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STABILITY OF GENERALIZED EQUATIONS GOVERNED BY COMPOSITE MULTIFUNCTIONS*

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Abstract: This paper deals with metric regularity of a parametrized epigraphical set-valued mapping associated with a parametrized composition set-valued mapping and also with semiregularity of composition of set-valued mappings. Then, we obtain Lipschitz-likeness, calmness of the implicit set-valued mappings which were defined by the associated generalized equation. Moreover, Robinson's metric regularity of implicit set-valued mappings associated to a composite mapping was also derived when the so-called "local composition stability" is imposed. Our work is new and generalizes the recent results on this topic by Durea, Strugariu [19], Ngai, Tron, Théra [33], Zheng, Ng [45], Durea, Strugariu [18, 20], Durea, Huynh, Nguyen, Strugariu [23], and Cibulka, Fabian, Kruger [9].

Key words: *generalized equation, metric regularity, Robinson metric regularity, semiregularity, calmness, Lipschitz-likeness, implicit set-valued mapping*

Mathematics Subject Classification: *49J52, 49J53, 90C30*

1 Introduction

An important topic in variational analysis concerns the study of so-called *generalized equations* (GEs) described by an inclusion of the form

$$y \in F(x),$$

where $F : X \rightrightarrows Y$ is a set-valued mapping defined between two metric spaces X and Y. In the literature, these generalized equations are also called inclusions or variational systems and their study was initiated by Robinson in 1970s ([36, 37]) since their large coverage. Indeed, they include many problems and phenomena such as equations/ equation systems, variational inequalities, complementary problems, dynamical systems, optimal control, and necessary/sufficient conditions for optimization and control problems, fixed point theory, coincidence point theory and so on. Nowadays, these generalized equations have attracted the interest and the study of many experts in the community of variational analysis and optimization (see, for instance [3, 16, 26, 30, 31, 36, 37, 41]).

Important contents in studying generalized equations are the existence and the behavior of their solution set when the data is perturbed. A key property ensuring these things is the so-called concept of *metric regularity*. This property is now considered to be a central

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concept of variational analysis and it has a crucial role in many areas of mathematics such as analysis of convergence of optimization algorithms, necessary/sufficient conditions for optimization problems and control problems, fixed point theory, coincidence point theory and so on.

Metric regularity goes back to the Banach open mapping theorem for linear operators; its extension to nonlinear operators known as the Lyusternik [29] and Graves theorem [24], and to closed convex set-valued mappings known as the Robinson-Ursescu theorem ([39,43]). These results were further extended to the case of the sum of a set-valued mapping with a single-valued one by Arutyunov [1,2], Dontchev, Lewis [13], Lewis, Dontchev, Rockafellar [14], Ioffe [26], Mordukhovich [30]. Subsequently, they were studied and developed recently by several authors. In [19], Durea and Strugariu established the openness for the sum of two set-valued mappings. Next, Ngai, Tron, Théra [33] considered metric regularity of the sum of two set-valued mappings. In [45], Zheng, Ng studied metric regularity of the composition of a set-valued mapping with a single-valued one. Later, Durea, Strugariu [18,20] obtained linear openness for nonparametric composition of set-valued mappings defined as in (2.1), and Durea, Huynh, Nguyen, Strugariu [23] established metric regularity for this mapping.

There are many variant versions of regularity properties, including Lipschitz-likeness, linear openness, metric subregularity, semiregularity, calmness, etc. For a detailed account, of these various regularities of set-valued mappings, as well as their diverse applications, the reader is referred to the works [5, 8, 11, 14, 26–28, 30, 40], and the references given therein.

The main goal of this article is to establish metric regularity, semiregularity of the parametrized epigraphical and composition set-valued mapping and to derive the stability of the solution set of generalized equations (called also implicit set-valued mapping). Concretely, we derive calmness, Lipschitz-likeness and Robinson metric regularity of implicit set-valued mappings associated to generalized equations. In recent years, the topic on stability of generalized equations has also attracted the interest and the study of many authors from community of variational analysis and many important results had been obtained. We refer the reader to various contributions such as for instance [5,6,16,19,25,26,30,32,34,35,44].

Our work, is an expository paper about sensitivity analysis of generalized equations that also recent results in this direction such as [18–20, 23, 33, 45] and the one given very recently by Cibulka, Fabian, Kruger [9]. Ideas and techniques in the paper benefitted from the contributions by Ngai, Tron, Théra [33] and Durea, Huynh, Nguyen, Strugariu [23].

The rest of this paper is organized in main sections. Section 2 introduces preliminaries and notations necessary for the next sections. The next one focuses on semiregularity of parametric composition set-valued mapping, and metric regularity of the parametrized epigraphical set-valued mapping. In the final section, we establish some types of regularities on implicit set-valued mappings such as Lipschitz-likeness, calmness, and Robinson metric regularity.

$\boxed{2}$ Premilinaries

In this section, we recall the necessary knowledge and notations used throughout the paper. Let X be a metric space, and let A be a nonempty subset of X. Given $x \in X$, the distance from x to the set A is denoted by $d(x, A)$ and is defined by $d(x, A) := \inf_{a \in A} d(x, a)$. The excess of a set C over another one D is given by $e(C, D) = \sup\{d(x, D) | x \in C\}$. When X being a normed space, we denote $\overline{B}_X$ to be closed unit in X. One of the important tools used in this paper is the so-called the strong slope of a lower semicontinuous function. For a lower semicontinuous function $h : X \to \mathbb{R} \cup \{+\infty\}$ defined on a metric space X, the strong slope of h at $\bar{x} \in \text{dom } h$ is defined by $|\nabla h|(\bar{x}) = 0$ if $\bar{x}$ is a local minimum of h and otherwise by

$$|\nabla h|(\bar{x}) := \limsup_{x \to \bar{x}} \frac{h(\bar{x}) - h(x)}{d(\bar{x}, x)}.$$

For $\bar{x} \notin \operatorname{dom} h$, we set $|\nabla h|(\bar{x}) = +\infty$. Given a set-valued mapping $T : X \rightrightarrows Y$, the lower semicontinuous envelope associated T is defined by $\varphi_T(x, y) := \liminf_{u \to x} d(y, T(u))$, and given a parametrized set-valued mapping $T : X \times P \rightrightarrows Y$, its lower envelope defined by $\varphi^p_T(x, y) := \liminf_{u \to x} d(y, T(u, p))$ will play a significant role in what follows. We next recall important concepts of regularities used in variational analysis.

Definition 2.1. A set- valued mapping $T : X \rightrightarrows Y$ defined between metric spaces X, Y is said to be *metrically regular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ with modulus $\tau > 0$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that for every $(x, y) \in U \times V$,

$$d(x, T^{-1}(y)) \leq \tau d(y, T(x)).$$

Parametrized version of this property can be similarly defined as follows:

Definition 2.2. Let X, Y be metric space, P is a topological space, $T : X \times P \rightrightarrows Y$ a set-valued mapping. T is said to be *metrically regular* around $(\bar{x}, \bar{p}, \bar{y}) \in \operatorname{gph} T$ with respect to (x, y), uniformly in p with modulus $\tau > 0$ if there exists a neighborhood $U \times V \times W$ of $(\bar{x}, \bar{y}, \bar{p})$ such that for every $(x, y, p) \in U \times V \times W$,

$$d(x, T_p^{-1}(y)) \leq \tau d(y, T(x, p)).$$

Here, the notation T_p^{-1} means by $x \in T_p^{-1}(y) \Longleftrightarrow y \in T(x, p)$.

Definition 2.3. $T : X \rightrightarrows Y$ is said to be *Lipschitz-like (or pseudo-Lipschitz or to has the Aubin property)* around $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ with modulus $\tau > 0$ if there exists a neighborhood $\mathcal{U} \times \mathcal{V}$ of $(\bar{x}, \bar{y})$ such that, for every $x, x' \in \mathcal{U}$,

$$T(x) \cap \mathcal{V} \subset T(x') + \tau d(x, x')\overline{B}_Y.$$

In this definition, if we fix x' by $\bar{x}$, one obtains a weaker property called the calmness property:

Definition 2.4. $T : X \rightrightarrows Y$ is said to be *calm* around $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ with modulus $\tau > 0$ if there exists a neighborhood $\mathcal{U} \times \mathcal{V}$ of $(\bar{x}, \bar{y})$ such that, for every $x \in \mathcal{U}$,

$$T(x) \cap \mathcal{V} \subset T(\bar{x}) + \tau d(x, \bar{x})\overline{B}_Y.$$

A variant of metric regularity was recently studied ([9]) in order to achieve the convergence of Newton's method for generalized equations.

Definition 2.5. $T : X \rightrightarrows Y$ is said to be *metrically semiregular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ with modulus $\tau > 0$ if there exists a neighborhood $\mathcal{V}$ of $\bar{y}$ such that, for every $y \in \mathcal{V}$,

$$d(\bar{x}, T^{-1}(y)) \leq \tau d(y, \bar{y}).$$

Our main motivation in this paper is to study various regularities mentioned above for the composition of set-valued mappings.

$$H_p(x) := H(x, p) = T(T_1(x), T_2(x, p)) = \bigcup_{(y_1, y_2) \in T_1(x) \times T_2(x, p)} T(y_1, y_2), \tag{2.1}$$

and parametrized variational system associated to this map

$$z \in H_p(x), \tag{2.2}$$

where $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ are set-valued mappings, X, Y_1, Y_2 are metric spaces, Z is a normed linear space, P is a topological space. In some applications, we need only to consider (2.2) with z a point which is fixed. In particular, without loss of generality, one considers the case (2.2) when $z \equiv 0$:

$$0 \in H_p(x). \tag{2.3}$$

We now consider *a parametrized epigraphical-type set-valued mapping* associated to H and defined by

$$\mathcal{E}_H^p(x, y_1, y_2) := \mathcal{E}_H((x,p), y_1, y_2) = \begin{cases} T(y_1, y_2) & \text{if } (y_1, y_2) \in T_1(x) \times T_2(x,p), \\ \emptyset & \text{otherwise.} \end{cases}$$

$\mathcal{E}_H$ is closed-graph when partial set-valued mappings are closed ones while the original mapping H in general fails to be closed. Hence, it is convenient to work with $\mathcal{E}_H$ instead of H. For every $(z,p) \in Z \times P$, let us define

$$\mathbb{S}_{\mathcal{E}_H}(z,p) := \{(x, y_1, y_2) \in X \times Y_1 \times Y_2 : z \in \mathcal{E}_H^p(x, y_1, y_2)\},$$

and

$$\mathbb{S}_H(z,p) := \{x \in X : z \in T(T_1(x), T_2(x,p)) = H_p(x)\},$$

and the lower semicontinuous function associated to $\mathcal{E}_H$ defined for every $(x, p, y_1, y_2, z) \in X \times P \times Y_1 \times Y_2 \times Z$ by,

$$\varphi_{\mathcal{E}_H}^p((x, y_1, y_2), z) = \begin{cases} \liminf\limits_{\substack{(v_1,v_2) \in T_1(u) \times T_2(u,p), \\ (u,v_1,v_2) \to (x,y_1,y_2)}} d(z, T(v_1, v_2)) & \text{if } (y_1, y_2) \in T_1(x) \times T_2(x,p), \\ +\infty & \text{otherwise.} \end{cases}$$

Next we recall the local composition-stability introduced by Durea and Strugariu in [20] with the aim to establish the Aubin property of the composition of two set-valued mappings.

Definition 2.6. Let $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ be two set-valued mappings between metric spaces. The pair (F, G) is said to be locally composition stable around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} \in G(\bar{y}), \bar{y} \in F(\bar{x})$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that for any $x \in B(\bar{x}, \delta)$ and any $z \in (G \circ F)(x) \cap B(\bar{z}, \delta)$, there exists $y \in F(x) \cap B(\bar{y}, \varepsilon)$ such that $z \in G(y)$.

Durea and Strugariu in [20] gave an example of a pair (F, G) which is not locally composition stable around $(\bar{x}, \bar{y}, \bar{z})$, such that the composite mapping $G \circ F$ fails to have the Aubin property even if F and G have the Aubin property. In the case of the sum of two closed set-valued mappings, Ngai, Tron, Théra ([33]) showed that the sum of a metrically regular set-valued mapping and a pseudo-Lipschitz one is not in general metrically regular without the sum-stability. Two interesting cases which ensure this property can be found in [23], Proposition 3.4 and 3.5.

$\boxed{3}$ Regularity of Parametrized Epigraphical and Composition Set-Valued Mappings

In this section, we will use a result given by Ngai, Tron, Théra ([33], Theorem 3. 2) to establish metric regularity of the parametrized epigraphical set-valued mapping associated to the general composition one mentioned above, and then we will obtain the semiregularity of $H_{\bar{p}}$ as defined in (2.2). The following proposition gives a characterization of a parametrized set-valued mapping through the strong slope of the lower envelope function associated to it. This characterization has an important role in the sequel. Before stating this proposition, we recall the definition of lower semicontinuity for a set-valued mapping. Let $F : X \rightrightarrows Y$ be a set-valued mapping between two topological spaces. We say that F is lower semicontinuous at $\bar{x} \in X$ if for any $\bar{y} \in F(\bar{x})$ and any neighborhood V of $\bar{y}$, there exists a neighborhood U of $\bar{x}$ such that

$$\forall x \in U, F(x) \cap V \neq \emptyset.$$

Proposition 3.1. *Let X be a complete metric space, P be a topological space and Y be a normed space, $T : X \times P \rightrightarrows Y$ be a set-valued mapping with $(\bar{x}, \bar{p}, \bar{y}) \in \operatorname{gph} T$ and satisfying:*

(a) *the set-valued mapping $p \rightrightarrows T(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;*

(b) *for any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows T(x, p)$ is a closed (i.e., has closed graphs).*

Then,

(i) *T is metrically regular around $(\bar{x}, \bar{p}, \bar{y})$ with respect to (x, y), uniformly in p with modulus $\tau > 0$ if and only if there exists a neighborhood $U \times W \times V$ of $(\bar{x}, \bar{p}, \bar{y})$ such that for all $(x, p, y) \in U \times W \times V$ with $0 < \varphi_T^p(x, y) < +\infty$, one has*

$$|\nabla \varphi_T^p(y)|(x) \geq \tau^{-1}; \tag{3.1}$$

(ii) *T is metrically regular around $(\bar{x}, \bar{p}, \bar{y})$ with respect to (x, y), uniformly in p with modulus $\tau > 0$ if and only if there exists a neighborhood $U \times W \times V$ of $(\bar{x}, \bar{p}, \bar{y})$ such that for any $(x, p, y) \in U \times W \times V$ with $0 < \varphi_T^p(x, y) < +\infty$ and for any $\varepsilon > 0$, for any sequence $\{x_n\} \subset X$ tending to x with $\liminf_{n\to\infty} d(y, T(x_n, p)) = \varphi_T^p(x, y)$, there exists a sequence $\{u_n\} \subset X$ with $\liminf_{n\to\infty} d(u_n, x) > 0$ such that*

$$\limsup_{n\to\infty} \frac{d(y, T(x_n, p)) - d(y, T(u_n, p))}{d(x_n, u_n)} > \frac{1}{\tau + \varepsilon}.$$

Remark 3.2. (i) We say a function $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ is epi-upper semicontinuous at $(\bar{x}, \bar{p}) \in X \times P$ if

$$\tilde{f}(\bar{x}, \bar{p}) \leq f(\bar{x}, \bar{p}),$$

where $\tilde{f}(\bar{x}, \bar{p})$ is defined by

$$\tilde{f}(\bar{x}, \bar{p}) := \sup_{\varepsilon>0} \inf_{\delta>0} \sup_{p\in B(\bar{p},\delta)} \inf_{x\in B(\bar{x},\varepsilon)} f(x, p).$$

In fact,

$$\tilde{f}(\bar{x}, \bar{p}) = \sup_{\varepsilon>0} \left(\limsup_{p\to\bar{p}} \left(\inf_{x\in B(\bar{x},\varepsilon)} f(x, p) \right) \right) = \lim_{\varepsilon\searrow 0} \left(\limsup_{p\to\bar{p}} \left(\inf_{x\in B(\bar{x},\varepsilon)} f(x, p) \right) \right).$$

Then, the assumption on (b) on the upper semicontinuity of the mapping $p \to f(\bar{x}, p)$ given in Theorem 2 by Ngai, Tron, Théra [33] to ensure the existence of a local uniform error bound for the parametric system

$$f(x,p) \leq 0$$

is replaced by the epi-upper semicontinuity of the mapping $(x,p) \to f(x,p)$ as given in Theorem 3.1 by Azé and Benahmed [6].

(ii) So, the conclusion of Proposition 3.1 still holds if we replace the assumption (a) : "the set-valued mapping $p \rightrightarrows T(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$" by a weaker one (a'): "the set-valued mapping $(x,p) \rightrightarrows T(x,p)$ is epi-lower semicontinuous at $(\bar{x}, \bar{p})$ " which is defined by

Definition 3.3. We say that the set-valued mapping $(x,p) \rightrightarrows T(x,p)$ is epi-lower semicontinuous at $(\bar{x}, \bar{p})$ with $\bar{y} \in T(\bar{x}, \bar{p})$ if for any neighborhood V of $\bar{y}$ with $V \cap T(\bar{x}, \bar{p}) \neq \emptyset$, for any $\varepsilon > 0$, there exists a neighborhood U of $\bar{p}$ such that for every $p \in U$, there is $x_p \in B(\bar{x}, \varepsilon)$ satisfying $V \cap T(x_p, p) \neq \emptyset$.

(iii) Noting that if the set-valued mapping $(x,p) \rightrightarrows T(x,p)$ is epi-lower semicontinuous at $(\bar{x}, \bar{p})$ then the distance function $(x,p) \to d(\bar{y}, T_p(x))$ is epi-upper semicontinuous at $(\bar{x}, \bar{p})$ (a similar result can be seen, e.g., Aubin, Ekeland, ([4], Corollary 20)).

The following two lemmas will be useful in the sequel.

Lemma 3.4. *Let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, X, Y_1, Y_2, Z be metric spaces, P be a topological space. Then, for every $(z,p) \in Z \times P$, one has*

$$\mathbb{S}_{\mathcal{E}_H}(z,p) := \{(x, y_1, y_2) \in X \times Y_1 \times Y_2 : z \in T(y_1, y_2), (y_1, y_2) \in T_1(x) \times T_2(x,p)\}.$$

Proof. The proof of this lemma follows directly from the definition of map $\mathbb{S}_{\mathcal{E}_H}$. □

Lemma 3.5. *Let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, X, Y_1, Y_2, Z be metric spaces, P be a topological space satisfying the following conditions for some $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times P \times Y_1 \times Y_2 \times Z$:*

(a) $(\bar{x}, \bar{y}_1, \bar{y}_2) \in \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p})$;

(b) *the set-valued mapping $(y_1, y_2) \rightrightarrows T(y_1, y_2)$ is lower semicontinuous at $(\bar{y}_1, \bar{y}_2)$, $x \rightrightarrows T_1(x)$ is lower semicontinuous at $\bar{x}$, $p \rightrightarrows T_2(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;*

(c) *the set-valued mappings T_1, T have closed graphs, and for any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows T_2(x,p)$ is a closed set-valued mapping.*

Then,

(i) *for ever p near $\bar{p}$, the epigraphical set-valued mapping $\mathcal{E}_H^p$ has a closed graph; and the set-valued mapping $\mathcal{E}_H^p$ is epi-lower semicontinuous at $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2)$ and $H_p(\bar{x})$ is lower semicontinuous at $\bar{p}$;*

(ii) *the function $(x, p, y_1, y_2) \mapsto \varphi_{\mathcal{E}_H}^p((x, y_1, y_2), \bar{z})$ is epi-upper semicontinuous at $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2)$;*

(iii) *for each* $(z,p) \in Z \times P$, *one has*

$$\{(x,y_1,y_2) \in X \times Y_1 \times Y_2 : \varphi^p_{\mathcal{E}_H}((x,y_1,y_2),z) = 0\} = \mathbb{S}_{\mathcal{E}_H}(z,p).$$

Proof. For (i), it is obvious that the epigraphical set-valued mapping $\mathcal{E}^p_H$ has a closed graph when the set-valued mappings T_1, T have closed graphs, and the set-valued mapping $x \rightrightarrows T_2(x,p)$ is a closed set-valued mapping for any p near $\bar{p}$. Using the conservation of the lower semicontinuity of composition mappings, we obtain the lower semicontinuity at $\bar{p}$ of the mapping $H_p(\bar{x})$. For proving the epi-lower semicontinuity of the mapping $\mathcal{E}^p_H$ at $(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2)$, we take an arbitrary neighborhood W of $\bar{z}$ with $W \cap \mathcal{E}^p_H(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2) \neq \emptyset$ and $\varepsilon > 0$. Since the set-valued mapping $(y_1,y_2) \rightrightarrows T(y_1,y_2)$ is lower semicontinuous at $(\bar{y}_1,\bar{y}_2)$, we can choose a positive real $\delta < \varepsilon$ such that

$$W \cap T(y_1,y_2) \neq \emptyset,\ \forall (y_1,y_2) \in B(\bar{y}_1,\delta) \times B(\bar{y}_2,\delta). \tag{3.2}$$

On the other hand, due to the lower semicontinuity of the mapping $p \rightrightarrows T_2(\bar{x},p)$ at $\bar{p}$, as Remark 3.2, the set-valued mapping $(x,p) \rightrightarrows T_2(x,p)$ is epi-lower semicontinuous at $(\bar{x},\bar{p})$. Thus, for any $\varepsilon > 0$ arbitrarily, there is a neighborhood U of $\bar{p}$ such that for every $p \in U$, there is $x_p \in B(\bar{x},\varepsilon)$ we have

$$B(\bar{y}_2,\delta) \cap T_2(x_p,p) \neq \emptyset. \tag{3.3}$$

Moreover, since $x \rightrightarrows T_1(x)$ is lower semicontinuous at $\bar{x}$, there is a neighborhood U_1 of $\bar{x}$ such that

$$T_1(x) \cap B(\bar{y}_1,\delta) \neq \emptyset,\ \forall x \in U_1. \tag{3.4}$$

From (3.3), we have that for every $p \in U$, there exists $y^p_2 \in B(\bar{y}_2,\delta) \cap T_2(x_p,p)$. Taking ε smaller if necessary, we can assume that $B(\bar{x},\varepsilon) \subset U_1$. Then, by (3.4), we get $T_1(x_p) \cap B(\bar{y}_1,\delta) \neq \emptyset$. Therefore, there is a $y^p_1 \in T_1(x_p) \cap B(\bar{y}_1,\delta)$. Next, according to (3.2), one has that

$$W \cap T(y^p_1,y^p_2) \neq \emptyset.$$

Consequently, $W \cap T(y^p_1,y^p_2) \neq \emptyset$ with $x_p \in B(\bar{x},\varepsilon)$, $y^p_1 \in T_1(x_p) \cap B(\bar{y}_1,\delta)$, $y^p_2 \in B(\bar{y}_2,\delta) \cap T_2(x_p,p)$. In other words, for every $p \in U$, one gets a point $(x_p,y^p_1,y^p_2) \in B(\bar{x},\varepsilon) \times B(\bar{y}_1,\delta) \times B(\bar{y}_2,\delta) \subset B(\bar{x},\varepsilon) \times B(\bar{y}_1,\varepsilon) \times B(\bar{y}_2,\varepsilon)$ such that

$$W \cap \mathcal{E}^p_H(x_p,y^p_1,y^p_2) \neq \emptyset,$$

which imples that the set-valued mapping $\mathcal{E}^p_H$ is epi-lower semicontinuous at $(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2)$.

For (ii), since the set-valued mapping $\mathcal{E}^p_H$ is epi-lower semicontinuous at $(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2)$ (by (i)), the function $(x,p,y_1,y_2) \to d(\bar{z},\mathcal{E}^p_H(x,y_1,y_2))$ is epi-upper semicontinuous at $(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2)$. Thus,

$$\begin{aligned}\limsup_{p\to\bar{p}}\left(\inf_{(x,y_1,y_2)\in B(\bar{x},\varepsilon)\times B(\bar{y}_1,\varepsilon)\times B(\bar{y}_2,\varepsilon)} \varphi^p_{\mathcal{E}_H}((x,y_1,y_2),\bar{z})\right) &\le \limsup_{p\to\bar{p}} d(\bar{z},\mathcal{E}^p_H(\bar{x},\bar{y}_1,\bar{y}_2))\\ &\le d(\bar{z},\mathcal{E}^{\bar{p}}_H(\bar{x},\bar{y}_1,\bar{y}_2)) = 0\\ &= \varphi^{\bar{p}}_{\mathcal{E}_H}((\bar{x},\bar{y}_1,\bar{y}_2),\bar{z}),\end{aligned}$$

which implies that

$$\begin{aligned}\tilde{\varphi}^{\bar{p}}_{\mathcal{E}_H}((\bar{x},\bar{y}_1,\bar{y}_2),\bar{z}) &= \inf_{\varepsilon>0}\limsup_{p\to\bar{p}}\left(\inf_{(x,y_1,y_2)\in B(\bar{x},\varepsilon)\times B(\bar{y}_1,\varepsilon)\times B(\bar{y}_2,\varepsilon)} \varphi^p_{\mathcal{E}_H}((x,y_1,y_2),\bar{z})\right)\\ &\le \varphi^{\bar{p}}_{\mathcal{E}_H}((\bar{x},\bar{y}_1,\bar{y}_2),\bar{z}).\end{aligned}$$

This proves (ii). (iii) is clear. The proof is completed. $\square$

Using the results above, we establish metric regularity of the parametrized epigraphical set-valued mapping $\mathcal{E}_H^p$ and semiregularity of $H_{\bar{p}}$. This is one of our main results in this paper. The proof is inspired from the work by Durea, Huynh, Nguyen, Strugariu [23]. However, our argument is somewhere simpler.

Theorem 3.6. *Let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, X, Y_1, Y_2 be complete metric spaces, Z be a normed linear space, P be a topological space satisfying conditions $(a), (b), (c)$ in Lemma 3.5 around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times P \times Y_1 \times Y_2 \times Z$. Suppose that*

(i) *T_1 is metrically regular around $(\bar{x}, \bar{y}_1)$ with modulus $m > 0$;*

(ii) *T_2 has the Lipschitz-like property around $((\bar{x}, \bar{p}), \bar{y}_2)$ with respect to (x, y_2), uniformly in p with modulus $l > 0$;*

(iii) *T is metrically regular around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to (y_1, z), uniformly in y_2 with modulus $\lambda > 0$;*

(iv) *T is Lipschitz-like around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to (y_2, z), uniformly in y_1 with modulus $\gamma > 0$;*

(v) *$\lambda m \gamma l < 1$.*

Assume that the product space $X \times Y_1 \times Y_2$ is endowed with the metric defined by

$$d((x, y_1, y_2), (u, v_1, v_2)) = \max\left\{d(x, u), md(y_1, v_1), l^{-1}d(y_2, v_2)\right\}.$$

Then,

(a) *$\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z})$ with respect to (x, y_1, y_2) uniformly in p with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$;*

(b) *$H_{\bar{p}}$ is metrically semiregular at $(\bar{x}, \bar{z})$ with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$.*

Proof. • For proving (a), according to (i), T_1 is metrically regular around $(\bar{x}, \bar{y}_1)$ with modulus $m > 0$, and therefore, there are $\delta_1 > 0$ such that

$$d(x, T_1^{-1}(y_1)) \leq md(y_1, T_1(x)) \text{ for all } (x, y_1) \in B(\bar{x}, \delta_1) \times B(\bar{y}_1, \delta_1). \tag{3.5}$$

By (ii), T_2 is Lipschitz-like around $((\bar{x}, \bar{p}), \bar{y}_2)$ with respect to (x, y_2), uniformly in p with modulus $l > 0$. Hence, there are $\delta_2 > 0$ and a neighborhood $\mathcal{U}$ of $\bar{p}$ such that

$$T_2(x, p) \cap B(\bar{y}_2, \delta_2) \subset T_2(u, p) + ld(x, u)\bar{B}_{Y_2}, \tag{3.6}$$

for all $x, u \in B(\bar{x}, \delta_2)$, $p \in \mathcal{U}$. Next, by (iv), there is $\delta_3 > 0$ such that

$$T(y_1, y_2) \cap B(\bar{z}, \delta_3) \subset T(y_1, y_2') + \gamma d(y_2, y_2')\bar{B}_Z, \tag{3.7}$$

for all $y_1 \in B(\bar{y}_1, \delta_3), (y_2, y_2') \in B(\bar{y}_2, \delta_3)$. Moreover, by (iii) and using Proposition 3.1 (i), there is $\delta_4 > 0$ such that

$$|\nabla \varphi_T^{y_2}(z)|(y_1) \geq \frac{1}{\lambda}$$

for all $(y_1, y_2, z) \in B(\bar{y}_1, \delta_4) \times B(\bar{y}_2, \delta_4) \times B(\bar{z}, \delta_4)$ with $0 < \varphi_T^{y_2}((y_1, y_2), z) < +\infty$. So, for any $\varepsilon > 0$, there exists $v_1 \in B(\bar{y}_1, \delta_4), v_1 \neq y_1$ such that

$$\frac{\varphi_T^{y_2}((y_1, y_2), z) - \varphi_T^{y_2}((v_1, y_2), z)}{d(y_1, v_1)} > \frac{1}{\lambda + \varepsilon}. \tag{3.8}$$

Setting $\delta := \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then, for every $(x, p, y_1, y_2, z) \in B(\bar{x}, \delta) \times \mathcal{U} \times B(\bar{y}_1, \delta) \times B(\bar{y}_2, \delta) \times B(\bar{z}, \delta)$, $z \notin T(y_1, y_2)$, $(y_1, y_2) \in T_1(x) \times T_2(x, p)$ with $0 < \varphi^p_{\mathcal{E}_H}((x, y_1, y_2), z) < +\infty$, any $\varepsilon > 0$, any $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x, $\{y_{1n}\}_{n\in\mathbb{N}} \subseteq Y_1$ converging to y_1, $\{y_{2n}\}_{n\in\mathbb{N}} \subseteq Y_2$ converging to y_2 with $z \notin T(y_{1n}, y_{2n})$, $(y_{1n}, y_{2n}) \in T_1(x_n) \times T_2(x_n, p)$ and

$$\lim_{n\to\infty} d(z, T(y_{1n}, y_{2n})) = \liminf_{\substack{(x', y_1', y_2')\to(x, y_1, y_2),\\ (y_1', y_2')\in T_1(x')\times T_2(x')}} d(z, T(y_1', y_2')) = \varphi^p_{\mathcal{E}_H}((x, y_1, y_2), z), \tag{3.9}$$

one derives by the definition of $\varphi^{y_2}_T$, there exists v_{1n} converging to v_1 such that $\lim_{n\to\infty} d(z, T(v_{1n}, y_2)) = \varphi_T((v_1, y_2), z)$, with $\liminf_{n\to\infty} d(v_{1n}, y_{1n}) > 0$, and from the relations (3.8) and (3.9) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{d(z, T(y_{1n}, y_2)) - d(z, T(v_{1n}, y_2))}{d(y_{1n}, v_{1n})} &\geq \frac{\varphi^{y_2}_T((y_1, y_2), z) - \varphi^{y_2}_T((v_1, y_2), z)}{d(y_1, v_1)}\\ \limsup_{n\to\infty} \frac{d(y_1, v_1)}{d(y_{1n}, v_{1n})} &> \frac{1}{\lambda + \varepsilon}.\end{aligned} \tag{3.10}$$

Since (v_{1n}) converges to y_1, without loss of generality, we can assume that $v_{1n} \in B(\bar{y}_1, \delta)$. Then, by (3.5), we see that $d(x_n, T_1^{-1}(v_{1n})) \leq md(v_{1n}, T_1(x_n)) \leq md(v_{1n}, y_{1n})$. It follows that there is $u_n \in T_1^{-1}(v_{1n})$ such that

$$d(x_n, u_n) \leq md(v_{1n}, y_{1n}). \tag{3.11}$$

Moreover, since (y_{2n}) converges to y_2, for sufficiently large n we have $y_{2n} \in B(\bar{y}_2, \delta)$. Note that $y_{2n} \in T_2(x_n, p)$; thus by (3.6), there exists $w_{2n} \in T_2(u_n, p)$ such that

$$d(y_{2n}, w_{2n}) \leq ld(x_n, u_n). \tag{3.12}$$

Furthermore, by (3.7), one has

$$\begin{aligned} d(z, T(v_{1n}, w_{2n})) &\leq d(z, T(v_{1n}, y_{2n})) + e(T(v_{1n}, y_{2n}), T(v_{1n}, w_{2n}))\\ &\leq d(z, T(v_{1n}, y_{2n})) + \gamma d(y_{2n}, w_{2n}). \end{aligned} \tag{3.13}$$

Thus, $\liminf_{n\to\infty} d\big((u_n, v_{1n}, w_{2n}), (x, y_1, y_2)\big) \geq \liminf_{n\to\infty} d(v_{1n}, y_1) > 0$, and by using the relations (3.9)-(3.13), for large n, one obtains that

$$\begin{aligned}
&\frac{d\left(z, T(y_{1n}, y_{2n})\right) - d(z, T(v_{1n}, w_{2n}))}{d((x_n, y_{1n}, y_{2n}), (u_n, v_{1n}, w_{2n}))}\\
&\geq \frac{d(z, T(y_{1n}, y_{2n})) - d(z, T(v_{1n}, y_{2n})) - \gamma d(y_{2n}, w_{2n})}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n}), l^{-1}d(y_{2n}, w_{2n})\}}\\
&\geq \frac{d(z, T(y_{1n}, y_{2n})) - d(z, T(v_{1n}, y_{2n}))}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n}), l^{-1}d(y_{2n}, w_{2n})\}}\\
&- \frac{\gamma d(y_{2n}, w_{2n})}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n}), l^{-1}d(y_{2n}, w_{2n})\}}\\
&\geq \frac{d(z, T(y_{1n}, y_{2n})) - d(z, T(v_{1n}, y_{2n}))}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n}), l^{-1}d(y_{2n}, w_{2n})\}} - \frac{\gamma d(y_2, w_{2n})}{l^{-1}d(y_{2n}, w_{2n})}\\
&= \frac{d(z, T(y_{1n}, y_{2n})) - d(z, T(v_{1n}, y_{2n}))}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n})\}} - l\gamma\\
&= \frac{d\left(z, T(y_{1n}, y_2)\right) - d(z, T(v_{1n}, y_2))}{d(y_{1n}, v_{1n})} \cdot \frac{d(y_{1n}, v_{1n})}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n})\}} - l\gamma
\end{aligned}$$

$$\geq \frac{1}{m(\lambda+\varepsilon)} - l\gamma.$$

Since $\lambda m\gamma l < 1$, by choosing $\varepsilon > 0$ sufficiently small, one has $\frac{1}{m(\lambda+\varepsilon)} - l\gamma > 0$. Therefore, by taking into account Lemma 3.5 and applying Proposition 3.1, (ii) for $\mathcal{E}_H^p$, we obtain that $\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z})$ with respect to (x, y_1, y_2) uniformly in p with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$.

• For proving (b), according to (a), there exist $r > 0$ and a neighborhood $\mathcal{V}$ of $\bar{p}$ such that

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z, p)) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^p_{\mathcal{E}_H}((x, y_1, y_2), z) \tag{3.14}$$

for all $(x, p, y_1, y_2, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(\bar{y}_1, r) \times B(\bar{y}_2, r) \times B(\bar{z}, r)$. Taking $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2)$ in (3.14), one gets the estimation

$$d((\bar{x}, \bar{y}_1, \bar{y}_2), \mathbb{S}_{\mathcal{E}_H}(z, \bar{p})) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{\bar{p}}_{\mathcal{E}_H}((\bar{x}, \bar{y}_1, \bar{y}_2), z).$$

Taking $(x', y_1', y_2') \in \mathbb{S}_{\mathcal{E}_H}(z, \bar{p})$ such that

$$d((\bar{x}, \bar{y}_1, \bar{y}_2), (x', y_1', y_2')) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{\bar{p}}_{\mathcal{E}_H}((\bar{x}, \bar{y}_1, \bar{y}_2), z), \tag{3.15}$$

(3.15) yields

$$\begin{aligned} d(\bar{x}, x') &\leq d((\bar{x}, \bar{y}_1, \bar{y}_2), (x', y_1', y_2')) \\ &\leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{\bar{p}}_{\mathcal{E}_H}((\bar{x}, \bar{y}_1, \bar{y}_2), z) \\ &\leq \frac{m\lambda}{1 - m\lambda l\gamma} d(z, T(\bar{y}_1, \bar{y}_2)) \\ &\leq \frac{m\lambda}{1 - m\lambda l\gamma} \|z - \bar{z}\|. \end{aligned}$$

By noting that $z \in T(y_1', y_2') \subset T(T_1(x'), T_2(x', \bar{p})) = H_{\bar{p}}(x')$, i.e., $x' \in H_{\bar{p}}^{-1}(z)$, we deduce that for all z near $\bar{z}$,

$$d(\bar{x}, H_{\bar{p}}^{-1}(z)) \leq \frac{m\lambda}{1 - m\lambda l\gamma} d(z, T(\bar{y}_1, \bar{y}_2)) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \|z - \bar{z}\|.$$

So, $H_{\bar{p}}$ is metrically semiregular at $(\bar{x}, \bar{z})$ with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$. The proof is completed. □

The following corollary was given recently by Durea, Huynh, Nguyen, Strugariu [23]. It could be considered as a nonparametric case of Theorem 3.6.

Corollary 3.7. *Let X, Y_1, Y_2 be complete metric spaces, Z be a normed space. Suppose that $T_1 : X \rightrightarrows Y_1$, $T_2 : X \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ are closed set-valued mappings satisfying for some $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times Y_1 \times Y_2 \times Z$ with $(\bar{x}, \bar{y}_1) \in \operatorname{gph} T_1$, $(\bar{x}, \bar{y}_2) \in \operatorname{gph} T_2$, $((\bar{y}_1, \bar{y}_2), \bar{z}) \in \operatorname{gph} T$ the five following conditions:*

(i) *T_1 is metrically regular around $(\bar{x}, \bar{y}_1)$ with modulus $m > 0$;*

(ii) *T_2 is Lipschitz-like around $(\bar{x}, \bar{y}_2)$ with modulus $l > 0$;*

(iii) *T is metrically regular around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to y_1, uniformly in y_2 with modulus $\lambda > 0$;*

(iv) *T is Lipschitz-like around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to y_2, uniformly in y_1 with modulus $\gamma > 0$;*

(v) $\lambda m \gamma l < 1$.

Assume that the product space $X \times Y_1 \times Y_2$ is endowed with the metric defined by

$$d((x, y_1, y_2), (u, v_1, v_2)) = \max\left\{d(x, u), md(y_1, v_1), l^{-1} d(y_2, v_2)\right\}.$$

Then,

(a) *there exists a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z} \subseteq X \times Y_1 \times Y_2 \times Z$ of $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$ such that*

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z)) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi_{\mathcal{E}_H}((x, y_1, y_2), z)$$

for all $(x, y_1, y_2, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z}$;

(b) *H is metrically semiregular around $(\bar{x}, \bar{z})$ with modulus $\frac{m\lambda}{1 - m\lambda l\gamma}$.*

Remark 3.8. (i) If T is metrically subregular around $(\bar{y}_1, \bar{y}_2, \bar{z})$ with respect to y_1, uniformly in y_2 with modulus $\lambda > 0$, then the mapping $\mathcal{E}_H$ is metrically subregular around $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$ with modulus $\frac{m\lambda}{1 - m\lambda l\gamma}$ (see, for instance [17]).

(ii) If in the theorem above we suppose further that $((T_1, T_2), T)$ is locally composition-stable around $(\bar{x}, (\bar{y}_1, \bar{y}_2), \bar{z})$, then H is metrically regular at $(\bar{x}, \bar{z})$ (see [18], [20], [23]).

A special case of the above result is when $Y_1 \equiv Y_2 \equiv Y$ with Y being a normed space, $T(y_1, y_2) = \{y_1 + y_2\}$, $H(x) = T_1(x) + T_2(x)$, we can reobtain metric regularity of the sum mapping as well as the one of associated epigraphical map through regularities of component mappings (see, for instance [22], [33]). Further, we achieve semiregularity of the sum set-valued mapping given recently in [9].

Corollary 3.9. *Let X be a complete metric space, Y be a normed space and let $T_1, T_2 : X \rightrightarrows Y$ be closed set-valued mapping. If T_1 is metrically regular around $(\bar{x}, \bar{y}_1) \in \operatorname{gph} T_1$ with modulus τ and T_2 is Lipschitz-like around $(\bar{x}, \bar{y}_2) \in \operatorname{gph} T_2$ with modulus λ such that $\tau\lambda < 1$, then $T_1 + T_2$ is metrically semiregular around $(\bar{x}, \bar{y}_1 + \bar{y}_2)$ with modulus $\left(\tau^{-1} - \lambda\right)^{-1}$.*

The next corollary is also a special case of Theorem 3.7 when $Y_1 \equiv Y_2 \equiv Y$, $T_1 \equiv S : X \rightrightarrows Y$ and $T \equiv Q : Y \rightrightarrows Z$.

Corollary 3.10. *Let X be a complete metric space, Y be a metric space, Z be a normed space. Suppose that $S : X \rightrightarrows Y$ and $Q : Y \rightrightarrows Z$ are closed set-valued mappings satisfying the following conditions for some $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} S$, $(\bar{y}, \bar{z}) \in \operatorname{gph} Q$:*

(i) *S is metrically regular around $(\bar{x}, \bar{y})$ with modulus $\tau > 0$;*

(ii) *Q is metrically regular around $(\bar{y}, \bar{z})$ with modulus $\lambda > 0$;*

Suppose that the product space $X \times Y$ is endowed with the metric defined by

$$d((x, y), (u, v)) = \max\left\{d(x, u), \tau d(y, v)\right\}.$$

Then,

(a) *the set-valued mapping* $\mathcal{E}_{Q\circ S}$ *is metrically regular at* $(\bar{x}, \bar{y}, \bar{z})$ *with modulus* $\tau\lambda$*;*

(b) $Q \circ S$ *is metrically semiregular around* $(\bar{x}, \bar{z})$ *with modulus* $\tau\lambda$.

In Corollary 3.10, if we consider the special case which the map $S : X \to Y$ is continuous, then the pair (S, Q) is locally composition-stable around the considerable point, we recover a result given by Zheng and Ng in [45].

Corollary 3.11. *Let* X *be a complete metric space,* Y *be a metric space,* Z *be a normed space. Suppose that* $S : X \to Y$ *is continuous and* $Q : Y \rightrightarrows Z$ *is closed set-valued mapping satisfying the following conditions for some* $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$ *with* $(\bar{x}, \bar{y}) \in \operatorname{gph} S$, $(\bar{y}, \bar{z}) \in \operatorname{gph} Q$:

(i) S *is metrically regular around* $(\bar{x}, \bar{y})$ *with modulus* $\tau > 0$*;*

(ii) Q *is metrically regular around* $(\bar{y}, \bar{z})$ *with modulus* $\lambda > 0$*;*

Then $Q \circ S$ *is metrically regular around* $(\bar{x}, \bar{z})$ *with modulus* $\tau\lambda$.

Proof. According to Corollary 3.10, one obtains that the set-valued mapping $\mathcal{E}_{Q\circ S}$ is metrically regular at $(\bar{x}, \bar{y}, \bar{z})$ with modulus $\tau\lambda$. Since (S, Q) is locally composition-stable around $(\bar{x}, \bar{y}, \bar{z})$, by taking into account Remark 3.8 (ii) above, it is metrically regular around $(\bar{x}, \bar{z})$ with modulus $\tau\lambda$. $\square$

4 Stability of Implicit Set-Valued Mappings

4.1 Stability of implicit set-valued mappings associated to epigraphical set-valued mapping

The goal of this section is to establish Lipschitz-likeness, calmness of the solution set-valued mapping $\mathbb{S}_{\mathcal{E}_H}$, when the space of parameters P is a metric space.

Theorem 4.1. *Let* $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ *and* $T : Y_1 \times Y_2 \rightrightarrows Z$ *be set-valued mappings,* X, Y_1, Y_2*, be complete metric spaces,* P *be a metric space,* Z *be a normed linear space satisfying conditions* $(a), (b), (c)$ *in Lemma 3.5 around* $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times P \times Y_1 \times Y_2 \times Z$*. Furthermore, suppose that*

(i) T_1 *is metrically regular around* $(\bar{x}, \bar{y}_1)$ *with modulus* $m > 0$*;*

(ii) T_2 *has the Lipschitz-like property around* $((\bar{x}, \bar{p}), \bar{y}_2)$ *with respect to* x*, uniformly in* p *with modulus* $l > 0$*;*

(iii) T *is metrically regular around* $((\bar{y}_1, \bar{y}_2), \bar{z})$ *with respect to* y_1*, uniformly in* y_2 *with modulus* $\lambda > 0$*;*

(iv) T *is Lipschitz-like around* $((\bar{y}_1, \bar{y}_2), \bar{z})$ *with respect to* y_2*, uniformly in* y_1 *with modulus* $\gamma > 0$*;*

(v) $\lambda m \gamma l < 1$*;*

(vi) T_2 *has Lipschitz-like property around* $((\bar{x}, \bar{p}), \bar{y}_2)$ *with respect to* p*, uniformly in* x *with modulus* $\theta > 0$.

Then $\mathbb{S}_{\mathcal{E}_H}$ *is Lipschitz-like around* $((\bar{z}, \bar{p}), (\bar{x}, \bar{y}_1, \bar{y}_2))$ *with modulus* $\theta\big(\frac{m\lambda\gamma}{1-m\lambda l\gamma} + 1\big) + \frac{m\lambda}{1-m\lambda l\gamma}$.

Proof. By using Theorem 3.6, one obtains that $\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, z)$ with respect to (x, y_1, y_2) uniformly in p with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$. Then, there is $\delta_1 > 0$ such that

$$d\left((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z, p)\right) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^p_{\mathcal{E}_H}((x, y_1, y_2), z), \tag{4.1}$$

for all $(x, p, y_1, y_2, z) \in B((\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, \bar{z}), \delta_1)$. By (vi), there is $\delta_2 > 0$ such that

$$T_2(x, p) \cap B(\bar{y}_2, \delta_2) \subset T_2(x, p') + \theta d(p, p')\bar{B}_{Y_2}, \tag{4.2}$$

for all $p, p' \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$. Moreover, by (iv), there exists $\delta_3 > 0$ such that for all $y_1 \in B(\bar{y}_1, \delta_3)$, and for all $y_2, y_2' \in B(\bar{y}_2, \delta_3)$, one has

$$T(y_1, y_2) \cap B(\bar{z}, \delta_3) \subset T(y_1, y_2') + \gamma d(y_2, y_2')\bar{B}_Z. \tag{4.3}$$

Set $\alpha := \min\left\{\frac{\delta_1}{2\theta+1}, \delta_2, \delta_3\right\}$. Fix $(z, p), (z', p') \in B(\bar{z}, \alpha) \times B(\bar{p}, \alpha)$ and take $(x, y_1, y_2) \in \mathbb{S}_{\mathcal{E}_H}(z, p)) \cap B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)$. Then $z \in T(y_1, y_2)$, $(y_1, y_2) \in T_1(x) \times T_2(x, p)$ and $(x, y_1, y_2) \in B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)$. Taking into account (4.2), select ${y_2}' \in T_2(x, p')$ such that

$$d(y_2, y_2') \leq \theta d(p, p') < 2\theta\alpha. \tag{4.4}$$

Therefore, ${y_2}' \in B(\bar{y}_2, \delta_1)$, and then, by (4.1), one gets

$$d((x, y_1, {y_2}'), \mathbb{S}_{\mathcal{E}_H}(z', p')) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{p'}_{\mathcal{E}_H}((x, y_1, {y_2}'), z') \tag{4.5}$$

$$\leq \frac{m\lambda}{1 - m\lambda l\gamma} d(z', T(y_1, {y_2}'))). \tag{4.6}$$

Using the relations (4.4)-(4.6) and observing that $z \in T(y_1, y_2)$, we deduce that

$$\begin{aligned} d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z', p')) &\leq d(y_2, y_2') + d((x, y_1, {y_2}'), \mathbb{S}_{\mathcal{E}_H}(z', p')) \\ &\leq \theta d(p, p') + \frac{m\lambda}{1 - m\lambda l\gamma} d(z', T(y_1, {y_2}')). \\ &\leq \theta d(p, p') + \frac{m\lambda}{1 - m\lambda l\gamma} \|z - z'\| \\ &\quad + \frac{m\lambda}{1 - m\lambda l\gamma} d(z, T(y_1, {y_2}')). \end{aligned} \tag{4.7}$$

On the other hand, since $z \in T(y_1, y_2)$, thus by (4.3),

$$d(z, T(y_1, {y_2}')) \leq e(T(y_1, y_2), T(y_1, {y_2}')) \tag{4.8}$$

$$\leq \gamma d(y_2, y_2') \tag{4.9}$$

$$\leq \gamma\theta d(p, p'). \tag{4.10}$$

Thus, by the relations (4.7)-(4.10), we derive that

$$\begin{aligned} d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z', p')) &\leq \theta d(p, p') + \frac{m\lambda}{1 - m\lambda l\gamma} \|z - z'\| + \frac{m\lambda\gamma}{1 - m\lambda l\gamma} \theta d(p, p') \\ &\leq \theta \left(\frac{m\lambda}{1 - m\lambda l\gamma} + 1\right) d(p, p') + \frac{m\lambda\gamma}{1 - m\lambda l\gamma} \|z - z'\|. \end{aligned}$$

It follows that

$$\mathbb{S}_{\mathcal{E}_H}(z, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)] \subset \mathbb{S}_{\mathcal{E}_H}(z', p') + Ld((p, z), (p', z'))\overline{B}_{X \times Y_1 \times Y_2},$$

where $L := \theta\left(\frac{m\lambda\gamma}{1-m\lambda l\gamma} + 1\right) + \frac{m\lambda}{1-m\lambda l\gamma}$. This completes the proof. $\square$

In the next result we establish the calmness of the implicit set-valued mapping $\mathbb{S}_{\mathcal{E}_H}$ if in Theorem 4.1 we replace the condition (vi) that T_2 has the calmness property.

Theorem 4.2. *Let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, X, Y_1, Y_2, be complete metric spaces, P be a metric space, Z be a normed linear space satisfying conditions $(a), (b), (c)$ in Lemma 3.5 around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2) \in X \times P \times Y_1 \times Y_2$. Furthermore, we assume that*

(i) *T_1 is metrically regular around $(\bar{x}, \bar{y}_1)$ with modulus $m > 0$;*

(ii) *T_2 is Lipschitz-like around $((\bar{x}, \bar{p}), \bar{y}_2)$ with respect to x, uniformly in p with modulus $l > 0$;*

(iii) *T is metrically regular around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to y_1, uniformly in y_2 with modulus $\lambda > 0$;*

(iv) *T is Lipschitz-like around $((\bar{y}_1, \bar{y}_2), \bar{z})$ with respect to y_2, uniformly in y_1 with modulus $\gamma > 0$;*

(v) *$\lambda m \gamma l < 1$.*

(vi) *T_2 is calm around $((\bar{x}, \bar{p}), \bar{y}_2)$ with respect to p, uniformly in x with modulus $\theta > 0$.*

Then $\mathbb{S}_{\mathcal{E}_H}$ is calm around $((\bar{z}, \bar{p}), (\bar{x}, \bar{y}_1, \bar{y}_2))$ with modulus $\theta\big(\frac{m\lambda\gamma}{1-m\lambda l\gamma} + 1\big) + \frac{m\lambda}{1-m\lambda l\gamma}$.

Proof. According to Theorem 3.6, there is $\delta_1 > 0$ such that

$$d\left((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z, p)\right) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{p}_{\mathcal{E}_H}((x, y_1, y_2), z), \tag{4.11}$$

for all $(x, p, y_1, y_2, z) \in B((\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, 0), \delta_1)$.

And, by (vi), there is $\delta_2 > 0$ such that

$$T_2(x, p) \cap B(\bar{y}_2, \delta_2) \subset T_2(x, \bar{p}) + \theta d(p, \bar{p}) \bar{B}_{Y_2}, \tag{4.12}$$

for all $p \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$.
Moreover, by (iv), there $\delta_3 > 0$ such that for all $y_1 \in B(\bar{y}_1, \delta_3)$, and for all $y_2, y_2' \in B(\bar{y}_2, \delta_3)$, we have that

$$T(y_1, y_2) \cap B(\bar{z}, \delta_3) \subset T(y_1, y_2') + \gamma d(y_2, y_2') \bar{B}_Z, \tag{4.13}$$

Set $\alpha := \min\big\{\frac{\delta_1}{\theta+1}, \delta_2, \delta_3\big\}$, and fix $(z, p) \in B(\bar{z}, \alpha) \times B(\bar{p}, \alpha)$. Taking $(x, y_1, y_2) \in \mathbb{S}_{\mathcal{E}_H}(z, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)]$, then $z \in T(y_1, y_2)$, $(y_1, y_2) \in T_1(x) \times T_2(x, p)$ and $(x, y_1, y_2) \in B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)$. By (4.12), select ${y_2}' \in T_2(x, \bar{p})$ such that

$$d(y_2, {y_2}') \leq \theta d(p, \bar{p}) < \theta\alpha. \tag{4.14}$$

Thus, ${y_2}' \in B(\bar{y}_2, \delta_1)$ and by (4.11), one obtains

$$d((x, y_1, {y_2}'), \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p})) \leq \frac{m\lambda}{1 - m\lambda l\gamma} \varphi^{\bar{p}}_{\mathcal{E}_H}((x, y_1, {y_2}'), \bar{z}) \tag{4.15}$$

$$\leq \frac{m\lambda}{1 - m\lambda l\gamma} d(\bar{z}, T(y_1, {y_2}')). \tag{4.16}$$

Noticing that $z \in T(y_1, y_2)$ and using the relations (4.14)-(4.16), one derives that

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p})) \le d(y_2, y_2') + d((x, y_1, y_2'), \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p})) \tag{4.17}$$

$$\le \theta d(p, \bar{p}) + \frac{m\lambda}{1 - m\lambda l\gamma} d(\bar{z}, T(y_1, y_2')) \tag{4.18}$$

$$\le \theta d(p, \bar{p}) + \frac{m\lambda}{1 - m\lambda l\gamma} \|z - \bar{z}\| \tag{4.19}$$

$$+ \frac{m\lambda}{1 - m\lambda l\gamma} d(z, T(y_1, y_2')). \tag{4.20}$$

On the other hand, since $z \in T(y_1, y_2)$, by (4.13) we have

$$d(z, T(y_1, y_2')) \le e(T(y_1, y_2), T(y_1, y_2')) \tag{4.21}$$

$$\le \gamma d(y_2, y_2') \tag{4.22}$$

$$\le \theta\gamma d(p, \bar{p}). \tag{4.23}$$

Thus, by combining the relations from (4.15) to (4.23), one gets

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p})) \le \theta d(p, p') + \frac{m\lambda}{1 - m\lambda l\gamma} \|z - \bar{z}\| + \frac{m\lambda\gamma}{1 - m\lambda l\gamma} \theta d(p, p') \tag{4.24}$$

$$\le \theta \left(\frac{m\lambda\gamma}{1 - m\lambda l\gamma} + 1 \right) d(p, p') + \frac{m\lambda}{1 - m\lambda l\gamma} \|z - \bar{z}\|. \tag{4.25}$$

It follows that

$$\mathbb{S}_{\mathcal{E}_H}(z, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{y}_1, \alpha) \times B(\bar{y}_2, \alpha)] \subset \mathbb{S}_{\mathcal{E}_H}(\bar{z}, \bar{p}) + Ld((p, z), (\bar{p}, \bar{z}))\overline{B}_{X \times Y_1 \times Y_2},$$

This completes the proof. □

$\boxed{4.2}$ Stability of the implicit set-valued mapping associated to a composite mapping

In Theorem 4.1 if we further impose hypothesis on the local composition-stability of the pair $((T_1, T_2), T)$ then we achieve Robinson's metric regularity as well as the Lipschitzian stability of the solution set mapping $\mathbb{S}_H(0, \cdot)$ as given in the next theorem. Here, let us recall concept of Robinson metric regularity studied by Robinson [36, 37].

Definition 4.3. We say that $\mathbb{S}_H(0, \cdot)$ is Robinson metrically regular around $(\bar{x}, \bar{p})$ with modulus τ, if there exist neighborhoods U, V of $\bar{x}, \bar{p}$, respectively, such that

$$d(x, \mathbb{S}_H(0, p)) \le \tau d(0, H_p(x)), \text{ for all } (x, p) \in U \times V.$$

The relationships between Robinson metric regularity and Lipschitz-likeness of implicit set-valued mappings can be found in the work by Chieu, Yao, Yen ([10]). Before establishing the main result, one needs the following propositions.

Proposition 4.4. *Let X, Y_1, Y_2 are metric spaces, Z is a normed linear space, P is a topological space, and let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings. If $\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, 0)$ with respect to (x, y_1, y_2) uniformly in p with modulus τ, then there exist neighborhoods $B(\bar{x}, r), \mathcal{V}, B(\bar{y}_1, r), B(\bar{y}_2, r), B(0, r)$ of the points $x, p, y_1, y_2, 0$, respectively such that*

$$d(x, H_p^{-1}(z)) \le \tau d(z, T(T_1(x) \cap B(\bar{y}_1, r), T_2(x, p) \cap B(\bar{y}_2, r))),$$
$$\forall\, (x, p, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(0, r). \tag{4.26}$$

Proof. Since $\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, 0)$ with respect to (x, y_1, y_2) uniformly in p with modulus τ, there exist neighborhoods $B(\bar{x}, r), \mathcal{V}, B(\bar{y}_1, r), B(\bar{y}_2, r), B(0, r)$ of the points $x, p, y_1, y_2, 0$, respectively such that

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z, p)) \leq \tau d(z, \mathcal{E}_H^p(x, y_1, y_2)) \tag{4.27}$$

for all $(x, p, y_1, y_2, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(\bar{y}_1, r) \times B(\bar{y}_2, r) \times B(0, r)$. We derives that

$$d((x, y_1, y_2), \mathbb{S}_{\mathcal{E}_H}(z, p)) \leq \tau d(z, T(y_1, y_2)) \tag{4.28}$$

for all $(x, p, y_1, y_2, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(\bar{y}_1, r) \times B(\bar{y}_2, r) \times B(0, r)$ and $(y_1, y_2) \in T_1(x) \times T_2(x, p)$. Take $(x, p, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(0, r)$ and fix $(y_1, y_2) \in [T_1(x) \cap B(\bar{y}_1, r)] \times [T_2(x, p) \cap B(\bar{y}_2, r)]$ such that $T(y_1, y_2) \neq \emptyset$. By (4.28), for every $\varepsilon > 0$, there is some $(x', y_1', y_2') \in \mathbb{S}_{\mathcal{E}_H}(z, p)$ such that

$$d((x, y_1, y_2), (x', y_1', y_2')) \leq \tau d(z, T(y_1, y_2)) + \varepsilon.$$

Therefore,

$$d(x, x') \leq \tau d(z, T(y_1, y_2)) + \varepsilon.$$

Since $z \in T(y_1', y_2'), (y_1', y_2') \in T_1(x') \times T_2(x', p)$, one has $z \in H_p(x')$. Thus, $x' \in H_p^{-1}(z)$ and as a result,

$$d(x, H_p^{-1}(z)) \leq \tau d(z, T(T_1(x) \cap B(\bar{y}_1, r), T_2(x, p) \cap B(\bar{y}_2, r))) + \varepsilon,$$
$$\forall\, (x, p, z) \in B(\bar{x}, r) \times \mathcal{V} \times B(0, r).$$

Taking $\varepsilon \to 0$, we get the conclusion. $\square$

Proposition 4.5. *Let X, Y_1, Y_2 be complete metric spaces, P be a topological space, Z be a normed linear space and let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings satisfying conditions $(a), (b), (c)$ in Lemma 3.5 around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, 0) \in X \times P \times Y_1 \times Y_2 \times Z$. If there exist neighborhoods $B(\bar{x}, r), \mathcal{V}, B(\bar{y}_1, r), B(\bar{y}_2, r)$ of the points x, p, y_1, y_2, respectively such that*

$$d(x, \mathbb{S}_H(0, p)) \leq \tau d(0, T(T_1(x) \cap B(\bar{y}_1, r), T_2(x, p) \cap B(\bar{y}_2, r))),\ \forall\, (x, p) \in B(\bar{x}, r) \times \mathcal{V}, \tag{4.29}$$

and $((T_1, T_2), T)$ is locally composition-stable around $((\bar{x}, \bar{p}), (\bar{y}_1, \bar{y}_2), 0)$, then $\mathbb{S}_H(0, \cdot)$ is Robinson metrically regular around $(\bar{x}, \bar{p})$ with modulus τ.

Proof. Suppose that (4.29) holds for every $(x, p) \in B(\bar{x}, r) \times \mathcal{V}$. Since $((T_1, T_2), T)$ is locally composition-stable around $((\bar{x}, \bar{p}), (\bar{y}_1, \bar{y}_2), 0)$, then there exists $\delta > 0$ such that every $(x, p) \in B(\bar{x}, r) \times \mathcal{V}$ and every $z \in (T \circ (T_1, T_2))(x, p) \cap B(0, \delta)$, there is $(y_1, y_2) \in (F_1(x) \cap B(\bar{y}_1, \delta)) \times (F_2(x, p) \cap B(\bar{y}_2, \delta))$ such that $z \in T(y_1, y_2)$. Taking δ smaller if necessary, we may assume that $\delta < r/2$. Fixing now $(x, p) \in B(\bar{x}, \delta/2) \times \mathcal{V}$, we consider two cases:

• Case 1. $d(0, H_p(x)) < \delta/2$. Choose $\gamma > 0$ small enough in order to get $d(0, H_p(x)) + \gamma < \delta/2$. It follows that there is a point $t \in H_p(x)$ such that $\|t\| < d(0, H_p(x)) + \gamma < \delta/2$. Therefore, $t \in (T \circ (T_1, T_2))(x, p) \cap B(0, \delta)$ and thus by the local composition stability, there is $(y_1, y_2) \in (F_1(x) \cap B(\bar{y}_1, \delta)) \times (F_2(x, p) \cap B(\bar{y}_2, \delta))$ such that $t \in T(y_1, y_2)$. As a result,

$$d(x, \mathbb{S}_H(0, p)) \leq \tau d(0, T(T_1(x) \cap B(\bar{y}_1, r), T_2(x, p) \cap B(\bar{y}_2, r))) \leq \tau \|t\| < \tau\big(d(0, H_p(x)) + \gamma\big).$$

Since $\gamma > 0$ is arbitrarily small, one gets that

$$d(x, \mathbb{S}_H(0, p)) \leq \tau d(0, H_p(x)).$$

Since (x,p) is arbitrary in $B(\bar{x},\delta/2)\times\mathcal{V}$, we obtain the conclusion.

• Case 2. $d(0,H_p(x)) \geq \delta/2$. By Lemma 3.5, (i) one has that the set-valued mapping $p \rightrightarrows H_p(\bar{x})$ is lower semicontinuous at $\bar{p}$. Hence, the distance function $p \to d(0,H_p(\bar{x}))$ is upper semicontinuous at $\bar{p}$ (see, e.g., Aubin, Ekeland, ([4], Corollary 20)), and therefore there exists a neighborhood W of $\bar{p}$ such that

$$d(0,H_p(\bar{x})) \leq \delta/4, \;\; \forall\, p \in W.$$

Shrinking W if necessary, one may assume that $W \subset \mathcal{V}$ and choose $0 < \delta_1 < \min\{\delta, \tau\delta/4\}$. Taking $(x,p) \in B(\bar{x},\delta_1)\times W$, then, by (4.29), for every $\varepsilon > 0$, there is $u \in \mathbb{S}_H(0,p)$ such that

$$d(\bar{x},u) < (1+\varepsilon)\tau d(0,H_p(\bar{x})).$$

Consequently,

$$\begin{aligned} d(x,u) \leq & d(x,\bar{x}) + d(\bar{x},u) < \delta_1 + (1+\varepsilon)\tau d(0,H_p(\bar{x})) \\ & < \tau\delta/4 + (1+\varepsilon)\tau\delta/4 \\ & \leq \tau/2 d(0,H_p(x)) + \tau/2(1+\varepsilon)d(0,H_p(x)). \end{aligned}$$

Since ε is arbitrarily, one gets that

$$d(x,\mathbb{S}_H(0,p)) \leq \tau d(0,H_p(x)),$$

establishing the proof. □

Using these propositions along with Theorem 3.6, one obtains Robinson's metric regularity and Lipschitz-likeness of the map $\mathbb{S}_H(0,\cdot)$ given in Theorem 4.6 below.

Theorem 4.6. *Let $T_1 : X \rightrightarrows Y_1$, $T_2 : X \times P \rightrightarrows Y_2$ and $T : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, X, Y_1, Y_2 be complete metric spaces, P be a metric space, Z be a normed linear space satisfying conditions (a), (b), (c) in Lemma 3.5 around $(\bar{x},\bar{p},\bar{y}_1,\bar{y}_2,0) \in X\times P\times Y_1\times Y_2\times Z$. Suppose that*

(i) *$((T_1,T_2),T)$ is locally composition-stable around $((\bar{x},\bar{p}),(\bar{y}_1,\bar{y}_2),0)$;*

(ii) *T_1 is metrically regular around $(\bar{x},\bar{y}_1)$ with modulus $m > 0$;*

(iii) *T_2 is Lipschitz-like around $((\bar{x},\bar{p}),\bar{y}_2)$ with respect to x, uniformly in p with modulus $l > 0$;*

(iv) *T_2 is Lipschitz-like around $((\bar{x},\bar{p}),\bar{y}_2)$ with respect to p, uniformly in x with modulus $\theta > 0$*

(v) *T is metrically regular around $((\bar{y}_1,\bar{y}_2),0)$ with respect to y_1, uniformly in y_2 with modulus $\lambda > 0$;*

(vi) *T is Lipschitz-like around $((\bar{y}_1,\bar{y}_2),0)$ with respect to y_2, uniformly in y_1 with modulus $\gamma > 0$;*

(vii) *$\lambda m\gamma l < 1$.*

Then $\mathbb{S}_H(0,\cdot)$ is Robinson metrically regular around $(\bar{x},\bar{p})$ with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$ and $\mathbb{S}_H(0,\cdot)$ is Lipschitz-like around $(\bar{x},\bar{p})$ with modulus $\frac{\gamma\theta m\lambda}{1-m\lambda l\gamma}$.

Proof. • Applying Theorem 3.6, yields that $\mathcal{E}_H^p$ is metrically regular around $(\bar{x}, \bar{p}, \bar{y}_1, \bar{y}_2, 0)$ with respect to (x, y_1, y_2) uniformly in p with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$. Then, by Proposition 4.4, one obtains the estimation (4.26). In this estimation, we replace z by 0, one has (4.29), and from Poposition 4.5 along with the local composition stability of $((T_1, T_2), T)$ around $((\bar{x}, \bar{p}), (\bar{y}_1, \bar{y}_2), 0)$, one obtains that $\mathbb{S}_H(0, \cdot)$ is Robinson metrically regular around $(\bar{x}, \bar{p})$ with modulus $\frac{m\lambda}{1-m\lambda l\gamma}$.

• By the definition of Robinson's metric regularity of $\mathbb{S}_H(0, \cdot)$, we derive the existence of some $\delta_1 > 0$ such that

$$d(x, \mathbb{S}_H(0, p)) \leq \frac{m\lambda}{1 - m\lambda l\gamma} d(0, H(x, p)) = \frac{m\lambda}{1 - m\lambda l\gamma} d\left(0, T(T_1(x), T_2(x, p))\right), \tag{4.30}$$

for all $(x, p) \in B((\bar{x}, \bar{p}), \delta_1)$. By (iv), since T_2 is Lipschitz-like around $((\bar{x}, \bar{p}), \bar{y}_2)$ with respect to p, uniformly in x with modulus $\theta > 0$, there is $\delta_2 > 0$ such that

$$T_2(x, p) \cap B(\bar{y}_2, \delta_2) \subset T_2(x, p') + \theta d(p, p') \overline{B}_{Y_2}, \tag{4.31}$$

for all $p, p' \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$.
Moreover, according to (vi), since T is Lipschitz-like around $((\bar{y}_1, \bar{y}_2), 0)$ with respect to y_2, uniformly in y_1 with modulus $\gamma > 0$, there exists $\delta_3 > 0$ such that

$$T(y_1, y_2) \cap B(0, \delta_3) \subset T(y_1, y_2') + \gamma d(y_2, y_2') \overline{B}_Z, \tag{4.32}$$

for all $y_1 \in B(\bar{y}_1, \delta_3)$, for all $y_2, y_2' \in B(\bar{y}_2, \delta_3)$. Using the local composition-stability of the pair $((T_1, T_2), T)$ around $((\bar{x}, \bar{p}), (\bar{y}_1, \bar{y}_2), 0)$ in (i), selecting $\delta_4 > 0$ such that for every $(x, p) \in B(\bar{x}, \delta_4) \times B(\bar{p}, \delta_4)$ and every $z \in T(T_1(x), T_2(x, p)) \cap B(0, \delta_4)$, there exists

$$(y_1, y_2) \in \Big(T_1(x) \cap B\Big(\bar{y}_1, \min\{\delta_2, \delta_3\}\Big)\Big) \times \Big(T_2(x, p) \cap B\Big(\bar{y}_2, \min\{\delta_2, \delta_3\}\Big)\Big),$$

such that $z \in T(y_1, y_2)$.

Setting $\alpha := \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, and take $p, p' \in B(\bar{p}, \alpha)$, and $x \in \mathbb{S}_H(0, p) \cap B(\bar{x}, \alpha)$. This means that, $0 \in T(T_1(x), T_2(x, p)) \subset T(T_1(x), T_2(x, p)) \cap B(0, \delta_4)$ and $x \in B(\bar{x}, \alpha)$. It follows that there exists

$$(y_1, y_2) \in \Big(T_1(x) \cap B\Big(\bar{y}_1, \min\{\delta_2, \delta_3\}\Big)\Big) \times \Big(T_2(x, p) \cap B\Big(\bar{y}_1, \min\{\delta_2, \delta_3\}\Big)\Big),$$

such that $0 \in T(y_1, y_2)$. Consequently, for $y_2' \in T_2(x, p')$,

$$d(x, \mathbb{S}_H(0, p')) \leq \frac{m\lambda}{1 - m\lambda l\gamma} d\Big(0, T\left(T_1(x), T_2(x, p')\right)\Big) \tag{4.33}$$

$$\leq \frac{m\lambda}{1 - m\lambda l\gamma} d(0, T(y_1, y_2'))). \tag{4.34}$$

So, by taking into account $0 \in T(y_1, y_2)$ and by using the estimations (4.33) and (4.34), we have

$$\begin{aligned} d(x, \mathbb{S}_H(0, p')) &\leq \frac{m\lambda}{1 - m\lambda l\gamma} e\left(T(y_1, y_2), T(y_1, y_2')\right) \\ &\leq \frac{\gamma m\lambda}{1 - m\lambda l\gamma} d(y_2, y_2') \\ &\leq \frac{\theta\gamma m\lambda}{1 - m\lambda l\gamma} d(p, p'), \end{aligned}$$

which implies that

$$\mathbb{S}_H(0,p) \cap B(\bar{x}, \alpha) \subset \mathbb{S}_H(0,p') + \frac{\theta\gamma m\lambda}{1 - m\lambda l\gamma} d(p,p')\overline{B}_X.$$

This means that $\mathbb{S}_H(0,\cdot)$ is Lipschitz-like around $(\bar{x},\bar{p})$ with modulus $\frac{\gamma\theta m\lambda}{1-m\lambda l\gamma}$, which completes the proof. $\square$

5 Conclusions

Regularity of composite set-valued mappings were studied by many experts in the community of variational analysis ([7], [18], [20], [23], [26], [45]) by using various tools of variational analysis. However, in this paper, we establish regularity of the parametrized epigraphical composition set-valued mapping and semiregularity of $H_{\bar{p}}$ in a slightly different way. Furthermore, using this result we have obtained some types of regularities for implicit set-valued mappings such as calmness, Lipschitz-likeness and Robinson metric regularity (see [18–20, 23, 33, 45])

Our results resumes recent works by Durea, Strugariu [19], Ngai, Tron, Théra [33], Zheng, Ng [45], Durea, Strugariu [18, 20], Durea, Huynh, Nguyen, Strugariu [23], Cibulka, Fabian, Kruger [9]. By obtaining semiregularity of the composite set-valued mappings we hope to obtain in a future work convergence of Newton-type methods for generalized equations of composite type as Cibulka, Fabian, Kruger in [9] did recently for the sum case. Ideas and techniques in the paper are inspired from the works by Ngai, Tron, Théra [33] and Durea, Huynh, Nguyen, Strugariu [23].

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