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HERMITIAN COMPLETELY POSITIVE TENSORS

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Abstract: In this paper, we introduce the (resp., real) Hermitian completely positive tensor, which has a (resp., real) Hermitian completely positive decomposition with all real and imaginary parts of the decomposition vectors being nonnegative (resp. real parts of the decomposition vectors being nonnegative but imaginary parts being zeros). Some properties of the (real) Hermitian completely positive tensors and (real) Hermitian copositive tensors are discussed. A semidefinite algorithm is also proposed for checking whether a Hermitian tensor is Hermitian completely positive or not. If a Hermitian tensor is not Hermitian completely positive, a certificate can be obtained; if it is, a Hermitian completely positive decomposition can be given.

Key words: *Hermitian completely positive tensors, real Hermitian completely positive tensors, truncated moment problem, semidefinite program*

Mathematics Subject Classification: *15A69, 15B57, 44A60, 90C22*

1 Introduction

Let $\mathbb{F} = \mathbb{C}$ (the complex field) or $\mathbb{R}$ (the real field). Given a positive integer n, denote the set $[n] := \{1, \ldots, n\}$. For positive integers m and $n_1, \ldots, n_m$, an mth order $(n_1, \ldots, n_m)$-dimensional tensor $\mathcal{T}$ is an array $\mathcal{T} = (\mathcal{T}_{i_1 \ldots i_m})$, with $i_k \in [n_k]$ for $k \in [m]$. Let $\mathbb{F}^{n_1 \times \cdots \times n_m}$ be the space of all such tensors of order m and dimension $(n_1, \ldots, n_m)$ with entries in $\mathbb{F}$. For a tensor $\mathcal{T}$, $\overline{\mathcal{T}}$ denotes its conjugate entrywise, $\mathcal{T}^{re}$ and $\mathcal{T}^{im}$ denote its real and imaginary parts, respectively. For a matrix or vector u, u^T denotes its transpose, and u^* denotes its conjugate transpose. For two tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{n_1 \times \cdots \times n_m}$, their inner product is defined as

$$\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1, \ldots, i_m = 1}^{n_1, \ldots, n_m} \mathcal{A}_{i_1 \ldots i_m} \overline{\mathcal{B}_{i_1 \ldots i_m}}. \tag{1.1}$$

The Hilbert–Schmidt norm of $\mathcal{A}$ is accordingly defined as $\|\mathcal{A}\| := \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}$. For vectors $w_k \in \mathbb{F}^{n_k}, k \in [m]$, we use $w_1 \otimes \cdots \otimes w_m$ to denote their tensor product, i.e.,

$$(w_1 \otimes \cdots \otimes w_m)_{i_1 \ldots i_m} = (w_1)_{i_1} \cdots (w_m)_{i_m}$$

for $i_k \in [n_k]$. Tensors like multiples of $w_1 \otimes \cdots \otimes w_m$ are called rank-1 tensors. The rank of $\mathcal{T}$, denoted as rank$(\mathcal{T})$, is the smallest r such that

$$\mathcal{T} = \sum_{i=1}^{r} u_1^i \otimes \cdots \otimes u_m^i, u_j^i \in \mathbb{C}^{n_j}. \tag{1.2}$$

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The decomposition above (1.2) is often called a CANDECOMP/PARAFAC or canonical polyadic decomposition.

A tensor $\mathcal{A} \in \mathbb{F}^{n_1 \times \cdots \times n_m}$ is called a square tensor, if $n_1 = \cdots = n_m = n$. Let $\mathbb{T}^m(\mathbb{F}^n)$ be the set of all mth order n-dimensional square tensors in the field $\mathbb{F}$. A tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{F}^n)$ is symmetric if $\mathcal{A}_{i_1,\ldots,i_m}$ is invariant for all permutations of $(i_1, \ldots, i_m)$. Let $\mathbb{S}^m(\mathbb{F}^n)$ be the set of all symmetric tensors in $\mathbb{T}^m(\mathbb{F}^n)$. For a vector $w \in \mathbb{F}^n$, rank-1 symmetric tensors are multiples of $w^{\otimes m} := w \otimes \cdots \otimes w$ (repeated m times). Similarly, the smallest number r such that $\mathcal{A} = \sum_{i=1}^r \lambda_i u_i^{\otimes m}$, with each $u_i \in \mathbb{C}^n$ and $\lambda_i \in \mathbb{C}$, is called the symmetric rank of $\mathcal{A}$. We refer to [4, 5, 13, 23, 24] for theories, methods and applications of tensor decompositions.

Let $\mathbb{R}_+^n$ be the nonnegative orthant. A symmetric tensor $\mathcal{A} \in \mathbb{S}^m(\mathbb{R}^n)$ is *completely positive* (CP), if there exist nonnegative vectors $w^1, \ldots, w^r \in \mathbb{R}_+^n$ such that

$$\mathcal{A} = \sum_{i=1}^r (w^i)^{\otimes m}, \tag{1.3}$$

where r is called the length of the decomposition (1.3) (cf. [30]). The smallest r in the above is called the CP-rank of $\mathcal{A}$. If $\mathcal{A}$ is completely positive, (1.3) is called a *completely positive (CP) decomposition* of $\mathcal{A}$. For $\mathcal{B} \in \mathbb{S}^m(\mathbb{R}^n)$, we define

$$\mathcal{B}x^m := \sum_{1 \le i_1,\ldots,i_m \le n} \mathcal{B}_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m}.$$

We call $\mathcal{B} \in \mathbb{S}^m(\mathbb{R}^n)$ a *copositive* (resp., *positive semidefinite*) tensor, if $\mathcal{B}x^m \ge 0$ for all $x \in \mathbb{R}_+^n$ (resp., for all $x \in \mathbb{R}^n$). The CP (resp., copositive) tensors are natural generalizations of the CP (resp., copositive) matrices. It is clear that both symmetric nonnegative tensors and positive semidefinite tensors are copositive tensors. Denote

$$\begin{aligned} \mathbf{CP}_{m,n} &= \{\mathcal{A} \in \mathbb{S}^m(\mathbb{R}^n) : \mathcal{A} \ \ \textit{is} \ \ \textit{completely} \ \ \textit{positive}\}, \\ \mathbf{COP}_{m,n} &= \{\mathcal{B} \in \mathbb{S}^m(\mathbb{R}^n) : \mathcal{B} \ \ \textit{is} \ \ \textit{copositive}\}. \end{aligned}$$

Both $\mathbf{CP}_{m,n}$ and $\mathbf{COP}_{m,n}$ are proper cones, i.e., closed, convex, pointed and full-dimensional. Moreover, they are dual to each other [29]. As we know, checking a completely positive matrix is NP-hard [8]. Thus, it is more complicated to check completely positive tensors. Completely positive tensors and decompositions have wide applications [1, 3, 26]. We refer the reader to [9, 14, 35] for numerical methods for CP tensor decompositions and CP optimization problems.

Recently, Zhou et al. [37] introduced the complex completely positive tensor which has a symmetric complex decomposition with all real and imaginary parts of the decomposition vectors being nonnegative. Let $\mathbb{C}_+^n$ be the set of complex vectors in $\mathbb{C}^n$ with nonnegative real and imaginary parts. A symmetric tensor $\mathcal{A} \in \mathbb{S}^m(\mathbb{C}^n)$ is *complex completely positive (CCP)*, if there exist vectors $w^1, \ldots, w^r \in \mathbb{C}_+^n$ such that

$$\mathcal{A} = \sum_{i=1}^r (w^i)^{\otimes m}, \tag{1.4}$$

where $w = u + \sqrt{-1}v \in \mathbb{C}^n$ with $u, v \in \mathbb{R}_+^n$, and r is called the length of the decomposition (1.4). The smallest r in the above is called the CCP-rank of $\mathcal{A}$. If $\mathcal{A}$ is complex completely positive, (1.4) is called a *complex completely positive (CCP) decomposition* of $\mathcal{A}$. Clearly, if a real symmetric tensor is completely positive, it must be complex completely positive, and its CP decompositions must be CCP decompositions.

Hermitian tensor is a generalization of Hermitian matrix. A $2m$th order tensor $\mathcal{H} \in \mathbb{C}^{n_1\times\cdots\times n_m\times n_1\times\cdots\times n_m}$ is called a *Hermitian tensor* [19], if

$$\mathcal{H}_{i_1\ldots i_m j_1\ldots,j_m} = \overline{\mathcal{H}_{j_1\ldots j_m i_1\ldots i_m}}$$

for each $i_k, j_k \in [n_k]$ and $k \in [m]$. If the entries of a Hermitian tensor are all real, it is called a real Hermitian tensor [25]. For a $2m$th order real tensor $\mathcal{A} \in \mathbb{R}^{n_1\times\cdots\times n_m\times n_1\times\cdots\times n_m}$ is called a *real skew-symmetric tensor*, if

$$\mathcal{A}_{i_1\ldots i_m j_1\ldots j_m} = -\mathcal{A}_{j_1\ldots j_m i_1\ldots i_m}$$

for each $i_k, j_k \in [n_k]$ and $k \in [m]$. Denote by $\mathbb{C}^{[n_1,\ldots,n_m]}$ (resp., $\mathbb{R}^{[n_1,\ldots,n_m]}$, $\mathbb{R}_{skew}^{[n_1,\ldots,n_m]}$) the set of all Hermitian (resp., real Hermitian, real skew-symmetric) tensors in $\mathbb{C}^{n_1\times\cdots\times n_m\times n_1\times\cdots\times n_m}$. It is easy to check that, for a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, its real part $\mathcal{H}^{re}$ is a real Hermitian tensor and its imaginary part $\mathcal{H}^{im}$ is a real skew-symmetric tensor.

For vectors $w_i \in \mathbb{C}^{n_i}, i \in [m]$, denote the tensor product of conjugate pairs

$$[w_1, \ldots, w_m]_{\otimes h} := w_1 \otimes \cdots \otimes w_m \otimes \overline{w_1} \otimes \cdots \otimes \overline{w_m}. \tag{1.5}$$

Clearly, the above tensor product is always a Hermitian tensor. In fact, every rank-1 Hermitian tensor must be in the form of $\lambda \cdot [w_1, \ldots, w_m]_{\otimes h}$ with a real scalar $\lambda \in \mathbb{R}$. As a generalization of Hermitian matrices, Hermitian tensors inherit some nice results from Hermitian matrices, although they have very different properties. Ni in [19] showed that every Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ always has a *Hermitian decomposition*, i.e., there exist vectors $u_j^i \in \mathbb{C}^{n_j}$ and real scalars $\lambda_i \in \mathbb{R}$, for $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_1^i, \ldots, u_m^i]_{\otimes h}, \tag{1.6}$$

where r is called the length of the decomposition (1.6). The smallest r in (1.6) is called the *Hermitian rank* of $\mathcal{H}$, which is denoted as $\operatorname{hrank}(\mathcal{H})$. However, it was shown in [25] that not all real Hermitian tensors are $\mathbb{R}$-*Hermitian decomposable*, i.e., there exist vectors $v_j^i \in \mathbb{R}^{n_j}$ and real scalars $\lambda_i \in \mathbb{R}$, for all $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [v_1^i, \ldots, v_m^i]_{\otimes h}. \tag{1.7}$$

In fact, a tensor $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ is $\mathbb{R}$-Hermitian decomposable if and only if it is partial-wise symmetric, i.e., $\mathcal{H}_{i_1\ldots i_m j_1\ldots j_m} = \mathcal{H}_{k_1\ldots k_m l_1\ldots l_m}$ for all labels such that $\{i_s, j_s\} = \{k_s, l_s\}, s = 1, \ldots, m$. The subspace of $\mathbb{R}$-Hermitian decomposable tensors in $\mathbb{R}^{[n_1,\ldots,n_m]}$ is denoted as $\mathbb{R}_D^{[n_1,\ldots,n_m]}$. Note that the dimension of $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ is $\prod_{k=1}^m \binom{n_k+1}{2}$, and the dimension of $\mathbb{R}^{[n_1,\ldots,n_m]}$ is $\binom{N+1}{2}$ with $N := n_1 \cdots n_m$. Thus, for $m > 1$ and $n_i > 1$, $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ is a proper subspace of $\mathbb{R}^{[n_1,\ldots,n_m]}$ (cf. [25]).

The set $\mathbb{C}^{[n_1,\ldots,n_m]}$ is a vector space over $\mathbb{R}$. For Hermitian tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, by the definition of inner product (1.1), it always holds that $\langle \mathcal{A}, \mathcal{B} \rangle$ is real. Hermitian decompositions (1.6) can be equivalently expressed by conjugate polynomials. For complex variables $x_k \in \mathbb{C}^{n_k}, k \in [m]$, denote $\mathbf{x} := (x_1, \ldots, x_m)$. Given a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, we denote by

$$\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}}) := \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle, \tag{1.8}$$

which is a conjugate symmertric polynomial in $\mathbf{x}$, i.e., $\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}}) = \overline{\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}})}$. The Hermitian decomposition (1.6) implies that

$$\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}}) = \sum_{i=1}^{r} \lambda_i |(u_1^i)^* x_1|^2 \cdots |(u_m^i)^* x_m|^2. \tag{1.9}$$

Hermitian tensors play an essential role in quantum physics. A Hermitian tensor can represent a mixed state, and the separability discrimination problem of mixed states can be regarded as the positive Hermitian decomposition problem of Hermitian tensors. Recently, Nie and Yang [25] studied some basic properties for Hermitian tensors, such as Hermitian decompositions and Hermitian ranks. For more results on Hermitian tensor and its applications, we refer the reader to [18, 19, 25] and the references therein.

In this paper, motivated by the important applications for CP tensor and Hermitian tensor, we introduce the (real) Hermitian completely positive tensors. They are different extensions of the even order CP tensors in the complex field. And it is important to study the properties of (real) Hermitian completely positive tensors both in academic and physics applications such as quantum entanglement problem. Throughout the paper, a complex vector is written as $w = u + \sqrt{-1}v$, where u and v are its real and imaginary parts, respectively.

Definition 1.1. A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is called *Hermitian completely positive (HCP)* if there exist vectors $w_j^i \in \mathbb{C}_+^{n_j}$ for all $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} [w_1^i, \ldots, w_m^i]_{\otimes h}, \tag{1.10}$$

where r is called the length of the decomposition (1.10). The smallest r in the above is called the HCP-rank of $\mathcal{H}$, which we denote as hcprank($\mathcal{H}$). If $\mathcal{H}$ is Hermitian completely positive, (1.10) is called a *Hermitian completely positive (HCP) decomposition* of $\mathcal{H}$.

Similarly, we can give the following definitions of real Hermitian completely positive tensors and (real) Hermitian copositive tensors.

Definition 1.2. A Hermitian tensor $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ is called *real Hermitian completely positive (RHCP)* if there exist nonnegative vectors $u_j^i \in \mathbb{R}_+^{n_j}$ for all $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} [u_1^i, \ldots, u_m^i]_{\otimes h}, \tag{1.11}$$

where r is called the length of the decomposition (1.11). The smallest r in the above is called the RHCP-rank of $\mathcal{H}$, denoted as $\text{hcprank}_R(\mathcal{H})$. If $\mathcal{H}$ is real Hermitian completely positive, (1.11) is called a *real Hermitian completely positive (RHCP) decomposition* of $\mathcal{H}$.

Definition 1.3. A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ (resp., $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]}$) is called *Hermitian copositive (HCOP)* (resp., *real Hermitian copositive (RHCOP)*) if $\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}}) \geq 0$ for all $\mathbf{x} := (x_1, \ldots, x_m)$ with $x_k \in \mathbb{C}_+^{n_k}$ (resp., with $x_k \in \mathbb{R}_+^{n_k}$), $k \in [m]$.

We denote by

$$\begin{aligned}
&\mathbf{HCP}_{[n_1,\ldots,n_m]} = \{\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]} : \mathcal{H} \text{ is Hermitian completely positive}\},\\
&\mathbf{HCOP}_{[n_1,\ldots,n_m]} = \{\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]} : \mathcal{H} \text{ is Hermitian copositive}\},\\
&\mathbf{RHCP}_{[n_1,\ldots,n_m]} = \{\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]} : \mathcal{H} \text{ is real Hermitian completely positive}\},\\
&\mathbf{RHCOP}_{[n_1,\ldots,n_m]} = \{\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]} : \mathcal{H} \text{ is real Hermitian copositive}\}.
\end{aligned}$$

For the case of $m = 1$, the HCP and HCOP tensors are reduced to the HCP and HCOP matrices, respectively. These concepts are introduced and studied by Zhou et al.[38]. For $m = 2$, the RHCP tensors are reduced to the completely positive separable matrices introduced by Zhou and Fan [36]. It is easy to check that $\mathbf{RHCP}_{[n_1,\ldots,n_m]} \subseteq \mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCP}_{[n_1,\ldots,n_m]} \subseteq \mathbf{HCOP}_{[n_1,\ldots,n_m]}$.

Contributions. In this paper, we study (real) Hermitian completely positive tensors and (real) Hermitian copositive tensors. Some properties of (real) Hermitian completely positive tensors cones and their dual cones are presented. We prove that the cone of Hermitian completely positive tensors is dual to the cone of Hermitian copositive tensors, and both of them are proper cones. For a Hermitian completely positive tensor, its conjugate tensor and real part tensor are both Hermitian completely positive. For a Hermitian copositive tensor, its real part must be real Hermitian copositive. However, the cone of real Hermitian completely positive tensors and its dual cone are not proper, generally. For a $\mathbb{R}$-Hermitian decomposable tensor, we show that it is Hermitian completely positive (resp., Hermitian copositive) if and only if it is real Hermitian completely positive (resp., real Hermitian copositive).

We prove that the Hermitian copositive tensor checking problem is equivalent to a conjugate symmertric polynomial optimization problem, which can be solved by a linear conic programming problem with the cone of Hermitian completely positive tensors. Hermitian completely positive tensors can be characterized in terms of truncated moment sequences. A semidefinite algorithm is also proposed for checking whether a Hermitian tensor is Hermitian completely positive or not. If it is, an HCP decomposition can be further obtained.

The paper is organized as follows. Section 2 studies the properties of (real) Hermitian completely positive tensors and (real) Hermitian copositive tensors. In Section 3, we study the certificate and decompositions for Hermitian completely positive tensors and Hermitian copositive tensors. A semidefinite algorithm is proposed for checking Hermitian completely positive tensors. Convergence properties of the algorithm are also discussed. Some numerical examples are illustrated in Section 4. Finally, we conclude the paper in Section 5.

Notation. The symbol $\mathbb{N}$ denotes the set of nonnegative integers. Let $\mathbb{R}^n_+$ and $\mathbb{C}^n_+$ be the nonnegative orthant and the set of complex vectors whose real and imaginary parts are both nonnegative, respectively. For all $k \in [m]$, x_k denotes the complex vector in $\mathbb{C}^{n_k}$. The tuple of all such complex vectors is denoted as $\mathbf{x} := (x_1, \ldots, x_m)$. Let $\mathcal{I} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ be the identity tensor, i.e., $\mathcal{I}(\mathbf{x}, \overline{\mathbf{x}}) = (x_1^* x_1) \cdots (x_m^* x_m)$. For a vector u in $\mathbb{F}^n$, $\|u\|$ denotes its standard Euclidean norm. For $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote the monomial power $\tilde{x}^\alpha := \tilde{x}_1^{\alpha_1} \cdots \tilde{x}_n^{\alpha_n}$. Let $\mathbb{R}[\tilde{x}] := \mathbb{R}[\tilde{x}_1, \ldots, \tilde{x}_n]$ be the ring of real polynomials in $\tilde{x}$ with coefficients in $\mathbb{R}$, and $\mathbb{R}[\tilde{x}]_d$ be the set of polynomials in $\mathbb{R}[\tilde{x}]$ with degrees at most d. The $\deg(p)$ denotes the degree of a polynomial p. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n \mid |\alpha| \leq d\}$. Let $\mathbb{R}^{\mathbb{N}^n_d}$ be the set of real vectors indexed by $\alpha \in \mathbb{N}^n_d$. For $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer not smaller than t. For a symmetric matrix X, $X \succeq 0$ means that X is positive semidefinite. The symbol $\otimes$ denotes the tensor product, while $\odot$ denotes the classical Kronecker product.

2 Properties of HCP and RHCP Tensors

In this section, we study properties of the cones of HCP tensors and RHCP tensors in the Hermitian tensor space $\mathbb{C}^{[n_1,\ldots,n_m]}$ over real field $\mathbb{R}$. Throughout the paper, we denote by $N := \prod_{i=1}^m n_i$ and $n := \sum_{i=1}^m n_i$ on the space $\mathbb{C}^{[n_1,\ldots,n_m]}$. We define the linear map

$\mathfrak{M} : \mathbb{C}^{[n_1,\ldots,n_m]} \to \mathbb{M}^N$ such that for all $w_i \in \mathbb{C}^{n_i}, i \in [m]$,

$$\mathfrak{M}([w_1, \ldots, w_m]_{\otimes h}) = (w_1 w_1^*) \odot (w_2 w_2^*) \odot \cdots \odot (w_m w_m^*), \tag{2.1}$$

where $\odot$ denotes the classical Kronecker product. The map $\mathfrak{M}$ is a bijection between $\mathbb{C}^{[n_1,\ldots,n_m]}$ and $\mathbb{M}^N \cong \mathbb{M}^{n_1} \odot \cdots \odot \mathbb{M}^{n_m}$. Then, the Hermitian decomposition $\mathcal{H} = \sum_{i=1}^{r} \lambda_i [w_i^1, \ldots, w_i^m]_{\otimes h}$ is equivalent to

$$\begin{aligned} \mathfrak{M}(\mathcal{H}) &= \sum_{i=1}^{r} \lambda_i (w_i^1 (w_i^1)^*) \odot \cdots \odot (w_i^m (w_i^m)^*) \\ &= \sum_{i=1}^{r} \lambda_i (w_i^1 \odot \cdots \odot w_i^m)(w_i^1 \odot \cdots \odot w_i^m)^*. \end{aligned} \tag{2.2}$$

The matrix $\mathfrak{M}(\mathcal{H})$ is called the Hermitian flattening matrix of $\mathcal{H}$ (cf. [25]).

2.1 Properties of HCP and HCOP tensors

Recall that a cone is said to be solid if it has a nonempty interior; it is said to be pointed if it does not contain any line through the origin; a cone is called proper if it is closed, convex, pointed and solid. Then, we have the following properties for the cones of HCP and HCOP tensors.

Proposition 2.1. *In the space $\mathbb{C}^{[n_1,\ldots,n_m]}$ over real field $\mathbb{R}$, the cones $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ are dual to each other, i.e.,*

$$\mathbf{HCOP}_{[n_1,\ldots,n_m]} = \mathbf{HCP}^*_{[n_1,\ldots,n_m]},\ \mathbf{HCP}_{[n_1,\ldots,n_m]} = \mathbf{HCOP}^*_{[n_1,\ldots,n_m]}. \tag{2.3}$$

Furthermore, they are both proper cones.

Proof. By the definition, it is easy to verify that $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is closed and convex. Note that $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ equals the conic hull of the compact set

$$W := \left\{[w_1, \ldots, w_m]_{\otimes h} : w_i \in \mathbb{C}_+^{n_i}, \|w_i\| = 1, \forall i \in [m]\right\}. \tag{2.4}$$

So it is also a closed convex cone [2].

Next, we show that the dual relationship (2.3) holds. A tensor $\mathcal{X} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ belongs to the dual cone of $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ if and only if $\langle \mathcal{X}, \mathcal{Y} \rangle \geq 0$ for all $\mathcal{Y} \in W$, which is equivalent to the fact that $\mathcal{X}$ is Hermitian copositive. Therefore, the dual cone of $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ is $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$. Since both $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ are closed convex cones, the dual cone of $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is equal to $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ by the biduality theorem [2]. Hence, $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ are dual to each other.

To show $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is solid, we need to prove that it has nonempty interior. Let $\mathcal{I} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ be the identity tensor. Clearly, the conjugate polynomial $\mathcal{I}(\mathbf{x}, \overline{\mathbf{x}})$ is one for all $x_k \in \mathbb{C}_+^{n_k}, \|x_k\| = 1$. Thus, $\mathcal{I} \in \mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is an interior point, and $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is solid. By duality, we also know that $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ is pointed.

Now, we prove $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ is solid, i.e., full-dimensional. By [38, Proposition 2.4], for each HCP matrix cone $\mathbf{HCP}_{[n_i]}, i \in [m]$, its dimension $dim\mathbf{HCP}_{[n_i]} = n_i^2$. Note that Hermitian flattening $\mathfrak{M}$ is a bijection between the cones $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCP}_{[n_1]} \odot \cdots \odot \mathbf{HCP}_{[n_m]}$. Thus, these two cones have the same dimension. In other words, we have $dim\mathbf{HCP}_{[n_1,\ldots,n_m]} = \prod_{i=1}^m n_i^2$, which is equal to the dimension of the Hermitian tensor space

$\mathbb{C}^{[n_1,\ldots,n_m]}$ over real field $\mathbb{R}$. This implies that $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ is full-dimensional. By duality, we also have $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ is pointed.

Therefore, the cones $\mathbf{HCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$ are both proper and dual to each other. This completes the proof. □

From the definition of HCP tensor, we have the following results on its conjugate tensor and real part tensor.

Proposition 2.2. *For a Hermitian completely positive tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, its conjugate $\overline{\mathcal{H}}$ is also Hermitian completely positive. Furthermore, its real part $\mathcal{H}^{re}$ is also Hermitian completely positive.*

Proof. Since $\mathcal{H}$ is Hermitian completely positive, there exist $w_j^i = u_j^i + \sqrt{-1}v_j^i \in \mathbb{C}_+^{n_j}$ with $u_j^i, v_j^i \in \mathbb{R}_+^{n_j}$ for all $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} [w_1^i, \ldots, w_m^i]_{\otimes h}.$$

Then we have

$$\begin{aligned}
\overline{\mathcal{H}} &= \sum_{i=1}^{r} [\overline{w_1^i}, \ldots, \overline{w_m^i}]_{\otimes h} \\
&= \sum_{i=1}^{r} [u_1^i - \sqrt{-1}v_1^i, \ldots, u_m^i - \sqrt{-1}v_m^i]_{\otimes h} \\
&= \sum_{i=1}^{r} (-\sqrt{-1})^m (\sqrt{-1})^m (v_1^i + \sqrt{-1}u_1^i) \otimes \cdots \otimes (v_m^i + \sqrt{-1}u_m^i) \\
&\quad \otimes (v_1^i - \sqrt{-1}u_1^i) \otimes \cdots \otimes (v_m^i - \sqrt{-1}u_m^i) \\
&= \sum_{i=1}^{r} (-\sqrt{-1})^m (\sqrt{-1})^m [v_1^i + \sqrt{-1}u_1^i, \ldots, v_m^i + \sqrt{-1}u_m^i]_{\otimes h} \\
&= \sum_{i=1}^{r} [v_1^i + \sqrt{-1}u_1^i, \ldots, v_m^i + \sqrt{-1}u_m^i]_{\otimes h}.
\end{aligned}$$

Thus, $\overline{\mathcal{H}}$ is Hermitian completely positive. Note that the real part of $\mathcal{H}$ can be expressed as

$$\mathcal{H}^{re} = \frac{\mathcal{H} + \overline{\mathcal{H}}}{2}.$$

This implies that $\mathcal{H}^{re}$ has the following HCP decomposition:

$$\mathcal{H}^{re} = \frac{1}{2} \sum_{i=1}^{r} \left([w_1^i, \ldots, w_m^i]_{\otimes h} + [v_1^i + \sqrt{-1}u_1^i, \ldots, v_m^i + \sqrt{-1}u_m^i]_{\otimes h} \right). \tag{2.5}$$

Therefore, $\mathcal{H}^{re}$ is also Hermitian completely positive. This completes the proof. □

Remark 2.3. Assume a Hermitian tensor $\mathcal{H}$ is HCP and has the HCP decomposition

$$\mathcal{H} = \sum_{i=1}^{r} [w_1^i, \ldots, w_m^i]_{\otimes h},$$

where $w_j^i = u_j^i + \sqrt{-1}v_j^i \in \mathbb{C}_+^{n_j}$ for all $i \in [r], j \in [m]$. From the proof of Proposition 2.2, it is clear that, for any vectors u_j^i and v_j^i in the HCP decomposition above changed to $-u_j^i$ or $-v_j^i$, the new tensor obtained is also HCP. This result is also verified by Example 4.7 in Section 4.

Recall that for a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\dots,n_m]}$, its real part $\mathcal{H}^{re}$ is real Hermitian and its imaginary part $\mathcal{H}^{im}$ is real skew-symmetric. Then we have the following results for HCOP tensors.

Proposition 2.4. *For a Hermitian copositive tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\dots,n_m]}$ with real part $\mathcal{H}^{re} \in \mathbb{R}^{[n_1,\dots,n_m]}$ and imaginary part $\mathcal{H}^{im} \in \mathbb{R}^{[n_1,\dots,n_m]}_{skew}$, $\mathcal{H}^{re}$ is real Hermitian copositive.*

Proof. Since $\mathcal{H}$ is Hermitian copositive, then it holds that

$$\begin{aligned}\langle \mathcal{H}, [x_1,\dots,x_m]_{\otimes h}\rangle &= \langle \mathcal{H}^{re} + \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle \\ &= \langle \mathcal{H}^{re}, [x_1,\dots,x_m]_{\otimes h}\rangle + \langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle \geq 0,\end{aligned}$$

for all $x_k \in \mathbb{R}^{n_k}_+$, $k \in [m]$. Note that $\mathcal{H}$ is Hermitian, then $\sqrt{-1}\mathcal{H}^{im}$ is also Hermitian and the inner product $\langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle$ must be real. Thus,

$$\begin{aligned}\langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle &= \overline{\langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle} \\ &= \langle -\sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle \\ &= -\langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle.\end{aligned}$$

This implies that $\langle \sqrt{-1}\mathcal{H}^{im}, [x_1,\dots,x_m]_{\otimes h}\rangle = 0$. So we have

$$\langle \mathcal{H}, [x_1,\dots,x_m]_{\otimes h}\rangle = \langle \mathcal{H}^{re}, [x_1,\dots,x_m]_{\otimes h}\rangle \geq 0,$$

for all $x_k \in \mathbb{R}^{n_k}_+$, $k \in [m]$. Thus, $\mathcal{H}^{re}$ is real Hermitian copositive. □

Remark 2.5. Proposition 2.4 is an extension for Hermitian copositive tensors of Proposition 2.5 in [38]. As it was shown in [38, Proposition 2.1], if a matrix A is HCP, then its real part must be a CP matrix. However, this result does not hold in the case of $\mathbf{HCP}_{[n_1,\dots,n_m]}$ for $m > 1$. In other words, for a tensor $\mathcal{H} \in \mathbf{HCP}_{[n_1,\dots,n_m]}$ with $m > 1$, its real part $\mathcal{H}^{re}$ may not be real Hermitian completely positive. For instance, consider the following tensor in $\mathbf{HCP}_{[2,2]}$

$$\mathcal{H} = \left[\begin{pmatrix} 1+\sqrt{-1} \\ 0+\sqrt{-1}\end{pmatrix}, \begin{pmatrix} \frac{1}{2}+\sqrt{-1} \\ 1+\frac{\sqrt{-1}}{2}\end{pmatrix}\right]_{\otimes h}.$$

Its real part $\mathcal{H}^{re}$ is given as following:

$$\begin{aligned}\mathcal{H}^{re}(:,:,1,1) &= \tfrac{1}{4}\begin{pmatrix} 10 & 8 \\ 5 & 7\end{pmatrix}, & \mathcal{H}^{re}(:,:,2,1) &= \tfrac{1}{4}\begin{pmatrix} 5 & 1 \\ 5 & 4\end{pmatrix}, \\ \mathcal{H}^{re}(:,:,1,2) &= \tfrac{1}{4}\begin{pmatrix} 8 & 10 \\ 1 & 5\end{pmatrix}, & \mathcal{H}^{re}(:,:,2,2) &= \tfrac{1}{4}\begin{pmatrix} 7 & 5 \\ 4 & 5\end{pmatrix}.\end{aligned}$$

It is clear that $\mathcal{H}^{re}$ is not $\mathbb{R}$-Hermitian decomposable, because $\mathcal{H}^{re}_{1122} = 7/4 \neq 1/4 = \mathcal{H}^{re}_{2112}$. This implies that $\mathcal{H}^{re}$ is not RHCP. However, as shown in Proposition 2.2, we know $\mathcal{H}^{re}$ must be HCP and it has the following HCP decomposition:

$$\mathcal{H}^{re} = \frac{15}{4}\left(\left[\begin{pmatrix} \frac{1+\sqrt{-1}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{4+3\sqrt{-1}}{5\sqrt{2}}\end{pmatrix}\right]_{\otimes h} + \left[\begin{pmatrix} \frac{1+\sqrt{-1}}{\sqrt{3}} \\ \frac{\sqrt{-1}}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix} \frac{4+3\sqrt{-1}}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix}\right]_{\otimes h}\right).$$

This example also illustrates that there exists a real Hermitian tensor $\mathcal{A} \in \mathbb{R}^{[n_1,\dots,n_m]} \setminus \mathbb{R}^{[n_1,\dots,n_m]}_D$ such that $\mathcal{A}$ is HCP.

For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, let k be an integer with $0 \le k < m$. Given indices $r_j \in [n_j]$ for all $j \in [m] \setminus [k]$, we define the partial tensor $\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m} \in \mathbb{C}^{[n_1,\ldots,n_k]}$ whose elements are given as following:

$$(\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m})_{i_1\ldots i_k j_1 \ldots j_k} = \mathcal{H}_{i_1\ldots i_k r_{k+1}\ldots r_m j_1\ldots j_k r_{k+1}\ldots r_m}, \forall i_l \in [n_l], l \in [k].$$

Then, we have $\mathcal{H}_{[0]}^{r_1,\ldots,r_m} = \mathcal{H}_{r_1\ldots r_m r_1 \ldots r_m}$. It is clear that if $\mathcal{H}$ is Hermitian, so is the partial tensor $\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m}$. In fact, this property also holds for the Hermitian completely positive tensors.

Proposition 2.6. *For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, let k be an integer with $0 \le k < m$. For any fixed indices $r_j \in [n_j]$ for all $j \in [m] \setminus [k]$, if $\mathcal{H}$ is Hermitian completely positive,so is the partial tensor $\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m}$.*

Proof. Since $\mathcal{H}$ is Hermitian completely positive, then there exist nonnegative vectors $w_j^i \in \mathbb{C}_+^{n_j}$ for $i \in [r], j \in [m]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} [w_1^i, \ldots, w_m^i]_{\otimes h}.$$

By the definition of the partial tensor $\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m}$, we have

$$\begin{aligned} \mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m} &= \sum_{i=1}^{r} w_1^i \otimes \cdots \otimes w_k^i \otimes (w_{k+1}^i)_{r_{k+1}} \otimes \cdots \otimes (w_m^i)_{r_m} \\ &\quad \otimes \overline{w_1^i} \otimes \cdots \otimes \overline{w_k^i} \otimes (\overline{w_{k+1}^i})_{r_{k+1}} \otimes \cdots \otimes (\overline{w_m^i})_{r_m} \\ &= \sum_{i=1}^{r} (\textstyle\prod_{j=k+1}^{m} |(w_j^i)_{r_j}|^2) w_1^i \otimes \cdots \otimes w_k^i \otimes \overline{w_1^i} \otimes \cdots \otimes \overline{w_k^i} \\ &= \sum_{i=1}^{r} (\textstyle\prod_{j=k+1}^{m} |(w_j^i)_{r_j}|^2) [w_1^i, \ldots, w_k^i]_{\otimes h}. \end{aligned}$$

Note that $\prod_{j=k+1}^{m} |(w_j^i)_{r_j}|^2 \ge 0$ for $i = 1, \ldots, r$. This implies that the partial tensor $\mathcal{H}_{[k]}^{r_{k+1},\ldots,r_m}$ is also an HCP tensor. □

2.2 Properties of RHCP and RHCOP tensors

Now, we show some properties on the cones of RHCP and RHCOP tensors.

Proposition 2.7. *For $m > 1$ and $n_1, \ldots, n_m > 1$, in the space $\mathbb{R}^{[n_1,\ldots,n_m]}$ over the real field $\mathbb{R}$, the cones $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ are both closed and convex, and they are dual to each other, i.e.,*

$$\begin{aligned} \left(\mathbf{RHCP}_{[n_1,\ldots,n_m]}\right)^* = \mathbf{RHCOP}_{[n_1,\ldots,n_m]}, \\ \left(\mathbf{RHCOP}_{[n_1,\ldots,n_m]}\right)^* = \mathbf{RHCP}_{[n_1,\ldots,n_m]}. \end{aligned}$$

Moreover, $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$ is pointed but not solid, while $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ is solid but not pointed.

Proof. For the first part, it can be similarly deduced from Proposition 2.1 that the cones $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$ and $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ are both closed and convex, and they are dual to each other. It is also easy to check that the identity tensor $\mathcal{I}$ is an interior point of the cone $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$. Thus, $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ is solid.

For $m > 1$ and $n_1, \ldots, n_m > 1$, it is clear that $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ is a proper subspace of $\mathbb{R}^{[n_1,\ldots,n_m]}$. Note that the set $\mathbf{RHCP}_{[n_1,\ldots,n_m]} \subseteq \mathbb{R}_D^{[n_1,\ldots,n_m]}$. So there exist $0 \neq \mathcal{Y} \in \mathbb{R}^{[n_1,\ldots,n_m]}$ that is orthogonal to the subspace $\mathbb{R}_D^{[n_1,\ldots,n_m]}$. Then for all $\mathcal{X} \in \mathbf{RHCP}_{[n_1,\ldots,n_m]}$ and $\lambda \in \mathbb{R}$, we have $\langle \lambda\mathcal{Y}, \mathcal{X}\rangle = 0$. This implies that $\lambda\mathcal{Y} \in \mathbf{RHCOP}_{[n_1,\ldots,n_m]}$, i.e., $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$ contains a line through the origin. Therefore, it is not pointed.

By the duality, $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$ is pointed but not solid [2]. The proof is completed. □

Proposition 2.8. *For $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$,$\mathcal{H}$ is RHCOP if and only if $\mathcal{H}$ is HCOP.*

Proof. The "if" direction is clear. Now, we prove the "only if" direction. For $w_j \in \mathbb{R}^{n_j}$ and $v_j \in \mathbb{C}_+^{n_j}$, write $v_j = x_j + \sqrt{-1}y_j$ with $x_j, y_j \in \mathbb{R}_+^{n_j}$. Then, we have

$$\langle [w_1,\ldots,w_m]_{\otimes h}, [v_1,\ldots,v_m]_{\otimes h}\rangle = \prod_{j=1}^m (w_j)^T v_j (w_j)^T \overline{v}_j = \prod_{j=1}^m |(w_j)^T v_j|^2$$

$$= \prod_{j=1}^m |(w_j)^T x_j|^2 + |(w_j)^T y_j|^2 = \sum_{z_j \in \{x_j, y_j\}} \langle [w_1,\ldots,w_m]_{\otimes h}, [z_1,\ldots,z_m]_{\otimes h}\rangle.$$

Note that $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, it is a sum of real multiples of rank-1 real Hermitian tensors. Thus,

$$\langle \mathcal{H}, [v_1,\ldots,v_m]_{\otimes h}\rangle = \sum_{z_j \in \{x_j, y_j\}} \langle \mathcal{H}, [z_1,\ldots,z_m]_{\otimes h}\rangle.$$

If $\mathcal{H}$ is RHCOP, then we have

$$\langle \mathcal{H}, [v_1,\ldots,v_m]_{\otimes h}\rangle = \sum_{z_j \in \{x_j, y_j\}} \langle \mathcal{H}, [z_1,\ldots,z_m]_{\otimes h}\rangle \geq 0$$

for all $v_j = x_j + \sqrt{-1}y_j \in \mathbb{C}_+^{n_j}$, with $x_j, y_j \in \mathbb{R}_+^{n_j}, j \in [m]$. Thus, $\mathcal{H}$ is HCOP. □

Remark 2.9. Note that if $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]} \setminus \mathbb{R}_D^{[n_1,\ldots,n_m]}$, then $\mathcal{H}$ may be RHCOP but not HCOP. Consider $\mathcal{H} \in \mathbb{R}^{[2,2]}$ such that

$$\mathcal{H}_{1111} = \mathcal{H}_{1122} = \mathcal{H}_{2211} = 1, \mathcal{H}_{1221} = \mathcal{H}_{2112} = -1,$$

and all other entries are zeros. It was shown in [25] that $\langle \mathcal{H}, [v_1, v_2]_{\otimes h}\rangle \geq 0$ for all vectors $v_1, v_2 \in \mathbb{R}^2$. So, $\mathcal{H}$ is RHCOP. However, $\mathcal{H}$ is not HCOP, because for $w_1 = w_2 = (\sqrt{-1}, 1)^T \in \mathbb{C}_+^2$, $\langle \mathcal{H}, [w_1, w_2]_{\otimes h}\rangle = -3 < 0$. Thus, it holds that

$$\begin{aligned}\mathbf{RHCOP}_{[n_1,\ldots,n_m]} &\supseteq \mathbb{R}^{[n_1,\ldots,n_m]} \cap \mathbf{HCOP}_{[n_1,\ldots,n_m]} \\ &\supseteq \mathbb{R}_D^{[n_1,\ldots,n_m]} \cap \mathbf{HCOP}_{[n_1,\ldots,n_m]} \\ &= \mathbb{R}_D^{[n_1,\ldots,n_m]} \cap \mathbf{RHCOP}_{[n_1,\ldots,n_m]}.\end{aligned}$$

The relationships among the sets $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}, \mathbf{HCOP}_{[n_1,\ldots,n_m]}, \mathbb{R}^{[n_1,\ldots,n_m]}$ and $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ are shown in Figure 1. For convenience, we use $\mathbf{RHCOP}, \mathbf{HCOP}, \mathbf{R}$ and $\mathbf{R_D}$ to represent the sets $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$, $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$, $\mathbb{R}^{[n_1,\ldots,n_m]}$ and $\mathbb{R}_D^{[n_1,\ldots,n_m]}$, respectively.

Proposition 2.10. *For $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, $\mathcal{H}$ is RHCP if and only if $\mathcal{H}$ is HCP.*

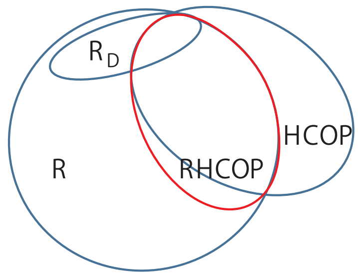

Figure 1: The relationships among the sets $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}, \mathbf{HCOP}_{[n_1,\ldots,n_m]}, \mathbb{R}^{[n_1,\ldots,n_m]}$ and $\mathbb{R}_D^{[n_1,\ldots,n_m]}$.

Proof. The "only if" direction is obvious. Now, we prove the "if" direction. Assume that $\mathcal{H}$ is HCP. Then (1.10) holds for some complex vectors $w_j^i = u_j^i + \sqrt{-1}v_j^i \in \mathbb{C}_+^{n_j}$ with $u_j^i, v_j^i \in \mathbb{R}_+^{n_j}$ for all $i \in [r], j \in [m]$. For all real vectors $x_j \in \mathbb{R}^{n_j}$, the inner product $\langle [w_1^i, \ldots, w_m^i]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h}\rangle = \prod_{j=1}^m |(w_j^i)^* x_j|^2$, which can be expanded as

$$\prod_{j=1}^m |(u_j^i)^T x_j|^2 + |(v_j^i)^T x_j|^2 = \sum_{z_j^i \in \{u_j^i, v_j^i\}} \langle [z_1^i, \ldots, z_m^i]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h}\rangle.$$

Then (1.10) implies that for all real vectors $x_j \in \mathbb{R}^{n_j}$,

$$\langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h}\rangle = \sum_{i=1}^r \sum_{z_j^i \in \{u_j^i, v_j^i\}} \langle [z_1^i, \ldots, z_m^i]_{\otimes h}, [x_1, \ldots, x_m]_{\otimes h}\rangle.$$

Since $\mathcal{H} \in \mathbb{R}_D^{[n_1,\ldots,n_m]}$, we have

$$\|\mathcal{H} - \sum_{i=1}^r \sum_{z_j^i \in \{u_j^i, v_j^i\}} [z_1^i, \ldots, z_m^i]_{\otimes h}\| = 0,$$

i.e., $\mathcal{H} = \sum_{i=1}^r \sum_{z_j^i \in \{u_j^i, v_j^i\}} [z_1^i, \ldots, z_m^i]_{\otimes h}$ with $u_j^i, v_j^i \in \mathbb{R}_+^{n_j}$ for $i \in [r], j \in [m]$. Thus, $\mathcal{H}$ is also RHCP. □

Remark 2.11. Note that if $\mathcal{H} \in \mathbb{R}^{[n_1,\ldots,n_m]} \setminus \mathbb{R}_D^{[n_1,\ldots,n_m]}$, then $\mathcal{H}$ must not be RHCP. Furthermore, as shown in Remark 2.5, it holds that

$$\begin{aligned} \mathbf{RHCP}_{[n_1,\ldots,n_m]} &= \mathbb{R}_D^{[n_1,\ldots,n_m]} \cap \mathbf{HCP}_{[n_1,\ldots,n_m]} \\ &\subseteq \mathbb{R}^{[n_1,\ldots,n_m]} \cap \mathbf{HCP}_{[n_1,\ldots,n_m]}. \end{aligned}$$

Combining with

$$\begin{aligned} &\mathbf{RHCP}_{[n_1,\ldots,n_m]} \subseteq \mathbf{RHCOP}_{[n_1,\ldots,n_m]}, \\ &\mathbf{HCP}_{[n_1,\ldots,n_m]} \subseteq \mathbf{HCOP}_{[n_1,\ldots,n_m]}, \end{aligned}$$

we further have

$$\begin{aligned} \mathbf{RHCP}_{[n_1,\ldots,n_m]} &= \mathbb{R}_D^{[n_1,\ldots,n_m]} \cap \mathbf{HCP}_{[n_1,\ldots,n_m]} \\ &\subseteq \mathbb{R}^{[n_1,\ldots,n_m]} \cap \mathbf{HCP}_{[n_1,\ldots,n_m]} (\text{or } \mathbb{R}_D^{[n_1,\ldots,n_m]} \cap \mathbf{HCOP}_{[n_1,\ldots,n_m]}) \\ &\subseteq \mathbb{R}^{[n_1,\ldots,n_m]} \cap \mathbf{HCOP}_{[n_1,\ldots,n_m]} \\ &\subseteq \mathbf{RHCOP}_{[n_1,\ldots,n_m]}. \end{aligned}$$

The relationships among the sets $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$, $\mathbf{HCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$, $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ and $\mathbb{R}^{[n_1,\ldots,n_m]}$ are shown in Figure 2. For convenience, we use **HCOP**, **HCP**, **RHCP**, **RHCOP**, $\mathbf{R_D}$ and **R** to represent the sets $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$, $\mathbf{HCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$, $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ and $\mathbb{R}^{[n_1,\ldots,n_m]}$, respectively.

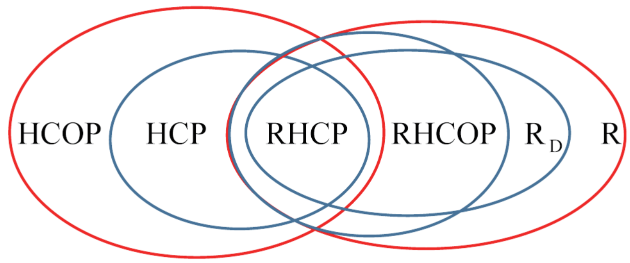

Figure 2: The relationships among the sets $\mathbf{HCOP}_{[n_1,\ldots,n_m]}$, $\mathbf{HCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCP}_{[n_1,\ldots,n_m]}$, $\mathbf{RHCOP}_{[n_1,\ldots,n_m]}$, $\mathbb{R}_D^{[n_1,\ldots,n_m]}$ and $\mathbb{R}^{[n_1,\ldots,n_m]}$.

3 Certificate and Decomposition for HCP and HCOP Tensors

In this section, we show how to check whether a Hermitian tensor is Hermitian completely positive or Hermitian copositive.

3.1 Certificate for HCOP tensors

Given a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, a natural question is how to certify it is Hermitian copositive or not? Consider the conjugate symmertric polynomial optimization problem

$$\begin{cases} p_1^* = \min & \mathcal{H}(\mathbf{x}, \overline{\mathbf{x}}) := \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle \\ \quad\quad s.t. & \|x_j\| = 1, x_j \in \mathbb{C}_+^{n_j}, j = 1, \ldots, m, \end{cases} \tag{3.1}$$

where $\mathcal{H}(\mathbf{x}, \overline{\mathbf{x}})$ is conjugate quadratic in each x_j. It has wide applications in signal processing and wireless communications [27, 31, 33, 34]. By the definition of HCOP tensors, it is clear that $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCOP if and only if the optimal value $p_1^* \geq 0$ for the optimization problem (3.1). In fact, we can also consider the linear conic optimization

$$\begin{cases} p_2^* = \min & \langle \mathcal{H}, \mathcal{X} \rangle \\ \quad\quad s.t. & \langle \mathcal{I}, \mathcal{X} \rangle = 1, \mathcal{X} \in \mathbf{HCP}_{[n_1,\ldots,n_m]}, \end{cases} \tag{3.2}$$

where $\mathcal{I}$ denotes the identity tensor in $\mathbb{C}^{[n_1,\ldots,n_m]}$. We call (3.2) the standard Hermitian completely positive programming. By the duality of the cones of HCP and HCOP tensors, we know that $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCOP if and only if the optimal value $p_2^* \geq 0$ for (3.2). The following theorem shows that the problems (3.1) and (3.2) are equivalent.

Theorem 3.1. *Given a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, the problems* (3.1) *and* (3.2) *are equivalent, so $p_1^* = p_2^*$.*

Proof. Since $\mathcal{I}$ is the identity tensor in $\mathbb{C}^{[n_1,\ldots,n_m]}$, we have $\mathcal{I}(\mathbf{x}, \overline{\mathbf{x}}) = (x_1^* x_1) \cdots (x_m^* x_m)$. Then, the problem (3.1) is equivalent to

$$\begin{cases} \widetilde{p}^* = \min & \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle \\ \quad\quad \text{s.t.} & \langle \mathcal{I}, [x_1, \ldots, x_m]_{\otimes h} \rangle = 1, \\ & \|x_j\| = 1, x_j \in \mathbb{C}_+^{n_j}, j = 1, \ldots, m. \end{cases} \tag{3.3}$$

Note that the problem (3.2) is a relaxation for the problem (3.3). So, $p_1^* = \widetilde{p}^* \geq p_2^*$.

Suppose $\mathcal{X}^*$ is an optimizer of (3.2). There exist $\rho_i > 0$ and $w_j^i \in \mathbb{C}_+^{n_j}$ with $\|w_j^i\| = 1$ for all $j \in [m], i \in [r]$ such that

$$\mathcal{X}^* = \sum_{i=1}^{r} \rho_i [w_1^i, \ldots, w_m^i]_{\otimes h}. \tag{3.4}$$

We order such that for all $1 \leq i \leq r-1$,

$$\langle \mathcal{H}, [w_1^i, \ldots, w_m^i]_{\otimes h} \rangle \leq \langle \mathcal{H}, [w_1^{i+1}, \ldots, w_m^{i+1}]_{\otimes h} \rangle.$$

Note that for each $w_j^i \in \mathbb{C}_+^{n_j}$ and $\|w_j^i\| = 1$, we have

$$1 = \langle \mathcal{I}, \mathcal{X}^* \rangle = \langle \mathcal{I}, \sum_{i=1}^{r} \rho_i [w_1^i, \ldots, w_m^i]_{\otimes h} \rangle = \sum_{i=1}^{r} \rho_i.$$

Thus,

$$\begin{aligned} \langle \mathcal{H}, \mathcal{X}^* \rangle &= \langle \mathcal{H}, \sum_{i=1}^{r} \rho_i [w_1^i, \ldots, w_m^i]_{\otimes h} \rangle \\ &\geq \sum_{i=1}^{r} \rho_i \langle \mathcal{H}, [w_1^1, \ldots, w_m^1]_{\otimes h} \rangle \\ &= \langle \mathcal{H}, [w_1^1, \ldots, w_m^1]_{\otimes h} \rangle. \end{aligned}$$

This implies that there exist $x_j = w_j^1 \in \mathbb{C}_+^{n_j}$ for all $j \in [m]$, such that $\|x_j\| = 1$ and $\langle \mathcal{H}, \mathcal{X}^* \rangle \geq \langle \mathcal{H}, [x_1, \ldots, x_m]_{\otimes h} \rangle$, i.e., $p_1^* = \widetilde{p}^* \leq p_2^*$.

Therefore, $p_1^* = \widetilde{p}^* = p_2^*$. So, (3.3) is equivalent to the linear conic optimization problem (3.2). The proof is completed. $\square$

To solve the problem (3.2), a natural question is how to check whether a tensor $\mathcal{X} \in \mathbf{HCP}_{[n_1,\ldots,n_m]}$ or not? This is NP-hard for the general case. Another problem is that if we obtain a minimizer $\mathcal{X}^*$ of the problem (3.2), how can we obtain a minimizer $(x_1^*, \ldots, x_m^*)$ of the problem (3.1) from $\mathcal{X}^*$? By the proof of Theorem 3.1, we know it is related to how to obtain an HCP decomposition of $\mathcal{X}^*$.

3.2 Certificate and decomposition for HCP tensors

In this subsection, we show how to check a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCP or not. If not, how to give a certificate; if it is, how to obtain an HCP decomposition. The method is mainly based on the polynomial optimization and truncated moment problem. In the following, we first introduce some basic concepts on polynomial optimization, moment and localizing matrices.

3.2.1 Polynomial optimization, moment and localizing matrices.

An ideal I in $\mathbb{R}[\tilde{x}]$ is a subset of $\mathbb{R}[\tilde{x}]$ such that $I \cdot \mathbb{R}[\tilde{x}] \subseteq I$ and $I + I \subseteq I$. For a polynomial tuple $h = (h_1, \dots, h_t)$ in $\mathbb{R}[\tilde{x}]$, denote the ideal

$$I(h) := h_1 \cdot \mathbb{R}[\tilde{x}] + \cdots + h_t \cdot \mathbb{R}[\tilde{x}].$$

Denote the k-th *truncation* of the ideal $I(h)$ as

$$I_k(h) := h_1 \cdot \mathbb{R}[\tilde{x}]_{k-\deg(h_1)} + \cdots + h_t \cdot \mathbb{R}[\tilde{x}]_{k-\deg(h_t)}. \tag{3.5}$$

Clearly, $I(h) = \cup_{k\in\mathbb{N}} I_k(h)$.

A polynomial p is said to be a sum of squares (SOS) if $p = q_1^2 + \cdots + q_l^2$ for some real polynomials $q_1, \dots, q_l$. The set of all SOS polynomials in $\tilde{x}$ is denoted as $\Sigma[\tilde{x}]$. For a degree d, denote the truncation

$$\Sigma[\tilde{x}]_d := \Sigma[\tilde{x}] \cap \mathbb{R}[\tilde{x}]_d.$$

For a polynomial tuple $g = (g_1, \dots, g_s)$, its *quadratic module* is the set

$$Q(g) := \Sigma[\tilde{x}] + g_1 \cdot \Sigma[\tilde{x}] + \cdots + g_s \cdot \Sigma[\tilde{x}].$$

The k-th truncation of $Q(g)$ is denoted as

$$Q_k(g) := \Sigma[\tilde{x}]_{2k} + g_1 \cdot \Sigma[\tilde{x}]_{d_1} + \cdots + g_s \cdot \Sigma[\tilde{x}]_{d_s}, \tag{3.6}$$

where each $d_i = 2k - \deg(g_i)$. Obviously, $Q(g) = \cup_{k\in\mathbb{N}} Q_k(g)$.

For the polynomial tuples h, g as above, denote

$$K(h, g) := \{\tilde{x} \in \mathbb{R}^n \mid h(\tilde{x}) = 0, g(\tilde{x}) \geq 0\}. \tag{3.7}$$

It is clear that if $f \in I(h) + Q(g)$, then $f \geq 0$ on the set $S(h, g)$. The set $I(h) + Q(g)$ is said to be archimedean if there exists $N > 0$ such that $N - \|\tilde{x}\|^2 \in I(h) + Q(g)$. Clearly, if $I(h) + Q(g)$ is archimedean, then the set $K(h, g)$ is compact. On the other hand, if $K(h, g)$ is compact, then $I(h) + Q(g)$ can be forced to be archimedean by adding the polynomial $N - \|\tilde{x}\|^2$ to the tuple g, for sufficiently large N. When $I(h) + Q(g)$ is archimedean, if $f > 0$ on $K(h, g)$, then $f \in I(h) + Q(g)$. This is called *Putinar's Positivstellensatz* in the literature (cf. [28]). Moreover, if $f \geq 0$ on $K(h, g)$, then $f \in I(h) + Q(g)$ under some general optimality conditions (cf. [22]). We refer to [15, 16, 17] for more details in polynomial optimization.

A vector in $\mathbb{R}^{\mathbb{N}_d^n}$ is called a *truncated multi-sequence* (tms) of degree d. For $p \in \mathbb{R}[\tilde{x}]_d$ and $y \in \mathbb{R}^{\mathbb{N}_d^n}$, define the scalar product

$$\Big\langle \sum_{\alpha\in\mathbb{N}_d^n} p_\alpha \tilde{x}^\alpha, y \Big\rangle := \sum_{\alpha\in\mathbb{N}_d^n} p_\alpha y_\alpha, \tag{3.8}$$

where each p_α is a coefficient of the polynomial p. We say that y admits a $K(h, g)$-representing measure if there exists a Borel measure μ such that its support, denoted as $\mathrm{supp}(\mu)$, is contained in the set $K(h, g)$ and

$$y_\alpha = \int \tilde{x}^\alpha \mathtt{d}\mu, \quad \forall \alpha \in \mathbb{N}_d^n.$$

For a polynomial $q \in \mathbb{R}[\tilde{x}]_{2k}$, the k-th *localizing matrix* of q, generated by a tms $y \in \mathbb{R}^{\mathbb{N}_{2k}^n}$, is the symmetric matrix

$$L_q^{(k)}(y) = \Big(\sum_\alpha q_\alpha y_{\alpha+\beta+\gamma}\Big)_{\beta,\gamma\in\mathbb{N}^n_{k-\lceil deg(q)/2\rceil}}.$$

When $q = 1$ (the constant one polynomial), $L_q^{(k)}(y)$ becomes a *moment matrix* and is denoted as

$$M_k(y) = (y_{\beta+\gamma})_{\beta,\gamma\in\mathbb{N}_k^n}.$$

For convenience, when $q = (q_1, \ldots, q_s)$ is a tuple of s polynomials, we denote

$$L_q^{(k)}(y) := \text{diag}\left(L_{q_1}^{(k)}(y), \ldots, L_{q_s}^{(k)}(y)\right). \tag{3.9}$$

In the above, $\text{diag}(X_1, \ldots, X_s)$ denotes the block diagonal matrix whose diagonal blocks are $X_1, \ldots, X_s$. We refer to [6, 21] for localizing and moment matrices.

3.2.2 Reformulations and Algorithm

In this subsection, we formulate the HCP tensors checking problem as a truncated moment problem, then propose a semidefinite algorithm for it.

For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, define the semialgebraic set

$$\mathbb{K} := \left\{(x_1, \ldots, x_m) \in \mathbb{C}_+^{n_1} \times \cdots \times \mathbb{C}_+^{n_m} : \|x_i\|^2 = 1, \forall i \in [m]\right\}. \tag{3.10}$$

It is clear that a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCP if and only if there exist positive scalars $\lambda_i > 0$ and vectors $(w_1^i, \ldots, w_m^i) \in \mathbb{K}$ for all $i \in [r]$, such that

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [w_1^i, \ldots, w_m^i]_{\otimes h}. \tag{3.11}$$

Let $\mu := \sum_{i=1}^r \lambda_i \delta_{(w_1^i,\ldots,w_m^i)}$ be the weighted sum of Dirac measures. Then (3.11) is equivalent to

$$\mathcal{H} = \int [x_1, \ldots, x_m]_{\otimes h} d\mu, \tag{3.12}$$

where the support $\text{supp}(\mu)$ of the measure μ is contained in $\mathbb{K}$. As shown in the proof of Theorem 5.9 of [17], if there is a nonnegative Borel measure μ supported in $\mathbb{K}$, then there must exist $\lambda_i > 0$ and vectors $(w_1^i, \ldots, w_m^i) \in \mathbb{K}$ satisfying (3.12). Therefore, if we denote by $\mathfrak{B}(\mathbb{K})$ the set of nonnegative Borel measures supported on $\mathbb{K}$, then a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCP if and only if there exists a Borel measure $\mu \in \mathfrak{B}(\mathbb{K})$ such that (3.12) holds. The task of checking existence of the above μ is a truncated moment problem.

Interestingly, Hermitian completely positive tensors can also be characterized by the Hermitian flattening map $\mathfrak{M}$. As in (2.2), the decomposition (3.11) is equivalent to

$$\mathfrak{M}(\mathcal{H}) = \sum_{i=1}^{r} \lambda_i (w_i^1 (w_i^1)^*) \odot \cdots \odot (w_i^m (w_i^m)^*), \tag{3.13}$$

for positive scalars $\lambda_i > 0$ and vectors $(w_1^i, \ldots, w_m^i) \in \mathbb{K}$. We call such a matrix $\mathfrak{M}(\mathcal{H})$ Hermitian completely positive separable, if it has a decomposition as in (3.13). Furthermore, we can easily obtain the following property.

Proposition 3.2. *A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is Hermitian completely positive if and only if there exist Hermitian completely positive matrices $A_{ij} \in \mathbf{HCP}_{[n_i]}$ for all $i \in [s]$ and $j \in [m]$ such that*

$$\mathfrak{M}(\mathcal{H}) = \sum_{i=1}^{s} A_{i1} \odot \cdots \odot A_{im}. \tag{3.14}$$

Recently, for the RHCP tensors with $m = 2$, the real completely positive separable matrices have been studied in [36]. In the following, we pay more attention to the HCP tensors in the space $\mathbb{C}^{[n_1,\ldots,n_m]}$ as well as their HCP decompositions.

For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$, it can be labeled as $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$ such that

$$\mathcal{H}_{IJ} = \mathcal{H}_{i_1\cdots i_m j_1\cdots j_m}. \tag{3.15}$$

For convenience, denote

$$S := \Big\{(i_1, \ldots, i_m) : i_1 \in [n_1], \ldots, i_m \in [n_m]\Big\}.$$

The cardinality of the label set S is $N := \prod_{i=1}^m n_i$. For two labeling tuples $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$ in S, define the ordering $I < J$ if the first nonzero entry of $I - J$ is negative. Denote the index set

$$\mathbb{E} := \{(I, J) \in S \times S : \mathbf{1} \le I \le J \le \mathbf{n}, \mathbf{n} = (n_1, \ldots, n_m)\}, \tag{3.16}$$

where $\mathbf{1} \in \mathbb{N}^m$ is a row vector with all ones. The cardinality of the set $\mathbb{E}$ is $\widetilde{N} := \binom{N+1}{2}$. Then, each Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ can be identified by a vector pair $\mathbf{h}^{re}, \mathbf{h}^{im} \in \mathbb{R}^{\mathbb{E}}$ satisfying

$$\mathbf{h}^{re}_{IJ} + \sqrt{-1}\mathbf{h}^{im}_{IJ} = \mathcal{H}_{IJ}, \ \forall\, (I, J) \in \mathbb{E}.$$

We call $(\mathbf{h}^{re}, \mathbf{h}^{im})$ the identifying vector pair of $\mathcal{H}$.

Denote $\mathbf{x} := (x_1, \ldots, x_m) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m}$. Let $\mathbf{x}^{re} := (x_1^{re}, \ldots, x_m^{re})$ and $\mathbf{x}^{im} := (x_1^{im}, \ldots, x_m^{im})$ in $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ be the real part and imaginal part vectors of $\mathbf{x}$, respectively. Clearly, the complex vector $\mathbf{x} \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m}$ can be identified by real vector $\tilde{x} := (\mathbf{x}^{re}, \mathbf{x}^{im}) \in \mathbb{R}^{2n}$ with $n := \sum_{i=1}^m n_i$. Then the set $\mathbb{K}$ as in (3.10) can be rewritten as, equivalently,

$$\overline{\mathbb{K}} := \left\{\tilde{x} = (\mathbf{x}^{re}, \mathbf{x}^{im}) \in \mathbb{R}^{2n}_+ : \|x_i^{re}\|^2 + \|x_i^{im}\|^2 = 1, i \in [m]\right\}. \tag{3.17}$$

For convenience, we denote

$$g(\tilde{x}) := (1, \tilde{x}), h(\tilde{x}) := (h_1(\tilde{x}), \ldots, h_m(\tilde{x})), \tag{3.18}$$

with each $h_i(\tilde{x}) = \|x_i^{re}\|^2 + \|x_i^{im}\|^2 - 1, i \in [m]$. Then, the set $\overline{\mathbb{K}}$ can also be expressed as

$$\overline{\mathbb{K}} := \left\{\tilde{x} \in \mathbb{R}^{2n} : h(\tilde{x}) = 0, g(\tilde{x}) \ge 0\right\}. \tag{3.19}$$

For each $(I, J) \in \mathbb{E}$, we can expand

$$([x_1, \ldots, x_m]_{\otimes h})_{IJ} = \quad R_{IJ}(\tilde{x}) + \sqrt{-1}T_{IJ}(\tilde{x}), \tag{3.20}$$

where $R_{IJ}(\tilde{x}), T_{IJ}(\tilde{x}) \in \mathbb{R}[\tilde{x}]$. Then,

$$\int ([x_1, \ldots, x_m]_{\otimes h})_{IJ} d\mu = \int R_{IJ}(\tilde{x}) d\mu + \sqrt{-1} \int T_{IJ}(\tilde{x}) d\mu. \tag{3.21}$$

Let $d \ge 2m$ be an even integer. Choose a generic SOS polynomial $F \in \Sigma[\tilde{x}]_d$:

$$F(x) = \sum_{\alpha \in \mathbb{N}^{2n}_d} F_\alpha \tilde{x}^\alpha.$$

Consider the optimization problem

$$\begin{cases} \min\limits_{\mu} & \int F(\tilde{x})d\mu \\ \text{s.t.} & \int R_{IJ}(\tilde{x})d\mu = \mathbf{h}_{IJ}^{re}, \quad (I,J) \in \mathbb{E}, \\ & \int T_{IJ}(\tilde{x})d\mu = \mathbf{h}_{IJ}^{im}, \quad (I,J) \in \mathbb{E}, \\ & \mu \in \mathfrak{B}(\overline{\mathbb{K}}). \end{cases} \tag{3.22}$$

We replace μ by the vector of its moments. Denote the moment cone

$$\mathcal{R}_d := \left\{ y \in \mathbb{R}^{\mathbb{N}_d^{2n}} \mid \exists\, \mu \in \mathfrak{B}(\overline{\mathbb{K}}) \text{ such that } y_\beta = \int \tilde{x}^\beta d\mu \text{ for } \beta \in \mathbb{N}_d^{2n} \right\}. \tag{3.23}$$

So, (3.22) is equivalent to the linear optimization problem

$$\begin{cases} \min\limits_{y} & \langle F, y\rangle \\ \text{s.t.} & \langle R_{IJ}, y\rangle = \mathbf{h}_{IJ}^{re}, \quad (I,J) \in \mathbb{E}, \\ & \langle T_{IJ}, y\rangle = \mathbf{h}_{IJ}^{im}, \quad (I,J) \in \mathbb{E}, \\ & y \in \mathcal{R}_d. \end{cases} \tag{3.24}$$

Obviously, a Hermitian tensor $\mathcal{H}$ is HCP if and only if the feasible set of the problem (3.24) is nonempty. The main difficulty to solve the problem (3.24) is how to deal with the moment cone $\mathcal{R}_d$. Fortunately, such moment cone $\mathcal{R}_d$ has some nice semidefinite relaxations.

As shown by Nie in [20], if $z \in \mathbb{R}^{\mathbb{N}_{2k}^{2n}}$ admits a $\overline{\mathbb{K}}$-measure, then

$$L_h^{(k)}(z) = 0 \;\text{ and }\; L_g^{(k)}(z) \succeq 0. \tag{3.25}$$

If z also satisfies the rank condition

$$\operatorname{rank} M_{k-1}(z) = \operatorname{rank} M_k(z), \tag{3.26}$$

then z admits a unique $\overline{\mathbb{K}}$-measure, which is $\operatorname{rank} M_k(z)$-atomic (cf. [6]). We say that z is flat if both (3.25) and (3.26) are satisfied.

Given two tms' $y \in \mathbb{R}^{\mathbb{N}_t^{2n}}$ and $z \in \mathbb{R}^{\mathbb{N}_e^{2n}}$, we say z is an extension of y, if $t \le e$ and $y_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}_t^{2n}$. For convenience, we denote by $z|_t$ the subvector $z|_{\mathbb{N}_t^{2n}}$. If z is flat and extends y, we say z is a flat extension of y. By [20], a tms $y \in \mathbb{R}^{\mathbb{N}_d^{2n}}$ admits $\overline{\mathbb{K}}$-measure if and only if it is extendable to a flat tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^{2n}}$ for some k. Therefore, checking whether a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is HCP is equivalent to checking whether its corresponding vector y, satisfying $\langle R_{IJ}, y\rangle = \mathbf{h}_{IJ}^{re}$ and $\langle T_{IJ}, y\rangle = \mathbf{h}_{IJ}^{im}$ for all $(I,J) \in \mathbb{E}$, has a flat extension.

Based on (3.25) and (3.26), the cone $\mathcal{R}_d$ can be approximated by semidefnite relaxations. For h, g as in (3.18), denote the cones

$$\Gamma^k := \left\{ z \in \mathbb{R}^{\mathbb{N}_{2k}^{2n}} \mid L_h^{(k)}(z) = 0, L_g^{(k)}(z) \succeq 0 \right\} \tag{3.27}$$

and

$$\Gamma_d^k := \left\{ y \in \mathbb{R}^{\mathbb{N}_d^{2n}} \mid \exists\, z \in \Gamma^k, y = z|_d \right\}. \tag{3.28}$$

Clearly, Γ_d^k is a projection of Γ^k. For all $k \ge d/2$, we have

$$\mathcal{R}_d \subseteq \Gamma_d^{k+1} \subseteq \Gamma_d^k \quad \text{and} \quad \mathcal{R}_d = \bigcap_{k \ge d/2} \Gamma_d^k.$$

This produces the hierarchy of semidefinite relaxations

$$\begin{cases} \min\limits_{z} & \langle F, z\rangle \\ \text{s.t.} & \langle R_{IJ}, z\rangle = \mathbf{h}^{re}_{IJ}, \quad (I,J)\in\mathbb{E}, \\ & \langle T_{IJ}, z\rangle = \mathbf{h}^{im}_{IJ}, \quad (I,J)\in\mathbb{E}, \\ & z\in\Gamma^k, \end{cases} \tag{3.29}$$

for $k = k_0, k_0+1, \dots (k_0 = \lceil d/2\rceil)$. Based on solving the hierarchy of (3.29), we propose a semidefinite algorithm for checking whether a Hermitian tensor is HCP or not and giving an HCP decomposition for it if it is.

Algorithm 3.3 (A semidefinite algorithm for checking HCP tensors)**.**

Initialization. Input a Hermitian tensor $\mathcal{H}\in\mathbb{C}^{[n_1,\dots,n_m]}$ with identifying vector pair $(\mathbf{h}^{re}, \mathbf{h}^{im})$. Set $d = 2m$.

Step 0. Choose a generic $F\in\Sigma[\tilde{x}]_d$, and let $k := d/2$.

Step 1. Solve (3.29). If (3.29) is infeasible, then A is not HCP, and stop. Otherwise, compute a minimizer $z^{*,k}$. Let $t := 1$.

Step 2. Let $\hat{z} := z^{*,k}|_{2t}$. If the rank condition (3.26) is satisfied, go to Step 4.

Step 3. If $t < k$, set $t := t+1$ and go to Step 2; otherwise, set $k := k+1$ and go to Step 1.

Step 4. Compute $\lambda_i > 0$, $(w_1^i, \dots, w_m^i)\in\mathbb{K}$ for each $i\in[r]$, where $r := \operatorname{rank} M_t(\hat{z})$. Output an HCP decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i [w_1^i, \dots, w_m^i]_{\otimes h},$$

and stop.

Remark 3.4. Algorithm 3.3 can be implemented by the software `GloptiPoly 3` [12], which solves the generalized problem of moments. In Step 0, we choose $F = [\tilde{x}]^T_{d/2} G^T G [\tilde{x}]_{d/2}$, where $[\tilde{x}]_d := (\tilde{x}^\alpha)_{\alpha\in\mathbb{N}^{2n}_d}$ and G is a random square matrix obeying Gaussian distribution. In Step 1, we solve the semidefinite relaxation problem (3.29) by the semidefinite programming solver `SeDuMi` [32]. In Step 2, we evaluate the rank of a matrix as the number of its singular values that are not smaller than 10^{-6}, which is a standard procedure in numerical linear algebra (see [7, 10]). In Step 4, we use Henrion and Lasserre's method [11] to compute the finitely atomic $\overline{\mathbb{K}}$-measure μ admitted by $\hat{z}$, which then produces an HCP decomposition of $\mathcal{H}$, if it exists.

Note that the set $\overline{\mathbb{K}}$ as in (3.19) is compact. Then, if a Hermitian tensor $\mathcal{H}$ is HCP, Algorithm 3.3 is always well-defined and has the asymptotic convergence. This can be deduced from Nie [20, Section 5]. For the sake of clarity, we omit the proof.

Theorem 3.5. *Let $\mathcal{H}\in\mathbb{C}^{[n_1,\dots,n_m]}$ with the identifying vector pair $(\mathbf{h}^{re}, \mathbf{h}^{im})$. If $\mathcal{H}$ is HCP, then for a generically generated $F\in\Sigma[\tilde{x}]_d$, Algorithm 3.3 has the following properties:*

(1) *for all $k\geq d/2$, the relaxation (3.29) has an optimizer;*

(2) *for all t big enough, the sequence $\{z^{*,k}|_{2t}\}$ is bounded and each accumulation point is a flat extension of $y\in\mathbb{R}^{\mathbb{N}^{2n}_d}$ with $\langle R_{IJ}, y\rangle = \mathbf{h}^{re}_{IJ}$, $\langle T_{IJ}, y\rangle = \mathbf{h}^{im}_{IJ}$, $(I,J)\in\mathbb{E}$.*

Recall that the cardinality of the set $\mathbb{E}$ as in (3.16) is $\widetilde{N}$. Denote the real polynomial vectors:

$$R(\tilde{x}) := (R_{IJ}(\tilde{x}))_{(I,J)\in\mathbb{E}}, \quad T(\tilde{x}) := (T_{IJ}(\tilde{x}))_{(I,J)\in\mathbb{E}},$$

where $R_{IJ}(\tilde{x}), T_{IJ}(\tilde{x})$ are given as in (3.20), with the length $\widetilde{N}$. It is clear that the vector $R(\tilde{x})$ contains the elements $R_{II}(\tilde{x}) = \prod_{k=1}^{m}(x_{i_k}^* x_{i_k})$ for all $I = (i_1, \ldots, i_m) \in S$. Thus, there exist vectors $q_1, q_2 \in \mathbb{R}^{\widetilde{N}}$ such that the polynomial $q_1^T R(\tilde{x}) + q_2^T T(\tilde{x}) > 0$ on $\overline{\mathbb{K}}$. So Algorithm 3.3 has the following properties by [20, Theorem 5.5].

Theorem 3.6. *Let $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ with its identifying vector pair $(\mathbf{h}^{re}, \mathbf{h}^{im})$. Then we have:*

(i) *If (3.29) is infeasible for some k, then $\mathcal{H}$ is not HCP.*

(ii) *If $\mathcal{H}$ is not HCP, then (3.29) is infeasible for all k big enough.*

Remark 3.7. From item (ii) of Theorem 3.6, if a Hermitian tensor $\mathcal{H}$ is not HCP, then (3.29) is infeasible for some k, for any F (we don't need $F \in \Sigma[\tilde{x}]_d$). This implies that our algorithm can always obtain a certificate for a Hermitian tensor $\mathcal{H}$ which is not HCP. As shown in [20], under some almost necessary and sufficient conditions, Algorithm 3.3 has finite convergence. In other words, it is very likely that Algorithm 3.3 has finite convergence. Therefore, if the Hermitian tensor $\mathcal{H}$ is HCP, an HCP decomposition can be obtained by Algorithm 3.3 in finite steps. Indeed, the finite convergence occurred in all our numerical experiments.

Remark 3.8. As shown in Subsection 3.1, the Hermitian copositive tensor checking problem is equivalent to solving a linear conic optimization problem (3.2) with the cone of HCP tensors. The main difficulty to solve linear conic programming (3.2) is how to deal with the cone of HCP tensors. Actually, it can be similarly solved by a hierarchy of semidefinite relaxations like (3.29), and a semidefinite relaxation method similar to Algorithm 3.3 can also be designed.

4 Numerical Experiments

In this section, we present some numerical experiments for checking whether a Hermitian tensor is HCP or not by Algorithm 3.3. If it is, we can obtain an HCP decomposition; if not, we will obtain a certificate for this. The computation is implemented on a Lenovo Laptop with Intel Core i7-6600U CPU@2.60 GHz and RAM 16.0 GB, using MATLAB R2014a. We use the software GloptiPoly 3 [12] and SeDuMi [32] to solve semidefinite relaxations (3.29).

Example 4.1. Consider the real Hermitian tensor $\mathcal{H} \in \mathbb{R}^{[2,2]}$ such that

$$\mathcal{H}_{ijkl} = i + j + k + l$$

for all $1 \le i, j, k, l \le 2$. As shown in [25, Example 3.3], it is $\mathbb{R}$-Hermitian decomposable. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 2 with $k = 2$, because (3.29) is infeasible. Thus, $\mathcal{H}$ is not HCP. It takes about 5 seconds.

Example 4.2. Consider the real Hermitian tensor $\mathcal{H} \in \mathbb{R}^{[n_1,n_2]}$ such that

$$\mathcal{H}_{ijkl} = \frac{1}{i + j + k + l}$$

for all $1 \leq i, k \leq n_1, 1 \leq j, l \leq n_2$. It is a Cauchy tensor and also $\mathbb{R}$-Hermitian decomposable.

Case 1: $n_1 = n_2 = 2$. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{3} \lambda_i [(u_1^i + \sqrt{-1} v_1^i), (u_2^i + \sqrt{-1} v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 1. Thus, $\mathcal{H}$ is HCP. It takes about 4 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	0.0052	0.1699	0.4835	0.4536	0.2384
		0.2847	0.8102	0.7601	0.3994
2	0.0879	0.5230	0.7248	0.2406	0.8608
		0.2624	0.3637	0.1207	0.4320
3	0.6153	0.3639	0.6545	0.2754	0.6965
		0.3219	0.5792	0.2436	0.6163

Table 1: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 1** in Example 4.2.

Case 2: $n_1 = 3, n_2 = 2$. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{4} \lambda_i [(u_1^i + \sqrt{-1} v_1^i), (u_2^i + \sqrt{-1} v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 2. Thus, $\mathcal{H}$ is HCP. It takes about 12 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	0.0146	0.5882	0.7386	0.0000	0.9492
		0.1949	0.2447	0.0000	0.3145
		0.0646	0.0811		
2	0.2084	0.5358	0.5741	0.0000	0.8352
		0.3528	0.3780	0.0000	0.5500
		0.2323	0.2489		
3	0.7008	0.0000	0.6242	0.4109	0.6103
		0.0000	0.5747	0.3783	0.5619
		0.0000	0.5292		
4	0.0095	0.0000	0.4647	0.4021	0.4931
		0.0000	0.5634	0.4875	0.5978
		0.0000	0.6830		

Table 2: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 2** in Example 4.2.

Example 4.3. Consider the real Hermitian tensor $\mathcal{H} \in \mathbb{R}^{[2,2]}$ such that

$$\mathcal{H}_{ijkl} = i * j + k * l$$

for all $1 \leq i, j, k, l \leq 2$. Note that it is not partial-wise symmetric. So it is not $\mathbb{R}$-Hermitian decomposable. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 2 with $k = 2$, because (3.29) is infeasible. Thus, $\mathcal{H}$ is not an HCP tensor. It takes about 3 seconds.

Example 4.4. Consider the real Hermitian tensor $\mathcal{H} \in \mathbb{R}^{[n_1,n_2]}$ such that

$$\mathcal{H}_{ijkl} = i * k + j * l$$

for all $1 \le i, k \le n_1, 1 \le j, l \le n_2$. Clearly, it is $\mathbb{R}$-Hermitian decomposable.

Case 1: $n_1 = n_2 = 2$. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{2} \lambda_i[(u_1^i + \sqrt{-1}v_1^i), (u_2^i + \sqrt{-1}v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 3. Thus, $\mathcal{H}$ is HCP. It takes about 4 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	10.0000	0.0000	0.7071	0.0000	0.4472
		0.0000	0.7071	0.0000	0.8944
2	10.0000	0.0000	0.4472	0.1681	0.6868
		0.0000	0.8944	0.1681	0.6868

Table 3: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 1** in Example 4.4.

Case 2: $n_1 = 2, n_2 = 3$. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{2} \lambda_i[(u_1^i + \sqrt{-1}v_1^i), (u_2^i + \sqrt{-1}v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 4. Thus, $\mathcal{H}$ is HCP. It takes about 8 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	15.0000	0.4311	0.1189	0.0000	0.5774
		0.8623	0.2377	0.0000	0.5774
				0.0000	0.5774
2	28.0000	0.5462	0.4491	0.2673	0.0000
		0.5462	0.4491	0.5345	0.0000
				0.8018	0.0000

Table 4: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 2** in Example 4.4.

Example 4.5. Consider the random tensor $\mathcal{H} \in \mathbb{C}^{[2,3]}$ with a Hermitian decomposition $\mathcal{H} = \sum_{i=1}^{2}[A(:,i), B(:,i)]_{\otimes h}$, where

$$A = \begin{bmatrix} 1+\sqrt{-1} & -1+\sqrt{-1} \\ 1-\sqrt{-1} & -1-\sqrt{-1} \end{bmatrix}, B = \begin{bmatrix} \sqrt{-1} & 0 \\ \sqrt{-1} & 2-\frac{1}{2}\sqrt{-1} \\ 1+\sqrt{-1} & -\sqrt{-1} \end{bmatrix},$$

and $A(:,i)$ is the i-th column of A, $B(:,i)$ is the i-th column of B. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{2} \lambda_i[(u_1^i + \sqrt{-1}v_1^i), (u_2^i + \sqrt{-1}v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 5. Thus, $\mathcal{H}$ is HCP. It takes about 8 seconds.

Example 4.6. Consider the random tensor $\mathcal{H} \in \mathbb{C}^{[3,3]}$ with an HCP decomposition $\mathcal{H} = \sum_{i=1}^{2}[A(:,i), B(:,i)]_{\otimes h}$, where

$$A = \begin{bmatrix} 1+\sqrt{-1} & 1 \\ 0 & \sqrt{-1} \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} \sqrt{-1} & 0 \\ \frac{1}{5}\sqrt{-1} & \frac{3}{5}+\sqrt{-1} \\ 1+\frac{1}{2}\sqrt{-1} & \frac{1}{2}\sqrt{-1} \end{bmatrix}.$$

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	21.0000	0.7071	0.0000	0.0000	0.0000
		0.0000	0.7071	0.0000	0.8997
				0.4234	0.1059
2	16.0000	0.0000	0.7071	0.3536	0.3536
		0.7071	0.0000	0.3536	0.3536
				0.7071	0.0500

Table 5: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for Example 4.5.

Clearly, $\mathcal{H}$ is HCP. We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{2} \lambda_i[(u_1^i + \sqrt{-1}v_1^i), (u_2^i + \sqrt{-1}v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 6. Thus, $\mathcal{H}$ is HCP. It takes about 27 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	3.2200	0.7071	0.0000	0.0000	0.0000
		0.0000	0.7071	0.9191	0.0000
		0.0000	0.0000	0.3379	0.2027
2	6.8700	0.0000	0.8165	0.2955	0.5911
		0.0000	0.0000	0.0591	0.1182
		0.4082	0.4082	0.7388	0.0000

Table 6: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for Example 4.6.

Example 4.7. Consider the random tensor $\mathcal{H} \in \mathbb{C}^{[3,3]}$ with a Hermitian decomposition $\mathcal{H} = \sum_{i=1}^{2}[A(:, i), B(:, i)]_{\otimes h}$, where

$$A = \begin{bmatrix} -1+\sqrt{-1} & -1 \\ 0 & -\sqrt{-1} \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} -\sqrt{-1} & 0 \\ -\frac{1}{5}\sqrt{-1} & \frac{3}{5}-\sqrt{-1} \\ 1-\frac{1}{2}\sqrt{-1} & -\frac{1}{2}\sqrt{-1} \end{bmatrix}.$$

Case 1. We apply Algorithm 3.3 to $\mathcal{H}$ and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives the HCP decomposition $\mathcal{H} = \sum_{i=1}^{2} \lambda_i[(u_1^i + \sqrt{-1}v_1^i), (u_2^i + \sqrt{-1}v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 7. Thus, $\mathcal{H}$ is HCP. It takes about 25 seconds. This example also verifies the results in Remark 2.3.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	3.2200	0.7071	0.0000	0.0000	0.0000
		0.0000	0.7071	0.0000	0.9191
		0.0000	0.0000	0.2027	0.3379
2	6.8700	0.5774	0.5774	0.5911	0.2955
		0.0000	0.0000	0.1182	0.0591
		0.0000	0.5774	0.0000	0.7388

Table 7: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 1** in Example 4.7.

Case 2. We denote the real part of $\mathcal{H}$ by $\mathcal{H}^{re}$. As shown in Proposition 2.2, we know that $\mathcal{H}^{re}$ is also HCP. By the proof of Proposition 2.2, $\mathcal{H}^{re}$ has an HCP decomposition as in (2.5). We apply Algorithm 3.3 to the tensor $\mathcal{H}^{re}$ and choose $d = 4$ and $k = 2$ in step 0. Algorithm 3.3 terminates at step 3 with $k = 2$ and gives a different HCP decomposition $\mathcal{H}^{re} = \sum_{i=1}^{4} \lambda_i [(u_1^i + \sqrt{-1} v_1^i), (u_2^i + \sqrt{-1} v_2^i)]_{\otimes h}$, where λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ are listed in Table 8. Thus, $\mathcal{H}^{re}$ is HCP. It takes about 32 seconds.

i	λ_i	u_1^i	v_1^i	u_2^i	v_2^i
1	3.4350	0.5774	0.5774	0.5911	0.2955
		0.0000	0.0000	0.1182	0.0591
		0.0000	0.5774	0.0000	0.7388
2	1.6100	0.7071	0.0000	0.0000	0.0000
		0.0000	0.7071	0.4426	0.8055
		0.0000	0.0000	0.3404	0.1985
3	1.6100	0.0000	0.7071	0.0000	0.0000
		0.7071	0.0000	0.5686	0.7221
		0.0000	0.0000	0.0497	0.3909
4	3.4350	0.0000	0.8165	0.0000	0.6608
		0.0000	0.0000	0.0000	0.1322
		0.4082	0.4082	0.6608	0.3304

Table 8: The values of λ_i and $u_1^i, v_1^i, u_2^i, v_2^i$ for **Case 2** in Example 4.7.

Example 4.8. Consider the random tensor $\mathcal{H} \in \mathbb{C}^{[5,4]}$ with a Hermitian decomposition $\mathcal{H} = \sum_{i=1}^{2} [A(:,i), B(:,i)]_{\otimes h}$, where

$$A = \begin{bmatrix} 1+\sqrt{-1} & \frac{1}{2}+\sqrt{-1} \\ \sqrt{-1} & \frac{1}{2}-\sqrt{-1} \\ 2+\sqrt{-1} & 1+2\sqrt{-1} \\ -\sqrt{-1} & -2\sqrt{-1} \\ 1+\sqrt{-1} & 2\sqrt{-1} \end{bmatrix}, B = \begin{bmatrix} -1+2\sqrt{-1} & -\frac{1}{2}\sqrt{-1} \\ -\sqrt{-1} & \frac{1}{2}+\frac{1}{2}\sqrt{-1} \\ 1+2\sqrt{-1} & \frac{1}{2}-\frac{1}{2}\sqrt{-1} \\ -\sqrt{-1} & -2\sqrt{-1} \end{bmatrix}.$$

We apply Algorithm 3.3 and choose $d = 4$ and $k = 2$ in Step 0. Algorithm 3.3 terminates at step 2 with $k = 2$, because (3.29) is infeasible. Thus, $\mathcal{H}$ is not HCP. It takes about 480 seconds.

5 Conclusions

In this paper, we studied (real) Hermitian completely positive tensors, (real) Hermitian completely positive decompositions and some related topics. We gave some basic properties and relationships of (real) Hermitian completely positive cone and (real) Hermitian copositive cone in the Hermitian tensor space. We formulated the Hermitian completely positive tensors checking and decomposition problems as a special truncated moment problem over nonnegative multispheres. A semidefinite algorithm was also proposed. If a Hermitian tensor is Hermitian completely positive, we can get a Hermitian completely positive decomposition; if not, a certificate can be obtained.

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