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PERTURBATION OF ERROR BOUNDS FOR VECTOR-VALUED FUNCTIONS*

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Abstract: Motivated by Zheng and Ng's paper (ESAIM Control Optim. Calc. Var. doi:10.1051/cocv/2018047) where they study the stability of error bounds for vector-valued functions under the assumption that the ordering cone has a nonempty interior, we consider the stability issues of error bounds for vector-valued functions under a weaker assumption that the ordering cone is dually compact. In particular, we provide several stable results on error bounds in term of the radius of error bounds which extend some results on the stability of error bounds in the literature.

Key words: *perturbation, error bound, conic inequality, optimization*

Mathematics Subject Classification: *49J52, 49J53, 90C31*

1 Introduction

The theory of error bounds plays an important role in optimization and has been widely used in different areas such as sensitivity analysis, convergence analysis of algorithms and penalty function methods. The origin of error bound theory can be traced back to the pioneering work by Hoffman [6]. Since Hoffman's work, the study of error bound theory has received extensive attention, and a great deal of works have been done on error bound issues (c.f. [2, 3, 5, 7, 12, 18, 19, 21, 23, 24, 25]). For a proper lower semicontinuous function φ defined on a real Banach space X, we denote by $S(\varphi,\bar{x}) := \{x \in X \mid \varphi(x) \leq \varphi(\bar{x})\}$ the solution set of the following inequality:

$$\varphi(x) \leq \varphi(\bar{x}). \tag{IE}$$

Recall that φ has (or admits) a local error bound at $\bar{x} \in \mathrm{dom}(\varphi)$ if there exist $\tau,\ \delta \in (0,+\infty)$ such that

$$\tau d(x, S(\varphi,\bar{x})) \leq [\varphi(x) - \varphi(\bar{x})]_+ \quad \forall x \in B(\bar{x},\delta), \tag{1.1}$$

where $d(x,S(\varphi,\bar{x})) = \inf\{\|x-v\| \mid v \in S(\varphi,\bar{x})\}$, $[\varphi(x)-\varphi(\bar{x})]_+ := \max\{0,\varphi(x)-\varphi(\bar{x})\}$ and $B(\bar{x},\delta)$ is the open ball with the center $\bar{x}$ and radius δ.

In many practical problems, the initial data we obtained may be inaccurate. Therefore, from the points of view of theory as well as application, it is important to study the behavior

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of the error bounds under data perturbations (c.f. [13, 32]). In 1994, Luo and Tseng [13] studied perturbation analysis of error bounds for linear inequality systems. Recently, Kruger, López and Théra [10] studied the perturbation of error bounds for real-valued convex functions. In paticular, they used the radius of error bounds to study the stability of error bounds. On the other hand, in many optimization problems, we need to deal with a family (finite or infinite) of inequalities rather than a single inequality. For example, in semi-infinite programming, there are infinitely many inequality constraints. Hence many authors studied the error bounds for inequality systems such as the inequality/equality systems and the semi-infinite constraint systems (see [9, 10, 11, 15, 16, 17, 26, 28, 33] and references therein for details).

Throughout this paper, let X and Y are Banach spaces, K is a closed convex cone in Y and ∞_Y is the abstract infinity in Y which satisfies:

$$\alpha\cdot\infty_Y=\infty_Y,\quad y+\infty_Y=\infty_Y,$$

$$0\cdot\infty_Y=0,\qquad y\leq_K\infty_Y,\quad \|\infty\|=+\infty$$

for any $\alpha\in\mathbb{R}_+$ and $y\in Y$. Let $Y^\bullet:=Y\cup\{\infty_Y\}$ and $F:X\to Y^\bullet$ be a vector-valued function. Very recently, Zheng and Ng [32] studied the stability of error bounds for the following conic inequality

$$F(x)\leq_K F(\bar{x}) \tag{CIE}$$

where Y is ordered by the cone K and $\bar{x}\in\mathrm{dom}(F):=\{x\in X\mid F(x)\in Y\}$. Recall that (CIE) has a local error bound at $\bar{x}$ if there exist $\tau,\ \delta\in(0,+\infty)$ such that

$$\tau d(x,S(F,\bar{x},K))\leq d(F(x)-F(\bar{x}),-K)\quad \forall x\in B(\bar{x},\delta), \tag{1.2}$$

where $S(F,\bar{x},K):=\{x\in X\mid F(x)\leq_K F(\bar{x})\}$ denotes the solution set of (CIE) and $d(\infty_Y,-K)$ is understood as $+\infty$. In the case when $Y=\mathbb{R}$ and $K=\mathbb{R}_+$, (1.2) reduces to (1.1). It is worth noting that the conic inequality model (CIE) contains many constraint systems in optimization. For example, when $Y=\mathbb{R}^{n+m},\ K=\mathbb{R}^n_+\times\{0\}\subset Y,\ F(x)=(f_1(x),\cdots,f_{n+m}(x))$ and $F(\bar{x})=0$, (CIE) reduces to

$$f_i(x)\leq 0 \text{ for } i=1,\cdots,n \text{ and } f_i(x)=0 \text{ for } i=n+1,\cdots,n+m.$$

When T is a compact metric space, $Y=C(T,\mathbb{R})$ is the Banach space of continuous functions $h:X\to R$. and $K=\{h\in C(T,\mathbb{R})\mid h(t)\geq 0\ \forall t\in T\}$, (CIE) reduces to semi-infinite constraint systems. The constraint systems in conic optimization problems can also be viewed as examples of (CIE).

Let $\mathrm{Er}(F)$ be the modulus of the error bound of the conic inequality (CIE):

$$\begin{aligned}\mathrm{Er}(F):=&\sup\{\tau>0\mid (1.2)\text{ holds for some }\delta>0\}\\ =&\liminf_{\substack{x\to\bar{x}\\ x\in\mathrm{dom}(F)\setminus S(F,\bar{x},K)}}\frac{d(F(x)-F(\bar{x}),-K)}{d(x,S(F,\bar{x},K))}.\end{aligned} \tag{1.3}$$

Clearly, F has an error bound at $\bar{x}$ if and only if $\mathrm{Er}(F)>0$. Recall [32] that (CIE) has a stable error bound at $\bar{x}\in\mathrm{dom}(F)$, if there exist $\eta,\ r\in(0,+\infty)$ such that $\mathrm{Er}(F+H)\geq\eta$ for any $H:X\to Y$ with

$$\|H\|_{\bar{x}}:=\limsup_{x\to\bar{x}}\frac{\|H(x)-H(\bar{x})\|}{\|x-\bar{x}\|}<r. \tag{1.4}$$

Let Y^* be the dual of Y and

$$K^+ := \{y^* \in Y^* \mid \langle y^*, x\rangle \geq 0 \ \forall x \in K\} \text{ and } \mathcal{I}_{K^+} := \{y^* \in K^+ : \|y^*\| = 1\}. \tag{1.5}$$

In order to study the stability of error bound for (CIE), Zheng and Ng [32] adopt the following subdifferential for the vector-valued function F at $\bar{x}$:

$$\partial_K F(\bar{x}) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\mathrm{epi}_K(F), (\bar{x}, F(\bar{x}))) \text{ for some } y^* \in \mathcal{I}_{K^+}\}$$

where $\mathrm{epi}_K(F) := \{(x, y) \in X \times Y : F(x) \leq_K y\}$ and $N(\mathrm{epi}_K(F), (\bar{x}, F(\bar{x})))$ is the Clarke normal cone to $\mathrm{epi}_K(F)$ at $(\bar{x}, F(\bar{x}))$ (see also [31]). In [32], using this kind of subdifferential for vector-valued functions, Zheng and Ng characterized the stability of error bounds for conic inequalities. For a closed subset A of a Banach space X, we define the boundary of A as $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(\mathrm{cl}(A))$. It is easy to see that $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$ if and only if $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$ or $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$. When $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$, we denote

$$D := d(0, \mathrm{bd}(\partial_K F(\bar{x}))). \tag{1.6}$$

In the case when $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$, we have

$$D = \sup\{r > 0 \mid rB_{X^*} \subset \mathrm{cl}(\partial_K F(\bar{x}))\} > 0.$$

In the case when $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$, we obtain

$$D = d(0, \partial_K F(\bar{x})) > 0.$$

Two main theorems extracted from [32] can be stated as follows (see [32, Theorem 4.1 and Theorem 4.4]):

Theorem 1.1. *Let $F \in \Gamma$. Suppose that F is w-quasi-subsmooth at $\bar{x}$ and $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$. Then for any $H : X \to Y$ with $F + H \in \Gamma$ and $\|H\|_{\bar{x}} < D$, and any $\varepsilon \in (0, +\infty)$, there exists $\delta > 0$ such that*

$$\|x - \bar{x}\| \leq \frac{1 + \varepsilon}{D - \|H\|_{\bar{x}}} d((F + H)(x) - (F + H)(\bar{x}), -K) \quad \forall\, x \in B(\bar{x}, \delta),$$

where the definitions of the notation Γ and the notion 'w-quasi-subsmooth' are given in section 3.

Theorem 1.2. *Suppose that K has a nonempty interior. Let $F \in \Gamma$ be such that $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$. Let $H : X \to Y$ be such that $F + H \in \Gamma$ and $\|H\|_{\bar{x}} < d(0, \partial_K F(\bar{x}))$. Then the following statements hold:*

(i) *If X, Y are Asplund spaces and $F + H$ is w-quasi-subsmooth at $\bar{x}$, then*

$$\mathrm{Er}(F + H) \geq \gamma_K(d(0, \partial_K F(\bar{x})) - \|H\|_{\bar{x}}),$$

where $\gamma_K := \sup\limits_{y \in K \cap S_Y} \{r > 0 \mid B(y, r) \subset K\}$.

(ii) *If $F + H$ is quasi-subsmooth at $\bar{x}$ and continuous on some neighborhood of $\bar{x}$, then*

$$\mathrm{Er}(F + H) \geq \gamma_K(d(0, \partial_K F(\bar{x})) - \|H\|_{\bar{x}}).$$

(iii) *If $\mathrm{epi}_K(F + H)$ is convex, then*

$$\mathrm{Er}(F + H) \geq \gamma_K(d(0, \partial_K F(\bar{x})) - \|H\|_{\bar{x}}).$$

Under the assumption that $\text{int}(K) \neq \emptyset$, Theorem 1.1 and Theorem 1.2 imply that $0 \notin \text{bd}(\partial_K F(\bar{x}))$ is a sufficient condition for F to have a stable error bound at $\bar{x}$. It is worth mentioning that, under the assumption $\text{int}(K) \neq \emptyset$ and some other mild assumptions, Zheng and Ng proved that $0 \notin \text{bd}(\partial_K F(\bar{x}))$ is also a necessary condition for F to have a stable error bound at $\bar{x}$. However, the assumption that $\text{int}(K) \neq \emptyset$ is restrictive sometimes. For example, when $Y = \mathbb{R}^{n+m},\ K = \mathbb{R}^n_+ \times \{0\} \subset Y$, it is easy to see that the interior of K is empty. Hence it is necessary to consider the stability of error bounds for (CIE) in the case when the interior of the ordering cone is not necessarily nonempty. On the other hand, in order to relax the condition that the ordering cone has a nonempty interior, Zheng and Ng [27] introduce the notion of the dually compact cone (see section 3 for definition). This is a broad class of cones including every closed convex cone in finite dimensional spaces and every closed convex cone with a nonempty interior in Banach spaces (see [27]). In this paper, motivated by [10] and [32], we consider the stability of error bounds for conic inequalities under the assumption that the ordering cone K is dually compact. Under this assumption, we establish some sufficient conditions of the stability of error bound for conic inequality (CIE), and provide the lower bound estimation for the radius of error bound. Our results extend some results in [32] and [10].

2 Preliminaries

Let X^* be the topological dual of X. Let B_X be the closed unit ball of X, and S_X be the unit sphere of X. We use $B(x,r)$ and $B[x,r]$ to denote the open and closed balls with the center x and radius r, respectively. For a set $M \subset X$, the support functional of M is defined by

$$\sigma_M(x^*) := \sup_{x \in M} \langle x^*, x\rangle \quad \forall\ x^* \in X^*.$$

For a closed subset A of X and a point $\bar{x} \in A$, the Clarke tangent cone to A at $\bar{x}$ is defined by

$$T(A,\bar{x}) := \{v \in X \mid \forall x_n \xrightarrow{A} \bar{x} \text{ and } \forall t_n \downarrow 0\ \exists v_n \to v\ s.t.\ x_n + t_n v_n \in A\ \ \forall n \in \mathbb{N}\},$$

and the Clarke normal cone to A at $\bar{x}$ is defined by

$$N(A,\bar{x}) := \{x^* \in X^* \mid \langle x^*, h\rangle \leq 0\ \forall h \in T(A,\bar{x})\}.$$

For $\varepsilon \geq 0$, we use $\widehat{N}_\varepsilon(A,\bar{x})$ to denote the set of Fréchet ε-normals to A at $\bar{x}$, which is defined by

$$\widehat{N}_\varepsilon(A,\bar{x}) := \left\{x^* \in X^* \mid \limsup_{x \xrightarrow{A} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq \varepsilon\right\}.$$

When $\varepsilon = 0$, $\widehat{N}_\varepsilon(A,\bar{x})$ is a convex cone which is called the Fréchet normal cone to A at $\bar{x}$. In this case, $\widehat{N}_\varepsilon(A,\bar{x})$ is denoted by $\widehat{N}(A,\bar{x})$. The Mordukhovich normal cone to A at $\bar{x}$ is defined by

$$\overline{N}(A,\bar{x}) := \{x^* \in X^* \mid \exists x_n \xrightarrow{A} \bar{x},\ \varepsilon_n \downarrow 0 \text{ and } x_n^* \xrightarrow{w^*} x^*\ s.t.\ x_n^* \in \widehat{N}_{\varepsilon_n}(A,x_n)\ \forall n \in \mathbb{N}\}.$$

For $\bar{x} \notin A$, $N(A,\bar{x})$, $\widehat{N}_\varepsilon(A,\bar{x})$, $\widehat{N}(A,\bar{x})$ and $\overline{N}(A,\bar{x})$ are the empty sets. For a proper lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, we use $\text{dom}(\varphi)$ to denote the domain of φ, and use epi(φ) to denote the epigraph of φ. The Clarke subdifferential of φ at $x \in \mathrm{dom}(\varphi)$ is defined by

$$\partial\varphi(x) := \{x^* \in X^* \mid (x^*, -1) \in N(\mathrm{epi}(\varphi), (x, \varphi(x)))\}.$$

The Fréchet subdifferential of φ at $x \in \mathrm{dom}(\varphi)$ is defined as

$$\widehat{\partial}\varphi(x) := \left\{x^* \in X^* \mid \liminf_{h\to 0} \frac{\varphi(x+h) - \varphi(x) - \langle x^*, h\rangle}{\|h\|} \geq 0\right\}.$$

The limiting subdifferential of φ at $x \in \mathrm{dom}(\varphi)$ is defined as

$$\overline{\partial}\varphi(x) := \{x^* \in X^* \mid (x^*, -1) \in \overline{N}(\mathrm{epi}(\varphi), (x, \varphi(x)))\}.$$

When $x \notin \mathrm{dom}(\varphi)$, $\partial\varphi(x)$, $\widehat{\partial}\varphi(x)$ and $\overline{\partial}\varphi(x)$ are the empty sets. It is well known that $\widehat{\partial}\varphi(x) \subset \overline{\partial}\varphi(x) \subset \partial\varphi(x)$,

$$\widehat{\partial}\varphi(x) = \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\mathrm{epi}(\varphi), (x, \varphi(x)))\},$$

and that when f is convex,

$$\widehat{\partial}\varphi(x) = \overline{\partial}\varphi(x) = \partial\varphi(x) = \{x^* \in X^* \mid \langle x^*, u\rangle \leq \varphi(x+u) - \varphi(x) \ \forall\ u \in X\}$$

for all $x \in \mathrm{dom}(\varphi)$. Let's recall that a Banach space X is an Asplund space if every continuous convex function on X is Fréchet differentiable on a dense G_δ-set of X. The following results on normal cones and subdifferentials are well known in variational analysis and are useful for us (cf. [14]).

Lemma 2.1. *Let A be a closed subset in X. Let $\varphi_1, \varphi_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions such that φ_1 is locally Lipschitz at $\bar{x} \in \mathrm{dom}(\varphi_1) \cap \mathrm{dom}(\varphi_2)$. Then the following statements hold:*

(i) $\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x})$

(ii) *If X is an Asplund space, then for any $x^* \in \widehat{\partial}(\varphi_1 + \varphi_2)(\bar{x})$ and $\varepsilon > 0$, there exist $x_1, x_2 \in B(\bar{x}, \varepsilon)$ such that $|\varphi_i(x_i) - \varphi_i(\bar{x})| < \varepsilon$ ($i = 1, 2$) and $x^* \in \widehat{\partial}\varphi_1(x_1) + \widehat{\partial}\varphi_2(x_2) + \varepsilon B_{X^*}$.*

(iii) *If X is an Asplund space, then $\overline{N}(A, a) = \limsup\limits_{v \xrightarrow{A} a} \widehat{N}(A, v)$ for all $a \in A$, and $N(A, a) = \bar{co}^{w*}\overline{N}(A, a)$.*

We define the preorder $\leq_K$ in Y as: $y_1 \ \leq_K \ y_2 \Leftrightarrow y_2 - y_1 \in K$. Let

$$K^+ := \{x^* \in Y^* \mid \langle x^*, y\rangle \geq 0 \ \forall y \in K\} \text{ and } \mathcal{I}_{K^+} = \{x^* \in K^+ \mid \|x^*\| = 1\}. \tag{2.1}$$

The epigraph of the vector-valued function $F : X \longrightarrow Y^\bullet$ with respect to the ordering cone K is defined by

$$\mathrm{epi}_K(F) := \{(x, y) \in X \times Y \mid F(x) \leq_K y\}.$$

We say F is K-convex, if $\mathrm{epi}_K(F)$ is convex. For the epigraph $\mathrm{epi}_K(F)$, we have the following results [32]:

Lemma 2.2. *Let $G : X \to Y^\bullet$ be a function such that $\mathrm{epi}_K(G)$ is closed. Then, for any $(x, y) \in \mathrm{epi}_K(G)$, the following statements hold:*

(i) $\widehat{N}(\mathrm{epi}_K(G),(x,y)) \subset X^* \times (-K^+)$.

(ii) *If G is continuous at x, then*

$$N(\mathrm{epi}_K(G),(x,y)) \subset N(\mathrm{epi}_K(G),(x,G(x))) \subset X^* \times (-K^+).$$

Following [31] and [32], for $x \in \mathrm{dom}(F)$, we adopt the following coderivative $D_e^*F(x) : Y^* \rightrightarrows X^*$ for F at x:

$$D_e^*F(x)(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in N(\mathrm{epi}_K(F),(x,F(x)))\} \ \ \forall y^* \in Y^*,$$

and the following subdifferential for F at x:

$$\begin{aligned}\partial_K F(x) &:= D_e^*F(x)(\mathcal{I}_{K^+}) \\ &= \{x^* \in X^* \mid (x^*,-y^*) \in N(\mathrm{epi}_K(F),(x,F(x))) \ \text{ for some } y^* \in \mathcal{I}_{K^+}\}.\end{aligned}$$

where $\mathcal{I}_{K^+}$ is defined as (2.1).

We adopt the following conventions. For $\emptyset \subset [0,+\infty)$, $\inf \emptyset$ is understood as $+\infty$. For a point z in a Banach space Z and the set $\emptyset$ in Z, $d(x,\emptyset)$ is understood as $+\infty$. For any $\alpha \in (0,+\infty]$, we set $\frac{\alpha}{0} = +\infty$.

3 Main Results

As a useful extension of the convexity and smoothness, Aussel, Daniilidis and Thibault introduced the notion of the subsmoothness of a closed set in [1]. Let A be a closed subset in X. Recall that A is subsmooth at $\bar{x} \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, u - v\rangle \le \varepsilon\|u - v\| \ \ \forall u, v \in A \cap B(\bar{x},\delta) \text{ and } \forall x^* \in N(A, v) \cap B_{X^*}. \tag{3.1}$$

In [33], motivated by (3.1) and the primal-lower-nice property introduced by Poliquin (c.f. [20]), Zheng and Wei introduced the quasi-subsmoothness for a proper lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$: φ is said to be quasi-subsmooth at $\bar{x} \in \mathrm{dom}(\varphi)$, if for any $\varepsilon,\ M \in (0,+\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - v\rangle \le \varphi(u) - \varphi(v) + \varepsilon\|u - v\| \ \ \forall u, v \in B(\bar{x},\delta) \text{ and } \forall x^* \in \partial\varphi(v) \cap MB_{X^*}.$$

In [32], Zheng and Ng further extended the subsmoothness from the real-valued function to the vector-valued function. Let $F : X \to Y^\bullet$ be a vector-valued function. Recall [32] that F is said to be quasi-subsmooth at $\bar{x} \in \mathrm{dom}(F)$ with respect to K if for any $\varepsilon,\ M \in (0,+\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - v\rangle \le \langle y^*, F(u) - F(v)\rangle + \varepsilon\|u - v\|$$

whenever $u,\ v \in B(\bar{x},\delta),\ y^* \in \mathcal{I}_{K^+}$ and $x^* \in D_e^*F(v)(y^*) \cap MB_{X^*}$ (where $\langle y^*, \infty_Y\rangle := \infty$) and that F is said to be w-quasi-subsmooth at $\bar{x} \in \mathrm{dom}(F)$ with respect to K if for any $\varepsilon,\ M \in (0,+\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - \bar{x}\rangle \le \langle y^*, F(u) - F(\bar{x})\rangle + \varepsilon\|u - \bar{x}\|$$

whenever $u \in B(\bar{x},\delta),\ y^* \in \mathcal{I}_{K^+}$ and $x^* \in D_e^*F(\bar{x})(y^*) \cap MB_{X^*}$. Having $\bar{x} \in X$ a fixed element, we adopt the following notations: $\Gamma := \{G : X \to Y^\bullet \mid \bar{x} \in \mathrm{dom}(G)$ and $\mathrm{epi}_K(G)$

is closed} and

$$\begin{aligned}\mathcal{F}_1 &:= \{G \in \Gamma \mid G \text{ is } w\text{-quasi-subsmooth at } \bar{x}\}\\ \mathcal{F}_2 &:= \{G \in \Gamma \mid G \text{ is quasi-subsmooth at } \bar{x} \text{ and}\\ &\qquad\qquad G \text{ is continuous on some neighborhood of } \bar{x}\}\\ \mathcal{F}_3 &:= \{G \in \Gamma \mid G \text{ is } K\text{-convex}\}.\end{aligned}$$

For $H : X \to Y$, following [32], we use the following notation

$$\|H\|_{\bar{x}} := \limsup_{x \to \bar{x}} \frac{\|H(x) - H(\bar{x})\|}{\|x - \bar{x}\|}.$$

Having now a fixed $F \in \Gamma$ and $\varepsilon \in [0, +\infty]$, following [10], we adopt the following notations:

$$\mathrm{P}_i(\varepsilon) := \{G \in \mathcal{F}_i \mid \exists\, H : X \to Y \ s.t.\ \|H\|_{\bar{x}} \le \varepsilon \text{ and } G = F + H\} \qquad (i = 1, 2, 3).$$

Then for $G \in \Gamma$, $\mathrm{Er}(G) := \sup\{\tau > 0 \mid (1.2) \text{ holds for } G \text{ and some } \delta > 0\}$. Clearly, G has an error bound at $\bar{x}$ if and only if $\mathrm{Er}(G) > 0$. For the special case that $K = X$, we have $\mathrm{Er}(G) = +\infty$. Indeed, in this case, we have $S(G, \bar{x}, K) = \mathrm{dom}(G)$. It follows that $d(x, S(G, \bar{x}, K)) = 0$ for every $x \in \mathrm{dom}(G)$. For $x \notin \mathrm{dom}(G)$, we have $d(G(x) - G(\bar{x}), -K) = +\infty$. Hence we have (1.2) holds for every $\tau > 0$ and $\mathrm{Er}(G) = +\infty$. Following [10], we denote

$$\mathrm{Er}_i(\varepsilon) := \inf\{\mathrm{Er}(G) \mid G \in \mathrm{P}_i(\varepsilon)\} \qquad (i = 1, 2, 3).$$

and define the radius of error bound of F at $\bar{x}$ with respect to the function classes $\mathcal{F}_i$ as follows:

$$\mathcal{R}_i := \inf\{\varepsilon > 0 \mid \mathrm{Er}_i(\varepsilon) = 0\} \qquad (i = 1, 2, 3).$$

When $\mathcal{R}_i > 0$ $(i = 1, 2, 3)$, for any $H : X \to Y$ with $\|H\|_{\bar{x}} < \mathcal{R}_i$ and $F + H \in \mathcal{F}_i$, we have $\mathrm{Er}(F + H) > 0$, and hence $F + H$ has a local error bound at $\bar{x}$. However, in practical applications, it is not enough only to know the existence of error bound for G, and we need to know the value of τ such that (1.2) holds. For this purpose, we consider the following radius of error bound of F at $\bar{x}$. For any positive real number $\kappa > 0$, we define the radius of error bound of F at $\bar{x}$ with κ by

$$\mathcal{R}_i(\kappa) := \inf\{\varepsilon > 0 \mid \mathrm{Er}_i(\varepsilon) < \kappa\} \qquad (i = 1, 2, 3).$$

When $\mathcal{R}_i(\kappa) > 0$ $(i = 1, 2, 3)$, for any $H : X \to Y$ with $\|H\|_{\bar{x}} < \mathcal{R}_i$ and $F + H \in \mathcal{F}_i$, we have $\mathrm{Er}(F + H) \ge \kappa$. This provides us not only with the existence of error bound of $F + H$ at $\bar{x}$, but also with the value of τ such that (1.2) holds. Recall [32] that F has a stable error bound at $\bar{x}$ with respect to the function class $\mathcal{F}_i$ $(i = 1, 2, 3)$, if there exist $\kappa, \delta \in (0, +\infty)$ such that

$$\mathrm{Er}(G) \ge \kappa \tag{3.2}$$

for any $G \in \mathrm{P}_i(\delta)$ holds. It is easy to see that F has a stable error bound at $\bar{x}$ with respect to the function class $\mathcal{F}_i$ if and only if there exists $\kappa > 0$ such that $\mathcal{R}_i(\kappa) > 0$ holds. In [32], Zheng and Ng established their stability results on error bounds for conic inequalities (CIE) under the assumption that the closed convex ordering cone K has a nonempty interior. Note that K has a nonempty interior if and only if there exists $y_0 \in Y$ such that

$$K^+ \subset \{y^* \mid \|y^*\| \le \langle y^*, y_0 \rangle\}. \tag{3.3}$$

In [27], in order to relax the condition that the ordering cone has a nonempty interior, Zheng and Ng introduced the notion of dually compact cone. Let K be a closed convex cone in a Banach space Y. Recall that K is said to be dually compact if there exists a compact subset C in Y such that

$$K^+ \subset \{y^* \mid \|y^*\| \le \sigma_C(y^*)\}. \tag{3.4}$$

When Y is a finite dimensional space, for every closed convex cone $K \subset Y$, we can take C as the unit sphere in Y which is a compact subset that satisfies (3.4). When Y is a general Banach space and K is a closed convex cone with a nonempty interior in Y, there exists $y_0 \in Y$ such that (3.3) holds. For this case, we can take C as $\{y_0\}$. Hence every closed convex cone in finite dimensional spaces and every closed convex cone with a nonempty interior in Banach spaces are dually compact. The rest of this article is devoted to the sufficient condition of stability of error bounds for conic inequalities under the assumption that the ordering cone is dually compact. To achieve this, we need the following lemmas established in [32, Lemma 4.3 and Proposition 3.9].

Lemma 3.1. *Let $G \in \Gamma$. Then for any $t > \mathrm{Er}(G)$, there exists $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times K \times (0, +\infty)$ such that*

$$G(\bar{x}_n) + \bar{c}_n \ne G(\bar{x}) \quad \forall n \in \mathbb{N}, \tag{3.5}$$

$$(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n, \eta_n) \to (\bar{x}, G(\bar{x}), 0) \tag{3.6}$$

and

$$(0,0) \in \widehat{\partial}(g + \delta_{\mathrm{epi}_K G} + t\| \cdot -(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n)(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \ \forall n \in \mathbb{N}, \tag{3.7}$$

where

$$g(x,y) := \|y - G(\bar{x})\| \text{ and } \|(x,y)\|_n := \|x\| + \eta_n\|y\| \quad \forall (x,y) \in X \times Y. \tag{3.8}$$

Lemma 3.2. *Let $H : X \to Y$ be such that $\|H\|_{\bar{x}} < +\infty$ and $F + H \in \Gamma$. Then*

$$\partial_K(F+H)(\bar{x}) \subset \partial_K F(\bar{x}) + \|H\|_{\bar{x}} B_{X^*}.$$

With the help of Lemma 3.1 and Lemma 3.2, we will establish our main results below. Noting that $K = Y$ is a dually compact closed convex cone, $\mathrm{Er}(G) = +\infty$ holds for all $G \in \Gamma$ in this special case. Hence, $\mathcal{R}_i = \mathcal{R}_i(\kappa) = +\infty$ $(i = 1, 2, 3)$ for any $\kappa > 0$. Therefore, we only need to consider the case when $K \ne Y$.

Theorem 3.3. *Let X and Y be Asplund spaces, K be dually compact and $K \ne Y$. Suppose that F is w-quasi-subsmooth at $\bar{x}$ and $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$. Then*

(i)

$$\mathcal{R}_1 \ge D, \tag{3.9}$$

where D is defined by (1.6).

(ii)

$$\mathcal{R}_1(\kappa) \ge D - \kappa M, \ \forall \ \kappa < \frac{D}{M}, \tag{3.10}$$

where M is a positive constant depending only on K and D is defined by (1.6). *Consequently, F has a stable error bound at $\bar{x}$ with respect to the function class $\mathcal{F}_1$.*

Proof. Since K is a dually compact closed convex cone, there exists a compact subset $C \subset Y$ such that (3.4) holds. We will prove (3.10) holds for $M := \max_{h\in C} \|h\|$. Let $\kappa < \frac{D}{M}$. For any $G := F + H \in \mathrm{P}_1(D - \kappa M)$, we claim that

$$\mathrm{Er}(G) \geq \kappa. \tag{3.11}$$

Assume that (3.11) holds. By the definitions of $\mathrm{Er}_1(D-\kappa M)$ and $\mathcal{R}_1(\kappa)$, we have $\mathrm{Er}_1(D-\kappa M) \geq \kappa$ and $\mathcal{R}_1(\kappa) \geq D - \kappa M$. It remains to prove (3.11). Since $K \neq Y$, we have $K^+ \neq \{0\}$ and $M := \max_{h\in C} \|h\| \geq 1$ (by (3.4)). By the assumption that $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$, we have $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$ or $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$ hold. For the first case, we have that $D = d(0, \partial_K F(\bar{x}))$. Since $G \in \mathrm{P}_1(D-\kappa M)$, we have $G = F + H \in \mathcal{F}_1$ and $\|H\|_{\bar{x}} \leq D - \kappa M$. For any $t > \mathrm{Er}(G)$, by Lemma 3.1, there exists $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times K \times (0, +\infty)$ such that (3.5), (3.6) and (3.7) hold. Combining this with Lemma 2.1 (ii), there exist (x_n, y_n) and $(\hat{x}_n, \hat{y}_n)$ converging to $(\bar{x}, G(\bar{x}))$ such that $\hat{y}_n \neq G(\bar{x})$ and

$$(0,0) \in \widehat{\partial} g(\hat{x}_n, \hat{y}_n) + \widehat{\partial}\delta_{\mathrm{epi}_K(G)}(x_n, y_n) + t\widehat{\partial}\| \cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n(\hat{x}_n, \hat{y}_n) + \frac{1}{n}(B_{X^*} \times B_{Y^*}),$$

where $g(x, y) := \|y - G(\bar{x})\|$ and $\|(x, y)\|_n := \|x\| + \eta_n\|y\|$ for all $(x, y) \in X \times Y$. It follows from the fact $\hat{y}_n \neq G(\bar{x})$ that

$$\begin{aligned}(0,0) &\in 0 \times S_{Y^*} + \widehat{N}(\mathrm{epi}_K(G), (x_n, y_n)) + t(B_{X^*} \times \eta_n B_{Y^*}) + \frac{1}{n}(B_{X^*} \times B_{Y^*}) \\ &= 0 \times S_{Y^*} + \widehat{N}(\mathrm{epi}_K(G), (x_n, y_n)) + (t + \frac{1}{n})B_{X^*} \times (t\eta_n + \frac{1}{n})B_{Y^*}.\end{aligned}$$

Thus there exist $x_n^* \in (t + \frac{1}{n})B_{X^*}$, $y_n^* \in S_{Y^*}$ and $z_n^* \in (t\eta_n + \frac{1}{n})B_{Y^*}$ such that

$$(x_n^*, -y_n^* - z_n^*) \in \widehat{N}(\mathrm{epi}_K(G), (x_n, y_n)). \tag{3.12}$$

It follows from Lemma 2.2 (i) that $y_n^* + z_n^* \in K^+$. By (3.4), we have that

$$\sigma_C(y_n^* + z_n^*) \geq \|y_n^* + z_n^*\| \geq 1 - t\eta_n - \frac{1}{n}.$$

Noting that C is compact, thus there exists $h_n \in C$ such that

$$\langle y_n^* + z_n^*, h_n \rangle \geq 1 - t\eta_n - \frac{1}{n}. \tag{3.13}$$

Since $\{x_n^*\}$, $\{y_n^*\}$ and $\{z_n^*\}$ are bounded, we can assume that $x_n^* \xrightarrow{w^*} x^*$ for some $x^* \in tB_{X^*}$ and that $y_n^* + z_n^* \xrightarrow{w^*} y^*$ for some $y^* \in K^+$ (taking a subsequence if necessary). Thanks to the compactness of C, we can assume that $h_n \to h$ for some $h \in C$ (taking a subsequence if necessary). Let $n \to \infty$ in (3.13), we have $1 \leq \langle y^*, h \rangle \leq M\|y^*\|$ and $\|y^*\| \geq \frac{1}{M}$. It follows from (3.12) and Lemma 2.1 (iii) that

$$(x^*, -y^*) \in \overline{N}(\mathrm{epi}_K(G), (\bar{x}, G(\bar{x}))) \subset N(\mathrm{epi}_K(G), (\bar{x}, G(\bar{x}))),$$

and hence $\frac{x^*}{\|y^*\|} \in D_e^* G(\bar{x})(\frac{y^*}{\|y^*\|}) \subset \partial_K G(\bar{x})$. Since $G \in \mathcal{F}_1$, Lemma 3.2 tells us that $\partial_K G(\bar{x}) = \partial_K(F + H)(\bar{x}) \subset \partial_K F(\bar{x}) + \|H\|_{\bar{x}} B_{X^*}$. Thus we have that $\frac{x^*}{\|y^*\|} \in \partial_K F(\bar{x}) + \|H\|_{\bar{x}} B_{X^*}$. It follows from the facts $\|H\|_{\bar{x}} \leq D - \kappa M$, $\|x^*\| \leq t$ and $\|y^*\| \geq \frac{1}{M}$ that

$$d(0, \partial_K F(\bar{x})) \leq \frac{\|x^*\|}{\|y^*\|} + \|H\|_{\bar{x}} \leq tM + D - \kappa M,$$

and so $\kappa \leq t$ (noting that $D = d(0, \partial_K F(\bar{x}))$). Let $t \to \mathrm{Er}(G)$, we have $\mathrm{Er}(G) \geq \kappa$. For the second case when $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$, by Theorem ZN1, we have that

$$\mathrm{Er}(G) = \mathrm{Er}(F + H) \geq \frac{D - \|H\|_{\bar{x}}}{1+\varepsilon} \geq \frac{\kappa M}{1+\varepsilon} \geq \frac{\kappa}{1+\varepsilon}$$

for every $\varepsilon \in (0, +\infty)$. Letting $\varepsilon \to 0$, we have $\mathrm{Er}(G) \geq \kappa$. Hence (3.11) holds. We complete the proof of (ii). For (i), noting that $\mathcal{R}_1 \geq \mathcal{R}_1(\kappa)$ for all $\kappa < \frac{D}{M}$ and letting $\kappa \to 0$ in (3.10), we have $\mathcal{R}_1 \geq D$. □

Remark 3.4. In Theorem 3.3, M can be taken as $\max_{h \in C} \|h\|$.

Note that, in Theorem 3.3, we assume that X and Y are Asplund spaces. We don't know whether or not Theorem 3.3 holds if the Asplund space assumption on X and Y is dropped. However, with $\mathcal{F}_2$ or $\mathcal{F}_3$ replacing $\mathcal{F}_1$, we have the following theorems.

Theorem 3.5. *Let X and Y be Banach spaces, K be dually compact and $K \neq Y$. If F is w-quasi-subsmooth at $\bar{x}$ and $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$, then*

(i)
$$\mathcal{R}_2 \geq D, \tag{3.14}$$

where D is defined by (1.6).

(ii)
$$\mathcal{R}_2(\kappa) \geq D - \kappa M, \ \forall\ \kappa < \frac{D}{M}, \tag{3.15}$$

where M is a positive constant depending only on K and D is defined by (1.6). *Consequently, F has a stable error bound at $\bar{x}$ with respect to the function class $\mathcal{F}_2$.*

Proof. Since (i) can be obtained by letting $\kappa \to 0$ in (ii), we only prove (ii). Take M as in Theorem 3.3. Similarly to the discussion in the proof of Theorem 3.3, we only need to prove that (3.11) holds for all $\kappa < \frac{D}{M}$ and all $G := F + H \in \mathrm{P}_2(D - \kappa M)$, and we only need to prove (3.11) holds for the case when $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$ (the case when $0 \in \mathrm{int}(\mathrm{cl}(\partial_K F(\bar{x})))$ is same as that of Theorem 3.3). Now, suppose that $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$. Since $G \in \mathrm{P}_2(D - \kappa M)$, we have $G = F + H \in \mathcal{F}_2$ and $\|H\|_{\bar{x}} \leq D - \kappa M$. For any $t > \mathrm{Er}(G)$, by Lemma 3.1, there exists $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times K \times (0, +\infty)$ such that (3.5), (3.6) and (3.7) hold. Combining this with Lemma 2.1 (i), we have that

$$\begin{aligned}(0,0) \in & \partial g(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) + \partial \delta_{\mathrm{epi}_K(G)}(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\ & + t\partial \| \cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\ \subset & 0 \times S_{Y^*} + N(\mathrm{epi}_K G, (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)) + t(B_{X^*} \times \eta_n B_{Y^*})\end{aligned}$$

where g and $\|\cdot\|_n$ are defined by (3.8), and the last inclusion follows from the fact $G(\bar{x}_n) + \bar{c}_n \neq G(\bar{x})$. Thus there exist $x_n^* \in tB_{X^*}$, $y_n^* \in S_{Y^*}$ and $z_n^* \in t\eta_n B_{Y^*}$ such that $(x_n^*, -y_n^* - z_n^*) \in N(\mathrm{epi}_K(G), (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n))$. Noting that $G \in \mathcal{F}_2$, from Lemma 2.2 (ii) and the fact G is continuous on some neighborhood of $\bar{x}$, it follows that $(x_n^*, -y_n^* - z_n^*) \in N(\mathrm{epi}_K(G), (\bar{x}_n, G(\bar{x}_n)))$ and $y_n^* + z_n^* \in K^+$ for all sufficiently large n. Hence we have $\frac{y_n^* + z_n^*}{\|y_n^* + z_n^*\|} \in \mathcal{I}_{K^+}$ and $\frac{x_n^*}{\|y_n^* + z_n^*\|} \in D_e^* G(\bar{x}_n)\left(\frac{y_n^* + z_n^*}{\|y_n^* + z_n^*\|}\right)$(noting that $\|y_n^* + z_n^*\| \geq 1 - t\eta_n$). Since

G is quasi-subsmooth at $\bar{x}$ (noting that $G \in \mathcal{F}_2$), $\bar{x}_n \to \bar{x}$ and $\left\{\frac{x_n^*}{\|y_n^*+z_n^*\|}\right\}$ is bounded, thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\langle \frac{x_n^*}{\|y_n^* + z_n^*\|}, x - \bar{x}_n \right\rangle \le \left\langle \frac{y_n^* + z_n^*}{\|y_n^* + z_n^*\|}, G(x) - G(\bar{x}_n) \right\rangle + \varepsilon \|x - \bar{x}_n\| \tag{3.16}$$

for all $x \in B(\bar{x}, \delta)$ and sufficiently large n. Since $\|\frac{x_n^*}{\|y_n^*+z_n^*\|}\| \le t \times \frac{1}{1-t\eta_n} = \frac{t}{1-t\eta_n} \to t$ and $\|\frac{y_n^*+z_n^*}{\|y_n^*+z_n^*\|}\| = 1$, by taking a subnet if necessary we can assume that $\frac{x_n^*}{\|y_n^*+z_n^*\|} \overset{w^*}{\to} x^*$ for some $x^* \in tB_{X^*}$ and that $\frac{y_n^*+z_n^*}{\|y_n^*+z_n^*\|} \overset{w^*}{\to} y^*$ for some $y^* \in K^+$. It follows from the fact $\frac{y_n^*+z_n^*}{\|y_n^*+z_n^*\|} \in K^+ \subset \{c^* \mid \|c^*\| \le \sigma_C(c^*)\}$ that $\sigma_C\left(\frac{y_n^*+z_n^*}{\|y_n^*+z_n^*\|}\right) \ge \|\frac{y_n^*+z_n^*}{\|y_n^*+z_n^*\|}\| = 1$. Thus there exists $h_n \in C$ such that

$$\left\langle \frac{y_n^* + z_n^*}{\|y_n^* + z_n^*\|}, h_n \right\rangle \ge 1. \tag{3.17}$$

Since C is compact, we can assume that $h_n \to h$ for some $h \in C$ (taking a subnet if necessary). It follows from (3.17) that $1 \le \langle y^*, h\rangle \le M\|y^*\|$ and $\|y^*\| \ge \frac{1}{M}$. By (3.16) and (3.6), we have $\langle x^*, x - \bar{x}\rangle \le \langle y^*, G(x) - G(\bar{x})\rangle + \varepsilon\|x - \bar{x}\|$ for all $x \in B(\bar{x}, \delta)$, and so

$$\begin{aligned} \langle x^*, x - \bar{x}\rangle - \langle y^*, y - G(\bar{x})\rangle &\le \langle x^*, x - \bar{x}\rangle - \langle y^*, G(x) - G(\bar{x})\rangle \\ &\le \varepsilon\|x - \bar{x}\| \\ &\le \varepsilon\|x - \bar{x}\| + \varepsilon\|y - G(\bar{x})\| \end{aligned}$$

for all $(x, y) \in B((\bar{x}, G(\bar{x})), \delta) \cap \mathrm{epi}_K(G)$. Hence $(x^*, -y^*) \in \widehat{N}(\mathrm{epi}_K(G), (\bar{x}, G(\bar{x}))) \subset N(\mathrm{epi}_K(G), (\bar{x}, G(\bar{x})))$ and $\frac{x^*}{\|y^*\|} \in D_e^* G(\bar{x})(\frac{y^*}{\|y^*\|}) \subset \partial_K G(\bar{x})$. Then, by Lemma 3.2, we have that $\frac{x^*}{\|y^*\|} \in \partial_K G(\bar{x}) = \partial_K(F + H)(\bar{x}) \subset \partial_K F(\bar{x}) + \|H\|_{\bar{x}} B_{X^*}$. From the facts $\|H\|_{\bar{x}} \le D - \kappa M$, $\|x^*\| \le t$ and $\|y^*\| \ge \frac{1}{M}$, it follows that

$$d(0, \partial_K F(\bar{x})) \le \frac{\|x^*\|}{\|y^*\|} + \|H\|_{\bar{x}} \le tM + D - \kappa M,$$

so $\kappa \le t$ (noting that $D = d(0, \partial_K F(\bar{x}))$). Let $t \to \mathrm{Er}(G)$, we have $\mathrm{Er}(G) \ge \kappa$. The proof is complete. □

Theorem 3.6. *Let X and Y be Banach spaces, K be dually compact and $K \ne Y$. If F is w-quasi-subsmooth at $\bar{x}$ and $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$, then*

(i)

$$\mathcal{R}_3 \ge D, \tag{3.18}$$

where D is defined by (1.6).

(ii)

$$\mathcal{R}_3(\kappa) \ge D - \kappa M, \ \forall\ \kappa < \frac{D}{M}, \tag{3.19}$$

where M is a positive constant depending only on K and D is defined by (1.6). *Consequently, F has a stable error bound at $\bar{x}$ with respect to the function class $\mathcal{F}_3$.*

Proof. Take M as in Theorem 3.3. We only need to prove that (3.11) holds for all $G := F + H \in \mathrm{P}_3(D - \kappa M)$ in the case when $0 \notin \mathrm{cl}(\partial_K F(\bar{x}))$. Since $G \in \mathrm{P}_3(D - \kappa M)$, we have $G = F + H \in \mathcal{F}_3$ and $\|H\|_{\bar{x}} \leq D - \kappa M$. For any $t > \mathrm{Er}(G)$, by Lemma 3.1, there exists $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times K \times (0, +\infty)$ such that (3.5), (3.6) and (3.7) hold. Combining this with Lemma 2.1 (i), we have that

$$\begin{aligned}(0,0) \in & \partial g(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) + \partial \delta_{\mathrm{epi}_K(G)}(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\ & + t\partial \| \cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\ \subset & 0 \times S_{Y^*} + N(\mathrm{epi}_K(G), (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)) + t(B_{X^*} \times \eta_n B_{Y^*})\end{aligned}$$

where g and $\|\cdot\|_n$ are defined by (3.8), and the last inclusion follows from the fact $G(\bar{x}_n) + \bar{c}_n \neq G(\bar{x})$. Thus there exist $x_n^* \in tB_{X^*}$, $y_n^* \in S_{Y^*}$ and $z_n^* \in t\eta_n B_{Y^*}$ such that $(x_n^*, -y_n^* - z_n^*) \in N(\mathrm{epi}_K(G), (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n))$. Since $G \in \mathcal{F}_3$, we have $\mathrm{epi}_K(G)$ is convex, $y_n^* + z_n^* \in K^+$(by Lemma 2.2 (i)) and

$$\langle x_n^*, x - \bar{x}_n \rangle - \langle y_n^* + z_n^*, y - G(\bar{x}_n) - \bar{c}_n \rangle \leq 0 \quad \forall (x, y) \in \mathrm{epi}_K(G). \tag{3.20}$$

Since $\{x_n^*\}$, $\{y_n^*\}$ and $\{z_n^*\}$ are bounded, by taking a subnet if necessary we can assume that $x_n^* \xrightarrow{w^*} x^*$ for some $x^* \in tB_{X^*}$ and that $y_n^* + z_n^* \xrightarrow{w^*} y^*$ for some $y^* \in K^+$. By (3.20) and (3.6), we have

$$\langle x^*, x - \bar{x} \rangle - \langle y^*, y - G(\bar{x}) \rangle \leq 0 \quad \forall (x, y) \in \mathrm{epi}_K(G),$$

and hence $(x^*, -y^*) \in N(\mathrm{epi}_K(G), (\bar{x}, G(\bar{x})))$. Then, by Lemma 3.2, we have that

$$\frac{x^*}{\|y^*\|} \in D_e^* G(\bar{x}) \left(\frac{y^*}{\|y^*\|} \right) \subset \partial_K G(\bar{x}) = \partial_K (F + H)(\bar{x}) \subset \partial_K F(\bar{x}) + \|H\|_{\bar{x}} B_{X^*}.$$

It follows from the fact $\|x^*\| \leq t$ that

$$d(0, \partial_K F(\bar{x})) \leq \frac{\|x^*\|}{\|y^*\|} + \|H\|_{\bar{x}} \leq \frac{t}{\|y^*\|} + \|H\|_{\bar{x}}. \tag{3.21}$$

On the other hand, by the fact $y_n^* + z_n^* \in K^+ \subset \{c^* \mid \|c^*\| \leq \sigma_C(c^*)\}$, we have that $\sigma_C(y_n^* + z_n^*) \geq \|y_n^* + z_n^*\| \geq 1 - t\eta_n$. Thus there exists $h_n \in C$ such that

$$\langle y_n^* + z_n^*, h_n \rangle \geq 1 - t\eta_n. \tag{3.22}$$

Since C is compact, we can assume that $h_n \to h$ for some $h \in C$ (taking a subnet if necessary). If follows from (3.22) that $1 \leq \langle y^*, h \rangle \leq M\|y^*\|$ and $\|y^*\| \geq \frac{1}{M}$. Hence, by (3.21) and the fact that $\|H\|_{\bar{x}} \leq D - \kappa M$, we have that

$$d(0, \partial_K F(\bar{x})) \leq tM + D - \kappa M,$$

which implies that $t \geq \kappa$. Let $t \to \mathrm{Er}(G)$, we have $\mathrm{Er}(G) \geq \kappa$. we complete the proof. $\square$

Remark 3.7. Our Theorem 3.3, Theorem 3.5 and Theorem 3.6 proved that $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$ is a sufficient condition for F to have a stable error bound at $\bar{x}$ under the assumption that the ordering cone K is dually compact. Zheng and Ng [32] proved the same conclusion under the assumption that the ordering cone K has a nonempty interior (see Theorem 1.1 and Theorem 1.2 in section1). Our results extend some results in [32] from the case that the ordering cone K has a nonempty interior to the case that the ordering cone K is dually compact. It is worth mentioning that, under the assumption $\mathrm{int}(K) \neq \emptyset$ and some other mild assumptions, Zheng and Ng proved that $0 \notin \mathrm{bd}(\partial_K F(\bar{x}))$ is also a necessary condition for F to have a stable error bound at $\bar{x}$.

Remark 3.8. In the real-valued and convex setting, for a proper convex lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, Kruger, López and Théra [10] proved that $\mathcal{R}_3 = D$. This implies that: if a lower semicontinuous function $h : X \to \mathbb{R}$ satisfies that $\|h\|_{\bar{x}} < d(0, \mathrm{bd}(\partial f(\bar{x})))$ and that $f + h$ is convex, then $f + h$ has a local error bound at $\bar{x}$. For the conic inequality setting (not necessarily convex), our Theorem 3.3 (i), Theorem 3.5 (i), Theorem 3.6 (i) show that the radius $\mathcal{R}_i$ $(i = 1, 2, 3)$ of error bound for (CIE) with respect to the function classes $\mathcal{F}_i$ $(i = 1, 2, 3)$ are no less than D. Furthermore, we considered another kind of radius of error bound for (CIE): $\mathcal{R}_i(\kappa)$ $(i = 1, 2, 3)$. We proved that $\mathcal{R}_i(\kappa)$ $(i = 1, 2, 3)$ are no less than $D - \kappa M$. This means that $\mathrm{Er}(F + H) \geq \kappa$ when $H : X \to Y$ satisfies $\|H\|_{\bar{x}} < D - \kappa M$ and $F + H \in \mathcal{F}_i$.

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