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# EXPECTED RESIDUAL MINIMIZATION METHOD FOR THE STOCHASTIC STACKELBERG GAME PROBLEMS IN SUPPLY CHAIN UNDER ASYMMETRIC INFORMATION*

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**Abstract:** In this paper, we consider a class of stochastic Stackelberg game problems and its optimality systems can be expressed as the two-stage stochastic complementary problems. By using the expected residual minimization (ERM) method, we reformulate this class of problems as stochastic constrained optimization problems. The error bound, solvability of the ERM problem and the convergence of its sample average approximation are established. Then, some numerical results in supply chain under asymmetric information are presented to verify the effectiveness of the proposed ERM method.

**Key words:** *stochastic Stackelberg game, asymmetric information, two-stage stochastic complementarity problems, NCP function, expected residual minimization method*

**Mathematics Subject Classification:** *90C15, 90C33*

## 1 Introduction

Information transparency is very important for supply chain operations. To gain competitive advantages in today's complex global markets, it is crucial for firms to effectively manage and utilize business information in supply chains. However, due to self-interests, firms are commonly reluctant to share or reveal their private information [9, 10, 11, 18, 19, 20]. Under the pattern of symmetric information, the firms have complete information about each other's behaviors. But, in a supply chain, the firms, being independent entities, have private information about various aspects of their businesses which are not common knowledge.

Corbett [9] investigated cycle stocks, safety stocks and consignment stocks in a two-stage supply chain under asymmetric information pattern. He proposed that the consignment stocks helped to reduce cycle stocks but at the same time, through incentives, the buyer was induced to increase safety stocks. In Cachon and Lariviere [2], a manufacturer had to face uncertain demand and contracted with a supplier who owned capacity in anticipation of demand realization. They studied both symmetric and asymmetric information, and found that the value of each type of contract varies with the information structure and compliance regime in place. In [20], the authors studied the risk management strategies

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when the disruptions of demand and cost were private information in supplier chains. They indicated linear contract menus to analyze the situations in supply chain under demand and cost disruptions being asymmetric information. They also pointed out that the optimal contract for the supplier, and showed how asymmetric disruption information affected the performance of the supplier, the retailer and the supply chain.

Motivated by the importance of information transparency in supply chain operations, it is necessary to investigate how supply chain members should make their decisions under asymmetric information. From now on, we focus on the pattern of asymmetric information in market. In what follows, a stochastic Stackelberg-Nash-Cournot equilibrium problem in supply chain model is introduced under an asymmetric information pattern. A key feature of this kind of model is that the players possess asymmetric information about the market. The model relies on the fact that the leader withholds certain information related to market shakes, and the behavior of the followers are known to the leader. The relationship between leader and followers is modeled by a stochastic non-cooperative Stackelberg games.

**A stochastic Stackelberg-Nash-Cournot equilibrium problem in supply chain under asymmetric information** Consider a supply side oligopoly market where $N+1$ firms compete to supply a homogeneous product in a non-cooperative manner. One of them, called leader, knows the behaviors of the others. The leader wants to obtain a optimal quantity to supply in order to maximize his total profit. The other firms, called followers, attempt to maximize their profits by supplying product under Cournot conjecture that the remaining firms will hold their supplies. It is well-known that such a problem can be described as the Stackelberg-Nash-Cournot game.

A Stackelberg-Nash-Cournot equilibrium is the situation where the leader chooses an optimal supply that maximizes his profit, given his knowledge of the followers' reaction to his choice of supply, the followers reaching a Nash-Cournot equilibrium where each firm cannot improve his profit by unilaterally changing his supply. The Stackelberg-Nash-Cournot equilibrium model has been studied extensively, see [27, 29] and reference therein. In Sherali et.al.[27], the authors applied a quadratic programming approach to analyze the followers' Nash-Cournot equilibrium and presented a numerical method to find the equilibrium. De Wolf and Smeers [29] considered a stochastic version of the Stackelberg-Nash-Cournot equilibrium model. The stochastic factors came from some uncertainties of market demand at the time when the leader made his decision on supply quantity. It was assumed that the leader knew the distribution of the stochastic variables. Since the stochastic demand is not realized at the time when the leader makes a decision, the leader had to maximize the expected profit based on his knowledge of the distribution of demand and the followers' reaction in each realization. The authors used a method proposed by Sherali et al. [27] to find the stochastic Stackelberg-Nash-Cournot equilibrium, and applied these results to the European gas market.

In our settings, we describe the market shakes with the inverse demand function $p(q, \xi)$, where $q$ is the total quantity of supply to the market, $\xi : \Omega \to \Xi \subset \mathbb{R}^n$ is a random shock with known distribution, and $p(q, \xi)$ is the market price. As we mentioned, the random shock reflected the price of product at the time when the leader makes a decision. Let $x$ denote the decision variable of the leader, that is, the product quantity of the leader to be supplied in the market; $q_i$ $(i = 1, \ldots, N)$ denote the decision variable of follower $i$, that is, the quantity supplied by firm $i$ to the market; and $c_i$ $(i = 0, 1, \ldots, N)$ denote the production cost function of the leader's and the followers', respectively.

The followers' decision problem. Assumed that each firm $i$ $(i = 0, 1, \ldots, N)$ is faced with a production cost function $c_i$, which depends, in general, only on firm $i$'s supply. In this case, the production cost of a particular firm depends only on his production output, not on those of the other firms.

If the quantity supplied by the leader is $x$, random shock of the price is $\xi$, and each firm $i$'s supply is $q_i$, then the market price of the product in this situation is

$$p\left(x+\sum_{i=1}^{N} q_i, \xi\right).$$

The total revenue of firm $i$ ($i = 1, \ldots, N$) is

$$q_i p\left(x+\sum_{i=1}^{N} q_i, \xi\right).$$

Then firm $i$'s profit can be formulated as

$$f_i\left(q_i\right) = q_i p\left(x+\sum_{i=1}^{N} q_i, \xi\right) - c_i\left(q_i\right).$$

Since the market price depends partly on $q_i$ (firm $i$ has his market power), firm $i$ should to choose an optimal $q_i$ in order to maximize his profit $f_i\left(q_i\right)$. Therefore, follower $i$'s profit maximization problem can be written as

$$\max_{q_i \geq 0} f_i\left(q_i\right) = q_i p\left(x+q_i+\sum_{k=1, k\neq i}^{N} q_k, \xi\right) - c_i\left(q_i\right). \tag{1.1}$$

For each firm $i$'s decision problem as in (1.1), firm $i$ treats the other firms' supplies as constant. A Nash-Cournot equilibrium among followers (in the leader's decision $x$ and random shock scenario $\xi$) is a situation where no firm can improve its profit by unilaterally changing its supply. We denote such an equilibrium by $(q_1(x,\xi), \ldots, q_N(x,\xi))$ where $q_i(x,\xi)$ is the global optimal solution of (1.1), if $p(\cdot,\xi)$ is concave with respect to $q_i$ and $c_i(\cdot)$ is convex with respect to $q_i$.

The leader's decision problem. Based on the preceding discussions, we can formulate the leader's decision problem as follow:

$$\max_{x \geq 0} f_0(x) = \mathbb{E}\left[xp\left(x+\sum_{i=1}^{N} q_i(x,\xi), \xi\right)\right] - c_0(x). \tag{1.2}$$

Note that unlike the followers' optimization problem, the leader's objective function is not necessarily concave in general. Therefore, the "maximum" of the leader's expected profit may not refer to the global maximum of (1.2).

We investigate the situation where the leader maximizes the expected profit while the followers reach a Nash-Cournot equilibrium under the leader's decision $x$ and every random shock scenario $\xi$.

A stochastic Stackelberg-Nash-Cournot equilibrium is to find strategies $(x^*, q_1(x^*,\cdot), \ldots,$ $q_N(x^*,\cdot))$ such that

$$f_0\left(x^*\right) = \max_{x \geq 0} \mathbb{E}\left[xp\left(x+\sum_{i=1}^{N} q_i(x,\xi), \xi\right)\right] - c_0(x), \tag{1.3}$$

where

$$q_i(x,\xi) \in \arg\max_{q_i \geq 0} \left( q_i p \left( x + q_i + \sum_{k=1,k\neq i}^{N} q_k(x,\xi), \xi \right) - c_i\left(q_i\right) \right). \tag{1.4}$$

In the rest of this paper, we will propose a new approach-ERM method, which is another main motivation, to investigate stochastic Stackelberg-Nash-Cournot equilibria. The approach is based on the optimality conditions of the equilibria, and consequently, the stochastic Stackelberg-Nash-Cournot equilibrium is transformed into a two-stage stochastic complementarity problem and further into an ERM problem.

The paper is organized as follows. In the following section, the framework of the two-stage SCP is introduced. In section 3, we investigate the convergence anlysis of the sample average approximations of ERM formulation. And we propose an ERM formulation of the two-stage SCP in section 4. The corresponding error bound and solvability are also obtained. Then, in section 5, we solve an simple example of stochastic Stackelberg game problem by applying the proposed ERM method and give some numerical experiments.

## 2 A Compact Reformulation of the Model

In this section, we investigate the optimality conditions of stochastic Stackelberg-Nash-Cournot equilibrium and reformulate it as a two-stage stochastic complementary problems framework. We need some assumptions for the study of the model.

**Assumption 2.1.** For $i = 0, 1, \ldots, N, c_i(q)$ is twice continuously differentiable, $c_i'(q) \geq 0$ and $c_i''(q) \geq 0$ for $q \geq 0$.

This is a standard assumption, and it requires that the cost function of each firm is sufficiently smooth and convex.

**Assumption 2.2.** The inverse demand function $p(q,\xi)$ satisfies the following:

(i) for $q \geq 0$ and a.e. $\xi \in \Xi$, $p(q,\xi)$ is twice continuously differentiable in $q$ and $p_q'(q,\xi) < 0$;

(ii) for $q \geq 0$ and a.e. $\xi \in \Xi$, $p_q'(q,\xi) + qp_{qq}''(q,\xi) \leq 0$.

This assumption is similar to an assumption made by Sherali et al. [27] and De Wolf and Smeers [29]. If the leader's decision is $x$ and the follower $i$'s decision is $q_i$ $(i = 1, \ldots, N)$, then the leader's revenue under random shock scenario $\xi$ is $x \cdot p(x + \sum_{i=1}^N q_i, \xi)$. Consider an extraneous supply $K \geq 0$. Then, the marginal revenue is $p(x + \sum_{i=1}^N q_i + K, \xi) + x \cdot p_x'(x + \sum_{i=1}^N q_i + K, \xi)$. The rate of change of this marginal revenue with respect to the increase in the extraneous supply $K$ is $p_x'(x + \sum_{i=1}^N q_i + K, \xi) + x \cdot p_x''(x + \sum_{i=1}^N q_i + K, \xi)$. Assumption 2.2 (ii) implies that this rate is non-positive when $K = 0$ for a.e. $\xi \in \Xi$. In other words, any extraneous supply will potentially reduce the leader's marginal revenue in any random shock scenario [27].

**Assumption 2.3.** There exists $q^u$, such that

$$c_i'(q) \geq p(q,\xi), \quad \text{for } q \geq q^u,\ a.e.\ \xi \in \Xi, \quad i = 0, 1, \ldots, N.$$

The assumption implies that each firm $i$'s marginal cost at total output quantity $q^u$ or above would exceed any possible market price. Therefore, none of the firms would wish to supply more than $q^u$. See discussions in [27, 29].

**Proposition 2.1** ([30])**.** *Under Assumptions 2.1, 2.2 and 2.3,*

(i) *for any $x \geq 0$, $f_0(x)$ is non-negative and bounded,*

(ii) *for any $x \geq 0$ and a.e. $\xi \in \Xi$, there exists an unique Nash-Cournot equilibrium $(q_1(x,\xi), \ldots, q_N(x,\xi))$ among followers, which solves* (1.4)*; moreover, $q_i(x,\xi) \in [0, q^u)$, for $i = 1, \ldots, N$.*

Our first step is to formulate the followers' Nash-Cournot equilibrium as a nonlinear complementarity problem. Under the leader supply $x$ and a random shock scenario $\xi$, the follower $i$'s Nash-Cournot equilibrium problem by considering the Karush-Kuhn-Tucker condition is:

$$\begin{gathered} p(x + q_i + \sum_{k=1, k\neq i}^{N} q_k, \xi) + q_i p'_{q_i}(x + q_i + \sum_{k=1, k\neq i}^{N} q_k, \xi) - c'_i(q_i) + \mu_i = 0, \\ \mu_i \geq 0, \quad q_i \geq 0, \quad \mu_i q_i = 0. \end{gathered} \tag{2.1}$$

This is a parameterized $N$-dimensional nonlinear complementarity problem where both $x$ and $\xi$ become parameters. Let $\mathbf{q} = (q_1, \ldots, q_N)^T, \mathbf{e} = (1, \ldots, 1)^T$, $\mathbf{c}(\mathbf{q}) = (c_1(q_1), \ldots, c_N(q_N))^T$. Let

$$G(\mathbf{q}, x, \xi) \equiv -p\left(x + \mathbf{q}^T\mathbf{e}, \xi\right)\mathbf{e} - p'_q\left(x + \mathbf{q}^T\mathbf{e}, \xi\right)\mathbf{q} + \nabla\mathbf{c}(\mathbf{q}).$$

Then the KKT condition (2.1) can be rewritten as

$$0 \leq \mathbf{q} \perp G(\mathbf{q}, x, \xi) \geq 0.$$

In addition, under suitable conditions, we can write down the leader's (weak) optimality condition, i.e.

$$\begin{gathered} \mathbb{E}\left[p\left(x + \sum_{i=1}^{N} q_i(x,\xi), \xi\right) + x\nabla_x p\left(x + \sum_{i=1}^{N} q_i(x,\xi), \xi\right) - \nabla c_0(x) - \lambda\right] = 0, \\ x \geq 0, \quad \lambda \geq 0, \quad \lambda \perp x. \end{gathered}$$

The above optimality system is weaker than the conventional saddle-point condition. More specifically, stationarity comes from the first order optimality condition, where the relation $\partial\mathbb{E}\left[\Phi(x,\xi)\right] \subseteq \mathbb{E}\left[\partial_x\Phi(x,\xi)\right]$ (in the sense of Aumman[1]) implies a weaker condition for optimality.

Rewrite the leader's and the followers' optimality systems together in a compact form as a two-stage stochastic complementary system, which is to find the strategy pair $(x, \mathbf{q}(\cdot)) \in \mathbb{R}_+ \times \mathcal{Y}_+$ satisfying

$$\begin{gathered} 0 \leq x \perp \mathbb{E}\left[p\left(x + \sum_{i=1}^{N} q_i(x,\xi), \xi\right) + x\nabla_x p\left(x + \sum_{i=1}^{N} q_i(x,\xi), \xi\right) - \nabla c_j(\overline{x}_j)\right] \geq 0, \\ 0 \leq \mathbf{q} \perp_{a.s.} G(\mathbf{q}, x, \xi) \geq 0, \end{gathered}$$

where $\mathbf{q}(\cdot) \in \mathcal{Y}_+$ with $\mathcal{Y}$ being the space of measurable functions from $\Omega$ to $\mathbb{R}^m$ such that the involved expectation is well defined; $\mathcal{Y}_+$ means that $y(\cdot) \in \mathcal{Y}$ and for a.e. $\xi \in \Xi$, $y(\xi) \geq 0$. "a.e." is the abbreviation for "almost every"; "a.s." means that almost surely or with probability one.

In the first stage, the leader aims to find the "here and now" decision $x \in \mathbb{R}^n$ before realizing the uncertainty $\xi$; in the second stage, as soon as the "here and now" variable $x$ has been given and the uncertainty $\xi$ has been realized, the followers determine their "wait and see" solution $q_i(x,\xi)$ to attempt to achieve their maximal profits.

Notice that when the involved function is linear w.r.t. $x$ and $\mathbf{q}$, the above stochastic complementary system becomes the special case of the canonical two-stage stochastic linear complementarity problem (SLCP), which is to find $x \in \mathbb{R}^n_+$ and $y(\cdot) \in \mathcal{Y}$ such that

$$0 \le x \quad \perp Ax + \mathbb{E}[B(\xi)y(\xi)] + q_1 \ge 0, \tag{2.2}$$

$$0 \le y(\xi) \perp N(\xi)x + M(\xi)y(\xi) + q_2(\xi) \ge 0, \; a.e. \; \xi \in \Xi, \tag{2.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B(\xi) : \Xi \to \mathbb{R}^{n\times m}$, $N(\xi) : \Xi \to \mathbb{R}^{m\times n}$, $M(\xi) : \Xi \to \mathbb{R}^{m\times m}$, $q_1 \in \mathbb{R}^n$, $q_2(\xi) : \Xi \to \mathbb{R}^m$. In the recent two decades, the stochastic complementary problems have been studied extensively, and widely used in modelling various problems under uncertainty, see [3, 14, 17, 21, 12, 28, 16] for details. Moreover, the recently proposed two-stage stochastic complementary problems can be applied to the case when the decision makers need to make decisions in two stages in stochastic environments [7, 4, 6, 5, 22, 25, 24].

## 3 Convergence Analysis

In this section, firstly, we shows the existence and the uniqueness of the solution set of the two-stage SLCP when the random vector follows either a continuous distribution or a discrete distribution.

**Assumption 3.1.** For a.e. $\xi \in \Xi$,

$$(z^T, u^T) \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} \ge 0, \quad \forall z \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m,$$

and $\mathbb{E}[\|B(\xi)\|] \le \infty$, $\mathbb{E}[\|M(\xi)\|] \le \infty$, $\mathbb{E}[\|N(\xi)\|] \le \infty$, $\mathbb{E}[\|q(\xi)\|] \le \infty$.

**Assumption 3.2.** There exists a positive continuous function $\kappa(\xi)$ such that $\mathbb{E}[\kappa(\xi)] < \infty$ and for a.e. $\xi \in \Xi_0 \subseteq \Xi$ with $\mathbb{P}(\Xi_0) > 0$,

$$(z^T, u^T) \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} \ge \; \kappa(\xi)(\|z\|^2 + \|u\|^2), \quad \forall z \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m.$$

**Lemma 3.1.** *When Assumption 3.1 and 3.2 are satisfied, the two-stage SLCP* (2.2) *and* (2.3) *has a unique solution.*

*Proof.* First, we prove monotonicity of the infinite complementarity system (2.2) and (2.3). Let $\langle \cdot, \cdot \rangle$ denote the scalar product in the Hilbert space $\mathbb{R}^n \times \mathcal{Y}$ equipped with $\mathcal{L}_2$-norm, that is, for $x, z \in \mathbb{R}^n$ and $y, u \in \mathcal{Y}$,

$$\langle (x,y),(z,u) \rangle := x^T z + \int_\Xi y(\xi)^T u(\xi) P(d\xi).$$

Then, for any $(x, y(\cdot))$, $(z, u(\cdot)) \in \mathbb{R}^n \times \mathcal{Y}$, we have

$$\begin{aligned} &\Big\langle \begin{pmatrix} A(x-z) + \mathbb{E}[B(\xi)(y(\xi)-u(\xi))]) \\ M(\xi)(y(\xi)-u(\xi)) + N(\xi)(x-z) \end{pmatrix}, \begin{pmatrix} x-z \\ y(\xi)-u(\xi) \end{pmatrix} \Big\rangle \\ =&\mathbb{E}\Big[ \begin{pmatrix} x-z \\ y(\xi)-u(\xi) \end{pmatrix}^T \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} x-z \\ y(\xi)-u(\xi) \end{pmatrix} \Big]. \end{aligned}$$

The existence of solution follows from the assumptions, see [15].

Suppose the two-stage SLCP admits two different solutions $(x, y(\cdot))$, $(z, u(\cdot)) \in \mathbb{R}^n \times \mathcal{Y}$. Then it follows

$$\left\langle \begin{pmatrix} Ax + \mathbb{E}[B(\xi)y(\xi)] + q_1 \\ M(\xi)y(\xi) + N(\xi)x + q_2(\xi) \end{pmatrix}, \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix} \right\rangle \leq 0,$$

and

$$\left\langle \begin{pmatrix} Az + \mathbb{E}[B(\xi)u(\xi)] + q_1 \\ M(\xi)u(\xi) + N(\xi)z + q_2(\xi) \end{pmatrix}, \begin{pmatrix} z - x \\ u(\xi) - y(\xi) \end{pmatrix} \right\rangle \leq 0.$$

Adding these two inequalities together, we get

$$\begin{aligned} &\mathbb{E}\Big[ \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix}^T \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix} \Big] \\ =& \int_{\Xi \setminus \Xi_0} \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix}^T \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix} P(d\xi) \\ &+ \int_{\Xi_0} \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix}^T \begin{pmatrix} A & B(\xi) \\ N(\xi) & M(\xi) \end{pmatrix} \begin{pmatrix} x - z \\ y(\xi) - u(\xi) \end{pmatrix} P(d\xi) \\ \leq& 0, \end{aligned}$$

which is a contradictions with the Assumptions 3.1 and 3.2. □

It is possible to reformulate the two-stage SLCP (2.2)-(2.3) in the following equivalent problem. Let $\hat{y}(x, \xi)$ be a solution function of the second stage problem (2.3) for $x \geq 0$ and a.e. $\xi \in \Xi$, i.e.,

$$0 \leq \hat{y}(x, \xi) \perp N(\xi)x + M(\xi)\hat{y}(x, \xi) + q_2(\xi) \geq 0, \ \textit{a.e.}\ \xi \in \Xi,$$

Then the first stage problem (2.2) becomes

$$0 \leq x \perp Ax + \mathbb{E}[B(\xi)\hat{y}(x, \xi)] + q_1 \geq 0. \tag{3.1}$$

If $x$ is a solution of (3.1), then $(x, \hat{y}(x, \cdot))$ is a solution of (2.2)-(2.3). Conversely, if $(x, \hat{y}(x, \cdot))$ is a solution of (2.2)-(2.3), then $x$ is a solution of (3.1).

In general, for some $x \geq 0$, there are more than one solution of the second stage problem (2.3). In these cases, the choice of $\hat{y}(x, \xi)$ is somewhat confused, which motivates the following assumption.

**Assumption 3.3.** For every $x \geq 0$ and a.e. $\xi \in \Xi$, the second stage problem (2.3) has a unique solution.

Assumption 3.3 holds when $M(\xi)$ is a P-matrix for every $x \geq 0$ and a.e. $\xi \in \Xi$. Under Assumption 3.3, suppose that $\hat{y}(x, \xi)$ is uniquely defined for all $x \geq 0$ and a.e. $\xi \in \Xi$, then the two-stage SLCP (2.2)-(2.3) can be written equivalently as (3.1).

Now, we consider the sample average approximation (SAA) problem of the two-stage SLCP (2.2)-(2.3), which is to find $(x^\nu, y(\xi^1), \ldots, y(\xi^\nu))$, such that the following collection of complementarity problems is satisfied:

$$
\begin{aligned}
0 \leq x^{\nu} \quad &\perp Ax + \frac{1}{\nu}\sum_{i=1}^{\nu}[B(\xi^i)y(\xi^i)] + q_1 \geq 0, \\
0 \leq y(\xi^1) &\perp N(\xi^1)x + M(\xi^1)y(\xi^1) + q_2(\xi^1) \geq 0, \\
&\vdots \\
0 \leq y(\xi^{\nu}) &\perp N(\xi^{\nu})x + M(\xi^{\nu})y(\xi^{\nu}) + q_2(\xi^{\nu}) \geq 0,
\end{aligned} \tag{3.2}
$$

where $\xi^1, \ldots, \xi^{\nu}$ are $\nu$ independent identically distributed samples of the random variable $\xi$.

If we know the distribution of $\xi$ and can integrate out the expected value explicitly, then the problem becomes deterministic, no discretization procedures are required. Unfortunately, the expected value of stochastic functions generally can not be calculated in a closed form or is difficult to evaluate exactly, so we will have to approximate it through discretization. The SAA method was studied in detail by Shapiro et.al.[26]. The idea of SAA method is that samples $\xi^1, \ldots, \xi^{\nu}$ of $\nu$ independent identically distributed samples of the random variable $\xi$ is generated and the involved expected value function is approximated by the corresponding sample average function. It has shown that SAA method is efficient for solving stochastic programming problems [26]. For the above interesting facts, this paper will establish convergence analysis for the two-stage SLCP by using sample average approximation method. To obtain the convergence analysis, we need the following boundedness condition.

**Assumption 3.4.** For every $x \geq 0$ and a.e. $\xi \in \Xi$, there is a neighborhood $\mathcal{V}$ of $x$ and a measurable function $v(\xi)$ such that $\|\hat{y}(x', \xi)\| \leq v(\xi)$ for all $x' \in \mathcal{V} \cap \mathbb{R}^n_+$.

Denote by $\mathcal{S}^*$ the set of solutions of the first stage problem (2.2) and by $\mathcal{S}_N$ the set of solutions of the first stage problem of the SAA problem (3.2).

**Theorem 3.2.** *Suppose that: (i) Assumptions* (3.3)-(3.4) *hold, (ii) there is a compact subset $X$ of $\mathbb{R}^n_+$ such that $\mathcal{S}^* \subset X$ and for all $N$ large enough the set $\hat{\mathcal{S}}_N$ is nonempty and is contained in $X$, almost surely, (iii) $\mathbb{E}[B(\xi)\hat{y}(x,\xi)]$ is dominated by an integrable function. Then,*

$$\mathbb{D}\left(\hat{\mathcal{S}}_N, \mathcal{S}^*\right) \to 0, \ \textit{w.p.1}, \ \textit{as } N \to \infty.$$

*Proof.* The proof is similar to [5, Theorem 2.4] and we omit the details. □

## 4 An ERM Formulation

In this section, we proceed with an ERM-formulation which provides a reasonable solution better suits the overall inequalities, see [3] for details.

**Definition 4.1.** [12] A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP function if it has the property

$$\phi(a,b) = 0 \ \Leftrightarrow \ a \geq 0, \ b \geq 0, \ ab = 0.$$

Two popular NCP functions are the "min" function

$$\phi(a,b) = \min(a,b)$$

and the Fischer-Burmeister (F-B) function

$$\phi(a,b) = a + b - \sqrt{a^2 + b^2}.$$

By using the NCP functions, (NCP) can be reformulated as a system of nonlinear equations

$$\Gamma(x, F(x)) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix} = 0$$

with $\phi$ being the "min" function or F-B function. All NCP functions, including the "min" function and Fischer-Burmeister function have the same solution set in the sense of reformulating any complementarity problem as a system of nonlinear equations. But in stochastic cases, we may not have such equivalence [3].

In order to deal with the second stage problem (2.3), one have to similarly introduce a collection of NCP functions whose properties would be similar to those in Definition 1.1 and ask that whether $\psi(x, y(\xi), \xi) = 0$, a.e. $\xi \in \Xi$ or not, if and only if the following parametric stochastic complementarity problems

$$0 \le y(\xi) \perp F(x, y(\xi), \xi) \ge 0, \ a.e. \ \xi \in \Xi$$

hold.

**Definition 4.2** (stochastic-NCP function)**.** Consider the following collection of stochastic nonlinear complementarity problems (SNCPs): find $x \in \mathbb{R}^n$ and $y(\cdot) \in \mathcal{Y}$, such that

$$0 \le y(\xi) \perp \ F(x, y(\xi), \xi) \ge 0, \ a.e. \ \xi \in \Xi,$$

A function $\psi : \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}$ is a Stochastic-NCP function for these problems if for any $(x, y(\cdot)) \in \mathbb{R}^n \times \mathcal{Y}$, it holds that

$$0 \le y(\xi) \perp F(x, y(\xi), \xi) \ge 0, \ a.e. \ \xi \in \Xi \iff \psi(x, y(\xi), \xi) = 0, \ a.e. \ \xi \in \Xi.$$

For the two-stage stochastic linear complementarity problem in this definition is in line with the second stage problem (2.3). Chen et.al.[7] first introduce second stage recourse variables into SVI, which is to find $x$ and $u(x, \xi)$ such that

$$0 \in f(u(x, \xi), \xi) + \mathcal{N}_{C(\xi)}(u(x, \xi)).$$

In [4], Chen et.al. extend the ERM formulation from one-stage SVI to two-stage SVI. The authors investigated the solvability, differentiability and convexity of the two-stage ERM formulation and the convergence of its sample average approximation are established. They also consider stochastic traffic assignments on arcs flow which is then formulated as a two-stage stochastic variational inequality based on Wardrop flow equilibrium and present numerical results for the corresponding two-stage stochastic programming with recourse by applying the Douglas-Rachford splitting method.

In this paper, we use NCP functions and stochastic-NCP functions to define residuals of (2.2) and (2.3), and minimize the expected residuals. This idea leads us to seeking a solution of the following stochastic programming

$$\begin{aligned} \min_{(x,y(\cdot)) \in \mathbb{R}^n \times \mathcal{Y}} \quad & \|\Phi(x, y(\xi))\|^2 + \lambda \, \mathbb{E}\|\Psi(x, y(\xi), \xi)\|^2 \\ s.t. \quad & x \ge 0, \\ & y(\xi) \ge 0, \ a.e. \ \xi \in \Xi, \end{aligned} \tag{4.1}$$

where

$$\Phi(x, y(\xi)) := \begin{pmatrix} \phi\big(x_1, (Ax + \mathbb{E}[B(\xi)y(\xi)] + q_1)\big)_1 \\ \vdots \\ \phi\big(x_n, (Ax + \mathbb{E}[B(\xi)y(\xi)] + q_1)\big)_n \end{pmatrix}, \tag{4.2}$$

$$\Psi(x, y(\xi), \xi) =: \begin{pmatrix} \psi\big(y(\xi)_1, (N(\xi)x + M(\xi)y(\xi) + q_2(\xi)\big)_1 \\ \vdots \\ \psi\big(y(\xi)_m, (N(\xi)x + M(\xi)y(\xi) + q_2(\xi)\big)_m \end{pmatrix}, \tag{4.3}$$

$\phi$ and $\psi$ are NCP functions and stochastic-NCP functions, respectively; $\lambda$ is a positive penalty factor which allows for a further adjustment of the weight to ascribe to the required recourse decisions and residuals. If $\lambda$ is relatively large, one will end up with a solution that essentially avoids few violations of the second stage problem (2.3).

It's easy to verify that the function $\|\Psi(x, y(\xi), \xi)\|$ defined in (4.3) can be served as a residual function defined in [7] for our collection SLCPs (2.3). The stochastic programming (4.1) includes the expected value (EV) and the expected residual minimization (ERM) for stochastic complementarity problems as special cases.

Next, we need to obtain the following global error bounds for the two-stage SLCP, that is, the residual function defined in (4.1) can be used to give some quantitative information about the distance between the objective function of (4.1) and the solution set of the two-stage SLCP (2.2) and (2.3). Although the "min" function and the F-B function have a number of nice properties respectively, the F-B function has much better properties in differentiability. Among others, a distinctive property from the "min" function is that $\|\Phi(x, y(\xi))\|^2$ and $\|\Psi(x, y(\xi), \xi)\|^2$ defined by the F-B function is continuously differentiable in $x$ everywhere. However, the F-B function has poor performance in dealing with the LCP. In this paper, we concentrate on the ERM formulation (4.1) defined by the "min" function.

The theorem below shows the existence of the solution of the two-stage SLCP when the random vector follows a discrete distribution and the error bound for the two-stage SLCP under mild conditions.

Consider the ERM formulation (4.1) in the case of $\Xi = \{\xi^1, \xi^2, \ldots, \xi^\nu\}$ with $P(\xi^i) = p_i > 0$, $i = 1, \ldots, \nu$,

$$\min_{(x, y(\xi^1), \ldots, y(\xi^\nu)) \in \mathbb{R}^n_+ \times \mathbb{R}^{m\nu}_+} \quad f^\nu(x, y(\xi), \xi) := \|\Phi(x, y)\|^2 + \lambda \cdot \sum_{i=1}^{\nu} p_i \|\Psi(x, y(\xi^i), \xi^i)\|^2 \tag{4.4}$$

where

$$\Phi(x, y) = \begin{pmatrix} \min(x_1, (Ax + \sum_{i=1}^{\nu} p_i B(\xi^i) y(\xi^i) + q_1))_1 \\ \vdots \\ \min(x_n, (Ax + \sum_{i=1}^{\nu} p_i B(\xi^i) y(\xi^i) + q_1))_n \end{pmatrix}, \tag{4.5}$$

$$\Psi(x, y(\xi^i), \xi^i) = \begin{pmatrix} \psi(y(\xi^i)_1, (N(\xi^i)x + M(\xi^i)y(\xi^i) + q_2(\xi^i)))_1 \\ \vdots \\ \psi(y(\xi^i)_m, (N(\xi^i)x + M(\xi^i)y(\xi^i) + q_2(\xi^i)))_m \end{pmatrix}. \tag{4.6}$$

**Theorem 4.3.** *Suppose $\Xi = \{\xi^1, \xi^2, \ldots, \xi^\nu\} \subseteq \mathbb{R}^q$ with $P(\xi^i) = p_i > 0$, $i = 1, \ldots, \nu$. If for any $z = (x, y(\xi^1), \ldots, y(\xi^\nu))$, at least one of the following statements holds:*

(i) $x^T(Ax + \sum_{i=1}^{\nu} p_i B(\xi^i)y(\xi^i)) > 0$,

(ii) *there exists at least one* $i \in \{1, \dots, \nu\}$, *such that* $y(\xi^i)^T(N(\xi^i)x + M(\xi^i)y(\xi^i)) > 0$.

*Then, there is a positve constant* $\beta_\nu$ *such that*

$$\|z - z^*\|^2 \le \ \beta_\nu \cdot (\|\Phi(z)\|^2 + \lambda \cdot \mathbb{E}[\|\Psi(z)\|^2])$$

*for any* $\lambda \ge \frac{1}{\min_i p_i}$, *where* $z^*$ *is the unique solution of the two-stage SLCP.*

*Proof.* Notice that when $\Xi$ is finite, the two-stage SLCP (2.2) and (2.3) can be written as

$$\begin{aligned} 0 \le x \quad &\perp Ax + \sum_{i=1}^{\nu} p_i[B(\xi^i)y(\xi^i)] + q_1 \ge 0, \\ 0 \le y(\xi^1) &\perp N(\xi^1)x + M(\xi^1)y(\xi^1) + q_2(\xi^1) \ge 0, \\ &\vdots \\ 0 \le y(\xi^\nu) &\perp N(\xi^\nu)x + M(\xi^\nu)y(\xi^\nu) + q_2(\xi^\nu) \ge 0, \end{aligned}$$

or equivalently,

$$0 \le \begin{pmatrix} x \\ y(\xi^1) \\ \vdots \\ y(\xi^\nu) \end{pmatrix} \perp \begin{pmatrix} A & p_1 B(\xi^1) & \cdots & p_\nu B(\xi^\nu) \\ N(\xi^1) & M(\xi^1) & & \\ \vdots & & \ddots & \\ N(\xi^\nu) & & & M(\xi^\nu) \end{pmatrix} \begin{pmatrix} x \\ y(\xi^1) \\ \vdots \\ y(\xi^\nu) \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2(\xi^1) \\ \vdots \\ q_2(\xi^\nu) \end{pmatrix} \ge 0, \tag{4.7}$$

for any $x \in \mathbb{R}^n$, and $y(\xi^i) \in \mathbb{R}^m, \ i = 1, 2, \dots, \nu$. Denote

$$Q_\nu := \begin{pmatrix} A & p_1 B(\xi^1) & \cdots & p_\nu B(\xi^\nu) \\ N(\xi^1) & M(\xi^1) & & \\ \vdots & & \ddots & \\ N(\xi^\nu) & & & M(\xi^\nu) \end{pmatrix}, \ q_\nu := \begin{pmatrix} q_1 \\ q_2(\xi^1) \\ \vdots \\ q_2(\xi^\nu) \end{pmatrix},$$

it's easy to verify that $Q$ is a P-matrix under the assumptions. Then, we can obtain that LCP$(Q, q)$ (4.7) has an unique solution $z^* = (x^*, y^*(\xi^1), \dots, y^*(\xi^\nu))$. And from [8], for any $z = (x, y(\xi^1), \dots, y(\xi^\nu))$, we have the following error bound

$$\|z - z^*\|^2 \le \max_{d \in [0,1]^n} \|(I - D + D\hat{Q}_\nu)^{-1}\|^2 \|\Theta(z)\|^2, \tag{4.8}$$

where $D$ is a diagonal matrix whose diagonal elements are $d := (d_1, \dots, d_{n+m\nu}) \in [0,1]^{n+m\nu}$, and

$$\Theta(z) = \begin{pmatrix} \min(x, \nu Ax + \nu \sum_{i=1}^{\nu} p_i B(\xi^i)y(\xi^i) + \nu q_1)) \\ \min(y(\xi^1), N(\xi^1)x + M(\xi^1)y(\xi^1) + q_2(\xi^1)) \\ \vdots \\ \min(y(\xi^\nu), N(\xi^\nu)x + M(\xi^\nu)y(\xi^\nu) + q_2(\xi^\nu)) \end{pmatrix}.$$

Since

$$\|\Theta(z)\|^2 = \|\Phi(z)\|^2 + \sum_{i=1}^{\nu} \|\Psi(z)\|^2,$$

if $\lambda \geq \frac{1}{\min_i p_i}$, then (4.8) implies that

$$\|z - z^*\|^2 \leq \max_{d\in[0,1]^n} \|(I - D + D\hat{Q}_\nu)^{-1}\|^2 \ (\|\Phi(z)\|^2 + \lambda\mathbb{E}[\|\Psi(z)\|^2]).$$

Let $\beta_\nu = \max_{d\in[0,1]^n} \|(I - D + D\hat{Q}_\nu)^{-1}\|^2$, which completes the proof. □

**Remark 4.4.** When the random vector follows a discrete distribution, the assumptions to guarantee the existence and the uniqueness of the solution set of the two-stage SLCP are much weaker than those in Lemma 3.1. In processes of dealing with many practical examples, the assumptions in Theorem 4.3 are satisfied by using the regularized method. In fact, the optimality conditions of two-stage stochastic linear program are as follows:

$$\begin{aligned} 0 \leq \alpha \quad &\perp \tilde{A}\alpha + \mathbb{E}[\tilde{B}(\xi)\beta(\xi)] + \tilde{q_1} \geq 0, \\ 0 \leq \beta(\xi) &\perp \tilde{N}(\xi)\alpha + \tilde{M}(\xi)\beta(\xi) + \tilde{q_2}(\xi) \geq 0, \ a.e.\ \xi \in \Xi, \end{aligned} \tag{4.9}$$

where the vector $\alpha, \beta(\xi), \tilde{q_1}, \tilde{q_2}(\xi)$ and the matrices $\tilde{A}, \tilde{B}(\xi), \tilde{N}(\xi), \tilde{M}(\xi)$ are defined the same as (2.2) and (2.3). It's obviously that the assumptions of Theorem 4.3, together with Assumption 2.1 in [6] may not be satisfied by these matrices, so we can not obtain the existence of the solution set of the two-stage SLCP (4.9). However, the regularized problem of the two-stage SLCP (4.9) satisfies the assumptions of Theorem 4.3, and hence, there is a unique regularized solution. Similar method can also be found in the two-stage stochastic game problems as mentioned in [6].

To find a solution of an ERM problem (4.1) numerically, it is necessary to study the objective function of ERM problem (4.1) defined by NCP functions. In what follows, we will investigate the properties of the ERM formulation (4.1).

**Proposition 4.5.** *If $\Xi = \{\xi^1, \xi^2, \ldots, \xi^\nu\} \subseteq \mathbb{R}^q$, then the solution set of the ERM formulation* (4.4) *is nonempty.*

*Proof.* Let $z = (x, y(\xi^1), \ldots, y(\xi^\nu))$, and consider the complementarity problem

$$0 \leq \begin{pmatrix} x \\ y(\xi^1) \\ \vdots \\ y(\xi^\nu) \end{pmatrix} \perp \begin{pmatrix} A & p_1 B(\xi^1) & \cdots & p_\nu B(\xi^\nu) \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} x \\ y(\xi^1) \\ \vdots \\ y(\xi^\nu) \end{pmatrix} + \begin{pmatrix} q_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \geq 0,$$

then the squared norm of the function $\Phi(z) = \min(z, \tilde{Q}z + q)$ can be represented as

$$\|\Phi(z)\|^2 = (\tilde{Q}^j z + q^j)^T (\tilde{Q}^j z + q^j), \ z \in P^j, \ j = 1, \ldots, k,$$

where

$$\tilde{Q} = \begin{pmatrix} A & p_1 B(\xi^1) & \cdots & p_\nu B(\xi^\nu) \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix},$$

$P^j$ are polyhedral convex sets comprising a partition of $\mathbb{R}_+^{n+\nu m}$, each $(\tilde{Q}^j, q^j)$ is a row representative of $\left( \begin{pmatrix} I \\ \tilde{Q} \end{pmatrix}, \begin{pmatrix} 0 \\ q_1 \end{pmatrix} \right)$, and $j \leq 2^{n+\nu m}$. Since $\tilde{Q}^{j^T} \tilde{Q}^j$ is positive semi-definite, $\|\Phi(z)\|^2$ is a piecewise quadratic function and nonnegative on $\mathbb{R}_+^{n+\nu m}$. Similar arguments can be applied to $\|\Psi(z)\|^2$. By the Frank-Wolfe Theorem [13], the objective function $f_1(x, y(\xi), \xi)$ defined in problem (4.1) attains its minimum on $\mathbb{R}_+^{n+\nu m}$. □

## 4.1 Numerical experiment

In this section, we conduct some numerical experiments to test the efficiency of our proposed ERM method to solve the stochastic Stackelberg game problems in the supply chain under asymmetric information.

To clarify how the stochastic Stackelberg game problem can be applied, let us consider two firms competing to supply similar products in an open market, say products 1 and 2. Each firm need to arrange the quantities of the two products, however, in a different way. Firm 1, possessing stronger market power, has to make a production plan now, without observing the stochastic shock of the prices of the products. But, he knows the amount of the products of firm 2 to release in market. Firm 2, called the follower, whose decision will be made after the stochastic shock of the prices and the behaviors of firm 1 are disclosed. Each firm wants to decide the amount of each product to supply to maximize their respective revenues.

Let the amounts of product 1 and product 2 to be supplied by firm 1 be $x_1$ and $x_2$, and firm 2 be $y_1$ and $y_2$, respectively. Following the notation in Section 1, we define, for a.e. $\xi \in \Xi$,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y(\xi) = \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \end{pmatrix}, \quad z(\xi) = \begin{pmatrix} x \\ y(\xi) \end{pmatrix}.$$

We consider a particular case when the inverse demand function $p(q,\xi) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is affine, that is,

$$p(q,\xi) = a - Bq + \xi,$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, and the cost function of firm $i$ $(i = 1, 2)$ is quadratic, in term of

$$c_i(q_i) = \alpha_i + \beta_i q_i + \frac{1}{2}\gamma_i q_i^2.$$

These assumptions guarantee the existence, uniqueness of the firm 2's optimal solution as well as the concavity of the firm 1's objective functions. Then, in such a relatively simple case, the decision problem (2.1) of firm 2 is quadratic, and the optimality condition can be simplified as

$$0 \le \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \perp \left[ \begin{pmatrix} \beta_2^1 \\ \beta_2^2 \end{pmatrix} - \left( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right) + \begin{pmatrix} 2b_1 + \gamma_2^1 & 0 \\ 0 & 2b_2 + \gamma_2^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right] \ge 0 \quad (4.10)$$

where $y$ may depend on $x$ and $\xi$. And, if for each $i = 1, 2$,

$$\beta_2^i < a_i - b_i x_i + \xi_i,$$

then the complementary problem (4.10) has an unique solution

$$y(x,\xi) = R^{-1}[(a - Bx) + \xi - \beta_2],$$

where $R = 2B + \Gamma_2$, $\Gamma_2 = \begin{pmatrix} \gamma_2^1 & 0 \\ 0 & \gamma_2^2 \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} \beta_2^1 \\ \beta_2^2 \end{pmatrix}$.

Moreover, firm 1's profit maximization problem is

$$\max_{x \ge 0} f_1(x) = x \cdot \mathbb{E}\left[a + \xi - B\left(x + R^{-1}\left((a - Bx) + \xi - \beta_2\right)\right)\right]$$
$$- \alpha_1 - \beta_1 x - \frac{1}{2} x^T \Gamma_1 x.$$

Note that when $f_1'(x) = (\Lambda - \Gamma_1)\, x + \eta + \zeta = 0$, where $\Lambda = -2B + 2BR^{-1}B$, $\eta = (I + BR^{-1})\ \mathbb{E}[\xi]$, and $\zeta = (a - \beta_1)\ (I - BR^{-1})$. It is easy to check that if $b_i(b_i + \gamma_1^i) \neq 0$, then $x^* = (\Gamma_1 - \Lambda)^{-1} (\eta + \zeta)$. and $y^*(\xi) = R^{-1}[(a - Bx^*) + \xi - \beta_2]$.

Assume that there are two scenarios of the stochastic shock with probabilities $p_1 = 0.6$ and $p_2 = 0.4$, respectively. The two scenarios are $\Xi = \{\xi^1, \xi^2\}$,

$$\xi_1 = \begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \xi_2^1 \\ \xi_2^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Then, the stochastic Stackelberg game problem is to find 6-dimension vector $z(\xi) = \begin{pmatrix} x \\ y(\xi_1) \\ y(\xi_2) \end{pmatrix}$, such that

$$0 \leq x \perp (-2B - \Gamma_1)\, x - B(p_1 y(\xi_1) + p_2 y(\xi_2)) + (a + \mathbb{E}[\xi] - \beta_1) \geq 0,$$

$$0 \leq y(\xi_1) \perp Bx + (2B + \Gamma_2) y(\xi_1) + (\beta_2 - a - \xi_1)) \geq 0,$$

$$0 \leq y(\xi_2) \perp Bx + (2B + \Gamma_2) y(\xi_2) + (\beta_2 - a - \xi_2)) \geq 0,$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} \gamma_{21}^1 & 0 \\ 0 & \gamma_1^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\beta_1 = \begin{pmatrix} \beta_1^1 \\ \beta_2^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} \beta_2^1 \\ \beta_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The game has an unique solution $x = \left(\frac{1}{2}, \frac{1}{2}\right)^T$, $y_1 = \left(\frac{5}{8}, \frac{7}{8}\right)^T$, $y_2 = \left(\frac{1}{8}, \frac{3}{8}\right)^T$. And the numerical result of our ERM method ends up as in the following Table 1. All numerical experiments are run in Matlab R2017a and on a PC with an Intel(R) Core(TM) i5-6200U CPU @2.30 GHz and 8 GB of RAM under the Windows 7 operating system. In our settings, we let $\lambda = 2.5$ and the starting point $z_0 = 0$.

Table 1: The numerical results for ERM method

| $x_1$ | $x_2$ | $y_1^1$ | $y_1^2$ | $y_1^2$ | $y_2^2$ | $f(z)$ |
|---|---|---|---|---|---|---|
| 0.4894 | 0.4894 | 0.6316 | 0.8751 | 0.2197 | 0.3751 | 4.9183e-07 |

## 5 Conclusions

In practical activities, stochastic Stackelberg game problems, which impose great impacts on production decisions between manufacturing companies, often arise in supply chain members' behaviors under asymmetric information. However, most existing studies on supply chain under asymmetric information focus on contracts in supply chain. Fewer studies pay close attention to the decisions of both leader and followers simultaneously. In order to address this gap, we propose a class of stochastic Stackelberg game problems where both leader and followers want to maximize his total profit under asymmetric information. This paper then proved that this kind of problems can be equivalently reformulated as a two-stage stochastic complementary problems by putting their individual optimality conditions together. Second, based on the development of stochastic programming, we bring the ERM method to deal with this kind of problems. By using the ERM method, we reformulate the two-stage stochastic complementary problems as stochastic constrained optimization problems. The error bound, solvability of the ERM problem and the convergence of its sample average approximation are established. Finally, we present some numerical results to show ERM is a practicable method to handle this kind of problem.

With regard to future research, this work can be extended in several directions. Although, in our paper, the ERM method is used to solve two-stage stochastic complementary problems, it also can be used in other forms of two-stage stochastic equilibrium problems. In addition, our method can be applied not only in game theory, but other stochastic hierarchical problems where individual decision maker is in different level of cooperation and coordination.

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