t)) dB(t), \quad t \geq 0; \\ R(0) = R_0, \end{cases}$$

which has been employed to study the vertical cooperative advertising in supply chain with one manufacturer and one retailer by Nie and Xiong [32].

(ii) If $\varphi_2 = \sigma(R(t)) = 0$, then SDE (2.1) reduces to

$$\begin{cases} dR(t) = [\varphi_{vc} d_{vc}(t) + \varphi_1 d_1(t) - \varepsilon R(t)]\, dt, \quad t \geq 0; \\ R(0) = R_0, \end{cases}$$

which has been used to expound the value of long-term, stable cooperation for entrepreneurship chains by Zhao et al. [46]. For some related works, we refer the reader to Luo [28] and the references therein.

(iii) If $\varphi_1 = \varphi_2 = \sigma(R(t)) = 0$, then SDE (2.1) reduces to

$$\begin{cases} dR(t) = [\varphi_{vc} d_{vc}(t) - \varepsilon R(t)]\, dt, \quad t \geq 0; \\ R(0) = R_0, \end{cases}$$

which is the classical Nerlove-Arrow model [31] used to investigate the optimal advertising policy under dynamic conditions.

Presupposition 2.2. Let the cost functions of venture capitalist and the ith entrepreneur $(i = 1, 2)$ in maintaining the entrepreneurial chain reputation level be convex functions of their respective efforts rate, i.e.,

$$C_{vc}(t) = \frac{\mu_{vc}}{2} d_{vc}^2(t), \quad C_i(t) = \frac{\mu_i}{2} d_i^2(t) \quad (i = 1, 2), \tag{2.2}$$

where μ_{vc} and μ_i are positive parameters. Such widely used cost functions imply increasing marginal costs of quality improvement and advertising efforts (see, for example, [27, 32, 35, 46] and the references therein).

Presupposition 2.3. Similar to the work [35], the reputation level is not only increasing with its own efforts rate but also influenced by the competitor's efforts rate. Let $O_i(t)$ be total profits of the ith venture project $(i = 1, 2)$ at time t. Suppose that $O_i(t)$ depends on the reputation level $R(t)$ and the efforts rate of venture capitalist and the ith entrepreneur $(i = 1, 2)$. The reputation level is the state variable and the efforts rate of venture capitalist and the ith entrepreneur are the control variables. Assume that there is a linear relationship between control variables, i.e.,

$$O_i^{d_i, d_j}(t) := O_i(t) = \alpha_i d_{vc}(t) + \beta_1 d_i(t) + \beta_2 (d_i(t) - d_j(t)) + \theta_i R(t), \quad i, j = 1, 2 (i \neq j),$$

where $\alpha_i > 0, \beta_1 > 0$ and $\theta_i > 0$ are effectiveness coefficients of the efforts rate of venture capitalist, the ith entrepreneur and the reputation level, β_2 is the competition intensity coefficient between entrepreneurs, and $d_i(t) - d_j(t)$ represents the difference of efforts rate between two entrepreneurs. When $d_i(t) - d_j(t)$ is a constant, the difference between total profits of the ith venture project and the jth venture project are increasing with the growth of the absolute value of β_2.

Presupposition 2.4. In the Stakelberg game case, total profits of the ith venture project are distributed between the ith $(i = 1, 2)$ entrepreneur and venture capitalist. Suppose that the ith entrepreneur gets $\omega_i \in (0, 1)$ and venture capitalist gains $1 - \omega_i$ in the ith venture project. The allocation proportion is given in advance (i.e., venture capitalist holds the share of the ith entrepreneur). Venture capitalist and the ith entrepreneur have the same discount rate r.

In the Stakelberg game case, the venture capitalist announce the her strategy (d_{vc}, η_1, η_2). Then the goal of the ith entrepreneur is to seek the optimal strategy $d_i^\dagger$ to maximize his profits within an infinite time horizon, i.e.,

$$\begin{aligned} V_i &= V_i(d_i^\dagger, d_j^\dagger, d_{vc}, \eta_i, \eta_j) \\ &= \max_{d_i} \left\{ \Pi_i = E\left\{ \int_0^{+\infty} e^{-rt} \left[\omega_i O_i^{d_i, d_j^\dagger}(t) - (1 - \eta_i(t)) C_i(t) \right] dt \right\} \right\} \\ &= \max_{d_i} \left\{ \Pi_i = E\left\{ \int_0^{+\infty} e^{-rt} \left[\omega_i \left(\alpha_i d_{vc}(t) + \beta_1 d_i(t) + \beta_2 (d_i(t) - d_j^\dagger(t)) + \theta_i R(t) \right) \right. \right. \right. \\ &\qquad \left. \left. \left. -(1 - \eta_i(t)) \frac{\mu_i}{2} d_i^2(t) \right] dt \right\} \right\}, \end{aligned} \tag{2.3}$$

where $R(t)$ is a solution to (2.1).

Considering the ith entrepreneur takes the strategy $d_i^\dagger$, the venture capitalist would like to seek the optimal strategy $(d_{vc}^*, \eta_i^*, \eta_j^*)$ to maximize her profits within an infinite time horizon, i.e.,

$$\begin{aligned} V_{vc} &= V_{vc}(d_i^\dagger, d_j^\dagger, d_{vc}^*, \eta_i^*, \eta_j^*) \\ &= \max_{d_{vc}, \eta_1, \eta_2} \left\{ \Pi_{vc} = E\left\{ \int_0^{+\infty} e^{-rt} \left[\sum_{i=1}^{2} (1 - \omega_i) O_i^{d_i^\dagger, d_j^\dagger}(t) - C_{vc}(t) - \sum_{i=1}^{2} \eta_i(t) C_i(t) \right] dt \right\} \right\} \\ &= \max_{d_{vc}, \eta_1, \eta_2} \left\{ \Pi_{vc} = E\left\{ \int_0^{+\infty} e^{-rt} \left[\sum_{i=1}^{2} (1 - \omega_i) \left(\alpha_i d_{vc}(t) + \beta_1 d_i^\dagger(t) + \beta_2 (d_i^\dagger(t) - d_j^\dagger(t)) \right. \right. \right. \right. \\ &\qquad \left. \left. \left. \left. + \theta_i R^\dagger(t) \right) - \frac{\mu_{vc}}{2} d_{vc}^2(t) - \sum_{i=1}^{2} \eta_i(t) \frac{\mu_i}{2} (d_i^\dagger)^2(t) \right] dt \right\} \right\} \end{aligned} \tag{2.4}$$

with $d_i^\dagger = d_i^\dagger(\eta_i, \eta_j, d_{vc})$. Here $R^\dagger(t)$ is a solution to the following equation:

$$\begin{cases} dR^\dagger(t) = \left[\varphi_{vc} d_{vc}(t) + \varphi_1 d_1^\dagger(t) + \varphi_2 d_2^\dagger(t) - \varepsilon R^\dagger(t) \right] dt + \sigma(R(t)) dB(t), \quad t \geq 0; \\ R^\dagger(0) = R_0, \end{cases}$$

and $\eta_i(t)$ is "incentive factor" of venture capitalist to the ith entrepreneur at time t. Setting $d_i^* = d_i^\dagger(\eta_i^*, \eta_j^*, d_{vc}^*)$, the Stackelberg solution for this problem is given by $(d_1^*, d_2^*, d_{vc}^*, \eta_1^*, \eta_2^*)$ and the optimal values of objective functions Π_{vc} and Π_i for venture capitalist and the ith entrepreneur are given by $V_{vc}^* = V_{vc}(d_1^*, d_2^*, d_{vc}^*, \eta_1^*, \eta_2^*)$ and $V_i^* = V_i(d_1^*, d_2^*, d_{vc}^*, \eta_1^*, \eta_2^*)$, respectively, $i = 1, 2$.

In the cooperative game case, three players jointly determine d_{vc} and d_i with the common goal of maximizing profits for the whole entrepreneurial chain, i.e.,

$$\begin{aligned} V^C &= \max_{d_{vc},d_1,d_2} \left\{\Pi = E\left\{\int_0^{+\infty} e^{-rt}\left[\sum_{i=1}^{2} O_i(t) - C_{vc}(t) - \sum_{i=1}^{2} C_i(t)\right] dt\right\}\right\} \\ &= \max_{d_{vc},d_1,d_2} \left\{\Pi = E\left\{\int_0^{+\infty} e^{-rt}\Big[(\alpha_1+\alpha_2)d_{vc}(t) + \beta_1(d_1(t)+d_2(t)) + (\theta_1+\theta_2)R(t)\right.\right. \\ &\qquad \left.\left. -\frac{\mu_{vc}}{2}d_{vc}^2(t) - \frac{\mu_1}{2}d_1^2(t) - \frac{\mu_2}{2}d_2^2(t)\Big]\, dt\right\}\right\}, \end{aligned} \tag{2.5}$$

where $R(t)$ is a solution of (2.1) and V^C is the optimal value of an objective function Π for the whole entrepreneurial chain.

Remark 2.1. In the infinite horizon problems of both the Stakelberg game and the cooperative game, it is important to assume that the coefficients $\varphi_{vc}, \varphi_1, \varphi_2, \varepsilon, \mu_{vc}, \mu_i, \alpha_i, \beta_i, \theta_i, \omega_i$ do not depend on time in order to get the stationarity of the problem, and so a value function independent of time [34].

Now we recall some useful notations for the admissible control sets as follows.

$L^2(\delta, \mathfrak{F})$: the space of all $\mathfrak{F}_t$-adapted processes $X(t)$ with $E\left[\int_0^\infty e^{-\delta t}X^2(t)dt\right] < \infty$;

$\mathcal{U}_1^\delta = \{X \in L^2(\delta,\mathfrak{F}) | X(t) > 0 \text{ a.s.}\}$: the admissible control set for d_1, d_2 and d_{vc};

$\mathcal{U}_2^\delta = \left\{X \in L^2(\delta,\mathfrak{F}) | \frac{1}{1-X} \in L^2(\delta,\mathfrak{F}),\ X(t) \in (0,1) \text{ a.s.}\right\}$: the admissible control set for η_1 and η_2.

For fixed $d_1, d_2, d_{vc} \in \mathcal{U}_1^\delta$ and $\eta_1, \eta_2 \in \mathcal{U}_2^\delta$, we require that the coefficient $\sigma(R(t))$ is progressively measurable such that (2.1) admits a unique solution in some proper space. For example, when $\sigma(R(t)) = \sigma R(t)$ or $\sigma(R(t)) = \sigma\sqrt{R(t)}$, there exists a unique solution $R(t) \in L^2(\delta,\mathfrak{F})$, where σ is some fixed constant. This fact can be easily derived from the results in [10].

In the sequel, the Stackelberg game and cooperative game will be investigated respectively.

3 The Stackelberg Game Case

3.1 The equilibrium strategies

In this subsection, we investigate the Stackelberg game case. Venture capitalist plays a leading role controls its investment and management efforts rate and incentive factor. As followers, entrepreneurs control their technology and market efforts rate. Venture capitalist and entrepreneurs pursue the maximization of their own profits.

Theorem 3.1. *Let $\Omega_1 = (1-\omega_1)\theta_1 + (1-\omega_2)\theta_2$, $\Omega_2 = (1-\omega_1)\alpha_1 + (1-\omega_2)\alpha_2$, $\Phi_i = (2-\omega_i)\beta_1 + (2\omega_j - \omega_i)\beta_2$ and $\Psi_i = (2-\omega_i)\theta_i + 2(1-\omega_j)\theta_j$ with $i,j = 1,2$ $(i \neq j)$. If*

$$0 < (r+\varepsilon)[\Phi_i - 2(\omega_i\beta_1 + \omega_j\beta_2)] + \varphi_i(\Psi_i - 2\omega_i\theta_i),$$

then in the Stackelberg game case, one has the following conclusions:

(a) *the optimal investment and management efforts rate and the optimal incentive factor strategies of venture capitalist are given by*

$$\begin{cases} d_{vc}^* = \dfrac{\varphi_{vc}\Omega_1}{\mu_{vc}(r+\varepsilon)} + \dfrac{\Omega_2}{\mu_{vc}}, \\ \eta_i^* = \dfrac{(r+\varepsilon)[\Phi_i - 2(\omega_i\beta_1 + \omega_j\beta_2)] + \varphi_i(\Psi_i - 2\omega_i\theta_i)}{(r+\varepsilon)\Phi_i + \varphi_i\Psi_i} \quad (i = 1, 2); \end{cases} \tag{3.1}$$

(b) *the optimal technology and market efforts rate strategy of the ith entrepreneur is given by*

$$d_i^* = \frac{\omega_i}{(1-\eta_i^*)\mu_i}\left[(\beta_1+\beta_2) + \frac{\varphi_i}{r+\varepsilon}\right] = \frac{\Phi_i}{2\mu_i} + \frac{\varphi_i\Psi_i}{2\mu_i(r+\varepsilon)} \quad (i = 1, 2); \tag{3.2}$$

(c) *the optimal profits functions of venture capitalist and the ith entrepreneur are respectively given by*

$$\begin{cases} V_{vc}^* = \dfrac{\Omega_1}{r+\varepsilon}R_0 + \dfrac{[\varphi_{vc}\Omega_1 + (r+\varepsilon)\Omega_2]^2}{2r(r+\varepsilon)^2\mu_{vc}} \\ \qquad + \dfrac{[(r+\varepsilon)\Phi_1 + 2\varphi_1\Omega_1]^2 - (\varphi_1\theta_1\omega_1)^2}{8r(r+\varepsilon)^2\mu_1} + \dfrac{[(r+\varepsilon)\Phi_2 + 2\varphi_2\Omega_1]^2 - (\varphi_2\theta_2\omega_2)^2}{8r(r+\varepsilon)^2\mu_2}, \\ V_i^* = \dfrac{\theta_i\omega_i}{r+\varepsilon}R_0 + \dfrac{\omega_i\left[(r+\varepsilon)\alpha_i + \varphi_{vc}\theta_i\right]\left[\varphi_{vc}\Omega_1 + (r+\varepsilon)\Omega_2\right]}{r(r+\varepsilon)^2\mu_{vc}} \\ \qquad + \dfrac{\omega_i\left[(\beta_1+\beta_2)(r+\varepsilon) + \varphi_i\theta_i\right]\left[(r+\varepsilon)\Phi_i + \varphi_i\Psi_i\right]}{4r(r+\varepsilon)^2\mu_i} \\ \qquad + \dfrac{\omega_i\left[\theta_i\varphi_j - (r+\varepsilon)\beta_2\right]\left[(r+\varepsilon)\Phi_j + \varphi_j\Psi_j\right]}{2r(r+\varepsilon)^2\mu_j} \\ \quad (i, j = 1, 2, i \neq j). \end{cases} \tag{3.3}$$

Moreover, the Stackelberg solution $(d_1^*, d_2^*, d_{vc}^*, \eta_1^*, \eta_2^*)$ *given by* (3.1) *and* (3.2) *is unique.*

Proof. We adopt the reverse induction method. Since the ith entrepreneur makes its technology and market efforts rate decision based on decisions of venture capitalist, we first solve the stochastic optimal control problem of the ith entrepreneur. Let

$$\tfrac{dV_i}{dR_0} = (V_i)', \quad \tfrac{d^2V_i}{dR_0^2} = (V_i)'', \quad i = 1, 2.$$

According to (2.1), (2.3) and the stochastic optimal control theory in infinite horizon case (see, for example, [34]), the Hamiltonian-Jacobi-Bellman equation of the ith entrepreneur is given by

$$\begin{aligned} rV_i(R_0) = &\max_{d_i}\Big\{\omega_i\left[\alpha_i d_{vc} + \beta_1 d_i + \beta_2(d_i - d_j) + \theta_i R_0\right] - (1-\eta_i)\frac{\mu_i}{2}d_i^2 \\ &+ (V_i)'(R_0)\left(\varphi_{vc}d_{vc} + \varphi_1 d_1 + \varphi_2 d_2 - \varepsilon R_0\right) + \frac{1}{2}(V_i)''(R_0)\sigma^2(R_0)\Big\} \quad (i, j = 1, 2, i \neq j). \end{aligned} \tag{3.4}$$

Clearly, the right side of (3.4) is a concave function of d_i, which gives that

$$d_i^\dagger(\eta_i, \eta_j, d_{vc}) = \frac{(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)}{\mu_i(1-\eta_i)} \quad (i, j = 1, 2, i \neq j). \tag{3.5}$$

Next we are going to find the optimal strategies of the venture capitalist. In fact, venture capitalist can decide its optimal efforts rate and incentive factor strategies in view of the feedback strategy $d_i^{\dagger}$ to meet the goal of maximizing its own profits. Let

$$\frac{\partial V_{vc}}{\partial R_0} = (V_{vc})', \quad \frac{\partial^2 V_{vc}}{\partial R_0^2} = (V_{vc})''.$$

Similarly, applying the stochastic optimal control theory in infinite horizon case (see, for example, [34]), it follows from (2.1) and (2.4) that the Hamiltonian-Jacobi-Bellman equation of venture capitalist is given by

$$\begin{aligned} & rV_{vc}(R_0) \\ = & \max_{d_{vc},\eta_1,\eta_2}\left\{\sum_{i=1}^{2}(1-\omega_i)\left[\alpha_i d_{vc} + \beta_1 d_i^{\dagger} + \beta_2(d_i^{\dagger} - d_j^{\dagger}) + \theta_i R_0\right] - \frac{\mu_{vc}}{2}d_{vc}^2 - \sum_{i=1}^{2}\eta_i\frac{\mu_i}{2}(d_i^{\dagger})^2\right. \\ & \left.+(V_{vc})'\left(\varphi_{vc}d_{vc} + \varphi_1 d_1^{\dagger} + \varphi_2 d_2^{\dagger} - \varepsilon R_0\right) + \frac{1}{2}(V_{vc})''(R_0)\sigma^2(R_0)\right\}. \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6), we have

$$\begin{aligned} & rV_{vc}(R_0) \\ = & \max_{d_{vc},\eta_1,\eta_2}\left\{\sum_{i=1}^{2}(1-\omega_i)\left[\alpha_i d_{vc} + \beta_1\frac{(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)}{\mu_i(1-\eta_i)}\right.\right. \\ & \left.+\beta_2\left(\frac{(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)}{\mu_i(1-\eta_i)} - \frac{(\beta_1+\beta_2)\omega_j + \varphi_j(V_j)'(R_0)}{\mu_j(1-\eta_j)}\right) + \theta_i R_0\right] \\ & -\frac{\mu_{vc}}{2}d_{vc}^2 - \sum_{i=1}^{2}\eta_i\frac{\mu_i}{2}\left[\frac{(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)}{\mu_i(1-\eta_i)}\right]^2 \\ & \left.+(V_{vc})'\left[\varphi_{vc}d_{vc} + \sum_{i=1}^{2}\varphi_i\frac{(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)}{\mu_i(1-\eta_i)} - \varepsilon R_0\right] + \frac{1}{2}(V_{vc})''(R_0)\sigma^2(R_0)\right\}. \end{aligned}$$

Performing the maximization of the right side of the above equation, one has

$$\begin{cases} d_{vc}^* = \dfrac{\sum_{i=1}^{2}(1-\omega_i)\alpha_i + \varphi_{vc}(V_{vc})'(R_0)}{\mu_{vc}}, \\ \eta_i^*(t) = \dfrac{[(2-3\omega_i)(\beta_1+\beta_2) - 2(1-\omega_j)\beta_2] + \varphi_i[2(V_{vc})'(R_0) - (V_i)'(R_0)]}{[(2-\omega_i)(\beta_1+\beta_2) - 2(1-\omega_j)\beta_2] + \varphi_i[(2(V_{vc})'(R_0) + (V_i)'(R_0))]} \quad (i=1,2). \end{cases} \tag{3.7}$$

Substituting (3.5) and (3.7) in (3.4) and (3.6), we obtain

$$\begin{cases} rV_i(R_0) \\ = (\theta_i\omega_i - \varepsilon(V_i)'(R_0))\, R_0 \\ \quad + \dfrac{(\alpha_i\omega_i + \varphi_{vc}(V_i)'(R_0))\,[(1-\omega_1)\alpha_1 + (1-\omega_2)\alpha_2 + \varphi_{vc}(V_{vc})'(R_0)]}{\mu_{vc}} \\ \quad + \dfrac{\begin{array}{l}[(\beta_1+\beta_2)\omega_i + \varphi_i(V_i)'(R_0)] \\ \qquad [(\beta_1+\beta_2)(2-\omega_i) - 2\beta_2(1-\omega_j) + \varphi_i\,(2(V_{vc})'(R_0) + (V_i)'(R_0))]\end{array}}{4\mu_i} \\ \quad + \dfrac{\begin{array}{l}[\varphi_j V_i'(R_0) - \beta_2\omega_i] \\ \qquad [(\beta_1+\beta_2)(2-\omega_j) - 2\beta_2(1-\omega_i) + \varphi_j\,(2(V_{vc})'(R_0) + (V_j)'(R_0))]\end{array}}{2\mu_j} \\ \quad + \dfrac{1}{2}(V_i)''(R_0)\sigma^2(R_0) \quad (i = 1, 2), \\ rV_{vc}(R_0) \\ = [\theta_1(1-\omega_1) + \theta_2(1-\omega_2) - \varepsilon(V_{vc})'(R_0)]\, R_0 \\ \quad + \dfrac{[\alpha_1(1-\omega_1) + \alpha_2(1-\omega_2) + \varphi_{vc}(V_{vc})'(R_0)]^2}{2\mu_{vc}} \\ \quad + \dfrac{[(\beta_1+\beta_2)(2-\omega_1) - 2\beta_2(1-\omega_2) + \varphi_1\,(2(V_{vc})'(R_0) + (V_1)'(R_0))]^2}{8\mu_1} \\ \quad + \dfrac{[(\beta_1+\beta_2)(2-\omega_2) - 2\beta_2(1-\omega_1) + \varphi_2\,(2(V_{vc})'(R_0) + (V_2)'(R_0))]^2}{8\mu_2} \\ \quad + \dfrac{1}{2}(V_{vc})''(R_0)\sigma^2(R_0). \end{cases} \tag{3.8}$$

In general, it is not easy to solve (3.8). Thus, similar to [40, 46], we may satisfy (3.8) by conjecturing linear value functions. Assume that

$$\begin{cases} V_i(R_0) = m_i R_0 + n_i, \quad i = 1, 2, \\ V_{vc}(R_0) = mR_0 + n, \end{cases} \tag{3.9}$$

where m_i, n_i, m, n $(i = 1, 2)$ are all constants. By inserting (3.9) and their derivatives into

(3.8), we have

$$\begin{cases} r(m_iR_0+n_i) = (\theta_i\omega_i-\varepsilon m_i)\,R_0 + \dfrac{(\alpha_i\omega_i+\varphi_{vc}m_i)\,[(1-\omega_1)\alpha_1+(1-\omega_2)\alpha_2+\varphi_{vc}m]}{\mu_{vc}} \\ \qquad + \dfrac{[(\beta_1+\beta_2)\omega_i+\varphi_i m_i]\,[(\beta_1+\beta_2)(2-\omega_i)-2\beta_2(1-\omega_j)+\varphi_i\,(2m+m_i)]}{4\mu_i} \\ \qquad + \dfrac{(\varphi_j m_i-\beta_2\omega_i)\,[(\beta_1+\beta_2)(2-\omega_j)-2\beta_2(1-\omega_i)+\varphi_j\,(2(m+m_j)]}{2\mu_j} \\ \qquad (i=1,2,i\neq j), \\ r(mR_0+n) = [\theta_1(1-\omega_1)+\theta_2(1-\omega_2)-\varepsilon m]\,R_0 + \dfrac{[\alpha_1(1-\omega_1)+\alpha_2(1-\omega_2)+\varphi_{vc}m]^2}{2\mu_{vc}} \\ \qquad + \dfrac{[(\beta_1+\beta_2)(2-\omega_1)-2\beta_2(1-\omega_2)+\varphi_1\,(2m+m_1)]^2}{8\mu_1} \\ \qquad + \dfrac{[(\beta_1+\beta_2)(2-\omega_2)-2\beta_2(1-\omega_1)+\varphi_2\,(2m+m_2)]^2}{8\mu_2}. \end{cases}$$

Comparing the coefficients of the same term between the left and the right sides of the above equations, one has

$$\begin{cases} m = \dfrac{\Omega_1}{r+\varepsilon}, \quad m_i = \dfrac{\theta_i\omega_i}{r+\varepsilon} \quad (i=1,2,i\neq j), \\ n = \dfrac{[\varphi_{vc}\Omega_1+(r+\varepsilon)\Omega_2]^2}{2r(r+\varepsilon)^2\mu_{vc}} \\ \qquad + \dfrac{[(r+\varepsilon)\Phi_1+2\varphi_1\Omega_1]^2-(\varphi_1\theta_1\omega_1)^2}{8r(r+\varepsilon)^2\mu_1} + \dfrac{[(r+\varepsilon)\Phi_2+2\varphi_2\Omega_1]^2-(\varphi_2\theta_2\omega_2)^2}{8r(r+\varepsilon)^2\mu_2}, \\ n_i = \dfrac{\omega_i\,[(r+\varepsilon)\alpha_i+\varphi_{vc}\theta_i]\,[\varphi_{vc}\Omega_1+(r+\varepsilon)\Omega_2]}{r(r+\varepsilon)^2\mu_{vc}} \\ \qquad + \dfrac{\omega_i\,[(\beta_1+\beta_2)(r+\varepsilon)+\varphi_i\theta_i]\,[(r+\varepsilon)\Phi_i+\varphi_i\Psi_i]}{4r(r+\varepsilon)^2\mu_i} \\ \qquad + \dfrac{\omega_i\,[\theta_i\varphi_j-(r+\varepsilon)\beta_2]\,[(r+\varepsilon)\Phi_j+\varphi_j\Psi_j]}{2r(r+\varepsilon)^2\mu_j} \\ \qquad (i=1,2,i\neq j). \end{cases} \tag{3.10}$$

By (3.10), (3.9) and (3.7), after simple calculations, we have the conclusion (a). Then it follows from (3.5) and the conclusion (a) that

$$d_i^* = d_i^\dagger(\eta_i^*,\eta_j^*,d_{vc}^*) = \frac{(\beta_1+\beta_2)\omega_i+\varphi_i m_i}{\mu_i(1-\eta_i^*)} \quad (i=1,2),$$

which indicates, by combining (3.10) and (3.9), that the conclusion (b) holds. Moreover, by conclusions (a) and (b), conclusion (c) follows (2.1) directly.

Finally, we verify that the strategies given by (3.1) and (3.2) are the unique optimal ones. To this end, we first consider that the ith entrepreneur takes the strategy

$$d_i^\dagger(\eta_i) = \frac{(\beta_1+\beta_2)\omega_i+\varphi_i m_i}{\mu_i(1-\eta_i)} \quad (i=1,2)$$

in (2.1). Then for fixed $d_{vc} \in \mathcal{U}_1^\delta$ and $\eta_1, \eta_2 \in \mathcal{U}_2^\delta$, there exists a unique solution $R(t)$ in some proper space. According to Theorem 3.5.3 in [34], $d_i^\dagger(\eta_i)$ is an optimal strategy for the ith entrepreneur. Moreover, taking expectation on both sides in (2.1) derives that

$$\begin{cases} dE[R(t)] = [\varphi_{vc}E[d_{vc}(t)] + \varphi_1 E[d_1(t)] + \varphi_2 E[d_2(t)] - \varepsilon E[R(t)]]\,dt, \quad t \geq 0; \\ \quad R(0) = R_0. \end{cases}$$

Clearly, $E[R(t)]$ is linear with respect to all of $E[d_{vc}(t)]$, $E[d_1(t)]$ and $E[d_2(t)]$. Owing to this fact, Π_i is uniformly concave with respect to d_1 and d_2. Therefore uniqueness of the optimal strategy $d_i^\dagger(\eta_i)$ follows from Proposition 5.4 in [42] immediately.

Analogously we know η_1^*, η_2^* and d_{vc}^* are optimal. Moreover, $E[R(t)]$ is linear with respect to $\frac{1}{1-\eta_1}$ and $\frac{1}{1-\eta_2}$. Notice that the quadratic term $-\eta_i(t)\frac{\mu_i}{2}(d_i^\dagger)^2(t)$ of Π_{vc} can be written as

$$-\eta_i(t)\frac{\mu_i}{2}(d_i^\dagger)^2(t) = \frac{((\beta_1+\beta_2)\omega_i + \varphi_i m_i)^2}{2\mu_i}\left[\frac{1}{1-\eta_i(t)} - \frac{1}{(1-\eta_i(t))^2}\right],$$

which indicates the uniform concavity of Π_{vc} with respect to $1-\eta_1(t)$, $1-\eta_2(t)$ and d_{vc}. Thus again by Proposition 5.4 in [42], we obtain the uniqueness of η_1^*, η_2^* and d_{vc}^*, and the uniqueness of the Stackelberg solution $(d_1^*, d_2^*, d_{vc}^*, \eta_i^*, \eta_j^*)$ follows. $\square$

Remark 3.1. If $\varphi_2 = \sigma(R(t)) = \beta_2 = d_j(t) = 0$, then Theorem 3.1 reduces to Theorem 2 in [46].

3.2 Characteristic functions of the entrepreneurial chain reputation level

In this subsection, we study mean and variance functions of the entrepreneurial chain reputation level when venture capitalist and entrepreneurs choose their optimal efforts rates. Taking optimal efforts rates (3.1) and (3.2) into the reputation level's state equation (2.1), we obtain

$$\begin{cases} dR(t) = [\Theta - \varepsilon R(t)]\,dt + \sigma(R(t))dB(t), \\ R(0) = R_0, \end{cases} \tag{3.11}$$

where $\Theta = \varphi_{vc}d_{vc}^* + \varphi_1 d_1^* + \varphi_2 d_2^*$. In order to obtain mean and variance functions of the entrepreneurial chain reputation level, similar to the work [29], we assume that $\sigma(R(t)) = \sigma\sqrt{R(t)}$, where σ is a constant. By applying the stochastic differential equation theory [33], mean and variance functions of the entrepreneurial chain reputation level and their stable values are obtained in Theorem 3.2 when three parties all invest the optimal efforts.

Theorem 3.2. *When venture capitalist invests the optimal investment and management efforts rate and incentive factor strategies, and entrepreneurs choose the optimal technology and market efforts rate strategy in the Stackelberg game case, mean and variance functions of the entrepreneurial chain reputation level are given as follows:*

$$E[R(t)] = \left(R_0 - \frac{\Theta}{\varepsilon}\right)e^{-\varepsilon t} + \frac{\Theta}{\varepsilon}, \tag{3.12}$$

$$D[R(t)] = \frac{\sigma^2}{2\varepsilon}\left[(\Theta - 2\varepsilon R_0)e^{-2\varepsilon t} - (\Theta - \varepsilon R_0)e^{-\varepsilon t} + \Theta\right]. \tag{3.13}$$

Moreover, stable values of mean and variance functions of the reputation level are as follows:

$$\lim_{t\to+\infty} E[R(t)] = \frac{\Theta}{\varepsilon}, \quad \lim_{t\to+\infty} D[R(t)] = \frac{\sigma^2\Theta}{2\varepsilon}. \tag{3.14}$$

Proof. Rewrite (3.11) as the following stochastic integral equation

$$R(t) = R_0 + \int_0^t [\Theta - \varepsilon R(s)]\, ds + \sigma\sqrt{R(t)}dB(s).$$

Calculating the expectation value of the above equation, we have

$$E[R(t)] = R_0 + \int_0^t [\Theta - \varepsilon E[R(s)]]\, ds.$$

The above equation is equivalent to an ordinary differential equation with respect to $E[R(t)]$, which has the initial condition $E[R(0)] = R_0$. Thus, we can obtain (3.12) by solving the ordinary differential equation with respect to $E[R(t)]$. Taking $t \to +\infty$ in (3.12), we obtain the first result of (3.14).

In order to give the variance function, we need to calculate $E[R^2(t)]$. Using Itô's formula [33] to the stochastic differential equation (3.11), one has

$$dR^2(t) = \left[(2\Theta + \sigma^2)R(t) - 2\varepsilon R^2(t)\right] dt + 2\sigma R^{\frac{3}{2}}(t)dB(t).$$

Taking expectation value at both sides of the above equation, we obtain

$$\begin{cases} dE[R^2(t)] = \left[(2\Theta + \sigma^2)E[R(t)] - 2\varepsilon E[R^2(t)]\right] dt, \\ E[R^2(0)] = R_0^2. \end{cases}$$

Substituting (3.12) in the equation above, we have

$$\begin{cases} dE[R^2(t)] = \left\{(2\Theta + \sigma^2)\left[\left(R_0 - \dfrac{\Theta}{\varepsilon}\right)e^{-\varepsilon t} + \dfrac{\Theta}{\varepsilon}\right] - 2\varepsilon E[R^2(t)]\right\} dt, \\ E[R^2(0)] = R_0^2. \end{cases}$$

Calculating the above ordinary differential equation with respect to $E[R^2(t)]$, one has

$$E[R^2(t)] \quad = \quad \left(R_0^2 + \frac{\Theta}{2\varepsilon^2} - \frac{1}{\varepsilon}\right)e^{-2\varepsilon t} + \left(\frac{R_0}{\varepsilon} - \frac{\Theta}{\varepsilon^2}\right)e^{-\varepsilon t} + \frac{\Theta(\Theta + \sigma^2)}{2\varepsilon^2}. \tag{3.15}$$

We can obtain (3.13) from $D[R(t)] = E[R^2(t)] - E[R(t)]^2$. Taking $t \to +\infty$ in (3.15), we have the second result of (3.14). This completes the proof. □

Remark 3.2. (a) In general, it is difficult for an enterprise to get accurate distribution functions of reputation level $R(t)$. Theorem 3.2 gives $E[R(t)]$ and $D[R(t)]$ and so confidence intervals of the entrepreneurial chain reputation level $R(t)$ can be given by

$$\left(E[R(t)] - 1.96\sqrt{D[R(t)]}, E[R(t)] + 1.96\sqrt{D[R(t)]}\right)$$

when the confidence level is 95 percent;

(b) The entrepreneurial chain reputation level is an increasing function of time t if $\varepsilon < \frac{\Theta}{R_0}$ and it is an decreasing function of time t if $\varepsilon > \frac{\Theta}{R_0}$. However, mean and variance functions of the reputation level are increasing with the growth of efforts rate if the decay rate of the reputation level is small ($\varepsilon < \frac{\Theta}{R_0}$).

4 The Cooperative Game Case

4.1 The equilibrium strategies

In this subsection, we discuss the cooperative game case. This means that venture capitalist and entrepreneurs make decisions at the same time to maximize the whole entrepreneurial chain's profits. Thus, we can obtain their respective optimal efforts rate strategies and the optimal profit of the whole entrepreneurial chain.

Theorem 4.1. *In the cooperative game case, one has the following conclusions:*

(a) *the optimal investment and management efforts rate of venture capitalist and the optimal technology and market efforts rate strategy of the ith entrepreneur are uniquely given by*

$$\begin{cases} d_{vc}^C(t) = \dfrac{\alpha_1+\alpha_2}{\mu_{vc}} + \dfrac{\varphi_{vc}(\theta_1+\theta_2)}{\mu_{vc}(r+\varepsilon)}, \\ d_i^C(t) = \dfrac{\beta_1}{\mu_i} + \dfrac{\varphi_i(\theta_1+\theta_2)}{\mu_i(r+\varepsilon)} \ (i=1,2); \end{cases} \tag{4.1}$$

(b) *the optimal profit function of the whole entrepreneurial chain is given by*

$$\begin{aligned} V^C(R_0) =& \frac{(\theta_1+\theta_2)}{r+\varepsilon}R_0 + \frac{\left[(\alpha_1+\alpha_2)(r+\varepsilon)+\varphi_{vc}(\theta_1+\theta_2)\right]^2}{2r(r+\varepsilon)^2\mu_{vc}} \\ &+ \frac{\left[\beta_1(r+\varepsilon)+\varphi_1(\theta_1+\theta_2)\right]^2}{2r(r+\varepsilon)^2\mu_1} + \frac{\left[\beta_1(r+\varepsilon)+\varphi_2(\theta_1+\theta_2)\right]^2}{2r(r+\varepsilon)^2\mu_2}. \end{aligned} \tag{4.2}$$

Proof. In the cooperative game case, venture capitalist and entrepreneurs co-determine the optimal investment and management efforts rate d_{vc}^R and the optimal technology and market efforts rate d_i^R to maximize the entrepreneurial chain profit $V^C(R_0)$. Let

$$\tfrac{dV^C}{dR_0} = (V^C)', \quad \tfrac{d^2V^C}{dR_0^2} = (V^C)''.$$

According to (2.1), (2.5) and the stochastic optimal control theory in infinite horizon case (see, for example, [34]), the Hamiltonian-Jacobi-Bellman equation of the entrepreneurial chain's objective function is given by

$$\begin{aligned} rV^C(R_0) \quad = \quad & \max_{d_{vc},d_1,d_2} \Big\{(\alpha_1+\alpha_2)d_{vc} + \beta_1(d_1+d_2) + (\theta_1+\theta_2)R_0 - \frac{\mu_1}{2}d_1^2 - \frac{\mu_2}{2}d_2^2 - \frac{\mu_{vc}}{2}d_{vc}^2 \\ & +(V^C)'(R_0)\left(\varphi_{vc}d_{vc}+\varphi_1 d_1+\varphi_2 d_2-\varepsilon R_0\right) + \frac{1}{2}(V^C)''(R_0)\sigma^2(R_0)\Big\}. \end{aligned} \tag{4.3}$$

Clearly, the right side of (4.3) is a concave function of d_{vc} and d_i $(i=1,2)$. Thus, it follows from (4.3) that

$$d_{vc}^C = \frac{(\alpha_1+\alpha_2)+\varphi_{vc}(V)'(R_0)}{\mu_{vc}}, \quad d_i^C = \frac{\beta_1+\varphi_i(V)'(R_0)}{\mu_i} \quad (i=1,2). \tag{4.4}$$

Substituting d_{vc}^C and d_i^C $(i=1,2)$ into (4.3), one has

$$rV^C(R_0) = \left(\theta_1+\theta_2-\varepsilon(V^C)'(R_0)\right)R_0 + \frac{\left[(\alpha_1+\alpha_2)+\varphi_{vc}(V^C)'(R_0)\right]^2}{2\mu_{vc}} + \frac{\left[\beta_1+\varphi_1(V^C)'(R_0)\right]^2}{2\mu_1} + \frac{\left[\beta_1+\varphi_2(V^C)'(R_0)\right]^2}{2\mu_2} + \frac{1}{2}(V^C)''(R_0)\sigma^2(R_0).$$

According to [40, 46], we can assume that V^C has the following linear form with respect to R_0,

$$V^C(R_0) = MR_0 + N, \tag{4.5}$$

where M, N are constants. By inserting (4.5) and their derivatives into (4.3), we have

$$r(MR_0+N) = (\theta_1+\theta_2-\varepsilon M)\,R_0 + \frac{[(\alpha_1+\alpha_2)+\varphi_{vc}M]^2}{2\mu_{vc}} + \frac{[\beta_1+\varphi_1 M]^2}{2\mu_1} + \frac{[\beta_1+\varphi_2 M]^2}{2\mu_2}.$$

By comparing the coefficients of the same term between the left and the right sides of the above equations, one has

$$\begin{cases} M = \dfrac{\theta_1+\theta_2}{r+\varepsilon}, \\ N = \dfrac{[(\alpha_1+\alpha_2)(r+\varepsilon)+\varphi_{vc}(\theta_1+\theta_2)]^2}{2r(r+\varepsilon)^2\mu_{vc}} + \dfrac{[\beta_1(r+\varepsilon)+\varphi_1(\theta_1+\theta_2)]^2}{2r(r+\varepsilon)^2\mu_1} \\ \qquad + \dfrac{[\beta_1(r+\varepsilon)+\varphi_2(\theta_1+\theta_2)]^2}{2r(r+\varepsilon)^2\mu_2}. \end{cases} \tag{4.6}$$

Substituting (4.6) in (4.4) and (4.5), we have the results (4.1) and (4.2) in Theorem 4.1. Through a similar arguments as the proof in (3.1), we obtain the uniqueness of $d_{vc}^C(t)$ and $d_i^C(t)$. This completes the proof. □

Remark 4.1. If $\varphi_2 = \sigma(R(t)) = \beta_2 = d_j(t) = 0$, then Theorem 4.1 reduces to Theorem 2 in [46].

4.2 Characteristic functions of the entrepreneurial chain reputation level

In this subsection, we study mean and variance functions of the entrepreneurial chain reputation level when venture capitalist and entrepreneurs choose their optimal efforts. Taking the optimal efforts rate (4.1) into the reputation level's state equation (2.1), we obtain

$$\begin{cases} d\overline{R}(t) = \left[\Lambda - \varepsilon\overline{R}(t)\right]dt + \sigma(\overline{R}(t))dB(t), \\ \overline{R}(0) = \overline{R}_0, \end{cases} \tag{4.7}$$

where $\Lambda = \varphi_{vc}d_{vc}^C + \varphi_1 d_1^C + \varphi_2 d_2^C$. In order to obtain mean and variance functions and their stable values of the entrepreneurial chain reputation level, similar to the work [29], we assume that $\sigma(\overline{R}(t)) = \sigma\sqrt{\overline{R}(t)}$, where σ is a constant. By applying the stochastic differential equation theory [33], mean and variance functions and their stable values of the entrepreneurial chain reputation level are obtained in Theorem 4.2 when three players all choose their optimal efforts rates.

Similar to the proof of Theorems 3.2, we can prove the following result.

Theorem 4.2. *When venture capitalist invests the optimal investment and management efforts rate strategy and entrepreneurs invest the optimal technology and market efforts rate strategy in cooperative game case, one has the following conclusions:*

(a) *mean and variance functions of the entrepreneurial chain reputation level are given by*

$$\begin{aligned} E[\overline{R}(t)] &= \left(\overline{R}_0 - \frac{\Lambda}{\varepsilon}\right) e^{-\varepsilon t} + \frac{\Lambda}{\varepsilon}, && (4.8)\\ D[\overline{R}(t)] &= \frac{\sigma^2}{2\varepsilon}\left[(\Lambda - 2\varepsilon\overline{R}_0)e^{-2\varepsilon t} - (\Lambda - \varepsilon\overline{R}_0)e^{-\varepsilon t} + \Lambda\right]; && (4.9) \end{aligned}$$

(b) *stable values of mean and variance functions of reputation level are given by*

$$\lim_{t\to+\infty} E[\overline{R}(t)] = \frac{\Lambda}{\varepsilon}, \quad \lim_{t\to+\infty} D[\overline{R}(t)] = \frac{\sigma^2\Lambda}{2\varepsilon}. \tag{4.10}$$

5 Discussions

5.1 Comparison and Analysis

In this subsection, we compare the optimal investment and management efforts rate strategy of venture capitalist, the optimal technology and market efforts rate strategy of the ith entrepreneur, the whole entrepreneurial chain's profits and mean and variance functions of the reputation levels under optimal efforts rate strategies in cases of Stackelberg game and cooperative game, respectively.

By using the main results presented in Sections 3 and 4, we have the following result.

Theorem 5.1. (a) $d_{vc}^C > d_{vc}^*$;

(b) $d_i^C > d_i^*$ *providing*

$$\frac{\omega_i}{\omega_j} > \frac{2\beta_2(r+\varepsilon)) - 2\varphi_i\theta_j}{(\beta_1+\beta_2)(r+\varepsilon) + \varphi_i\theta_i};$$

(c) *If* $N > n + n_1 + n_2$, *then* $V^C > V_{vc}^* + V_1^* + V_2^*$;

(d) *If* $2\omega_i > \omega_j$ *and* $\beta_2 < \frac{\omega_i\beta_1}{2\omega_j - \omega_i}$, *then*

$$\begin{cases} E[\overline{R}(t)] > E[R(t)], \quad D[\overline{R}(t)] > D[R(t)], \\ \lim\limits_{t\to+\infty} E[\overline{R}(t)] > \lim\limits_{t\to+\infty} E[R(t)], \quad \lim\limits_{t\to+\infty} D[\overline{R}(t)] > \lim\limits_{t\to+\infty} D[R(t)]. \end{cases}$$

Proof. In order to obtain results (a)-(c) of Theorem 5.1, we need to compare optimal strategies and optimal profits of entrepreneurial chain in Theorems 3.1 and 4.1. It follows from (3.1) and (4.1) that

$$\begin{aligned} d_{vc}^C - d_{vc}^* =& \frac{\alpha_1+\alpha_2}{\mu_{vc}} + \frac{\varphi_{vc}(\theta_1+\theta_2)}{\mu_{vc}(r+\varepsilon)} - \frac{(1-\omega_1)\alpha_1 + (1-\omega_2)\alpha_2}{\mu_{vc}} \\ &- \frac{\varphi_{vc}[(1-\omega_1)\theta_1 + (1-\omega_2)\theta_2]}{\mu_{vc}(r+\varepsilon)} \\ =& \frac{\omega_1\alpha_1 + \omega_2\alpha_2}{\mu_{vc}} + \frac{\varphi_{vc}(\omega_1\theta_1 + \omega_2\theta_2)}{\mu_{vc}(r+\varepsilon)} > 0 \end{aligned}$$

and

$$\begin{aligned}
d_i^C - d_i^* =& \frac{\beta_1}{\mu_i} + \frac{\varphi_i(\theta_1+\theta_2)}{\mu_i(r+\varepsilon)} - \frac{\omega_i}{(1-\eta_i^*)\mu_i}\left[(\beta_1+\beta_2) + \frac{\varphi_i}{r+\varepsilon}\right] \\
=& \frac{\beta_1}{\mu_i} + \frac{\varphi_i(\theta_1+\theta_2)}{\mu_i(r+\varepsilon)} - \frac{(2-\omega_i)(\beta_1+\beta_2) - 2(1-\omega_j)\beta_2}{2\mu_i} \\
& - \frac{\varphi_i[(2-\omega_i)\theta_i + 2(1-\omega_j)\theta_j]}{2\mu_i(r+\varepsilon)} \\
=& \frac{[(\beta_1+\beta_2)+\varphi_i\theta_i]\,\omega_i}{2\mu_i(r+\varepsilon)} - \frac{[\beta_2(r+\varepsilon) - \varphi_i\theta_j]\,\omega_j}{\mu_i(r+\varepsilon)} < 0, \\
& \text{if} \quad \frac{\omega_i}{\omega_j} < \frac{2\beta_2(r+\varepsilon)) - 2\varphi_i\theta_j}{(\beta_1+\beta_2)(r+\varepsilon)+\varphi_i\theta_i}.
\end{aligned}$$

Thus, conclusions (a) and (b) are true.

Moreover, it follows from (3.3) and (4.2) that

$$V^C - (V_{vc}^* + V_1^* + V_2^*) = N - n - n_1 - n_2 > 0, \quad \text{if} \quad N > n + n_1 + n_2$$

and so conclusion (c) holds.

In order to prove the last result of Theorem 5.1, we need to compare mean and variance functions of the reputation levels under the optimal efforts rate strategies in Theorems 3.2 and 4.2. It follows from (3.12) and (4.8) that

$$\begin{aligned}
E[\overline{R}(t)] - E[R(t)] =& \frac{\Lambda - \Theta}{\varepsilon}(1 - e^{-\varepsilon t}) \\
=& \left\{ \frac{\varphi_{vc}}{\mu_{vc}}\left[(\omega_1\alpha_1 + \omega_2\alpha_2) + \varphi_{vc}(\omega_1\theta_1 + \omega_2\theta_2)\right] \right. \\
& + \frac{\varphi_1}{2\mu_1}\left[\omega_1\beta_1 - (2\omega_2 - \omega_1)\beta_2 + \frac{\varphi_1}{r+\varepsilon}(\omega_1\theta_1 + 2\omega_2\theta_2)\right] \\
& \left. + \frac{\varphi_2}{2\mu_2}\left[\omega_2\beta_1 - (2\omega_1 - \omega_2)\beta_2 + \frac{\varphi_2}{r+\varepsilon}(\omega_2\theta_2 + 2\omega_1\theta_1)\right]\right\}(1 - e^{-\varepsilon t}) > 0
\end{aligned}$$

when $2\omega_i > \omega_j$ and $\beta_2 < \frac{\omega_i\beta_1}{2\omega_j - \omega_i}$ $\quad (i = 1, 2, i \neq j)$. Similarly, it follows from (3.13), (3.14), (4.9) and (4.10) that

$$\lim_{t\to+\infty} E[\overline{R}(t)] > \lim_{t\to+\infty} E[R(t)], \quad D[\overline{R}(t)] > D[R(t)], \quad \lim_{t\to+\infty} D[\overline{R}(t)] > \lim_{t\to+\infty} D[R(t)].$$

This completes the proof. $\square$

Remark 5.1. By comparing results between two games, Theorem 5.1 shows that (i) values of the optimal efforts rate strategies of entrepreneurs, profits of entrepreneurship chain and characteristic functions of entrepreneurship chain reputation levels in the cooperative game are all higher than values in the Stakelberg game under certain conditions; (ii) optimal efforts rate strategies of entrepreneurs, profits of three participants as well as the entrepreneurship chain reputation level are all related to the competitive coefficient β_2 between entrepreneurs in the Stakelberg game. Theorem 5.1 also indicates that mean and variance functions of the entrepreneurial chain reputation level under the cooperative game are higher than ones under the Stackelberg game when $\beta_2 < \frac{\omega_i\beta_1}{2\omega_j - \omega_i}$ and $\omega_j < 2\omega_i$.

$\boxed{5.2}$ Numerical experiments

This subsection illustrates that how competition intensity coefficient β_2 between two competitive entrepreneurs effects on the mean value of entrepreneurship chain's profits, the incentive strategy of venture capitalist for entrepreneurs, as well as mean and variance functions of the entrepreneurship chain reputation level in the Stakelberg game. Unless otherwise stated, the related parameters are set as follows: $\varphi_{vc} = 0.4$, $\varphi_1 = 0.5$, $\varphi_2 = 0.6$, $\varepsilon = 0.05$, $\mu_1 = 1$, $\mu_2 = 1$, $\mu_{vc} = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta_1 = 1$, $\theta_1 = 0.3$, $\theta_2 = 0.4$, $\omega_1 = 0.6$, $\omega_2 = 0.7$, $r = 0.05$, $R_0 = 1$. These parameters values are chosen from the previous studies (see [11, 27]). Figs. 1-3 provide changes of profits for venture capitalist and two competitive entrepreneurs with β_2 and time t. Fig. 4 describes the variation of entrepreneurship chain's profits with β_2 and time t. Figs. 5 and 6 capture changes of the venture capitalist's incentive factors to entrepreneurs with β_2. Figs. 7 and 8 show comparing results of mean and variance functions of the entrepreneurship chain reputation level with β_2.

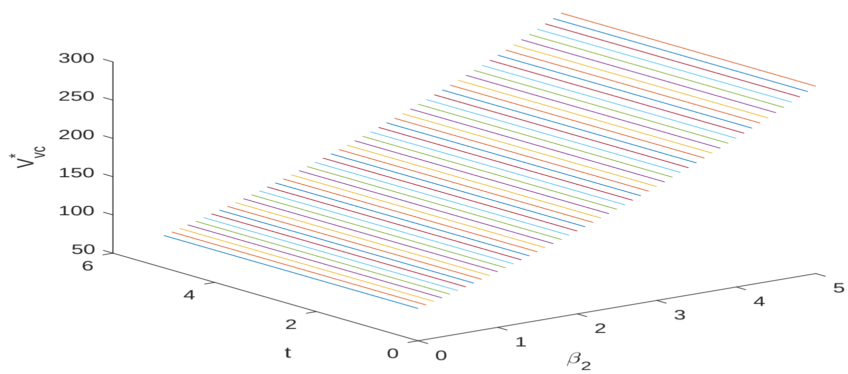

Figure 5.1: Change in V^*_{vc} with β_2 and time t.

Fig. 5.1 shows venture capitalist's profits V^*_{vc} is increasing with the growth of competition intensity coefficient β_2 and time t. If β_2 is fixed, V^*_{vc} increases gradually and then levels off as time is increasing. On the other hand, from Figs. 5.5 and 5.6, we also observe that venture capitalist gives the the ith entrepreneur a lower incentive factor η^*_1 η^*_2 when β_2 is increasing. This indicates that venture capitalist has a strong profit incentive to promote competition between two entrepreneurs.

Figs. 5.2 and 5.3 illustrate that the i entrepreneur' profits V^*_i decreases as competition intensity coefficient β_2 increases. V^*_i is increasing gradually and then is leveling off as time increases when β_2 value is fixed. At the same time, Figs. 5.5 and 5.6 show that the i entrepreneur gets a lower incentive factor η^*_1 η^*_2 when β_2 is increasing.

From Fig. 5.4, we observe that the mean value of entrepreneurial chain' profits $V^*_{vc}+V^*_1+V^*_2$ increases as competition intensity coefficient β_2 increases. $V^*_{vc} + V^*_1 + V^*_2$ is increasing gradually and then is leveling off as time increases when β_2 value is fixed. Figs. 5.1-5.6 illustrate that profits of the venture capitalist and the entrepreneurial chain system is increasing with the intensification of competition between the two entrepreneurs, while the profits of the two entrepreneurs gradually is decreasing. The greater the intensity of competition, the greater the difference in mean values of profits resulting from the difference

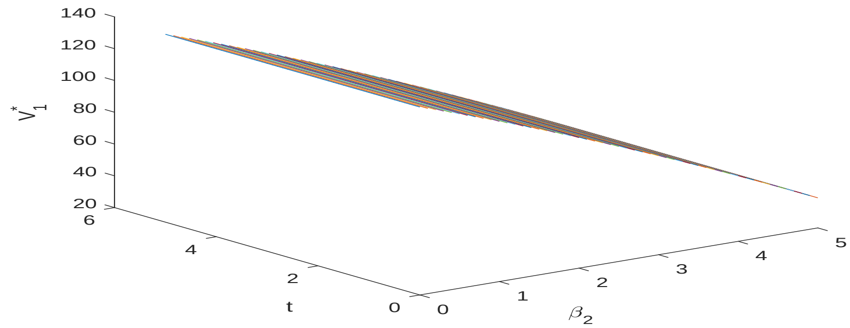

Figure 5.2: Change in V_1^* with β_2 and time t.

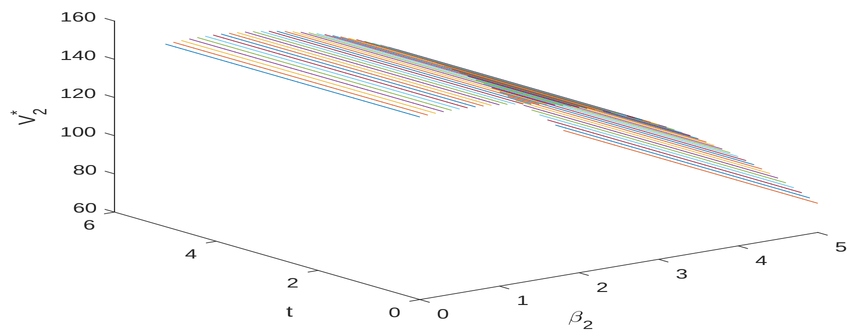

Figure 5.3: Change in V_2^* with β_2 and time t.

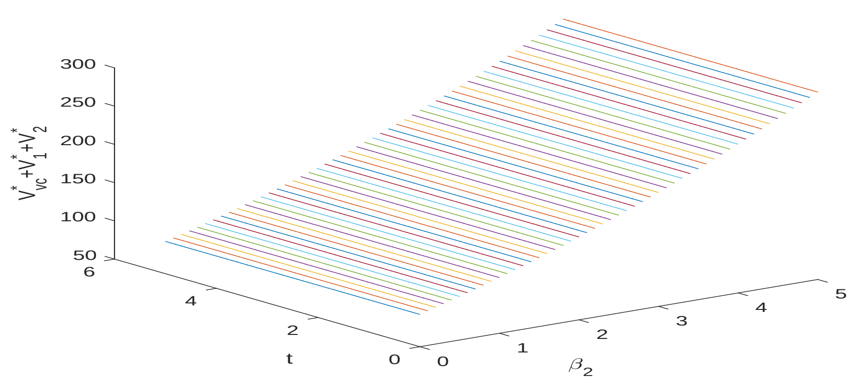

Figure 5.4: Change in $V_{vc}^* + V_1^* + V_2^*$ with β_2 and time t.

in effort between two entrepreneurs is. In order to gain more investments, both entrepreneurs will make more efforts which increases the costs for each entrepreneur. As a result, if competition intensity coefficient β_2 increases, the mean value of venture capitalist's profit goes up and the mean value of entrepreneurs' profit go down. The increase of the mean value of the venture capitalist's profit exceeds the decrease of mean values of entrepreneurs' profits, thus the mean value of entrepreneurial chain system profit increases gradually.

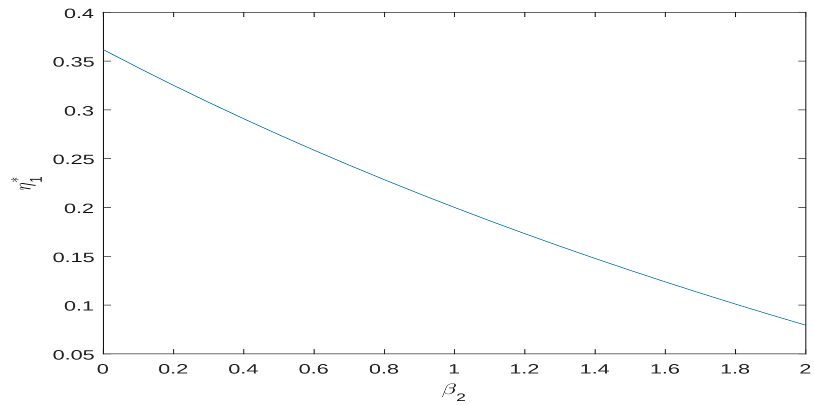

Figure 5.5: Change in η_1^* with β_2.

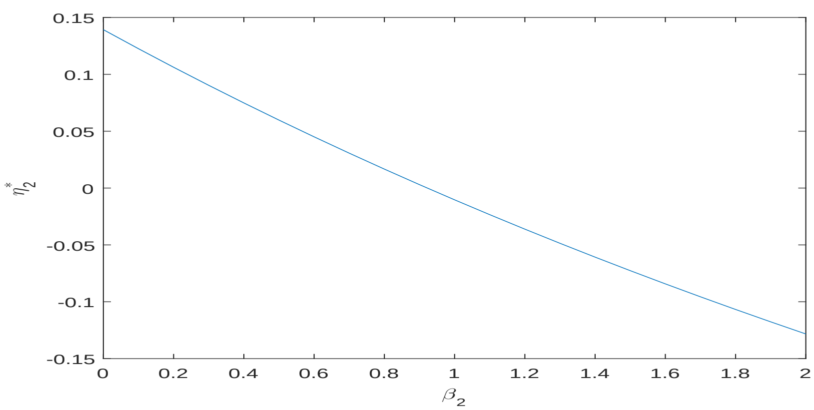

Figure 5.6: Change in η_2^* with β_2.

Fig. 5.7 shows that the mean function of the reputation level $E[R(t)]$ increases with the growth of the competition intensity coefficient β_2. This indicates that the entrepreneurship chain reputation level is increasing when the competition intensity coefficient β_2 is increasing.

Fig. 5.8 illustrates that the variance function of the reputation level $D[R(t)]$ firstly decreases and then increases with the growth of competition intensity coefficient β_2. This indicates that although the mean function of the entrepreneurship chain reputation level increases, the variance function of the reputation level also increases when the competition intensity coefficient β_2 is large enough. The higher the entrepreneurship chain reputation level, the more reputation risk of the entrepreneurship chain takes.

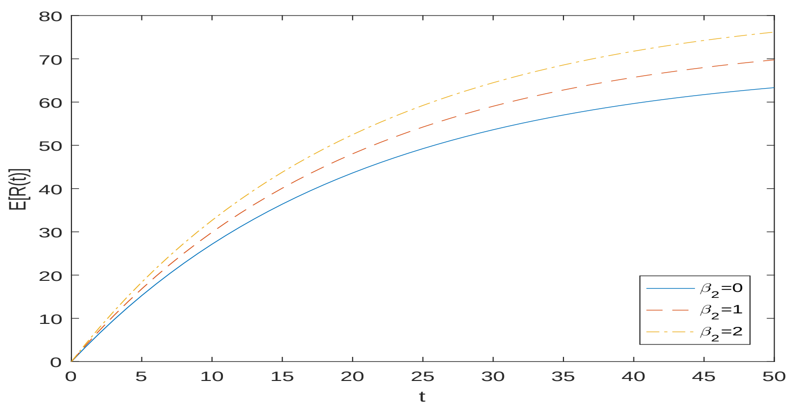

Figure 5.7: The trajectory of $E[R(t)]$ about different values of β_2.

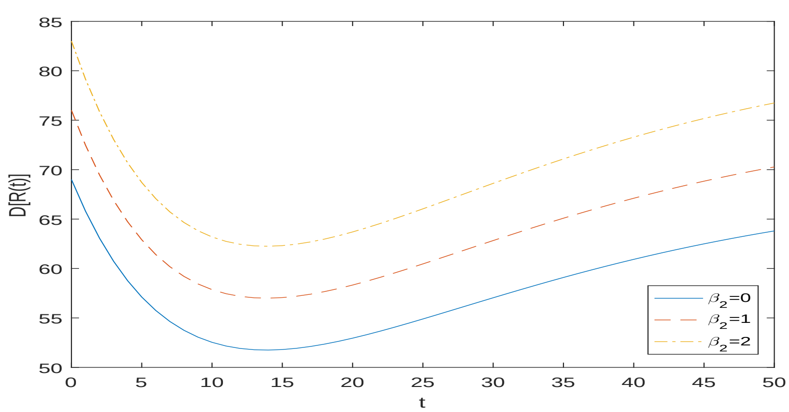

Figure 5.8: The trajectory of $D[R(t)]$ about different values of β_2.

6 Concluding Remarks

This paper investigates a vertical cooperative reputation model with one venture capital and two competitive entrepreneurs, which can be applied to solve some practical problems arising in supply chain [32]. Optimal efforts rate strategies of venture capitalist and entrepreneurs as well as mean and variance functions for the entrepreneurship chain reputation level are obtained in cases of Stakelberg game and cooperative game, respectively. Finally, numerical experiments are given for the Stakelberg game to illustrate how competition intensity coefficient between two competitive entrepreneurs effects on the mean value of entrepreneurship chain's profits, the incentive strategy of venture capitalist for entrepreneurs, as well as mean and variance functions of the entrepreneurship chain reputation level.

It is worth mentioning that appropriate prices of products will significantly influence sales rates and profits of the firms. Thus, it would be important to take prices of entrepreneurship chain products into account objective functions. Besides, in some practical situations, it is necessary to consider the case that total profits $O_i(t)$ of the ith venture project is quadratic with respect to the state $R(t)$ [6, 30]. We also note that the current paper is restricted to consider one venture capitalist, whereas many real markets are need to deal with two or more competitive venture capitalists [43, 44]. Therefore, it is important and necessary to consider the model involving two or more competitive venture capitalists and entrepreneurs. We leave these problems for our future work.

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