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GLOBALLY CONVERGENT INVERSE ITERATION ALGORITHM FOR FINDING THE LARGEST EIGENVALUE OF A NONNEGATIVE WEAKLY IRREDUCIBLE TENSOR*

Zhou Sheng and Qin Ni

Abstract: In this paper, we propose an inverse iteration algorithm for finding the largest eigenvalue of a nonnegative weakly irreducible tensor. The positive property of approximate eigenvector is preserved at each iteration for any initial positive vector, as we all know, this is crucial during the computation. The proposed algorithm involves a multilinear equation at each iteration, which can be solved by the Newton method. An important part of the paper consists of proving that the algorithm is globally convergent. Numerical examples are reported to illustrate the proposed algorithm is efficient and promising. We show an application of this algorithm to determine the positive definiteness of a weakly irreducible $\mathcal{Z}$-tensor, which is done on this $\mathcal{Z}$-tensor directly. The numerical results indicated that it is capable of testing the positive definiteness of weakly irreducible $\mathcal{Z}$-tensors.

Key words: *nonnegative tensors, Inverse iteration, largest eigenvalue, global convergence, $\mathcal{Z}$-tensors*

Mathematics Subject Classification: *65F15, 15A18, 15A69*

1 Introduction

In this paper, we consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$ as follows:

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_m}), \quad a_{i_1 i_2 \ldots i_m} \in \mathbb{R}, \quad 1 \le i_1, i_2, \ldots, i_m \le n,$$

where $\mathbb{R}$ is the real field. Tensors have applications in many areas such as spectral hypergraph theory [8], higher order network analysis [3, 24, 2, 7], higher order Markov chains [18], and the stability study of nonlinear autonomous systems [20]. The set $\mathbb{R}^{[m,n]}$ stands for all m-order n-dimensional real-valued tensors. For a given $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and an n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, we define a homogeneous polynomial of m-th degree as follows:

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}. \tag{1.1}$$

It is easy to see that $\mathcal{A}\mathbf{x}^m$ is a scalar.

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For any $\mathbf{x} \in \mathbb{R}^n$, we define n-dimensional column vectors $\mathcal{A}\mathbf{x}^{m-1} \in \mathbb{R}^n$ and $n \times n$ matrices $\mathcal{A}\mathbf{x}^{m-2} \in \mathbb{R}^{n\times n}$, respectively, as follows:

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m} \right)_{1\le i\le n}, \tag{1.2}$$

$$\mathcal{A}(\mathbf{x}) \overset{\text{def}}{=} \mathcal{A}\mathbf{x}^{m-2} = \left(\sum_{i_3,\ldots,i_m=1}^{n} a_{iji_3\ldots i_m} x_{i_3} \ldots x_{i_m} \right)_{1\le i,j\le n}. \tag{1.3}$$

Qi [22] and Lim [14] independently first defined eigenpairs of tensors in 2005. Let $\mathcal{A}$ be an m-order n-dimensional tensor, we say $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$, if there exists $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{1.4}$$

a vector $\mathbf{x}$ is an eigenvector associated with the eigenvalue λ, and $(\lambda, \mathbf{x})$ is called an eigenpair, where $\mathbf{x}^{[m-1]}$ is a vector, whose i-th entry is defined by $(\mathbf{x}^{[m-1]})_i = x_i^{m-1}$. The largest eigenvalue (spectral radius) of $\mathcal{A}$ is the maximum modulus of the eigenvalues of $\mathcal{A}$, which is denoted by $\rho(\mathcal{A})$.

An m-order n-dimensional tensor is called nonnegative (resp., positive) if all entries are nonnegative (resp., positive), that is, $a_{i_1i_2\ldots i_m} \ge 0$ (resp., > 0) for all $1 \le i_1, i_2, \ldots, i_m \le n$. The set of all real nonnegative tensors of order m and dimension n is denoted by $\mathbb{R}_+^{[m,n]}$. In addition, a tensor $\mathcal{A}$ is called weakly reducible [11] if there exists a nonempty proper index subset $\Omega \subset \{1, 2, \ldots, n\}$ such that

$$\mathcal{A}_{i_1i_2\ldots i_m} = 0, \text{ whenever } i_1 \in \Omega \text{ and at least one index in } \{i_2, \ldots, i_m\} \text{ does not belong to } \Omega.$$

If $\mathcal{A}$ is not weakly reducible, then we call it weakly irreducible.

The Perron-Frobenius theorem is extended to a nonnegative weakly irreducible tensor in [11]. For convenience, we denote $\max\{\frac{\mathbf{x}}{\mathbf{y}}\} \overset{\text{def}}{=} \max_{1\le i\le n}\{\frac{x_i}{y_i}\}$ and $\min\{\frac{\mathbf{x}}{\mathbf{y}}\} \overset{\text{def}}{=} \min_{1\le i\le n}\{\frac{x_i}{y_i}\}$ for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$.

Theorem 1.1 ([11, Theorem 4.1])**.** *Let $\mathcal{A}$ be a weakly irreducible tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. Then there exists a unique positive eigenvector $\mathbf{x}_* > \mathbf{0}$ corresponding to the largest eigenvalue $\rho(\mathcal{A})$ up to a multiplicative constant. Moreover, for any $\mathbf{x} > \mathbf{0}$, we have*

$$\min\left\{\frac{\mathcal{A}\mathbf{x}^{m-1}}{\mathbf{x}^{[m-1]}}\right\} \le \rho(\mathcal{A}) \le \max\left\{\frac{\mathcal{A}\mathbf{x}^{m-1}}{\mathbf{x}^{[m-1]}}\right\}.$$

In [18], an algorithm for computing the largest eigenvalue of nonnegative tensors, extending the Collatz's method for nonnegative matrices, was proposed by Ng, Qi and Zhou, so we often called it NQZ for short. In Chang, Pearson and Zhang [5], the authors proved the convergence of NQZ for primitive tensors. Its linear convergence was studied in [28, 12]. As its variation, Liu, Zhou and Ibrahim [17] proposed an always convergent algorithm for finding the largest eigenvalue of any irreducible nonnegative tensors by using shift technique. Ni and Qi [19] presented the Newton method for finding the largest eigenvalue of nonnegative tensors by revealing the relation between the homogenous polynomial map and its associated semi-symmetric tensor. Based on nonsingular $\mathcal{M}$ equations, Ding and Wei [9] generalized inverse iteration (or Noda iteration) [21] for solving the largest eigenvalue of nonnegative tensors. By combining the idea of the Newton method with Noda iteration, Liu, Guo and Lin [15, 16] presented a positivity preserving Newton-Noda iteration (NNI for short) algorithm, for computing the largest eigenvalue of a nonnegative (weakly) irreducible tensor. In their algorithm, the authors used practical procedure for choosing θ_k to guarantee the global convergence. In [23], we have presented a local quadratically convergent and positivity preserving algorithm, by combining inverse iteration and the Newton method, to compute the largest eigenvalue of nonnegative weakly irreducible tensors. Some recent papers on the largest eigenvalue of nonnegative tensors, we refer to [25, 6, 30, 26].

We stress that, different from [23], the goal there is to present a globally convergent inverse iteration algorithm, to find the largest eigenvalue of a nonnegative weakly irreducible tensor, whereas the algorithm in [23] does not have global convergence theory, the algorithm in [23] is essentially a modified Newton's method. The presentation of the proposed inverse iteration algorithm of this paper is also different from the inverse iteration algorithm in Ding and Wei [9]. As an important topic of this paper, we present that, by reformulating (1.4), a differently global inverse iteration algorithm for finding the largest eigenvalue of a nonnegative weakly irreducible tensor.

In this paper, we present an inverse iteration algorithm for finding the largest eigenvalue of a nonnegative weakly irreducible tensor, which is a positivity preserving algorithm. We show that a series of properties on the proposed algorithm, and also use the Newton method to get a positive solution of the $\mathcal{M}$-like equation produced by the proposed algorithm at each iteration. We prove that it is a globally convergent algorithm, and apply its derived algorithm to test the positive definiteness of an even order weakly irreducible $\mathcal{Z}$-tensor. Some results illustrate that they are stable and fast convergence in numerical experiments.

The paper is organized as follows. In Section 2, we present an inverse iteration algorithm and prove a nice property for this algorithm. In Section 3, we reformulate $\mathcal{M}$-like equation as a nonlinear system of equations, and apply the Newton method to solve it. In Section 4, the global convergence of the proposed algorithm is established. In Section 5, we present a derived algorithm to determine the positive definiteness of an even order weakly irreducible $\mathcal{Z}$-tensor. Some numerical results are reported in Section 6. Finally, some concluding remarks are given in Section 7.

Throughout the paper, $\|\cdot\|$ denotes the 2-norm or its induced matrix norm, and all matrices are $n\times n$ unless specified otherwise, the superscript T denotes the transpose of a vector or matrix.

$\boxed{2}$ Inverse Iteration Algorithm

In this section, we will present an inverse iteration algorithm for computing the largest eigenvalue of a nonnegative weakly irreducible tensor $\mathcal{A}$. Let $\mathrm{Diag}(\mathbf{x})$ be a diagonal matrix generated by $\mathbf{x}\in\mathbb{R}^n$, that is, $\mathrm{Diag}(\mathbf{x}) \overset{\text{def}}{=} \mathrm{Diag}\{x_1,\dots,x_n\}$.

For the purpose of computing the largest eigenvalue of $\mathcal{A}$, it follows from Theorem 1.1 that its associated eigenvector $\mathbf{x}_*$ is positive. For any $\mathbf{x}>\mathbf{0}$, then we give an equivalent formulation for (1.4), that is,

$$\mathrm{Diag}(\mathbf{x}^{[2-m]})\mathcal{A}\mathbf{x}^{m-1}=\lambda\mathbf{x}. \tag{2.1}$$

This is a vital step in the development of our inverse iteration algorithm for finding the largest eigenvalue of $\mathcal{A}$.

A matrix $\mathbf{A}\in\mathbb{R}^{n\times n}$ is a nonnegative (resp., positive) matrix, if its all entries are nonnegative (resp., positive), denoted by $\mathbf{A}\geq\mathbf{0}$ (resp., $A>\mathbf{0}$). $\mathbf{A}\in\mathbb{R}^{n\times n}$ is called a *Z-matrix* if its all off-diagonal entries are non-positive. Any a Z-matrix A can be written as form of

$\mathbf{A} = s\mathbf{I} - \mathbf{B}$ with $\mathbf{B} \geq \mathbf{0}$. If $s > \rho(\mathbf{B})$, then $\mathbf{A}$ is called a *nonsingular M-matrix*, and A is called a singular M-matrix if $s = \rho(\mathbf{B})$, where $\rho(\mathbf{B})$ is the largest eigenvalue of $\mathbf{B}$. From Berman and Plemmons [4], we give the following theorems for a M-matrix.

Theorem 2.1 ([4, pp. 134-138])**.** *For a Z-matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) $\mathbf{A}$ *is a nonsingular M-matrix;*

(ii) $\mathbf{A}\mathbf{x} > \mathbf{0}$ *for some* $\mathbf{x} > \mathbf{0}$*;*

(iii) $\mathbf{A}^{-1} \geq \mathbf{0}$.

Theorem 2.2 ([4, Theorem 2.7])**.** *The following two statements are true:*

(i) *an irreducible Z-matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if for some $\mathbf{x} > \mathbf{0}$ the vector $\mathbf{A}\mathbf{x}$ is nonnegative and nonzero;*

(ii) $\mathbf{A}$ *is an irreducible nonsingular M-matrix if and only if* $\mathbf{A}^{-1} > \mathbf{0}$.

From Ni and Qi [19], for a given $\mathcal{A}$, there always exists a semi-symmetric tensor $\mathcal{A}_s$ such that $\mathcal{A}\mathbf{x}^{m-1} = \mathcal{A}_s\mathbf{x}^{m-1}$ for any $\mathbf{x} \in \mathbb{R}^n$. So we assume that $\mathcal{A}$ is semi-symmetric (see [19, Definition 2.1]) in this paper. The following two lemmas give some basic properties on (2.1), the detailed proofs can be seen in [23, Lemmas 2 and 3].

Lemma 2.3. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. Then, Diag$(\mathbf{x}^{[2-m]})\mathcal{A}(\mathbf{x})$ and $\mathcal{A}(\mathbf{x})$ are nonnegative irreducible matrices for any $\mathbf{x} > \mathbf{0}$, where $\mathcal{A}(\mathbf{x})$ is defined in* (1.3)*.*

Lemma 2.4. *If $\rho(\mathcal{A})$ is the largest eigenvalue of a weakly irreducible tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, its associated eigenvector is $\mathbf{x}_*$, then $\mathbf{x}_*$ is also an eigenvector corresponding to the largest eigenvalue $\rho($Diag$(\mathbf{x}_*^{[2-m]})\mathcal{A}(\mathbf{x}_*))$ of Diag$(\mathbf{x}_*^{[2-m]})\mathcal{A}(\mathbf{x}_*)$, and vice versa.*

In what follows, we prove that a nice property of $\mathrm{Diag}(\mathbf{x}^{[2-m]})\mathcal{A}(\mathbf{x})$, which shows it is an irreducible nonsingular M-matrix for every $\mathbf{x} > \mathbf{0}$.

Theorem 2.5. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. If $s > \rho(\mathcal{A})$, then $s\mathbf{I} -$ Diag$(\mathbf{x}^{[2-m]})\mathcal{A}(\mathbf{x})$ is an irreducible nonsingular M-matrix for every $\mathbf{x} > \mathbf{0}$.*

Proof. For convenience, we denote $\mathbf{G}(\mathbf{x}) := s\mathbf{I} - \mathrm{Diag}(\mathbf{x}^{[2-m]})\mathcal{A}(\mathbf{x})$. It follows from Lemma 2.3 that $\mathrm{Diag}(\mathbf{x}^{[2-m]})\mathcal{A}(\mathbf{x})$ is a nonnegative irreducible matrix for any $\mathbf{x} > \mathbf{0}$, then $\mathbf{G}(\mathbf{x})$ is an irreducible Z-matrix. It is also easy to compute that $\mathrm{Diag}(\mathbf{x}^{[m-2]})\mathbf{G}(\mathbf{x}) = s \cdot \mathrm{Diag}(\mathbf{x}^{[m-2]}) - \mathcal{A}(\mathbf{x})$.

According to $\mathbf{G}(\mathbf{x})$ is a Z-matrix, and note that $\mathrm{Diag}(\mathbf{x}^{[m-2]})$ is a positive diagonal matrix, then we have $\mathrm{Diag}(\mathbf{x}^{[m-2]})G(\mathbf{x})$ is also a Z-matrix. Moreover,

$$\mathbf{g}(\mathbf{x}) := \mathrm{Diag}(\mathbf{x}^{[m-2]})\mathbf{G}(\mathbf{x})\mathbf{x} = s\mathbf{x}^{[m-1]} - \mathcal{A}\mathbf{x}^{m-1}. \tag{2.2}$$

Note that $s > \rho(\mathcal{A})$, then we get

$$s\mathbf{x}^{[m-1]} - \mathcal{A}\mathbf{x}^{m-1} > \rho(\mathcal{A})\mathbf{x}^{[m-1]} - \mathcal{A}\mathbf{x}^{m-1}.$$

Now we prove $\mathbf{g}(\mathbf{x})$ is a nonnegative and nonzero vector for every $\mathbf{x} > \mathbf{0}$. If it is not true, then $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$, that is, $sx_i^{m-1} \leq (\mathcal{A}\mathbf{x}^{m-1})_i$ for any $i \in \{1, 2, \ldots, n\}$, which implies $s \leq \frac{(\mathcal{A}\mathbf{x}^{m-1})_i}{x_i^{m-1}}$. Thus, $s \leq \min\{\frac{\mathcal{A}\mathbf{x}^{m-1}}{\mathbf{x}^{[m-1]}}\}$, contradictory to $s > \rho(\mathcal{A}) \geq \min\{\frac{\mathcal{A}\mathbf{x}^{m-1}}{\mathbf{x}^{[m-1]}}\}$. Hence,

$\mathbf{g}(\mathbf{x})$ is a nonnegative and nonzero vector for every $\mathbf{x} > \mathbf{0}$, this means that $\mathrm{Diag}(\mathbf{x}^{[m-2]})\mathbf{G}(\mathbf{x})$ is a nonsingular M-matrix from Theorem 2.2 (i).

Since $\mathrm{Diag}(\mathbf{x}^{[2-m]})$ is also a positive diagonal matrix, and $\mathbf{g}(\mathbf{x})$ is a nonnegative and nonzero vector for every $\mathbf{x} > \mathbf{0}$, it follows from (2.2) that $\mathbf{G}(\mathbf{x})\mathbf{x} = \mathrm{Diag}(\mathbf{x}^{[2-m]})\mathbf{g}(\mathbf{x}) > \mathbf{0}$ is a nonnegative and nonzero vector for every $\mathbf{x} > \mathbf{0}$. Together with Theorem 2.2 (i) again, we have that $\mathbf{G}(\mathbf{x})$ is an irreducible nonsingular M-matrix for every $\mathbf{x} > \mathbf{0}$, and thus, the proof is completed. □

By Theorem 2.5 and inverse iteration algorithm for nonnegative irreducible matrices, we give the detailed steps of inverse iteration algorithm for nonnegative weakly irreducible tensors, see Algorithm 1.

Algorithm 1 Inverse iteration algorithm for nonnegative weakly irreducible tensors.

1: **Initialization:** Given an initial point $\mathbf{x}_0 > \mathbf{0}$ with $\|\mathbf{x}_0\| = 1$ and tol > 0, compute $\overline{\lambda}_0 = \max\left\{\frac{\mathcal{A}\mathbf{x}_0^{m-1}}{\mathbf{x}_0^{[m-1]}}\right\}$ and $\underline{\lambda}_0 = \min\left\{\frac{\mathcal{A}\mathbf{x}_0^{m-1}}{\mathbf{x}_0^{[m-1]}}\right\}$.

2: **repeat**

3: Compute $\mathbf{y}_k$ by solving $\left[\overline{\lambda}_k\mathbf{I} - \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k)\right]\mathbf{y}_k = \mathbf{x}_k$.

4: Normalize the vector $\mathbf{x}_{k+1} = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}$.

5: Compute $\overline{\lambda}_{k+1} = \max\left\{\frac{\mathcal{A}\mathbf{x}_{k+1}^{m-1}}{\mathbf{x}_{k+1}^{[m-1]}}\right\}$ and $\underline{\lambda}_{k+1} = \min\left\{\frac{\mathcal{A}\mathbf{x}_{k+1}^{m-1}}{\mathbf{x}_{k+1}^{[m-1]}}\right\}$.

6: **until** $\frac{\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1}}{\overline{\lambda}_{k+1}} \leq \mathrm{tol}$.

7: **Output:** The largest eigenvalue $\rho(\mathcal{A}) \leftarrow \overline{\lambda}_{k+1}$ and its associated eigenvector $\mathbf{x}_* \leftarrow \mathbf{x}_{k+1}$.

It follows from the step 4 of Algorithm 1 and (1.3) that

$$
\begin{aligned}
\mathrm{Diag}(\mathbf{x}_{k+1}^{[2-m]})\mathcal{A}(\mathbf{x}_{k+1}) &= \mathrm{Diag}\left(\frac{\mathbf{y}_k^{[2-m]}}{\|\mathbf{y}_k\|^{2-m}}\right)\mathcal{A}\left(\frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}\right) \\
&= \frac{1}{\|\mathbf{y}_k\|^{2-m}\|\mathbf{y}_k\|^{m-2}}\mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k) \\
&= \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k).
\end{aligned} \tag{2.3}
$$

Note that, as shown in Theorem 2.5, if $\mathbf{y}_k$ is not an eigenvector of $\mathcal{A}$, $\overline{\lambda}_k\mathbf{I} - \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k)$ is an irreducible nonsingular M-matrix provided that $\mathbf{y}_k > \mathbf{0}$ at each iteration of Algorithm 1, which also ensures the positive property of approximate eigenvector. So, for the multi-linear equations in the step 3 of Algorithm 1, we call it as $\mathcal{M}$-like equation, which is different with the multi-linear $\mathcal{M}$ equation discussed in [9]. We will give a method (Newton's method) to get a positive solution by solving $\mathcal{M}$-like equation in the next section. We also remark that, the given inverse iteration algorithm in [9] need to solve an $\mathcal{M}$ equation at each iteration, but it is unknown whether or not the Newton method still work for solving $\mathcal{M}$ equation, when the given tensor is not symmetric, as the authors have pointed out in [9]. However, we can employ the Newton method to obtain a positive solution by solving $\mathcal{M}$-like equation at each iteration of Algorithm 1 for any nonnegative weakly irreducible tensors, the symmetric property of $\mathcal{A}$ is not required.

$\boxed{3}$ Solving $\mathcal{M}$-Like Equation Via the Newton Method

Consider $\mathcal{M}$-like equation $\big[s\mathbf{I}-\mathrm{Diag}(\mathbf{w}^{[2-m]})\mathcal{A}(\mathbf{w})\big]\mathbf{w}=\mathbf{b}$ for every $\mathbf{b}>\mathbf{0}$, where $s>\rho(\mathcal{A})$. This $\mathcal{M}$-like equation is equivalent to the following nonlinear system of equations

$$\begin{aligned}\mathbf{F}(\mathbf{w}) &\overset{\text{def}}{=} \big[s\mathbf{I}-\mathrm{Diag}(\mathbf{w}^{[2-m]})\mathcal{A}(\mathbf{w})\big]\mathbf{w}-\mathbf{b}\\ &= s\mathbf{w}-\mathrm{Diag}(\mathbf{w}^{[2-m]})\mathcal{A}\mathbf{w}^{m-1}-\mathbf{b}\\ &= \mathbf{0}.\end{aligned} \tag{3.1}$$

Lemma 3.1 ([23, Lemma 4])**.** *Let $\mathcal{A}\in\mathbb{R}^{[m,n]}$, for any $\mathbf{w}>\mathbf{0}$, then*

$$\begin{aligned}\mathbf{J}(\mathbf{w}) &\overset{def}{=} \frac{\partial}{\partial\mathbf{w}}(Diag(\mathbf{w}^{[2-m]})\mathcal{A}\mathbf{w}^{m-1})\\ &= (m-1)Diag(\mathbf{w}^{[2-m]})\mathcal{A}\mathbf{w}^{m-2}-(m-2)Diag(\mathbf{w}^{[1-m]})Diag(\mathcal{A}\mathbf{w}^{m-1}),\end{aligned} \tag{3.2}$$

and

$$\mathbf{J}(\mathbf{w})\mathbf{w}=Diag(\mathbf{w}^{[2-m]})\mathcal{A}(\mathbf{w})\mathbf{w}=Diag(\mathbf{w}^{[2-m]})\mathcal{A}\mathbf{w}^{m-1}. \tag{3.3}$$

We use the Newton method to solve nonlinear system of equations (3.1). Note from (3.2) in Lemma 3.1 that, we have the following Newton's iteration:

$$\nabla\mathbf{F}(\mathbf{w}_k)(\mathbf{w}_{k+1}-\mathbf{w}_k)=[s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)](\mathbf{w}_{k+1}-\mathbf{w}_k)=-\mathbf{F}(\mathbf{w}_k),\quad k=0,1,\ldots. \tag{3.4}$$

By some calculations on (3.4), we get

$$\begin{aligned}&[s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)]\mathbf{w}_{k+1}\\ =\;&[s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)]\mathbf{w}_k-\mathbf{F}(\mathbf{w}_k)\\ =\;&s\mathbf{w}_k-\mathbf{J}(\mathbf{w}_k)\mathbf{w}_k-s\mathbf{w}_k+\mathrm{Diag}(\mathbf{w}_k^{[2-m]})\mathcal{A}\mathbf{w}_k^{m-1}+\mathbf{b}\\ =\;&\mathbf{b},\end{aligned} \tag{3.5}$$

where the last equality is obtained from (3.3). Now, we need to show that $s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)$ is an irreducible nonsingular M-matrix for any $\mathbf{w}_k>\mathbf{0}$.

Theorem 3.2. *Let $\mathcal{A}\in\mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. If $s>\rho(\mathcal{A})$, then $s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)$ is an irreducible nonsingular M-matrix for every $\mathbf{w}_k>\mathbf{0}$.*

Proof. From (3.2) in Lemma 3.1 and the fact that $\mathrm{Diag}(\mathbf{w}_k^{[2-m]})\mathcal{A}(\mathbf{w}_k)$ is a nonnegative irreducible matrix for any $\mathbf{w}_k>\mathbf{0}$, we then have $s\mathbf{I}-J(\mathbf{w}_k)$ is an irreducible Z-matrix since its all off-diagonal entries are non-positive. Moreover, it follows from Theorem 2.5 and (3.3) that

$$\begin{aligned}&[s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)]\mathbf{w}_k\\ =\;&s\mathbf{w}_k-\mathrm{Diag}(\mathbf{w}_k^{[2-m]})\mathcal{A}(\mathbf{w}_k)\mathbf{w}_k\\ =\;&[s\mathbf{I}-\mathrm{Diag}(\mathbf{w}_k^{[2-m]})\mathcal{A}(\mathbf{w}_k)]\mathbf{w}_k\\ >\;&\mathbf{0},\end{aligned}$$

which, together with Theorem 2.1, means that $s\mathbf{I}-\mathbf{J}(\mathbf{w}_k)$ is an irreducible nonsingular M-matrix. The proof has been completed. □

Algorithm 2 The Newton method for solving (3.1).

1: **Initialization:** Given an initial point $\mathbf{w}_0 > \mathbf{0}$ and tol > 0, compute $\mathbf{F}(\mathbf{w}_0)$ and $\nabla\mathbf{F}(\mathbf{w}_0) = s\mathbf{I} - \mathbf{J}(\mathbf{w}_0)$.
2: **repeat**
3: Compute $\mathbf{w}_{k+1}$ by solving $[s\mathbf{I} - \mathbf{J}(\mathbf{w}_k)]\mathbf{w}_{k+1} = \mathbf{b}$.
4: **until** $\|\mathbf{F}(\mathbf{w}_{k+1})\| \leq$ tol.
5: **Output:** $\mathbf{w}_{k+1}$ as a solution of (3.1).

Hence, note that Theorem 2.2 (ii), (3.5) can be rewritten as $\mathbf{w}_{k+1} = [s\mathbf{I} - \mathbf{J}(\mathbf{w}_k)]^{-1}\mathbf{b} > \mathbf{0}$ provided that $\mathbf{w}_k > \mathbf{0}$. Further, we can give a detailed step to solve $\mathcal{M}$-like equation, see Algorithm 2.

It should be pointed out that the computational cost can be saved, if we use LU decomposition of matrix $s\mathbf{I} - J(\mathbf{w}_k)$, instead of computing its inversion directly. In the procedure of Algorithm 2, we will always have $\mathbf{w}_{k+1} > \mathbf{0}$ by setting the initial point $\mathbf{w}_0 > \mathbf{0}$. Therefore, Algorithm 2 is a positivity preserving algorithm in the sense of the computed vector $\mathbf{w}_{k+1}$.

In what follows, it is easy to show that the positive property of the sequence $\{\mathbf{x}_k\}$ produced by Algorithm 1, with an initial point $\mathbf{x}_0 > \mathbf{0}$.

Lemma 3.3. *Let $\{\overline{\lambda}_k, \mathbf{x}_k\}$ be generated by Algorithm 1, with an initial point $\mathbf{x}_0 > \mathbf{0}$. If we employ Algorithm 2 to solve $\mathcal{M}$-like equations in the step 3 of Algorithm 1, then $\mathbf{x}_k > \mathbf{0}$ for all $k = 1, 2, \ldots$.*

Proof. Note that $\overline{\lambda}_0 > \rho(\mathcal{A})$ and $\mathbf{x}_0 > \mathbf{0}$, then the solution output by Algorithm 2 as $\mathbf{y}_0 > \mathbf{0}$ because we employ Algorithm 2 to solve $[\overline{\lambda}_0\mathbf{I} - \mathrm{Diag}(\mathbf{y}_0^{[2-m]})\mathcal{A}(\mathbf{y}_0)]\mathbf{y}_0 = \mathbf{x}_0 > \mathbf{0}$. According to the step 4 of Algorithm 1, we have that $\mathbf{x}_1 = \frac{\mathbf{y}_0}{\|\mathbf{y}_0\|} > \mathbf{0}$. Continuing this iterative procedure $k = 2, 3, \ldots$, we also have $\mathbf{x}_k > \mathbf{0}$, and thus, the proof is completed. □

4 Convergence Analysis

In this section, we will investigate the global convergence of Algorithm 1. The $\mathcal{M}$-like equation of Algorithm 1 is solved by Algorithm 2 in this paper, and thus the positive property of $\mathbf{x}_k$ is ensured from Lemma 3.3.

Now, we will prove that Algorithm 1 is a globally convergent algorithm. The following theorem shows that the sequence $\{\overline{\lambda}_k\}$ is monotonically decreasing and bounded below by $\rho(\mathcal{A})$.

Theorem 4.1. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. If $\{\overline{\lambda}_k, \mathbf{x}_k, \mathbf{y}_k\}$ is generated by Algorithm 1, then the sequence $\{\overline{\lambda}_k\}$ is monotonically decreasing and bounded by $\rho(\mathcal{A})$ from below.*

Proof. From Lemma 3.3, it follows that the vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ satisfy $\mathbf{x}_k > \mathbf{0}$ and $\mathbf{y}_k > \mathbf{0}$

for all $k = 0, 1, \ldots$. Then, by the definition of $\overline{\lambda}_{k+1}$, and (1.3), we have

$$\begin{aligned}
\overline{\lambda}_{k+1} &= \max\left\{\frac{\mathcal{A}\mathbf{x}_{k+1}^{m-1}}{\mathbf{x}_{k+1}^{[m-1]}}\right\} \\
&= \max\left\{\frac{\operatorname{Diag}(\mathbf{x}_{k+1}^{[2-m]})\mathcal{A}(\mathbf{x}_{k+1})\mathbf{x}_{k+1}}{\mathbf{x}_{k+1}}\right\} \\
&= \max\left\{\frac{\operatorname{Diag}(\mathbf{y}_{k}^{[2-m]})\mathcal{A}(\mathbf{y}_{k})\mathbf{y}_{k}}{\mathbf{y}_{k}}\right\} \\
&= \max\left\{\frac{\overline{\lambda}_k\mathbf{y}_k - \mathbf{x}_k}{\mathbf{y}_k}\right\} \\
&= \overline{\lambda}_k - \min\left\{\frac{\mathbf{x}_k}{\mathbf{y}_k}\right\} \\
&< \overline{\lambda}_k,
\end{aligned} \tag{4.1}$$

where the third and fourth equalities are obtained by the steps 4 and 3 of Algorithm 1, respectively, the last inequality follows from $\mathbf{x}_k > \mathbf{0}$ and $\mathbf{y}_k > \mathbf{0}$ for all $k = 0, 1, \ldots$. This means that the sequence $\{\overline{\lambda}_k\}$ is monotonically decreasing. It follows from Theorem 1.1 that $\overline{\lambda}_k > \overline{\lambda}_{k+1} \geq \rho(\mathcal{A})$. The proof has been completed. □

In the following, we show that some properties of the sequence $\{\mathbf{x}_k\}$ produced by Algorithm 1, which will help us to establish the global convergence of Algorithm 1.

Lemma 4.2. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. If $\{\overline{\lambda}_k, \mathbf{x}_k\}$ is generated by Algorithm 1, then for any convergent subsequence of $\{\mathbf{x}_k\}$, its limit is positive.*

Proof. For ease of notation, we assume that a convergent subsequence $\{\mathbf{x}_{k_i}\}$ of $\{\mathbf{x}_k\}$, and denote $\mathbf{z} := \lim_{i\to\infty} \mathbf{x}_{k_i}$. It follows from Lemma 3.3 that $\mathbf{z} \geq \mathbf{0}$. By the definition of $\overline{\lambda}_k$ and Theorem 4.1, we have

$$\overline{\lambda}_0 > \overline{\lambda}_{k_i} = \max\left\{\frac{\mathcal{A}\mathbf{x}_{k_i}^{m-1}}{\mathbf{x}_{k_i}^{[m-1]}}\right\} \geq \frac{(\mathcal{A}\mathbf{x}_{k_i}^{m-1})_j}{(\mathbf{x}_{k_i}^{[m-1]})_j}, \tag{4.2}$$

for each $j \in \{1, 2, \ldots, n\}$. On the other hand, we have

$$\lim_{i\to\infty} \frac{(\mathcal{A}\mathbf{x}_{k_i}^{m-1})_j}{(\mathbf{x}_{k_i}^{[m-1]})_j} = \frac{(\mathcal{A}\mathbf{z}^{m-1})_j}{(\mathbf{z}^{[m-1]})_j}. \tag{4.3}$$

Thus, from (4.2) and (4.3), it follows that

$$(\mathcal{A}\mathbf{z}^{m-1})_j < \overline{\lambda}_0(\mathbf{z}^{[m-1]})_j, \tag{4.4}$$

for any $j \in \{1, 2, \ldots, n\}$. Further, assume that $z_t = 0$ for some $t \in \{1, 2, \ldots, n\}$, by setting $j = t$ on the above inequality (4.4), then we have

$$\sum_{j_2,\ldots,j_m=1, j_2,\ldots,j_m\neq t}^{n} a_{tj_2\ldots j_m} z_{j_2} \ldots z_{j_m} < \overline{\lambda}_0(z_t)^{m-1} = 0,$$

which contradicts to $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, and therefore, $\mathbf{z} > \mathbf{0}$. This completes the proof. □

Lemma 4.3. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. If $\{\mathbf{x}_k\}$ is generated by Algorithm 1, then there is a constant δ such that*

$$\lim_{k\to\infty} \min\{\mathbf{x}_k\} = \delta > 0.$$

Proof. By Lemma 3.3, we have $\delta \geq 0$. If $\delta = 0$, then there exists a subsequence $\{\mathbf{x}_{k_i}\}$, with $\lim_{i\to\infty} \min\{\mathbf{x}_{k_i}\} = 0$. Since $\|\mathbf{x}_k\| = 1$, that is, $\{\mathbf{x}_k\}$ is a bounded sequence, then we may assume that $\lim_{i\to\infty} \mathbf{x}_{k_i} = \mathbf{z}$. Note that $\mathbf{z} > \mathbf{0}$ from Lemma 4.2, but, $\lim_{i\to\infty} \min\{\mathbf{x}_{k_i}\} = \min\{\mathbf{z}\} = \mathbf{0}$, which is a contradiction. Therefore, there is a constant δ such that $\lim_{k\to\infty} \min\{\mathbf{x}_k\} = \delta > 0$, and thus, the proof is completed. □

It follows from Theorem 4.1 that the sequence $\{\overline{\lambda}_k\}$ produced by Algorithm 1 is monotonically decreasing and bounded below by $\rho(\mathcal{A})$, and thus we have $\lim_{k\to\infty} \overline{\lambda}_k = s \geq \rho(\mathcal{A})$. In what follows, we will prove that $s = \rho(\mathcal{A})$ based on the previous results.

Theorem 4.4. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible tensor. Suppose that $\{\overline{\lambda}_k, \mathbf{x}_k, \mathbf{y}_k\}$ is generated by Algorithm 1. Then, the monotonically decreasing sequence $\{\overline{\lambda}_k\}$ converges to $\rho(\mathcal{A})$, $\lim_{k\to\infty} \overline{\lambda}_k = \rho(\mathcal{A})$, and the positive sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_*$, $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_*$.*

Proof. From (4.1) it follows that

$$\begin{aligned} \overline{\lambda}_k - \overline{\lambda}_{k+1} = \min\left\{\frac{\mathbf{x}_k}{\mathbf{y}_k}\right\} &\geq \frac{\min\{\mathbf{x}_k\}}{\max\{\mathbf{y}_k\}} \\ &\geq \frac{\min\{\mathbf{x}_k\}}{\|\mathbf{y}_k\|}, \end{aligned} \tag{4.5}$$

where the last inequality is obtained from $\|\mathbf{y}_k\| \geq \max\{\mathbf{y}_k\}$. Note that the sequence $\{\overline{\lambda}_k\}$ is monotonically decreasing and bounded below by $\rho(\mathcal{A})$, that is, $\lim_{k\to\infty} \overline{\lambda}_k - \overline{\lambda}_{k+1} = 0$, it then follows from (4.5) that

$$\lim_{k\to\infty} \frac{\min\{\mathbf{x}_k\}}{\|\mathbf{y}_k\|} = 0.$$

By Lemma 4.3, we obtain that $\lim_{k\to\infty} \frac{1}{\|\mathbf{y}_k\|} = 0$. There exists a convergent subsequence $\{\mathbf{x}_{k_i+1}\}$ of $\{\mathbf{x}_k\}$ such that $\lim_{i\to\infty} \mathbf{x}_{k_i+1} = \mathbf{z} > \mathbf{0}$ according to Lemma 4.2 and $\|\mathbf{x}_k\| = 1$. We next prove that $\lim_{i\to\infty} \overline{\lambda}_{k_i} = s = \rho(\mathcal{A})$. Note that the step 3 of Algorithm 1 and $\|\mathbf{x}_{k_i}\| = 1$, we have

$$\begin{aligned} \|\mathbf{y}_{k_i}\| &= \|[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})]^{-1}\mathbf{x}_{k_i}\| \\ &\leq \|[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})]^{-1}\|, \end{aligned}$$

then

$$\frac{1}{\|\mathbf{y}_{k_i}\|} \geq \frac{1}{\|[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})]^{-1}\|} \geq 0.$$

Therefore, from $\lim_{k\to\infty} \frac{1}{\|\mathbf{y}_k\|} = 0$, it implies that $\lim_{i\to\infty} \|[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})]^{-1}\| = \infty$. Let $\frac{1}{\sigma_{k_i}}$ be the largest singular value of $[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})]^{-1} \in \mathbb{R}^{n\times n}$, then $\lim_{i\to\infty} \frac{1}{\sigma_{k_i}} = \infty$. Due to the fact that σ_{k_i} is the samllest singular value of $\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i}) \in \mathbb{R}^{n\times n}$, then we have that $\lim_{i\to\infty} \sigma_{k_i} = 0$, meaning that $\overline{\lambda}_{k_i}\mathbf{I} -$

$\mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i})$ tends to a singular matrix. From (2.3), it follows that $\mathrm{Diag}(\mathbf{y}_{k_i}^{[2-m]})\mathcal{A}(\mathbf{y}_{k_i}) = \mathrm{Diag}(\mathbf{x}_{k_i+1}^{[2-m]})\mathcal{A}(\mathbf{x}_{k_i+1})$. If $\lim_{i\to\infty}\overline{\lambda}_{k_i} = s > \rho(\mathcal{A})$, according to $\lim_{i\to\infty}[\overline{\lambda}_{k_i}\mathbf{I} - \mathrm{Diag}(\mathbf{x}_{k_i+1}^{[2-m]})\mathcal{A}(\mathbf{x}_{k_i+1})] = s\mathbf{I} - \mathrm{Diag}(\mathbf{z}^{[2-m]})\mathcal{A}(\mathbf{z})$ and Theorem 2.5, we obtain that $s\mathbf{I} - \mathrm{Diag}(\mathbf{z}^{[2-m]})\mathcal{A}(\mathbf{z})$ is a nonsingular M-matrix, which leads a contradiction. Hence, according to $\{\overline{\lambda}_k\}$ is a monotonically decreasing sequence and bounded below by $\rho(\mathcal{A})$, we have that $s = \rho(\mathcal{A})$, and

$$\begin{aligned}\rho(\mathcal{A}) &= \lim_{i\to\infty}\overline{\lambda}_{k_i} = \lim_{i\to\infty}\overline{\lambda}_{k_i+1}\\ &= \lim_{i\to\infty}\max\left\{\frac{\mathcal{A}\mathbf{x}_{k_i+1}^{m-1}}{\mathbf{x}_{k_i+1}^{[m-1]}}\right\}\\ &= \max\left\{\frac{\mathcal{A}\mathbf{z}^{m-1}}{\mathbf{z}^{[m-1]}}\right\}.\end{aligned}$$

Thus, $\mathbf{z} = \mathbf{x}_* > \mathbf{0}$ from Theorem 1.1. Furthermore, it follows from $\mathcal{A}\mathbf{x}_*^{m-1} = \rho(\mathcal{A})\mathbf{x}_*^{[m-1]}$ that

$$\mathrm{Diag}(\mathbf{x}_*^{[2-m]})\mathcal{A}(\mathbf{x}_*)\mathbf{x}_* = \rho(\mathcal{A})\mathbf{x}_*.$$

From Lemma 2.3 and the Perron-Frobenius theorem for nonnegative irreducible matrices, we also have that $\rho(\mathcal{A}) = \rho(\mathrm{Diag}(\mathbf{x}_*^{[2-m]})\mathcal{A}(\mathbf{x}_*))$. The proof has been completed. □

5 Testing the Positive Definiteness of an Even Order Weakly Irreducible $\mathcal{Z}$-tensor

In this section, we consider how to determine the positive definiteness of an even order weakly irreducible $\mathcal{Z}$-tensor. By the definition of $\mathcal{Z}$-tensor $\mathcal{B}$ in [29], we have the form $\mathcal{B} = \mu\mathcal{I} - \mathcal{A}$, where $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, μ is a scalar and $\mathcal{I}$ is a unit tensor whose entries are defined by

$$\mathcal{I}_{i_1 i_2 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.}\end{cases}$$

If $\mathcal{A}$ is a nonnegative weakly irreducible tensor, then $\mathcal{B}$ is a weakly irreducible tensor. For all $\mathbf{x} \in \mathbb{R}^n$, if we have $\mathcal{B}\mathbf{x}^m \geq 0$, then $\mathcal{B}$ is called positive semidefinite, in particular, we call $\mathcal{B}$ is positive definite if $\mathcal{B}\mathbf{x}^m > 0$ when $\mathbf{x} \neq \mathbf{0}$. For computing the largest eigenvalue of an even order nonnegative weakly irreducible tensor $\mathcal{A}$ by using Algorithm 1, as have been proved previously, it generates a monotonically decreasing sequence $\{\overline{\lambda}_k\}$ tends to $\rho(\mathcal{A})$. Then the smallest eigenvalue of $\mathcal{B}$ can be computed, which is $\mu_{\min} = \mu - \rho(\mathcal{A})$, and thus we can determine whether $\mathcal{B}$ is positive definite or not.

For Algorithm 1, in order to obtain the smallest eigenvalue of $\mathcal{B}$, we need to solve the following $\mathcal{M}$-like equation at the k-th iteration and normalize the vector $\mathbf{y}_k$

$$\begin{aligned}&[\overline{\lambda}_k\mathbf{I} - \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k)]\mathbf{y}_k = \mathbf{x}_k,\\ &\mathbf{x}_{k+1} = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}.\end{aligned} \tag{5.1}$$

By the form $\mathcal{B} = \mu\mathcal{I} - \mathcal{A}$ and $\mathbf{y}_k > \mathbf{0}$, we have

$$\mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{B}(\mathbf{y}_k) = \mu\mathbf{I} - \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k),$$

where $\mathcal{B}(\mathbf{y}_k) \overset{\text{def}}{=} \mathcal{B}\mathbf{y}_k^{m-2}$, and thus

$$\overline{\lambda}_k\mathbf{I} - \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{A}(\mathbf{y}_k) = \mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{B}(\mathbf{y}_k) - \mu_k\mathbf{I}, \tag{5.2}$$

where $\mu_k = \mu - \overline{\lambda}_k = \min\left\{\frac{\mathcal{B}\mathbf{x}_k^{m-1}}{\mathbf{x}_k^{[m-1]}}\right\}$. Hence, (5.1) is equivalent to

$$\begin{aligned} &[\mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{B}(\mathbf{y}_k) - \mu_k\mathbf{I}]\mathbf{y}_k = \mathbf{x}_k, \\ &\mathbf{x}_{k+1} = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}. \end{aligned} \tag{5.3}$$

It follows from Theorem 2.5 and (5.2) that $\mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{B}(\mathbf{y}_k) - \mu_k\mathbf{I}$ is also an irreducible nonsingular M-matrix. Similar to the previous analysis of Section 4, we also employ the Newton method to solve $\mathcal{M}$-like equation in (5.3), from a positive initial point we have that $\mathbf{y}_k > \mathbf{0}$. Thus $\mathbf{x}_{k+1} > \mathbf{0}$, which means that the positive property of $\mathbf{x}_k$ is guaranteed for each k provided that $\mathbf{x}_0 > \mathbf{0}$.

By the definition of μ_k, $\mathbf{x}_k > \mathbf{0}$ and $\mathbf{y}_k > \mathbf{0}$, we have

$$\mu_{k+1} = \mu_k + \min\left\{\frac{\mathbf{x}_k}{\mathbf{y}_k}\right\} > \mu_k,$$

this implies the sequence $\{\mu_k\}$ is monotonically increasing. Based on the previous analysis, we can get an derived algorithm of Algorithm 1 for computing the smallest eigenvalue of a weakly irreducible $\mathcal{Z}$-tensor, its detailed steps as follows.

Algorithm 3 Inverse iteration algorithm for weakly irreducible $\mathcal{Z}$-tensors.

1: **Initialization:** Given an initial point $\mathbf{x}_0 > \mathbf{0}$ with $\|\mathbf{x}_0\| = 1$ and tol > 0, compute $\mu_0 = \min\left\{\frac{\mathcal{B}\mathbf{x}_0^{m-1}}{\mathbf{x}_0^{[m-1]}}\right\}$.
2: **repeat**
3: Compute $\mathbf{y}_k$ by solving $\left[\mathrm{Diag}(\mathbf{y}_k^{[2-m]})\mathcal{B}(\mathbf{y}_k) - \mu_k\mathbf{I}\right]\mathbf{y}_k = \mathbf{x}_k$.
4: Normalize the vector $\mathbf{x}_{k+1} = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}$.
5: Compute $\mu_{k+1} = \min\left\{\frac{\mathcal{B}\mathbf{x}_{k+1}^{m-1}}{\mathbf{x}_{k+1}^{[m-1]}}\right\}$.
6: **until** $\|\mathcal{B}\mathbf{x}_{k+1}^{m-1} - \mu_{k+1}\mathbf{x}_{k+1}^{[m-1]}\|_\infty \le$ tol.
7: **Output:** The smallest eigenvalue $\mu_{\min} \leftarrow \mu_{k+1}$ and its associated eigenvector $\mathbf{x}_* \leftarrow \mathbf{x}_{k+1}$.

It is worth pointing out that, Algorithm 3 is independent of μ and does not use the expression $\mathcal{B} = \mu\mathcal{I} - \mathcal{A}$ in the procedure. Finally, we also give the main results of Algorithm 3 as follows.

Theorem 5.1. *Let $\mathcal{B} \in \mathbb{R}^{[m,n]}$ be an even order weakly irreducible $\mathcal{Z}$-tensor. Suppose that $\{\mu_k, \mathbf{x}_k\}$ is generated by Algorithm 3. Then the monotonically increasing sequence $\{\mu_k\}$ converges to $\mu_{\min}$, with $\lim_{k\to\infty}\mu_k = \mu_{\min}$, and the positive sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_*$, $\lim_{k\to\infty}\mathbf{x}_k = \mathbf{x}_*$.*

6 Numerical Results

We report some numerical results in this section to illustrate the effectiveness of Algorithm 1 and Algorithm 3, and compare them with NQZ [18] and NNI [16]. In our experiments, we take the parameter tol $= 10^{-8}$ and the number of maximum iteration steps 10,000 for Algorithm 1, Algorithm 3, NQZ and NNI. The relative residual is denoted by $\frac{\overline{\lambda}_k-\underline{\lambda}_k}{\overline{\lambda}_k}$ at the k-th iteration. We set initial points $\mathbf{x}_0 = \frac{(1,1,\ldots,1)^T}{\sqrt{n}}$ for Algorithm 1, Algorithm 3, NQZ and NNI, $\mathbf{w}_0 = (1,1,\ldots,1)^T$ for Algorithm 2 in all test examples of this section.

All numerical tests are done using MATLAB R2010b and the Tensor Toolbox version 2.6 [1]. The numerical experiments are performed on a PC with an Intel(R) Core(TM) 2 Duo CPU T6600 at 2.20 GHz and 4.00 GB of RAM under the Windows 7 operating system.

In the following tables, "m, n" denote the order and the dimension of test tensor, respectively, "Iter." refers to the number of iteration steps, "$\rho(\cdot)$" denotes the value of $\overline{\lambda}_k$ at the final iteration, "Res." denotes the residual error value of $\|\mathcal{A}\mathbf{x}_k^{m-1} - \overline{\lambda}_k\mathbf{x}_k^{m-1}\|_\infty$ when the iterative algorithms are terminated and "$\mu_{\min}$" denotes the value of μ_k at the final iteration.

We first employ Algorithm 1 to find the largest eigenvalue of the adjacency tensors and the signless Laplacian tensors [8] of loose paths,

Example 6.1. Consider an m-uniform hypergraph with r edges $\mathcal{G} = (V, E)$, which is called a loose path [27], if its vertex set as

$$V = \{i_{(1,1)}, \ldots, i_{(1,m)}, i_{(2,2)}, \ldots, i_{(2,m)}, \ldots, i_{(r-1,2)}, \ldots, i_{(r-1,m)}, i_{(r,2)}, \ldots, i_{(r,m)}\},$$

and edge set is

$$E = \big\{\{i_{(1,1)}, i_{(1,2)}, \ldots, i_{(1,m)}\}, \{i_{(1,m)}, i_{(2,2)}, \ldots, i_{(2,m)}\}, \ldots, \{i_{(r-1,m)}, i_{(r,2)}, \ldots, i_{(r,m)}\}\big\},$$

where r is the length of the loose path. We know that this loose path has $n = r(m-1)+1$ vertices. Then its the adjacency tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is a symmetric tensor, whose entries are defined by

$$a_{i_1 i_2 \ldots i_m} = \begin{cases} \dfrac{1}{(m-1)!}, & (i_1, i_2, \ldots, i_m) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

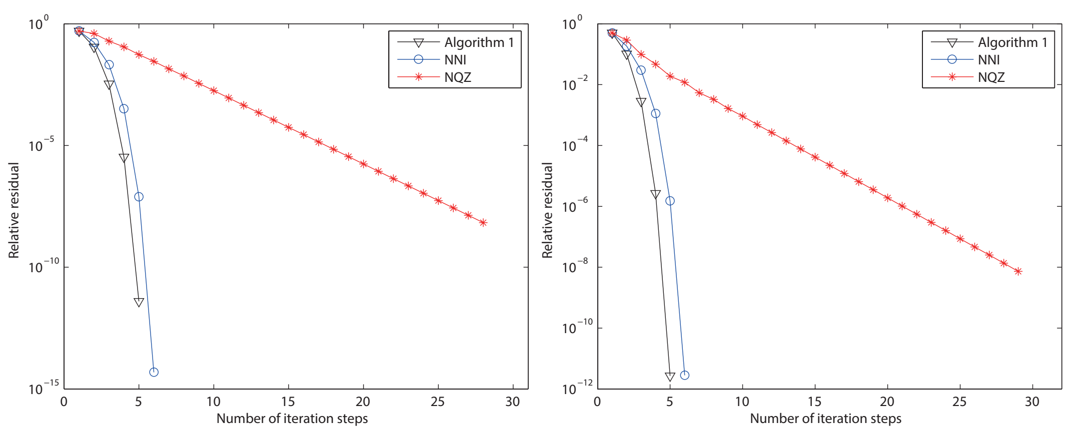
Example 6.2. Consider the signless Laplacian tensor $\mathcal{S}$ of an m-uniform loose path with r edges $\mathcal{G} = (V, E)$, with $\mathcal{S} = \mathcal{A} + \mathcal{D}$, where its the adjacency tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is defined in Example 6.1, and the diagonal tensor $\mathcal{D}$ with its diagonal element $d_{i,\ldots,i}$ equal to the degree of vertex i for all $i \in \{1, 2, \ldots, n\}$.

Tables 1 and 2 report the numerical results obtained by Algorithm 1, NQZ and NNI, for finding the largest eigenvalue of the adjacency tensors and the signless Laplacian tensors of loose paths in Examples 6.1 and 6.2, respectively. As we have observed, these two tables 1-2 show that the performance of Algorithm 1 is a little better than that of NQZ and NNI, and the number of iterations is at most 6 for Algorithm 1. We can also see that our results are to satisfy some results with theory of Theorems 1-4 in Yue, Zhang and Lu [27]. Figures 1 and 2 depict the relationship between the relative residual and the number of iterations on Examples 6.1 and 6.2 for Algorithm 1, NQZ and NNI, respectively. As we see, they indicate that Algorithm 1 , NQZ and NNI are monotonically decreasing. They also illustrate that the convergence rate of NQZ and NNI appear to be linear and quadratic, as confirmed some results with theory of their linear and quadratic convergence rate (see [28, 12, 16]), respectively. Moreover, from the tables 1-2 and figures 1-2, we can see that the Algorithm 1 is quadratic convergence. As we known, inverse iteration algorithm is quadratically convergent

This toolbox was downloaded from `http://www.sandia.gov/~tgkolda/TensorToolbox/`.

Table 1: The largest eigenvalue of adjacency tensors on Example 6.1

m	r	n	$\rho(\mathcal{A})$	Algorithm 1		NQZ		NNI	
				Iter.	Res.	Iter.	Res.	Iter.	Res.
3	3	7	1.3782	5	5.9583e-13	28	1.9399e-09	6	7.2164e-16
	20	41	1.5766	6	9.4022e-15	382	2.4383e-10	8	5.5511e-17
	50	101	1.5855	6	1.8645e-12	1923	1.0502e-10	8	4.5969e-12
	100	201	1.5869	6	3.6805e-12	6704	5.3626e-11	9	5.0247e-14
4	3	10	1.2720	5	8.4759e-14	29	2.2995e-10	6	8.8984e-14
	20	61	1.4070	6	1.2950e-15	565	2.1942e-11	8	1.6503e-12
5	3	13	1.2123	5	6.8548e-15	41	4.2660e-11	6	4.2909e-13
	4	17	1.2457	5	7.0655e-14	59	2.8393e-11	7	1.5613e-17
6	3	16	1.1740	5	5.4362e-16	52	8.5783e-12	6	6.0217e-13
	4	21	1.2009	5	4.4187e-15	75	3.7571e-12	7	4.8084e-17

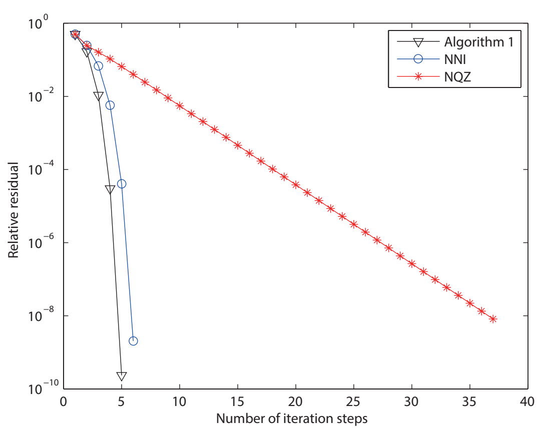


(a) $(m, r) = (3, 3)$ (b) $(m, r) = (4, 3)$

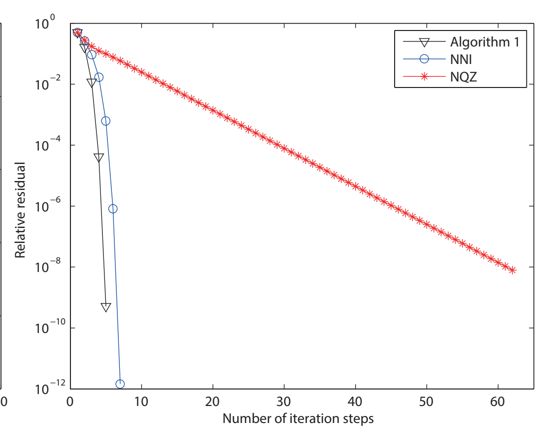
Figure 1: The comparison of Algorithm 1, NQZ and NNI for Example 6.1

Table 2: The largest eigenvalue of signless Laplacian tensors on Example 6.2

m	r	n	$\rho(\mathcal{A})$	Algorithm 1		NQZ		NNI	
				Iter.	Res.	Iter.	Res.	Iter.	Res.
3	3	7	2.9701	5	5.0972e-11	37	1.7638e-09	6	4.4224e-10
	20	41	3.3029	6	5.4944e-13	716	6.1341e-10	8	1.6659e-12
	50	101	3.3126	6	1.8374e-11	3767	2.5476e-10	9	9.7679e-14
	100	201	3.3141	6	3.6541e-11	10001	2.5361e-09	10	1.2681e-15
4	3	10	2.7549	5	2.2972e-11	62	3.5242e-10	7	6.3845e-14
	20	61	2.9923	6	1.5338e-13	1084	6.5733e-11	12	4.6543e-13
5	3	13	2.6256	5	4.3351e-12	87	8.1571e-11	7	3.0574e-12
	4	17	2.7004	5	2.5521e-11	110	5.6250e-11	8	3.4694e-16
6	3	16	2.5385	5	6.0630e-13	112	1.7154e-11	8	9.1940e-17
	4	21	2.6012	5	3.0291e-12	139	9.7894e-12	8	1.6731e-14



(a) $(m, r) = (3, 3)$



(b) $(m, r) = (4, 3)$

Figure 2: The comparison of Algorithm 1, NQZ and NNI for Example 6.2

for computing the largest eigenvalue of a nonnegative irreducible matrix [10, 13]. In [9], the authors conjectured the given inverse iteration algorithm converges quadratically as well. We conjecture Algorithm 1 is also quadratically convergent algorithm, however, we now can not prove this result in theory.

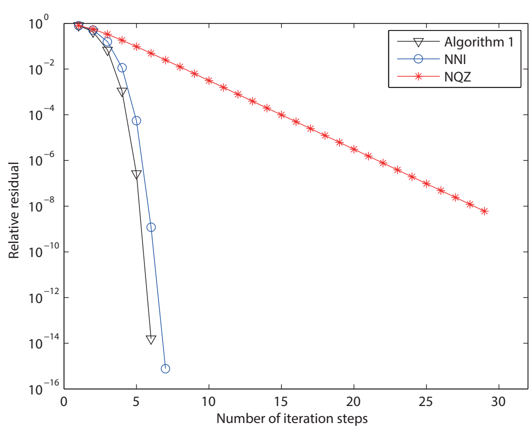
As a real application, eigenvector centrality is a standard network analysis tool for determining the importance of entities in a connected system that is represented by a graph or hypergraph. For hypergraph models of such multirelational data, Benson [2] proposed three hypergraph eigenvector centralities: clique motif eigenvector centrality (CEC), Z-eigenvector centrality (ZEC) and H-eigenvector centrality (HEC). In order to illustrate the effectiveness of Algorithm 1 for computing the HEC scores, in our tests we used the sunflower hypergraph with singleton core of different size coming from [2]. The sunflower hypergraph $\mathcal{G}=(V,E)$ with singleton core $\{1\}$ is an m-uniform, r-petal hypergraph, then its vertex set as $V=\{1,2\ldots,(m-1)r+1\}$, and edge set is

$$E=\big\{\{1,2,\ldots,m\},\{1,m+1,\ldots,2m-1\},\ldots,\{1,(m-1)r-m+3,\ldots,(m-1)r+1\}\big\}.$$

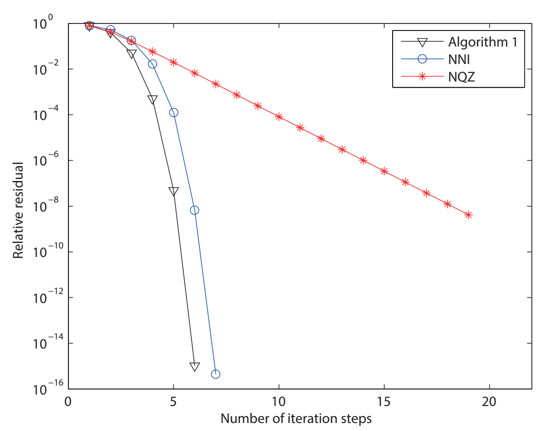
An illustration of a 4-uniform, 5-petal sunflower hypergraph with core $\{\mu\}$ can be found in the left part of [2, Figure 1]. The detailed results are shown in Table 3.

Table 3: HEC of the sunflower hypergraph with singleton core

m	r	HEC	Algorithm 1		NQZ		NNI	
			Iter.	Res.	Iter.	Res.	Iter.	Res.
3	5	1.7100	6	2.0262e-15	29	7.9328e-10	7	1.1102e-16
4	5	1.4953	6	2.7756e-17	19	8.6811e-11	7	2.7756e-17



(a) $(m,r)=(3,5)$



(b) $(m,r)=(4,5)$

Figure 3: The comparison of Algorithm 1, NQZ and NNI for sunflower hypergraph

We observe that the resulting HEC scores are correct from Table 3, as expected, Algorithm 1, NQZ and NNI perform well for the sunflower hypergraph with singleton core. The resulting HEC scores are also consistent with the analysis in [2, subsection 2.4], which are equal to $r^{\frac{1}{m}}$ for different size m and r. Figure 3 also depicts the relationship between the relative residual and the number of iterations on the sunflower hypergraph with singleton core for Algorithm 1, NQZ and NNI.

We then apply Algorithm 3 to determine whether an even order $\mathcal{Z}$-tensor is positive definite or not. For NQZ, to test the positive definiteness of an even order $\mathcal{Z}$-tensor, we need to apply it on $\mathcal{C} = \mu\mathcal{I} - \mathcal{B}$ with $\mathcal{B}$ is a $\mathcal{Z}$-tensor, where $\mu \geq \max_i\{b_{i,i,\ldots,i}\}$ for all $i \in \{1, 2, \ldots, n\}$. However, Algorithm 3 and NNI [16, section 6] can work on $\mathcal{B}$ directly, since they do not involve μ in the procedure.

Example 6.3. Consider a $\mathcal{Z}$-tensor $\mathcal{B} = 10\mathcal{D} - \mathcal{A} \in \mathbb{R}^{[m,n]}$, where $\mathcal{A}$ and $\mathcal{D}$ are the adjacency tensor and the diagonal tensor of an m-uniform loose path with r edges $\mathcal{G} = (V, E)$ with $n = r(m-1) + 1$, which are defined in Examples 6.1 and 6.2, respectively.

Table 4: Testing the positive definiteness of $\mathcal{Z}$-tensors on Example 6.3

m	n	$\mu_{\min}$	Algorithm 3		NQZ		NNI		Positivity
			Iter.	Res.	Iter.	Res.	Iter.	Res.	
4	10	9.5091	5	1.0547e-15	462	9.7749e-09	6	9.6657e-11	Yes
	13	9.5296	5	5.6011e-14	660	9.9024e-09	10	3.5961e-12	Yes
	61	9.5358	5	1.1468e-13	823	9.9263e-09	55	6.1062e-16	Yes
6	16	9.3454	4	1.0617e-11	431	9.8882e-09	6	1.3284e-11	Yes
	21	9.3625	4	7.7832e-11	620	9.9192e-09	11	2.8691e-14	Yes

Example 6.4. Consider a random $\mathcal{Z}$-tensor $\mathcal{B} \in \mathbb{R}^{[m,n]}$. Generate a random vector $\mathbf{z}$ whose elements are randomly distributed in the interval $(0, 1)$, let $b_{i,i,\ldots,i} = \alpha + z_i$ with $\alpha \in \{0, 1\}$ is a scalar, for all $i \in \{1, \ldots, n\}$, let $b_{i,i+1,\ldots,i+1} = -z_i$ for $1 \leq i \leq n-1$ and $b_{n1\ldots1} = -1$, otherwise, $b_{i_1,i_2,\ldots,i_m} = 0$ for $1 \leq i_1, i_2, \ldots, i_m \leq n$.

Table 5: Testing the positive definiteness of $\mathcal{Z}$-tensors on Example 6.4

m	n	α	$\mu_{\min}$	Algorithm 3		NQZ		NNI		Positivity
				Iter.	Res.	Iter.	Res.	Iter.	Res.	
4	10	0	-0.0732	5	8.2232e-09	211	9.4707e-09	7	3.8867e-11	No
	10	1	0.9908	5	9.3232e-11	144	9.9262e-09	6	3.6199e-10	Yes
4	50	0	-0.0014	5	4.8057e-10	777	9.8755e-09	7	4.2414e-12	No
	50	1	0.9925	6	2.3178e-09	2014	9.9412e-09	10	1.9082e-17	Yes
6	10	0	-0.0237	4	8.6772e-13	205	8.9229e-09	5	4.0588e-13	No
	10	1	0.9679	5	9.3330e-12	112	9.2917e-09	7	2.8125e-11	Yes
6	20	0	-0.0106	5	3.3754e-09	355	9.6312e-09	8	6.3701e-12	No
	20	1	0.9932	4	1.4733e-09	315	9.7967e-09	7	5.9631e-18	Yes

For NQZ, we take $\mu = 20$ and $\mu = \alpha + 1$ in Examples 6.3 and 6.4, respectively. Tables 4 and 5 list the numerical results obtained by Algorithm 3, NQZ and NNI, to test whether even order $\mathcal{Z}$-tensors are positivity or not in Examples 6.3 and 6.4, respectively. From $\mu_{\min}$ is computed by Algorithm 3 and NQZ for each $\mathcal{Z}$-tensor, if we have $\mu_{\min} > 0$, then we can determine it is positivity. These two tables 4-5 show that the improvement of Algorithm 3 over the existing NQZ and NNI in term of the number of iterations, and the number of iterations is at most 8 for Algorithm 3. But NQZ and NNI use much less cpu time than Algorithm 3 in numerical test. Finally, it is worth pointing out that, we also observe that the efficiency of the NQZ is dependent on the choice of μ in our experiments.

In summary, Algorithm 1 and Algorithm 3 are promising for finding the largest eigenvalue of nonnegative weakly irreducible tensors and testing the positive definiteness of even order weakly irreducible $\mathcal{Z}$-tensors, respectively, the numerical results illustrate that the overall good performances of both Algorithm 1 and Algorithm 3.

7 Conclusions

We have presented a positivity preserving inverse iteration algorithm with an initial positive vector for finding the largest eigenvalue of nonnegative weakly irreducible tensors, by reformulating (1.4) as (2.1) for any $\mathbf{x} > \mathbf{0}$ in this paper. We have analyzed some properties of Algorithm 1, and have established the global convergence results. At each iteration of Algorithm 1, we used the Newton method to compute $\mathcal{M}$-like equation, which always preserve $\mathbf{y}_k > \mathbf{0}$ for any initial positive point. In numerical, we have illustrated that the promising behavior of Algorithm 3 for testing the positive definiteness of even order $\mathcal{Z}$-tensors. The numerical results indicated that the efficiency of both Algorithm 1 and Algorithm 3. The quadratic convergence of inverse iteration algorithm is an attractive topic for future work. Reformulated (1.4) as (2.1), we obtain a matrix-based method, how to prove this result by combining with some results [10, 13] is still under investigation.

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Zhou Sheng
School of Mathematics & Physics
Anhui University of Technology
Maanshan, Anhui 243002, China
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing, Jiangsu 211106, China
E-mail address: szhou03@live.com

Qin Ni
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing, Jiangsu 211106, China
E-mail address: niqfs@nuaa.edu.cn