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ϵ-STRONG EFFICIENCY OF A SET AND ITS APPLICATIONS IN ORDERED LINEAR SPACES

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Abstract: In this paper, we investigate ϵ-strong efficiency of a set in ordered linear spaces. Firstly, a new conception of ϵ-strongly efficient point of a set is introduced. Secondly, some properties and the existence of ϵ-strongly efficient points of a set are studied. Finally, as the applications, the linear scalarization theorems of the set-valued optimization problem with generalized cone subconvexlikeness are obtained in the sense of ϵ-strong efficiency. Some examples are given to illustrate the results obtained in this paper.

Key words: *ϵ-strong efficient point, set-valued maps, generalized cone subconvexlikeness, scalarization*

Mathematics Subject Classification: *90C26, 90C29, 90C30*

1 Introduction

Recently, many scholars have paid attention to set-valued optimization in the study of optimization theory and applications. Under different kinds of generalized convexity, optimization conditions of set-valued optimization problems are established in the sense of different kinds of efficiency of solutions (see [11, 12, 14, 19, 20] and the references therein). However, from computational point of view, the algorithms used to solve optimization problems often give rise to approximate solutions. Rong and Wu [13] introduced ϵ-weakly efficient solution of the set-valued optimization problem and established a series of optimality conditions. Based on [13], many scholars investigated different classes of approximate proper efficiency and obtained some interesting results such as scalarization theorems, Lagrangian multiplier theorems, saddle point theorems and duality theorems (see [9, 16, 18] and the references therein). In the above references mentioned, the proper efficiency of the set-valued optimization problem was mainly studied in the topological vector spaces. It is possible that the properly efficient element of a set in real ordered linear spaces exist, but it does not exist in topological vector spaces (see Example 4.1 in [23]). Therefore, how to extend some results from topological vector spaces to the ordered linear spaces is interesting. In the ordered linear spaces, Li [10] established an alternative theorem of the subconvexlike

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set-valued map and obtained Kuhn-Tucker conditions in the sense of weak efficiency. Following the line of Li [10], some scholars [4, 21–23] explored some (approximate) efficiency of set-valued optimization problems in ordered linear spaces. Especially, Gutiérrez et al. [7] studied the nonconvex separation functional in ordered linear spaces with applications to vector equilibria. However, to the best of our knowledge, there are only a few scholars to study approximate properly efficient solutions of set-valued optimization problems in ordered linear spaces. Hence, investigating approximate proper efficiency of set-valued optimization problems in ordered linear spaces is interesting. Motivated by references [4, 7, 10, 21–23], we will research ϵ-strong efficiency of set-valued optimization problems in ordered linear spaces in this paper.

This paper is organized as follows. In Section 2, we give some preliminaries including basic concepts and lemmas. In Section 3, we investigate some properties and existence conditions of ϵ-strongly efficient points of the set. In Section 4, we establish scalarization theorems of an unconstrained set-valued optimization problem in the sense of ϵ-strong efficiency.

2 Preliminaries

In this paper, we suppose that X and Y are two real linear spaces. 0 stands for the zero element of every space. Let C be a nonempty subset in Y. The generated cone of C is denoted by $\text{cone}(C) := \{\lambda c | c \in C, \lambda \geq 0\}$. C is called a convex cone iff $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \geq 0$. A cone C is said to be pointed iff $C \cap (-C) = \{0\}$. C is said to be nontrivial iff $C \neq \{0\}$ and $C \neq Y$. The ordering of Y associated with a convex cone C is the relation defined by:

$$x \preceq y \Leftrightarrow y - x \in C.$$

From now on, we suppose that Y is a real ordered linear spaces with nontrivial, pointed and convex cone C. The algebraic dual of Y is denoted by Y^*. The algebraic dual cone C^+ of C is defined as $C^+ := \{y^* \in Y^* | \langle y, y^* \rangle \geqslant 0, \forall y \in C\}$, where $\langle y, y^* \rangle$ denotes the value of the linear functional y^* at the point y. Let $a, b \in Y$ with $a \preceq b$. The set $[a, b] := \{y \in Y | a \preceq y \preceq b\} = \{y \in Y | y - a \in C \text{ and } b - y \in C\}$ is called an order-interval with respect to C. The ordered algebraic dual space Y^{bd} of Y is defined as

$$Y^{bd} := \{y^* \in Y^* | y^* \text{ is bounded on any order-interval } [a, b] \subseteq Y\}.$$

If $Y := \mathbb{R}^2$ and $C := \{x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0 \text{ and } x_2 \geq 0\}$. It is easy to verify that $(1, 1) \in Y^{bd}$.

Definition 2.1 ([8])**.** Let $K \subseteq Y$. K is called ordering-bounded iff there exist $a, b \in Y$ with $b - a \in C$ such that $K \subseteq [a, b]$.

Definition 2.2 ([17])**.** Let K be a nonempty subset in Y. The algebaic interior of K is the set

$$\text{cor}(K) := \{k \in K | \forall k' \in Y, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], k + \lambda k' \in K\}.$$

Definition 2.3 ([1])**.** Let K be a nonempty subset in Y. The vector closure of K is the set

$$\text{vcl}(K) := \{k \in Y | \exists k' \in Y, \forall \lambda' > 0, \exists \lambda \in]0, \lambda'], k + \lambda k' \in K\}.$$

Remark 2.4. It is easy to show that, if K_1 and K_2 are two nonempty subsets in Y, then $\text{vcl}(K_1 \cap K_2) = \text{vcl}(K_1) \cap \text{vcl}(K_2)$.

Definition 2.5 ([15])**.** Let K be a nonempty subset in Y. K is called balanced iff, $\forall k \in K, \forall \lambda \in [-1, 1], \lambda k \in K$. K is called absorbent iff $0 \in \text{cor}(K)$.

Definition 2.6 ([23])**.** Let B be a nonempty convex subset in Y. B is a base of C iff $C = \text{cone}(B)$ and there exists a balanced, absorbent and convex set V such that $0 \notin B + V$ in Y.

Let B be a base of C. Write $B^{st} := \{y^* \in Y^* | \text{there exists } t > 0 \text{ such that } \langle b, y^* \rangle \geq t, \forall b \in B\}$ and $B^{bd} := \{y^* \in Y^{bd} | \text{there exists } t > 0 \text{ such that } \langle b, y^* \rangle \geq t, \forall b \in B\}$.

Remark 2.7. Let $y^* \in Y^* \setminus \{0\}$. Then, $y^* \in B^{st}$ iff there exists a balanced, absorbent and convex set V such that $\langle y, y^* \rangle < 0$ for any $y \in V - B$.

Remark 2.8. By the definitions of C^+ and B^{st}, it is easy to verify that $B^{st} + C^+ = B^{st}$.

Let $K \subseteq Y$ be a nonempty set. Write $\langle K, y^* \rangle := \{\langle k, y^* \rangle | k \in K\}$.

Lemma 2.9 ([8])**.** *Let C_1 and C_2 be two convex cones in Y and $y^* \in Y^*$. Then, the following two statements are equivalent:*

(1) $y^* \in C_1^+ - C_2^+$;

(2) *There exist a balanced, absorbent and convex set $U \subseteq Y$ and a real number $\alpha > 0$ such that $\langle y, y^* \rangle \geq -\alpha$ for any $y \in C_1 \cap (U - C_2)$.*

Lemma 2.10 ([2])**.** *Let K be a nonempty subset in Y. Then $K^+ = (\text{vcl}(K))^+$.*

Remark 2.11. Let $K \subseteq Y$. It follows from Lemma 2.10 that $y^* \in Y^* \setminus \{0\}$ is bounded on K iff y^* is bounded on $\text{vcl}(K)$.

Lemma 2.12 ([15])**.** *Let $\{p_i\}_{i \in I}$ be a family of seminorms on the linear space Y, where I is an index set. Then, there exists a most coarse topology defined on Y, which coincides with the linear structure and makes every p_i be continuous on the Y. For the above topology, Y is locally convex and the sets with the form $\{y \in Y | \max_{1 \leqslant k \leqslant n} p_{i_k}(y) < \delta\}$ formulate an open neighborhood basis of 0, where $\delta > 0, n \in \mathbb{N}, i_k \in I (k = 1, 2, ..., n)$.*

3 ϵ-Strongly Efficient Point

In [3], Cheng and Fu investigated strong efficiency in a locally convex space. Now, we introduce a new notion of ϵ-strongly efficient point of a set in the ordered linear space.

From now on, we suppose that B is a basis of C unless otherwise specified.

Definition 3.1. Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-strongly efficient point of K with respect to B (denoted by $\overline{y} \in \epsilon\text{-GE}(K, B)$) iff, for any $y^* \in Y^{bd}$, there exist two balanced, absorbent and convex sets U and V such that $\langle \text{vcl}(\text{cone}(K + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is bounded.

Remark 3.2. By Remarks 2.4 and 2.11, $\overline{y} \in \epsilon\text{-GE}(K, B)$ iff , for any $y^* \in Y^{bd}$, there exist two balanced, absorbent and convex sets U and V such that $\langle \text{cone}(K + \epsilon - \overline{y}) \cap (U - \text{cone}(V + B)), y^* \rangle$ is bounded.

Definition 3.3 ([23])**.** Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-weakly efficient point of K (denoted by $\overline{y} \in \epsilon\text{-WE}(K, C)$) iff $(K + \epsilon - \overline{y}) \cap (-\text{cor}C) = \emptyset$.

Remark 3.4. It is easy to verify that $\epsilon\text{-GE}(K, B) \subseteq \epsilon\text{-WE}(K, C)$. However, the following example shows that the inclusion relation $\epsilon\text{-WE}(K, C) \subseteq \epsilon\text{-GE}(K, B)$ does not hold.

Example 3.5. Let $K := \{(y_1, y_2)|y_1 \leq 2, y_2 \geq 0\}, \epsilon = (1, 0), \overline{y} = (2, 0), C := \{(y_1, y_2)|y_1 \geq 0, y_2 \geq 0\}$ and $B := \{(y_1, y_2)|y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$. It is easy to check that $\overline{y} \in \epsilon$-WE$(K, C)$. However, there exists $y^* = (-1, 1) \in Y^{bd}$ such that $\langle \text{vcl}(\text{cone}(K + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is unbounded for any balanced, absorbent and convex sets U and V. Hence, $\overline{y} \notin \epsilon$-GE$(K, B)$. Thus, the inclusion relation ϵ-WE$(K, C) \subseteq \epsilon$-GE$(K, B)$ does not hold.

The following proposition will be used to derive the scalarization theorem of the set-valued optimization problem in Section 4.

Proposition 3.6. *Let $\epsilon \in C, \overline{y} \in \epsilon$-GE$(K, B)$ and $y^* \in B^{bd}$. Then, there exist two balanced, absorbent and convex sets U and V such that $\langle \text{vcl}(\text{cone}(K+C+\epsilon-\overline{y}))\cap(U-\text{cone}(V+B)), y^* \rangle$ is bounded.*

Proof. We suppose that, for any balanced, absorbent and convex sets U and V, $\langle \text{vcl}(\text{cone}(K+C + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is unbounded. According to Remarks 2.4 and 2.11,

$$\langle \text{cone}(K + C + \epsilon - \overline{y}) \cap (U - \text{cone}(V + B)), y^* \rangle \tag{3.1}$$

is unbounded for any balanced, absorbent and convex sets U and V. Let $V = U$ in (3.1). Then, $\langle \text{cone}(K + C + \epsilon - \overline{y}) \cap (U - \text{cone}(U + B)), y^* \rangle$ is unbounded for any balanced, absorbent and convex set U. Since $\overline{y} \in \epsilon$-GE$(K, B)$, there exist two balanced, absorbent and convex sets U' and V' such that $\langle \text{cone}(K + \epsilon - \overline{y}) \cap (U' - \text{cone}(V' + B)), y^* \rangle$ is bounded. Write $W := U' \cap V'$. Clearly, $\langle \text{cone}(K + \epsilon - \overline{y}) \cap (W - \text{cone}(W + B)), y^* \rangle$ is bounded. Let $p(y) := |\langle y, y^* \rangle|$ for any $y \in Y$. It is easy to check that p is a seminorm on Y. By Lemma 2.12, there exists a topology τ induced by the seminorm p such that (τ, Y) is locally convex and $\{y \in Y | \langle y, y^* \rangle \in \delta U''\}_{\delta > 0}$ formulates an open neighborhood basis of 0, where $U'' = (-1, 1)$. Write $U_n := \{y \in Y | \langle y, y^* \rangle \in \frac{1}{n} U''\}$ for any $n \in \mathbb{N}$. Therefore,

$$\langle \text{cone}(K + C + \epsilon - \overline{y}) \cap (U_n - \text{cone}(U_n + B)), y^* \rangle$$

is unbounded for any $n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, there exists $r_n \in \langle \text{cone}(K + C + \epsilon - \overline{y}) \cap (U_n - \text{cone}(U_n + B)), y^* \rangle$ such that $|r_n| > n$. Hence, there exists

$$y_n \in \text{cone}(K + C + \epsilon - \overline{y}) \cap (U_n - \text{cone}(U_n + B)) \tag{3.2}$$

such that $|\langle y_n, y^* \rangle| = |r_n| > n$. Thus, we have

$$\lim_{n \to \infty} |\langle y_n, y^* \rangle| = +\infty. \tag{3.3}$$

By (3.2), there exist $\lambda_n \geq 0, k_n \in K, c_n \in C, u_n \in U_n, \beta_n \geq 0, v_n \in U_n$ and $b_n \in B$ such that

$$y_n = \lambda_n(k_n + c_n + \epsilon - \overline{y}) = u_n - \beta_n(v_n + b_n), \forall n \in \mathbb{N}. \tag{3.4}$$

It follows from (3.4) that

$$\langle y_n, y^* \rangle = \langle u_n, y^* \rangle - \beta_n(\langle v_n, y^* \rangle + \langle b_n, y^* \rangle), \forall n \in \mathbb{N}. \tag{3.5}$$

Clearly,

$$\lim_{n \to \infty} \langle u_n, y^* \rangle = \lim_{n \to \infty} \langle v_n, y^* \rangle = 0. \tag{3.6}$$

Since $y^* \in B^{bd}$ and B is an ordering-bounded basis of C, it follows from (3.3),(3.5) and (3.6) that the sequence $\{\beta_n\}$ is unbounded.

Because $c_n \in C$, there exist $\rho_n \geq 0$ and $b'_n \in B$ such that $c_n = \rho_n b'_n$. Thus, it follows from (3.4) that

$$\lambda_n(k_n + \epsilon - \overline{y}) = u_n - \beta_n(v_n + b_n) - \lambda_n \rho_n b'_n, \forall n \in \mathbb{N}. \tag{3.7}$$

Case one. If some $\beta_n = 0$, it follows from (3.7) that

$$\lambda_n(k_n + \epsilon - \overline{y}) \in U_n - C = U_n - \text{cone}(B) \subseteq U_n - \text{cone}(U_n + B).$$

Case two. If some $\beta_n \neq 0$, we write $\alpha_n := \frac{\beta_n + \lambda_n \rho_n}{\beta_n} \geq 1$. Since B is a convex set and $\frac{1}{\alpha_n} + \frac{\lambda_n \rho_n}{\alpha_n \beta_n} = 1$, it follows from (3.7) that

$$\begin{aligned}\lambda_n(k_n + \epsilon - \overline{y}) &= u_n - \beta_n \alpha_n \Big[\frac{1}{\alpha_n} v_n + \Big(\frac{1}{\alpha_n} b_n + \frac{\lambda_n \rho_n}{\alpha_n \beta_n} b'_n\Big)\Big] \\ &\in U_n - \beta_n \alpha_n (U_n + B) \subseteq U_n - \text{cone}(U_n + B).\end{aligned} \tag{3.8}$$

Cases one and two show that

$$\lambda_n(k_n + \epsilon - \overline{y}) \in U_n - \text{cone}(U_n + B), \forall n \in \mathbb{N}. \tag{3.9}$$

Clearly,

$$\lambda_n(k_n + \epsilon - \overline{y}) \in \text{cone}(K + \epsilon - \overline{y}), \forall n \in \mathbb{N}. \tag{3.10}$$

According to (3.9) and (3.10), we have

$$\lambda_n(k_n + \epsilon - \overline{y}) \in \text{cone}(K + \epsilon - \overline{y}) \cap (U_n - \text{cone}(U_n + B)), \forall n \in \mathbb{N}. \tag{3.11}$$

On the other hand, there exists $N \in \mathbb{N}$ such that

$$U_n \subseteq W, \forall n > N. \tag{3.12}$$

Since $\langle \text{cone}(K + \epsilon - \overline{y}) \cap (W - \text{cone}(W + B)), y^* \rangle$ is bounded, it follows from (3.11) and (3.12) that the sequence $\{\langle \lambda_n(k_n + \epsilon - \overline{y}), y^* \rangle\}$ is bounded.

As the sequence $\{\beta_n\}$ is unbounded, there exists $N' \in \mathbb{N}$ such that

$$\beta_n > 0, \forall n > N'.$$

Therefore, there exists $b''_n \in B$ such that

$$b''_n = \frac{1}{\alpha_n} b_n + \frac{\lambda_n \rho_n}{\alpha_n \beta_n} b'_n, \forall n > N'. \tag{3.13}$$

By (3.8) and (3.13), we have

$$\lambda_n(k_n + \epsilon - \overline{y}) = u_n - \beta_n \alpha_n \Big(\frac{1}{\alpha_n} v_n + b''_n\Big), \forall n > N'. \tag{3.14}$$

Since $y^* \in B^{bd}$ and the sequence $\{\beta_n\}$ is unbounded, it follows from (3.6) and (3.14) that

$$\lim_{n\to\infty} \langle \lambda_n(k_n + \epsilon - \overline{y}), y^* \rangle = \lim_{n\to\infty} \Big\langle u_n - \beta_n \alpha_n \Big(\frac{1}{\alpha_n} v_n + b''_n\Big), y^* \Big\rangle = \infty,$$

which contradicts that the sequence $\{\langle \lambda_n(k_n + \epsilon - \overline{y}), y^* \rangle\}$ is bounded. Hence, $\langle \text{vcl}(\text{cone}(K + C + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is bounded. $\square$

Remark 3.7. Though $\langle \text{vcl}(\text{cone}(K+C+\epsilon-\overline{y}))\cap(U-\text{cone}(V+B)),y^*\rangle$ is bounded in Proposition 3.6, we cannot assert that $\overline{y}\in\epsilon\text{-GE}(K+C,B)$. The following two examples shows that both $\epsilon\text{-GE}(K,B)\subseteq\epsilon\text{-GE}(K+C,B)$ and $\epsilon\text{-GE}(K+C,B)\subseteq\epsilon\text{-GE}(K,B)$ do not hold.

Example 3.8. Let $K:=[1,2]\times[1,2], C:=\{(y_1,y_2)|y_1\in\mathbb{R},y_2>0\}\cup\{(0,0)\}$ and $B:=\{(y_1,y_2)|y_1\in\mathbb{R},y_2=1\}$. Take $\epsilon=(0,1)$ and $\overline{y}=(1,1)$. For any $y^*\in Y^{bd}$, there exist balanced, absorbent and convex sets $U:=\{(y_1,y_2)|y_1^2+y_2^2=0.01\}$ and $V=U$ such that $\langle\text{vcl}(\text{cone}(K+\epsilon-\overline{y}))\cap(U-\text{cone}(V+B)),y^*\rangle$ is bounded. Hence, $\overline{y}\in\epsilon\text{-GE}(K,B)$. On the other hand, there exists $y^*=(-1,0)\in Y^{bd}\setminus B^{bd}$ such that $\langle\text{vcl}(\text{cone}(K+C+\epsilon-\overline{y}))\cap(U-\text{cone}(V+B)),y^*\rangle$ is unbounded for any balanced, absorbent and convex sets U and V. Therefore, $\overline{y}\notin\epsilon\text{-GE}(K+C,B)$. Thus, the inclusion relation $\epsilon\text{-GE}(K,B)\subseteq\epsilon\text{-GE}(K+C,B)$ does not hold.

Example 3.9. Let $K:=\{(1,1)\},\epsilon=(1,1),\overline{y}=(2,2),C:=\{(y_1,y_2)|y_1\geq 0,y_2\geq 0\}$ and $B:=\{(y_1,y_2)|y_1+y_2=1,y_1\geq 0,y_2\geq 0\}$. It is easy to check that $\overline{y}\in\epsilon\text{-GE}(K+C,B)$. However, $\overline{y}\notin\epsilon\text{-GE}(K,B)$. Hence, the inclusion relation $\epsilon\text{-GE}(K+C,B)\subseteq\epsilon\text{-GE}(K,B)$ does not hold.

Theorem 3.10. *Let $\epsilon\in C$ and K be a nonempty convex set in Y. Let $\overline{y}\in\epsilon\text{-GE}(K,B)$. Then, for any $y^*\in Y^{bd}\setminus(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$, there exist $y_1^*\in(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$ and $y_2^*\in B^{st}$ such that $y^*=y_1^*-y_2^*$.*

Proof. Since $\overline{y}\in\epsilon\text{-GE}(K,B)$ and $y^*\in Y^{bd}$, there exist two balanced, absorbent and convex sets U and V such that $\langle\text{vcl}(\text{cone}(K+\epsilon-\overline{y}))\cap(U-\text{cone}(V+B)),y^*\rangle$ is bounded. It follows from Lemma 2.9 that there exist $y_1^*\in(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$ and $y_2^*\in(\text{cone}(V+B))^+$ such that

$$y^*=y_1^*-y_2^*. \tag{3.15}$$

Now, we will show that $y_2^*\in B^{st}$. According to $y_2^*\in(\text{cone}(V+B))^+$, we have

$$\langle b+v,y_2^*\rangle\geq 0,\forall b\in B,\forall v\in V. \tag{3.16}$$

Since V is balanced, it follows from (3.16) that

$$\langle b,y_2^*\rangle\geq\langle v,y_2^*\rangle,\forall b\in B,\forall v\in V. \tag{3.17}$$

As $y^*\in Y^{bd}\setminus(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$, it follows from (3.15) that $y_2^*\neq 0$. Since V is absorbent, there exists $v'\in V$ such that

$$\langle v',y_2^*\rangle>0. \tag{3.18}$$

Using (3.17) and (3.18), we obtain

$$\langle b,y_2^*\rangle\geq\sup_{v\in V}\{\langle v,y_2^*\rangle\}\geq\langle v',y_2^*\rangle>0,\forall b\in B,$$

which implies $y_2^*\in B^{st}$. □

Let $K\subseteq Y,\epsilon\in C$ and $y^*\in Y^+$. Write $\epsilon\text{-Min}(K,y^*):=\{\overline{y}\in K|\langle\overline{y},y^*\rangle\leq\langle y,y^*\rangle+\langle\epsilon,y^*\rangle,\forall y\in K\}$.

Theorem 3.11. *Let $\epsilon\in C$ and K be a nonempty convex set in Y. Let $\overline{y}\in\epsilon\text{-GE}(K,B)$. Then, for any $y^*\in Y^{bd}\setminus(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$, there exists $y_1^*\in(\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+$ such that $y_1^*\in y^*+B^{st}$ and $\overline{y}\in\epsilon\text{-Min}(K,y_1^*)$*

Proof. Since $\overline{y} \in \epsilon$-GE$(K, B)$ and $y^* \in Y^{bd}$, it follows from Theorem 3.10 that, for any $y^* \in Y^{bd} \setminus (\text{vcl}(\text{cone}(K + \epsilon - \overline{y})))^+$, there exist $y_1^* \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y})))^+$ and $y_2^* \in B^{st}$ such that (3.15) holds. Hence, $y_1^* \in y^* + B^{st}$ and $\langle \text{vcl}(\text{cone}(K + \epsilon - \overline{y})), y_1^* \rangle \geq 0$. Clearly, $\langle K + \epsilon - \overline{y}, y_1^* \rangle \geq 0$, i.e.,

$$\langle \overline{y}, y_1^* \rangle \leq \langle y, y_1^* \rangle + \langle \epsilon, y_1^* \rangle, \forall y \in K,$$

which implies $\overline{y} \in \epsilon$-Min$(K, y_1^*)$. □

Theorem 3.12. *Let $\epsilon \in C, \overline{y} \in Y$ and K be a nonempty convex set in Y. For any $y^* \in Y^{bd}$, there exist $y_1^* \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y})))^+$ and $y_2^* \in B^{st}$ such that $y^* = y_1^* - y_2^*$. Then, $\overline{y} \in \epsilon$-GE$(K, B)$.*

Proof. Since $y_2^* \in B^{st}$, it follows from Remark 2.7 that there exists a balanced, absorbent and convex set V_1 such that

$$\langle y, y_2^* \rangle \leq 0, \forall y \in V_1 - B. \tag{3.19}$$

As V_1 is balanced, it follows from (3.19) that

$$\langle y, y_2^* \rangle \geq 0, \forall y \in V_1 + B,$$

i.e., $y_2^* \in (V_1+B)^+ \subseteq (\text{cone}(V_1+B))^+$. Hence, $y^* \in (\text{vcl}(\text{cone}(K+\epsilon-\overline{y})))^+ - (\text{cone}(V_1+B))^+$. By Lemma 2.9, there exist a balanced, absorbent and convex set U_1 and a real number $\alpha > 0$ such that

$$\langle y, y^* \rangle \geq -\alpha, \forall y \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y}))) \cap (U_1 - \text{cone}(V_1 + B)). \tag{3.20}$$

On the other hand, $-y^* \in Y^{bd}$. Therefore, there exist two balanced, absorbent and convex set U_2 and V_2 and a real number $\beta > 0$ such that

$$\langle y, -y^* \rangle \geq -\beta, \forall y \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y}))) \cap (U_2 - \text{cone}(V_2 + B)). \tag{3.21}$$

Write $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. Using (3.20) and (3.21), we obtain

$$-\alpha \leq \langle y, y^* \rangle \leq \beta, \forall y \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y}))) \cap (U - \text{cone}(V + B)).$$

Hence, $\overline{y} \in \epsilon$-GE$(K, B)$. □

Theorem 3.13. *Let $\epsilon \in C$ and K be a nonempty convex set in Y. For any $y^* \in Y^{bd}$, there exists $y_1^* \in Y^*$ such that $y_1^* \in y^* + B^{st}$ and $\overline{y} \in \epsilon$-Min$(K, y_1^*)$. Then, $\overline{y} \in \epsilon$-GE$(K, B)$.*

Proof. By $\overline{y} \in \epsilon$-Min$(K, y_1^*)$, we have

$$\langle y + \epsilon - \overline{y}, y_1^* \rangle \geq 0, \forall y \in K,$$

which implies that

$$\langle \lambda(y + \epsilon - \overline{y}), y_1^* \rangle \geq 0, \forall \lambda \geq 0, y \in K. \tag{3.22}$$

It follows from (3.22) that

$$y_1^* \in (\text{cone}(K + \epsilon - \overline{y}))^+. \tag{3.23}$$

According to (3.23) and Lemma 2.10, $y_1^* \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y})))^+$. On the other hand,

$$y_1^* \in y^* + B^{st}. \tag{3.24}$$

By (3.24), there exists $y_2^* \in B^{st}$ such that $y^* = y_1^* - y_2^*$. Thus, the conditions of Theorem 3.12 are satisfied. Therefore, $\overline{y} \in \epsilon$-GE$(K, B)$. □

Theorem 3.14. *Let $\epsilon \in C, \overline{y} \in Y$ and K be a nonempty convex set in Y. Let B be an ordering-bounded basis of C. If $(\mathrm{vcl}(\mathrm{cone}(K + \epsilon - \overline{y})))^+ \cap B^{st} \neq \emptyset$, then $\overline{y} \in \epsilon$-$\mathrm{GE}(K, B)$.*

Proof. Since $(\mathrm{vcl}(\mathrm{cone}(K+\epsilon-\overline{y})))^+ \cap B^{st} \neq \emptyset$, there exists $y_1^* \in (\mathrm{vcl}(\mathrm{cone}(K+\epsilon-\overline{y})))^+ \cap B^{st}$. By $y_1^* \in (\mathrm{vcl}(\mathrm{cone}(K + \epsilon - \overline{y})))^+$, it is easy to verify that

$$\overline{y} \in \epsilon\text{-}\mathrm{Min}(K, y_1^*). \tag{3.25}$$

According to $y_1^* \in B^{st}$, we assert that, for any $y^* \in Y^{bd}$, there exists $N \in \mathbb{N}$ such that

$$y_1^* - \frac{1}{N} y^* \in C^+. \tag{3.26}$$

Otherwise, there exists $y_2^* \in Y^{bd}$, for any $n \in \mathbb{N}$, we have $y_1^* - \frac{1}{n} y_2^* \notin C^+$. Thus, there exists $y_n \in C$ such that

$$\left\langle y_n, y_1^* - \frac{1}{n} y_2^* \right\rangle < 0, \forall n \in \mathbb{N}. \tag{3.27}$$

Since $y_n \in C$, there exist $\lambda_n > 0$ and $b_n \in B$ such that $y_n = \lambda_n b_n$. It follows from (3.27) that

$$\left\langle \lambda_n b_n, y_1^* - \frac{1}{n} y_2^* \right\rangle < 0, \forall n \in \mathbb{N}.$$

Clearly,

$$\left\langle b_n, y_1^* - \frac{1}{n} y_2^* \right\rangle < 0, \forall n \in \mathbb{N}. \tag{3.28}$$

Because $y_2^* \in Y^{bd}$ and B is an ordering-bounded basis of C,

$$\lim_{n\to\infty} \left\langle b_n, \frac{1}{n} y_2^* \right\rangle = \lim_{n\to\infty} \frac{1}{n} \langle b_n, y_2^* \rangle = 0. \tag{3.29}$$

It follows from $y_1^* \in B^{st}$ that there exists $t > 0$ such that

$$\langle b, y_1^* \rangle > t, \forall b \in B. \tag{3.30}$$

For the above t, it follows from (3.29) that there exists $N_1 \in \mathbb{N}$ such that

$$\left\langle b_n, \frac{1}{n} y_2^* \right\rangle < t, \forall n > N_1. \tag{3.31}$$

According to (3.28) and (3.31), we obtain

$$\langle b_n, y_1^* \rangle < \left\langle b_n, \frac{1}{n} y_2^* \right\rangle < t, \forall n > N_1,$$

which contradicts (3.30). Therefore, our assertion is correct. By (3.26) and Remark 2.8, we have

$$(N + 1)y_1^* - y^* = (Ny_1^* - y^*) + y_1^* \in C^+ + B^{st} = B^{st},$$

i.e.,

$$(N + 1)y_1^* \in y^* + B^{st}. \tag{3.32}$$

By (3.25), we have

$$\overline{y} \in \epsilon - \mathrm{Min}(K, (N + 1)y_1^*). \tag{3.33}$$

By (3.32) and (3.33), the conditions of Theorem 3.13 are satisfied. Hence, $\overline{y} \in \epsilon$-$\mathrm{GE}(K, B)$. □

Remark 3.15. The following example shows that Theorem 3.14 does not hold when the conditions that B is an ordering-bounded basis of C is deleted.

Example 3.16. Let $K := \{(y_1, y_2)|y_1 \leq 2, y_2 \geq 0\}, \epsilon = (1, 1), \overline{y} = (2, 1), C := \{(y_1, y_2)|y_1 \geq 0, y_2 > 0\} \cup \{(0, 0)\}$ and $B := \{(y_1, y_2)|y_1 \geq 0, y_2 = 2\}$. Clearly, $\epsilon \in C$ and K is a nonempty convex set in Y. Moreover, $(0, 2) \in (\text{vcl}(\text{cone}(K + \epsilon - \overline{y})))^+ \cap B^{st}$. Obviously, B is not an ordering-bounded basis of C. However, there exists $y^* = (-1, 0) \in Y^{bd}$ such that $\langle \text{vcl}(\text{cone}(K + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is unbounded for any balanced, absorbent and convex sets U and V. Hence, $\overline{y} \notin \epsilon\text{-GE}(K, B)$.

4 Scalarization

In this section, we will establish the scalarization theorems of an unconstrainted set-valued optimization problem in the sense of ϵ-strong efficiency.

From now on, let A be a nonempty set in X. Let $F : A \rightrightarrows Y$ be a set-valued map on A. Write $\langle F(x), y^* \rangle := \{\langle y, y^* \rangle | y \in F(x)\}, F(A) := \bigcup_{x \in A} F(x)$ and $\langle F(A), y^* \rangle := \bigcup_{x \in A} \langle F(x), y^* \rangle$.

Definition 4.1 ([5])**.** A set-valued map $F : A \rightrightarrows Y$ is called generalized C-subconvexlike on A iff $\text{cone}(F(A)) + \text{cor}(C)$ is a convex set in Y.

Remark 4.2. When $\text{cor}(C) \neq \emptyset$, the set-valued map F is generalized C-subconvexlike on A iff $\text{vcl}(\text{cone}(F(A) + C))$ is a convex set in Y (see Proposition 3.1 in [21]).

Let $F : A \rightrightarrows Y$ a set-valued map from A to Y. we consider the following set-valued optimization problem:

$$\text{(VP)} \quad \text{Min } F(x) \quad \text{subject to} \quad x \in A.$$

Definition 4.3. Let $\epsilon \in C$. $\overline{x} \in A$ is called an ϵ-strongly efficient solution of (VP) iff there exists $\overline{y} \in F(\overline{x})$ such that $\overline{y} \in \epsilon\text{-GE}(F(A), B)$. The pair $(\overline{x}, \overline{y})$ is called an ϵ-strongly efficient element of (VP).

The scalar minimization problem of (VP) is defined as follows:

$$\text{(VP)}_{y^*} \qquad \text{Min } \langle F(x), y^* \rangle \quad \text{subject to} \quad x \in A,$$

where $y^* \in Y^* \setminus \{0\}$.

Definition 4.4 ([13])**.** Let $\epsilon \in C$. $\overline{x} \in A$ is called an ϵ-optimal solution of (VP)_{y^*} iff there exists $\overline{y} \in F(\overline{x})$ such that

$$\langle \overline{y}, y^* \rangle \leq \langle y, y^* \rangle + \langle \epsilon, y^* \rangle, \forall x \in A, \forall y \in F(x).$$

The pair $(\overline{x}, \overline{y})$ is called an ϵ-optimal element of (VP)_{y^*}.

Theorem 4.5. *Let $\epsilon \in C, \overline{x} \in A, \overline{y} \in F(\overline{x})$ and $B^{bd} \neq \emptyset$. The set-valued map $F + \epsilon - \overline{y}$ is generalied C-subconvexlike on A. If $(\overline{x}, \overline{y})$ is an ϵ-strongly efficient element of (VP), then there exists $y^* \in B^{st}$ such that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(VP)_{y^*}$.*

Proof. Since $(\overline{x}, \overline{y})$ is an ϵ-strongly efficient element of (VP), $\overline{y} \in \epsilon\text{-GE}(F(A), B)$. Let $y_1^* \in B^{bd}$. According to Proposition 3.6, there exist two balanced, absorbent and convex sets U_1 and V_1 such that $\langle \text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})) \cap (U_1 - \text{cone}(V_1 + B)), y_1^* \rangle$ is bounded. Thus, it follows from Lemma 2.9 that there exist $y_2^* \in (\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})))^+$ and $y_3^* \in (\text{cone}(V_1 + B))^+$ such that $y_2^* = y_1^* + y_3^*$. Now, we prove that $y_2^* \in B^{st}$. It follows from the absorption of V_1 that

$$B \subseteq \text{cone}(V_1 + B). \tag{4.1}$$

(4.1) and $y_3^* \in (\text{cone}(V_1 + B))^+$ imply that

$$\langle b, y_3^* \rangle \geq 0, \forall b \in B,$$

which means that

$$\langle b, y_2^* \rangle = \langle b, y_1^* \rangle + \langle b, y_3^* \rangle \geq \langle b, y_1^* \rangle, \forall b \in B. \tag{4.2}$$

Since $y_1^* \in B^{bd}$, it follows from (4.2) that $y_2^* \in B^{st}$. According to Remark 2.7, there exists a balanced, absorbent and convex set U_2 such that

$$\langle y, y_2^* \rangle < 0, \forall y \in U_2 - B. \tag{4.3}$$

Because $y_2^* \in (\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})))^+$, it follows from (4.3) that

$$\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})) \cap (U_2 - B) = \emptyset.$$

Clearly,

$$\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})) \cap \text{cor}(U_2 - B) = \emptyset.$$

Since U_2 is absorbent and convex, $U_2 - B$ is a convex set with $\text{cor}(U_2 - B) \neq \emptyset$ according to Lemma 2.1 in [6]. On the other hand, as the set-valued map $F + \epsilon - \overline{y}$ is generalied C-subconvexlike on A, it follows from Remark 4.2 that $\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y}))$ is a convex set in Y. Thus, the conditions of the separation theorem of the convex sets are satisfied. Therefore, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y_1, y^* \rangle \geq \langle y_2, y^* \rangle, \forall y_1 \in \text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})), \forall y_2 \in U_2 - B. \tag{4.4}$$

As $\text{vcl}(\text{cone}(K + C + \epsilon - \overline{y}))$ is a cone in Y, it follows from (4.4) that

$$\langle y_1, y^* \rangle \geq 0, \forall y_1 \in \text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})). \tag{4.5}$$

Since $0 \in C$, it follows from (4.5) that

$$\langle \overline{y}, y^* \rangle \leq \langle y, y^* \rangle + \langle \epsilon, y^* \rangle, \forall x \in A, \forall y \in F(x). \tag{4.6}$$

By (4.4), we obtain

$$\langle y_2, y^* \rangle \leq 0, \forall y_2 \in U_2 - B. \tag{4.7}$$

Since U_2 is absorbent and $y^* \in Y^* \setminus \{0\}$, there exists $u' \in U_2$ such that $\langle u', y^* \rangle > 0$. According to (4.7), we have

$$\langle b, y^* \rangle \geq \sup_{u \in U_2} \langle u, y^* \rangle \geq \langle u', y^* \rangle > 0, \forall b \in B,$$

which means $y^* \in B^{st}$. (4.6) shows that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(\text{VP})_{y^*}$. □

The following example is used to illustrate Theorem 4.5.

Example 4.6. Let $Y := \mathbb{R}^2, A := [0, 2] \times \{0\} \subseteq \mathbb{R}^2$ and $C := \{(y_1, y_2) | y_1 \geq 0, y_2 \geq 0\} \subseteq Y$. The set-valued map $F : A \rightrightarrows Y$ is defined as follows:

$$F(x_1, x_2) = \begin{cases} \{(y_1, y_2) | y_1 = x_1, 1 \leq y_2 \leq 2 - x_1\} & \text{if} \quad (x_1, x_2) \in [0, 1[\times \{0\}, \\ \{(y_1, y_2) | y_1 = x_1, 0 \leq y_2 \leq 2 - x_1\} & \text{if} \quad (x_1, x_2) \in [1, 2] \times \{0\}. \end{cases}$$

Let $\epsilon = (1,0), \overline{x} = (1,0)$ and $\overline{y} = (1,0) \in F(\overline{x})$. Clearly, the set-valued map $F + \epsilon - \overline{y}$ is generalied C-subconvexlike on A. Now, let $B := \{(y_1, y_2)|y_1 + y_2 = 2, y_1 \geq 0, y_2 \geq 0\}$. Obviously, $(1,1) \in B^{bd} \neq \emptyset$. For any $y^* \in Y^{bd}$, there exist two balanced, absorbent and convex sets $U = V =: \{(y_1, y_2)|y_1^2 + y_2^2 \leq 0.01\}$ such that $\langle \text{vcl}(\text{cone}(K + \epsilon - \overline{y})) \cap (U - \text{cone}(V + B)), y^* \rangle$ is bounded. So, $(\overline{x}, \overline{y})$ is an ϵ-strongly efficient element of (VP). Thus, all conditions of Theorem 4.5 are satisfied. Hence, there exists $y^* = (1,2) \in B^{st}$ such that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(\text{VP})_{y^*}$.

Theorem 4.7. *Let $\epsilon \in C, \overline{x} \in A, \overline{y} \in F(\overline{x})$. Let B be an ordering-bounded basis of C. The set-valued map $F + \epsilon - \overline{y}$ is generalied C-subconvexlike on A. If there exists $y^* \in B^{st}$ such that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(VP)_{y^*}$, then $(\overline{x}, \overline{y})$ is an ϵ-strongly efficient element of (VP).*

Proof. Since $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(\text{VP})_{y^*}$, we have

$$\langle \overline{y}, y^* \rangle \leq \langle y, y^* \rangle + \langle \epsilon, y^* \rangle, \forall x \in A, \forall y \in F(x). \tag{4.8}$$

By $y^* \in B^{st}$, we obtain

$$\langle c, y^* \rangle \geq 0, \forall c \in C. \tag{4.9}$$

According to (4.8) and (4.9), we have

$$\langle y, y^* \rangle \geq 0, \forall y \in \text{cone}(F(A) + C + \epsilon - \overline{y}),$$

i.e.,

$$y^* \in (\text{cone}(F(A) + C + \epsilon - \overline{y}))^+. \tag{4.10}$$

It follows from (4.10) and Lemma 2.10 that

$$y^* \in (\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})))^+. \tag{4.11}$$

Since $y^* \in B^{st}$, it follows from the proof of Theorem 3.14 that, for any $y_1^* \in Y^{bd}$, there exists $N \in \mathbb{N}$ such that $y^* - \frac{1}{N} y_1^* \in C^+$. Clearly, $Ny^* - y_1^* \in C^+$. By Remark 2.8,

$$y_2^* := (N+1)y^* - y_1^* \in B^{st}. \tag{4.12}$$

According to Remark 2.4, there exists a balanced, absorbent and convex set V_1 such that

$$\langle y, y_2^* \rangle < 0, \forall y \in V_1 - B. \tag{4.13}$$

Since V_1 is balanced, it follows from (4.13) that

$$y_2^* \in (\text{cone}(V_1 + B))^+. \tag{4.14}$$

(4.11) implies that

$$(N+1)y^* \in (\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})))^+. \tag{4.15}$$

Using (4.12),(4.14) and (4.15), we obtain

$$y_1^* \in (\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y})))^+ - (\text{cone}(V_1 + B))^+. \tag{4.16}$$

Since the set-valued map $F + \epsilon - \overline{y}$ is generalied C-subconvexlike on A, it follows from Remark 4.2 that $\text{vcl}(\text{cone}(F(A) + C + \epsilon - \overline{y}))$ is a convex cone in Y. It follows from (4.16) and Lemma 2.9 that there exist a balanced, absorbent and convex set $U_1 \subseteq Y$ and a real number $\alpha > 0$ such that

$$\langle y, y_1^* \rangle \geq -\alpha, \forall y \in \mathrm{vcl}(\mathrm{cone}(F(A) + C + \epsilon - \overline{y})) \cap (U_1 - \mathrm{cone}(V_1 + B)). \tag{4.17}$$

Clearly, $-y_1^* \in Y^{bd}$. Therefore, there exist two balanced, absorbent and convex sets $U_2 \subseteq Y, V_2 \subseteq Y$ and a real number $\beta > 0$ such that

$$\langle y, -y_1^* \rangle \geq -\beta, \forall y \in \mathrm{vcl}(\mathrm{cone}(F(A) + C + \epsilon - \overline{y})) \cap (U_2 - \mathrm{cone}(V_2 + B)). \tag{4.18}$$

Let $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. It follows from (4.17) and (4.18) that

$$-\alpha \leq \langle y, y_1^* \rangle \leq \beta, \forall y \in \mathrm{vcl}(\mathrm{cone}(F(A) + C + \epsilon - \overline{y})) \cap (U - \mathrm{cone}(V + B)),$$

which means that $(\overline{x}, \overline{y})$ is an ϵ-strongly efficient element of (VP). □

5 Conclusions

In this paper, we extend ϵ-strongly efficient point of the set from topological spaces to ordered linear spaces. Some properties and existence conditions of strongly efficient points are investigated. Under the generalized cone subconvexlikeness of set-valued maps, we establish the relationship between the ϵ-strongly efficient element of an unconstrainted set-valued optimization problem and the ϵ-optimal solution of the scalarization problem. Note that we only establish linear scalarization theorems of the ϵ-strongly efficient element of an unconstrainted set-valued optimization problem. Following the line of [7], whether the nonlinear scalarization theorems of the ϵ-strongly efficient element can be obtained is interesting.

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