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FINDING A NONNEGATIVE SOLUTION TO AN M-TENSOR EQUATION*

DONG-HUI LI, HONG-BO GUAN AND XIAO-ZHOU WANG

Abstract: We are concerned with the tensor equation with an M-tensor, which we call the M-tensor equation. We first derive a necessary and sufficient condition for an M-tensor equation to have nonnegative solutions. We then develop a monotone iterative method to find a nonnegative solution to an M-tensor equation. The method can be regarded as an approximation to Newton's method for solving the equation. At each iteration, we solve a system of linear equations. An advantage of the proposed method is that the coefficient matrices of the linear systems are independent of the iteration. We show that if the initial point is appropriately chosen, then the sequence of iterates generated by the method converges to a nonnegative solution of the M-tensor equation monotonically and linearly. At last, we do numerical experiments to test the proposed methods. The results show the efficiency of the proposed methods.

Key words: *M-tensor equation, iterative method, monotone convergence*

Mathematics Subject Classification: *65H10,65K10,90C30*

1 Introduction

Consider the tensor equation

$$\mathcal{A}x^{m-1} = b, \tag{1.1}$$

where $x, b \in R^n$ and $\mathcal{A}$ is an mth-order n-dimensional tensor that takes the form

$$\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \quad a_{i_1 i_2 \dots i_m} \in R, \quad 1 \le i_1, i_2, \cdots, i_m \le n, \tag{1.2}$$

and $\mathcal{A}x^{m-1} \in R^n$ with elements

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} a_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \dots, n. \tag{1.3}$$

Tensor equation is also called multilinear equation. It appears in many practical fields including data mining and numerical partial differential equations [3, 6–8, 12, 13, 15, 24].

We denote the set of all mth-order n-dimensional tensors by $\mathcal{T}(m, n)$. A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathcal{T}(m, n)$ is called symmetric tensor if its elements $a_{i_1 i_2 \dots i_m}$ are invariant under any permutation of their indices $(i_1, i_2, \cdots, i_m)$. The set of all mth-order n-dimensional

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symmetric tensors is denoted by $\mathcal{ST}(m,n)$. In the case $\mathcal{A} \in \mathcal{ST}(m,n)$, it holds that $\nabla(\mathcal{A}x^m) = m\mathcal{A}x^{m-1}$. Consequently, the tensor equation (1.1) is just the first order necessary condition of the optimization problem

$$\min f(x) = \frac{1}{m}\mathcal{A}x^m - b^T x. \tag{1.4}$$

Tensor equation is a special system of nonlinear equations. It may be solved by existing numerical methods such as Newton and quasi-Newton methods for solving nonlinear equations. However, in some cases, Newton's method may fail or be only linearly convergent if the Jacobian matrix of F is singular.

Example 1.1. Consider the quadratic polynomial equation

$$F(x) = \begin{pmatrix} x_1^2 - 1 \\ x_1^2 + x_2^2 - 1 \\ -x_1^2 + x_2^2 + x_3^2 + 1 \end{pmatrix} = 0. \tag{1.5}$$

It corresponds to the tensor equation (1.1) with $b = (1,1,-1)^T$ and $\mathcal{A} = (a_{ijk})$ whose elements are

$$a_{111} = a_{222} = a_{333} = 1,\ a_{211} = a_{322} = 1,\ a_{311} = -1. \tag{1.6}$$

and all other elements are zeros. The equation have two solutions $x^* = (1,0,0)^T$ and $\tilde{x}^* = (-1,0,0)^T$.

The Jacobian of F in (1.5) at x is

$$F'(x) = 2\begin{pmatrix} x_1 & 0 & 0 \\ x_1 & x_2 & 0 \\ -x_1 & x_2 & x_3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix}. \tag{1.7}$$

It is singular if $x_1 = 0$, $x_2 = 0$ or $x_3 = 0$. In particular, $F'(x^*)$ and $F'(\tilde{x}^*)$ are singular.

Let x be the current iterate that has no zero elements. The next iterate generated by Newton's method x^+ is determined by

$$\begin{aligned} x^+ &= x - (F'(x))^{-1}F(x) \\ &= x - \frac{1}{2}\begin{pmatrix} x_1^{-1} & & \\ & x_2^{-1} & \\ & & x_3^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1^{-1} \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

It implies

$$x^+ - x^* = \frac{1}{2}\begin{pmatrix} (x_1 - x_1^*) + (x_1^{-1} - (x_1^*)^{-1}) \\ x_2 - x_2^* \\ x_3 - x_3^* \end{pmatrix}, \tag{1.8}$$

which shows the linear convergence of $\{x_k\}$.

On the other hand the tensor equation (1.5) can be written as

$$A\begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix} = b, \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \tag{1.9}$$

It can be solved easily by solving the system of linear equation $Ay = b$ first and then get the solution of the tensor equation by letting $x = \pm y^{1/2}$.

Example 1.2 Let us consider a more general equation:

$$F(x) = \frac{1}{m-1} M x^{[m-1]} - b = 0, \tag{1.10}$$

where $M \in R^{n\times n}$ is nonsingular, $b \in R^n$ and $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. It is a tensor equation in the form (1.1) where the elements of $\mathcal{A}$ are

$$a_{ij\ldots j} = m_{ij}, \quad i, j = 1, 2, \ldots, n \tag{1.11}$$

and all other elements of $\mathcal{A}$ are zeros. It is easy to see that the Jacobian of F at x is

$$F'(x) = M \ \mathrm{diag}(x_1^{m-2}, x_2^{m-2}, \ldots x_n^{m-2}). \tag{1.12}$$

It is singular even if M is nonsingular, if there exists an $x_i = 0$. As a result, Newton's method may fail to work if some iterate x_k has zero elements.

On the other hand, if M is nonsingular, the equation can be solved easily by solving the system $\frac{1}{m-1} My = b$ first and then letting $x = y^{[1/(m-1)]}$.

The last two examples indicate that it is important to develop special iterative methods for solving the tensor equation (1.1) using the special structure of the tensor $\mathcal{A}$. It is the major purpose of the paper.

In this paper, we will pay particular attention to tensor equations (1.1) where the coefficient tensor $\mathcal{A}$ is an M-tensor. Tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathcal{T}(m, n)$ is called a Z-tensor if it can be written as

$$\mathcal{M} = s\mathcal{I} - \mathcal{B}, \tag{1.13}$$

where $s > 0$, $\mathcal{I}$ is the identity tensor whose diagonals are all ones and all off diagonal elements are zeros, and $\mathcal{B} \geq 0$ is a nonnegative tensor in the sense that all its elements are nonnegative. Tensor $\mathcal{A}$ is called M-tensor if it can be written as (1.13) and satisfies $s \geq \rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of tensor $\mathcal{B}$, that is

$$\rho(\mathcal{B}) = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{B}\}. \tag{1.14}$$

If $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong or nonsingular M-tensor [5]. In the case $\mathcal{A}$ is an M-tensor, we call the equation (1.1) M-tensor equation and abbreviate it as M-TEQ. An interesting property of the M-TEQ is that if $\mathcal{A}$ is a strong M-tensor, then for every positive vector b the tensor equation (1.1) has a unique positive solution, namely, all entries of the solution are positive [6].

The study in the numerical methods for solving tensor equation has begun only a few years ago. Most of them focus on solving the M-TEQ. Ding and Wei [6] extended the classical iterative methods for solving system of linear equations including the Jacobi method, the Gauss-Seidel method, and successive overrelaxation method to solve the M-TEQ. The methods are linearly convergent to a nonnegative solution of the equation. Han [10] proposed a homotopy method for finding the unique positive solution of the M-TEQ with strong M-tensor and a positive right side vector b. Liu, Li and Vong [17] proposed a tensor splitting method for solving the M-TEQ. An advantage of the method is that at each iteration, only a system of linear equation needs to be solved. The coefficient matrix of the linear system does not depend on the iteration. The sequence of iterates $\{x_k\}$ generated by the method converges to a nonnegative solution of the equation monotonically if the initial point x_0 is restricted to the set $\{x \in R^n \mid 0 < \mathcal{A}x^{m-1} \le b\}$. Quite recently, He, Ling, Qi and Zhou [11] proposed a Newton-type method to solve the M-TEQ and established its quadratic convergence. Li, Xie and Xu [13] also extended the classic splitting methods for solving system of linear equations to solving symmetric tensor equations. The methods in [13] can solve those equations (1.1) where $\mathcal{A}$ are not necessary M-tensors. Li, Dai and Gao [16] developed an alternating projection method for a class of tensor equations and established its global convergence and linear convergence. Wang, Che and Wei [22] proposed continuous time neural network and modified continuous time neural networks for solving M-TEQ. Related works can also be found in [3,15,19,23,24,26]. Just when this article is about to be completed, we knew the new report by Bai, He, Ling and Zhou [2] where the authors proposed an iterative method for finding a nonnegative solution to M-tensor equations.

The existing methods for solving M-TEQ focus on finding a positive solution of the equation under the restriction $b > 0$. In this paper, we further study numerical methods for solving M-TEQ (1.1). Our purpose is to find the largest nonnegative solution of the equation without the requirement of $b > 0$. We first give a necessary and sufficient condition for the existence of the nonnegative solution of the M-TEQ. The result reveals an interesting property that if an M-TEQ has a nonnegative solution, then it has a largest nonnegative solution.

We then propose an iterative method to solve the M-TEQ with strong M-tensor. We first split the coefficient M-tensor into two parts. Based on the splitting form, we develop an approximation Newton method. At each iteration, we solve a system of linear equations. The coefficient matrix of linear system is independent of the iteration. We show that if the initial point is appropriately chosen, then the generated sequence of iterates converges to a nonnegative solution of the equation without the restriction that b is positive. We also do numerical experiments to test the proposed method. The results show that the methods are very efficient.

The rest of this paper is organized as follows. In Section 2, we derive a necessary and sufficient condition for an M-tensor equation to have a nonnegative solution. It particularly shows an interesting result that if an M-tensor equation has more than one nonnegative solutions, then it has a largest nonnegative solution. In Section 3, we propose a monotone iterative method for solving the M-TEQ with strong M-tensor and establish its monotone convergence. In Section 4, we make an improvement to the method proposed in Section 3 and establish its monotone convergence. At last, we do some numerical experiments to test the proposed method in Section 5. We conclude the paper by giving some final remarks.

2 The Existence of Nonnegative Solutions to the M-Tensor Equation

Consider the tensor equation

$$F(x) = \mathcal{M}x^{m-1} - b = 0, \tag{2.1}$$

where $\mathcal{M} = (m_{i_1 i_2 \dots i_m}) \in \mathcal{T}(m,n)$ and $b \in R^n$. We are interested in the nonnegative solution of an M-tensor equation. It was proved by Ding and Wei [6] that if $\mathcal{M}$ is a strong M-tensor and b is positive, then the equation has a unique positive solution. Liu, Li and Vong [17] also obtained some existence and uniqueness of the positive/nonnegative solutions for the M-TEQ via tensor splitting technique.

Similar to Theorem 5.4 in [9], we now derive a sufficient condition for the existence and uniqueness of the nonnegative solutions to an M-TEQ. We first introduce the concept of the so called majorization matrix of a tensor $\mathcal{M} \in \mathcal{T}(m,n)$ [14, 17, 20]. It is a matrix $M = (m_{ij}) \in R^{n\times n}$ whose elements are

$$m_{ij} = m_{ij\dots j}, \quad i,j = 1,2,\dots,n. \tag{2.2}$$

The theorem about the existence and uniqueness of the nonnegative solution of an M-TEQ is stated and proved below.

Theorem 2.1. *Suppose $\mathcal{M} = (m_{i_1\dots i_m}) \in \mathcal{T}(m,n)$ is a strong M-tensor such that for each index i,*

$$m_{ii_2\dots i_m} = 0, \quad \text{whenever } \; i_j \neq i_k \; \text{ for some } j \neq k. \tag{2.3}$$

Then, the M-TEQ has a unique nonnegative solution if and only if the unique solution of the M-LEQ

$$My = b \tag{2.4}$$

is nonnegative, where matrix $M = (m_{ij}) \in R^{n\times n}$ is the majorization matrix of $\mathcal{M}$.

Proof. It is not difficult to see that the M-TEQ can be written as the following equation

$$Mx^{[m-1]} - b = 0. \tag{2.5}$$

Since $\mathcal{M}$ is a strong M-tensor, its majorization matrix M is a strong M-matrix [17]. It is nonsingular and satisfies $M^{-1} \geq 0$. The results of the theorem become obvious. □

In many cases where b is not positive, the M-tensor equation may have no nonnegative solutions or multiple nonnegative solutions. For example, if $m = 2$ and $b < 0$, then for any strong M-matrix M, the unique solution of the equation (2.1) is negative. Consequently, it has no nonnegative solution. The following example suggested by Prof. C. Ling via private communication shows that an M-TEQ with a strong M-tensor $\mathcal{M}$ may have more than one nonnegative solutions.

Example 2.1 Given tensor $\mathcal{M} \in \mathcal{T}(4,2)$ in the form $\mathcal{M} = 3\mathcal{I} - \mathcal{B}$, where $\mathcal{I}$ is the identity tensor and $\mathcal{B}$ is a nonnegative tensor with elements

$$b_{1111} = b_{2222} = 0, \quad b_{1122} = \frac{3}{2}, \quad b_{1222} = \frac{1}{2} \tag{2.6}$$

and all other elements of $\mathcal{B}$ are zeros. It is easy to see that the unique eigenvalue of $\mathcal{B}$ is zero. Consequently, $\mathcal{M}$ is an M-tensor. Let $b = (-7, 24)^T$. Then the tensor equation $\mathcal{M}x^{m-1} - b = 0$ is written as

$$\begin{cases} 3x_1^3 - \frac{3}{2}x_1x_2^2 - \frac{1}{2}x_2^3 = -7, \\ 3x_2^3 = 24. \end{cases} \tag{2.7}$$

Clearly, the solutions of the following system are all the solutions of the M-TEQ:

$$\begin{cases} x_1^3 - 2x_1 + 1 = 0, \\ x_2 = 2. \end{cases} \tag{2.8}$$

The last system has two nonnegative solutions

$$x^{(1)} = (1,2)^T \quad \text{and} \quad x^{(2)} = (\frac{\sqrt{5}-1}{2}, 2)^T. \tag{2.9}$$

The following example shows that an M-tensor equation with an odd order strong M-tensor may also have multiple nonnegative solutions.

Example 2.2 Given tensor $\mathcal{M} \in \mathcal{T}(3,2)$ in the form $\mathcal{M} = \mathcal{I} - \mathcal{B}$, where $\mathcal{I}$ is the identity tensor and $\mathcal{B}$ is a nonnegative tensor with elements

$$b_{111} = b_{222} = 0, \quad b_{112} = \frac{3}{2}, \quad b_{122} = 1 \tag{2.10}$$

and all other elements of $\mathcal{B}$ are zeros. It is easy to see that the unique eigenvalue of $\mathcal{B}$ is zero. Consequently, $\mathcal{M}$ is a strong M-tensor. Let $b = (-6, 4)^T$. Then the tensor equation $\mathcal{M}x^{m-1} - b = 0$ is written as

$$\begin{cases} x_1^2 - \frac{3}{2}x_1x_2 - x_2^2 = -6, \\ x_2^2 = 4. \end{cases} \tag{2.11}$$

The last system has two nonnegative solutions

$$x^{(1)} = (1,2)^T \quad \text{and} \quad x^{(2)} = (2,2)^T. \tag{2.12}$$

Recently, Xu, Gu and Huang [25] investigated some interesting properties of the tensor equations. In this section, we investigate the existence of the nonnegative solutions and the largest nonnegative solution of the M-tensor equation.

To this end, we first introduce the concept of the so-called join semi-sublattice. A set $\mathcal{S} \subseteq R^n$ is called a join semi-sublattice if $x, y \in \mathcal{S}$, then there join, i.e., $\max\{x, y\}$, is also in $\mathcal{S}$. Here the vector operation $\max\{x, y\}$ is defined by elements.

The following theorem can be regarded as a dual theorem of Theorem 3.11.5 in [4]. The proof is similar to that of Theorem 3.11.5 in [4].

Theorem 2.2. *If $\mathcal{S}$ is a nonempty and bounded closed join semi-sublattice, then it has a largest element.*

Proof. Given an arbitrary positive vector c and any $\bar{x} \in \mathcal{S}$. Consider the following program problem:

$$\begin{cases} \max & \phi(x) = c^T x \\ \text{s.t.} & x \in \mathcal{S}, \ c^T x \geq c^T \bar{x}. \end{cases} \tag{2.13}$$

Clearly, the feasible set of the problem is closed and bounded from above. Hence, the problem has a solution $\tilde{x}$. For any $x \in \mathcal{S}$, we have $z = \max\{x, \tilde{x}\} \in \mathcal{S}$. So, we have

$$c^T(\tilde{x} - z) \geq 0. \tag{2.14}$$

Since $\tilde{x} \leq z$ and c is positive, the last inequality implies $\tilde{x} = z$. By the arbitrariness of x, we claim that $\tilde{x}$ is the largest element of $\mathcal{S}$. □

In the proof of the last theorem, we used the vector inequality $x \leq y$ with $x, y \in R^n$. It is defined by elements. The vector inequality $x \geq y$ can be defined similarly. In particular, the inequality $x \geq 0$ means that x is nonnegative.

Define the set

$$\mathcal{S} = \{x \in R_+^n \mid \mathcal{M}x^{m-1} \leq b\} = \{x \in R_+^n \mid F(x) \leq 0\}, \tag{2.15}$$

where $R_+^n = \{x \in R^n \mid x \geq 0\}$ and $F(x) = \mathcal{M}x^{m-1} - b$. The set $\mathcal{S}$ looks very similar to the feasible set

$$\tilde{\mathcal{S}} = \{x \in R_+^n \mid \mathcal{M}x^{m-1} \geq b\} = \{x \in R_+^n \mid F(x) \geq 0\} \tag{2.16}$$

of the tensor complementarity problem

$$x \geq 0 \quad F(x) \geq 0, \quad x^T F(x) = 0. \tag{2.17}$$

It was proved by Luo, Qi and Xiu [18] that if $\mathcal{M}$ is a Z-tensor, then the feasible set $\tilde{\mathcal{S}}$ has a least element. Moreover, the least element of $\tilde{\mathcal{S}}$ is a solution of the tensor complementarity problem. In what follows, we will establish a similar result to the M-TEQ. Specifically, we will show that if $\mathcal{M}$ is a strong M-tensor, then $\mathcal{S}$ has a largest element as long as it is not empty. Moreover, the largest element of the set is the largest nonnegative solution to the M-TEQ.

Denote for $x \in R^n$ and real number p

$$x^{[p]} = (x_1^p, x_2^p, \ldots, x_n^p)^T. \tag{2.18}$$

The lemma bellow shows that $\mathcal{S}$ has a largest element.

Lemma 2.3. *If $\mathcal{M}$ is a strongly M-tensor, and the set $\mathcal{S}$ defined by (2.15) is not empty, then $\mathcal{S}$ is a bounded join semi-sublattice. As a result, $\mathcal{S}$ has a largest element.*

Proof. We first show that $\mathcal{S}$ is a join semi-sublattice.

For $x = (x_1, \ldots, x_n)^T, y = (y_1, \ldots, y_n)^T \in S$, we let $z = \max\{x, y\}$. We have for each $i = 1, 2, \ldots, n$, if $z_i = x_i$,

$$\begin{aligned}
(\mathcal{M}z^{m-1})_i &= \sum_{i_2,\ldots,i_m} m_{ii_2\ldots i_m} z_{i_2} \cdots z_{i_m} \\
&= m_{ii\ldots i} x_i^{m-1} + \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} m_{ii_2\ldots i_m} z_{i_2} \cdots z_{i_m} \\
&\leq m_{ii\ldots i} x_i^{m-1} + \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} m_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} \\
&= (\mathcal{M}x^{m-1})_i \leq b_i,
\end{aligned}$$

where the first inequality follows from the fact $m_{ii_2\ldots i_m} \leq 0$, $\forall (i, i_2, \ldots, i_m) \neq (i, i, \ldots, i)$. Similarly, we can show that $(\mathcal{M}z^{m-1})_i \leq b_i$ for $z_i = y_i$. Consequently, $\mathcal{S}$ is a join semi-sublattice.

Next, we show that $\mathcal{S}$ is bounded. Suppose on the contrary that there is an unbounded sequence $\{x_k\} \subset \mathcal{S}$. That is, $\|x_k\|_\infty \to \infty$ as $k \to \infty$ and

$$x_k \geq 0, \quad \mathcal{M}x_k^{m-1} - b \leq 0. \tag{2.19}$$

Denote $u_k = x_k / \|x_k\|_\infty$. Then $\{u_k\}$ is bounded and hence has a limit point. Without loss of generality, we let $\{u_k\} \to u$. It is easy to get

$$u \geq 0, \qquad \mathcal{M}u^{m-1} \leq 0. \tag{2.20}$$

We write $\mathcal{M} = s\mathcal{I} - \mathcal{B}$ with $s > \rho(\mathcal{B})$. It follows from the last inequality that

$$s u_i^{m-1} \leq \Big(\mathcal{B}u^{m-1}\Big)_i, \quad \forall i = 1, 2, \ldots, n, \tag{2.21}$$

which implies

$$s \leq \min_{u_i \neq 0} \frac{\Big(\mathcal{B}u^{m-1}\Big)_i}{u_i^{m-1}} \leq \rho(\mathcal{B}), \tag{2.22}$$

where the last inequality follows from [21, Theorem 3.25]. The last inequality yields a contradiction. Consequently, $\mathcal{S}$ is bounded.

At last, by Theorem 2.2, $\mathcal{S}$ has a largest element. The proof is complete. □

Based on the last lemma and Theorem 2.2, we can derive a condition for the existence of nonnegative solution of the M-TEQ.

Theorem 2.4. *Let $\mathcal{M}$ be a strong M-tensor. Then the following statements are true.*

(i) *The M-tensor equation (2.1) has a nonnegative solution if and only if the set $\mathcal{S}$ defined by (2.15) is not empty. In the case $\mathcal{S} \neq \emptyset$, its largest element is the largest nonnegative solution of the equation. In particular, if $M^{-1}b \geq 0$, then (2.1) has a nonnegative solution.*

(ii) *The equation (2.1) has a positive solution if and only if the set $\mathcal{S}$ contains a positive element. In particular, if $M^{-1}b > 0$, then (2.1) has a positive solution.*

Proof. Conclusion (ii) follows from conclusion (i) directly. We only need to verify (i).

Since the solution set of (2.1) is a subset of $\mathcal{S}$ defined by (2.15), 'only if' part is obvious. We turn to the 'if' part.

Suppose $\mathcal{S} \neq \emptyset$. Theorem 2.2 and Lemma 2.3 ensure that $\mathcal{S}$ has a largest element $\bar{x}$, i.e.,

$$x \leq \bar{x} \quad \forall x \in \mathcal{S}. \tag{2.23}$$

It suffices to show that $\bar{x}$ is a solution of (2.1).

Suppose on the contrary that $\bar{x}$ is not a solution of (2.1). There must be an index i such that

$$F_i(\bar{x}) = (\mathcal{M}\bar{x}^{m-1} - b)_i < 0. \tag{2.24}$$

Let $\mathcal{D}$ be the diagonal part subtensor of $\mathcal{M}$ and $\mathcal{B} = \mathcal{M} - \mathcal{D}$. By the definition of M-tensor, we have $\mathcal{B} \leq 0$. Let D be a diagonal matrix whose diagonals are diagonals of $\mathcal{M}$ or $\mathcal{D}$, i.e.,

$$D = \mathrm{diag}(m_{1\ldots1}, \ldots, m_{n\ldots n}). \tag{2.25}$$

Define $\tilde{x}$ with elements

$$\tilde{x}_j = \Big(\bar{x}_j^{m-1} - \frac{1}{2}(D^{-1}F(\bar{x}))_j\Big)^{1/(m-1)}, \quad j = 1, 2, \ldots, n. \tag{2.26}$$

It implies

$$D\tilde{x}^{[m-1]} = D\bar{x}^{[m-1]} - \frac{1}{2}F(\bar{x}). \tag{2.27}$$

It is easy to see that $0 \leq \bar{x} \leq \tilde{x}$ and

$$\begin{aligned} F(\tilde{x}) &= \mathcal{D}\tilde{x}^{m-1} + \mathcal{B}\tilde{x}^{m-1} - b \\ &\leq \mathcal{D}\tilde{x}^{m-1} + \mathcal{B}\bar{x}^{m-1} - b \\ &= D\tilde{x}^{[m-1]} + \mathcal{B}\bar{x}^{m-1} - b \\ &= D\bar{x}^{[m-1]} - \frac{1}{2}F(\bar{x}) + \mathcal{B}\bar{x}^{m-1} - b \\ &= \frac{1}{2}F(\bar{x}) \leq 0. \end{aligned}$$

Consequently, $\tilde{x} \in \mathcal{S}$. However, by the definition of $\tilde{x}$, we obviously have $\tilde{x} \geq \bar{x}$. Moreover, there are at least one indices i such that $\tilde{x}_i > \bar{x}_i$, which contracts the fact that $\bar{x}$ is the largest element of $\mathcal{S}$. The contradiction shows that $\bar{x}$ must be a solution of (2.1).

Since the solution set of (2.1) is a subset of $\mathcal{S}$, the largest element of $\mathcal{S}$ is the largest solution of (2.1). The proof is complete. □

3 A Monotone Iterative Method

In this section, we develop an iterative method for solving the M-tensor equation. Our purpose is to find a nonnegative solution of the equation. So we assume throughout this section that tensor $\mathcal{M}$ in (2.1) is a strong M-tensor and that the set $\mathcal{S}$ defined by (2.15) is not empty.

The method can be regarded as an approximation to Newton's method. Notice that for each $i = 1, 2, \ldots, n$, function $(\mathcal{M}x^{m-1})_i$ is a homogeneous polynomial. It can be written as $\mathcal{M}_i x^{m-1}$ with symmetric tensor $\mathcal{M}_i \in \mathcal{ST}(m-1, n)$.

Let us consider the Newton iteration:

$$(m-1)\mathcal{M}x_k^{m-2}(x_{k+1} - x_k) + F(x_k) = 0, \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

An attractive property of Newton's method is its quadratic convergence if the Jacobian at the solution x^*, $F'(x^*) = (m-1)\mathcal{M}(x^*)^{m-2}$ is nonsingular. However, the method may be failure if $\mathcal{M}x_k^{m-2}$ is singular.

We are going to develop an approximation Newton method. To this end, we split the tensor $\mathcal{M}$ as

$$\mathcal{M} = \widetilde{\mathcal{M}} + \overline{\mathcal{M}}, \tag{3.2}$$

where $\widetilde{\mathcal{M}}$ is the sub-tensor of $\mathcal{M}$ whose possibly nonzero elements are $m_{ij\ldots j}$, $i, j = 1, 2, \ldots, n$ while other elements are zeros, $\overline{\mathcal{M}} = \mathcal{M} - \widetilde{\mathcal{M}}$. The sub-tensor $\widetilde{\mathcal{M}}$ contains all diagonal elements of $\mathcal{M}$. So, $\overline{\mathcal{M}}$ is a non-positive tensor, i.e., $\overline{\mathcal{M}} \leq 0$.

Let $M = (m_{ij})$ be majorization matrix defined by (2.2). It is an (a strong) M-matrix if $\mathcal{M}$ is an (a strong) M-tensor [17].

Using the above splitting form to $\mathcal{M}$, we write the Newton iteration as

$$\begin{aligned}
0 &= \mathcal{M}x_k^{m-2}x_{k+1} - \mathcal{M}x_k^{m-1} + \frac{1}{m-1}F(x_k) \\
&= Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \frac{1}{m-1}F(x_k) + [\mathcal{M}x_k^{m-2}x_{k+1} - Mx_{k+1}^{[m-1]} - \overline{\mathcal{M}}x_k^{m-1}] \\
&= Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \frac{1}{m-1}F(x_k) + \widetilde{\mathcal{M}}(x_k^{m-2} - x_{k+1}^{m-2})x_{k+1} + \overline{\mathcal{M}}x_k^{m-2}(x_{k+1} - x_k) \\
&= Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \frac{1}{m-1}F(x_k) + O(\|x_{k+1} - x_k\|), \quad k = 0, 1, 2, \ldots.
\end{aligned}$$

By neglecting the term $O(\|x_{k+1} - x_k\|)$, we get an approximate Newton iterative scheme

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \frac{1}{m-1}M^{-1}F(x_k), \quad k = 0, 1, 2, \ldots. \tag{3.3}$$

More generally, we give the following iterative scheme:

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k M^{-1}F(x_k), \quad k = 0, 1, 2, \ldots, \tag{3.4}$$

or

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} + \alpha_k d_k, \quad M d_k + F(x_k) = 0, \tag{3.5}$$

where $\alpha_k \in (0, 1]$. We call the above iterative scheme sequential M-matrix linear equation method (S-MEQM) because the subproblem (3.5) is a system of linear equation in d_k with an M-matrix as the coefficient matrix. Since M is independent of the iteration k, we can solve the system by LU decomposition at the beginning of the method. So, at each iteration, the computation cost for solving the system of linear equations (3.5) is $O(n^2)$ except for the initial step.

Recently Liu, Li and Vong [17] proposed a so-called tensor splitting (TS) method for solving the M-tensor equation. The TS method corresponds to $\alpha_k \equiv 1$. It is monotonically and linearly convergent if the initial point x_0 is chosen in the set

$$\mathcal{S}_1 = \{x \in R_+^n \mid 0 < \mathcal{M}x^{m-1} \le b\}. \tag{3.6}$$

Specifically, they proved the following theorem [17].

Theorem 3.1. *Let $\mathcal{M}$ be a strong M-tensor and b be positive. Then the sequence $\{x_k\}$ generated by the TS method with $x_0 \in \mathcal{S}_1$ converges to the unique positive solution of (2.1) monotonically in the sense*

$$x_{k+1} \ge x_k \ge 0. \tag{3.7}$$

Moreover, the convergence rate of $\{x_k\}$ is linear.

The condition $x_0 \in \mathcal{S}_1$ in the above convergence theorem is restrictive. It makes the method suitable for those problems with $b > 0$ only. The theorem below shows that the S-MEQM will retain monotone convergence if the initial point is in $\mathcal{S}$.

Theorem 3.2. *Starting from any $x_0 \in \mathcal{S}$, the sequence $\{x_k\}$ generated by the S-MEQM (3.5) is contained in $\mathcal{S}$ and satisfies*

$$x_{k+1} \ge x_k \ge 0, \quad k = 0, 1, 2, \ldots. \tag{3.8}$$

Moreover, $\{x_k\}$ converges to a nonnegative solution of the equation (2.1).

Proof. We first prove $\{x_k\} \subset \mathcal{S}$ and (3.8) by induction.

For $k = 0$, we have $F(x_0) \le 0$ and

$$M x_1^{[m-1]} = M x_0^{[m-1]} - \alpha_0 F(x_0) \ge M x_0^{[m-1]}. \tag{3.9}$$

Since M is an M-matrix, the last inequality yields (3.8) with $k = 0$. Moreover, we have

$$\begin{aligned} F(x_1) &= \mathcal{M}x_1^{m-1} - b = M x_1^{[m-1]} + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= M x_0^{[m-1]} - \alpha_0 F(x_0) + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= (1 - \alpha_0) F(x_0) + \overline{\mathcal{M}}(x_1^{m-1} - x_0^{m-1}) \\ &\le (1 - \alpha_0) F(x_0) \le 0, \end{aligned}$$

where the inequality holds because $x_1 \ge x_0$ and $\overline{\mathcal{M}} \le 0$.

Suppose the inequalities $x_k \ge x_{k-1} \ge 0$ and $F(x_k) \le 0$ hold for some $k \ge 1$. It follows from (3.5) that

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k M^{-1} F(x_k) \ge x_k^{[m-1]}. \tag{3.10}$$

Moreover, we can get

$$F(x_{k+1}) = Mx_{k+1}^{[m-1]} + \overline{\mathcal{M}}x_{k+1}^{m-1} - b \le (1-\alpha_k)F(x_k) \le 0. \tag{3.11}$$

By the principle of induction, we claim that the inequalities in (3.8) hold and $F(x_k) \le 0$ for all $k \ge 0$.

Since $\{x_k\}$ is bounded from above, then $\{x_k\}$ converges. Taking limits in both sizes of (3.5), it is easy to see that the limit of $\{x_k\}$ is a nonnegative solution of the tensor equation (2.1). The proof is complete. □

We turn to the convergence rate of the method. Suppose $\{x_k\} \to \bar{x}$. Clearly, we have $x_k \le x_{k+1} \le \bar{x}$. If for some i, $\bar{x}_i = 0$, then $(x_k)_i = 0$ for all $k \ge 0$. So, without loss of generality, we assume $\bar{x} > 0$.

Define

$$\phi(x) = \Big(x^{[m-1]} - \alpha M^{-1}F(x)\Big)^{1/(m-1)}. \tag{3.12}$$

The iterative scheme can be written as

$$x_{k+1} = \phi(x_k) \tag{3.13}$$

and the limit point $\bar{x}$ satisfies

$$\bar{x} = \phi(\bar{x}). \tag{3.14}$$

It follows from (3.12) that

$$M\phi(x)^{[m-1]} = Mx^{[m-1]} - \alpha F(x). \tag{3.15}$$

Without loss of generality, we suppose that $\mathcal{M}$ is symmetric. Taking derivative in both sizes of the equality, we obtain

$$\begin{aligned} M\ \mathrm{diag}(\phi(x)^{[m-2]})\phi'(x) &= M\ \mathrm{diag}(x^{[m-2]}) - \frac{1}{m-1}\alpha F'(x) \\ &= M\ \mathrm{diag}(x^{[m-2]}) - \alpha\mathcal{M}x^{m-2}. \end{aligned}$$

At the limit $\bar{x}$, the last equality yields

$$M\ \mathrm{diag}(\bar{x}^{[m-2]})\phi'(\bar{x}) = M\ \mathrm{diag}(\bar{x}^{[m-2]}) - \alpha\mathcal{M}\bar{x}^{m-2}. \tag{3.16}$$

It implies

$$\begin{aligned} \phi'(\bar{x}) &= I - \alpha\ \mathrm{diag}(\bar{x}^{[-(m-2)]})M^{-1}\mathcal{M}\bar{x}^{m-2} \\ &= (1-\alpha)I - \alpha\ \mathrm{diag}(\bar{x}^{[-(m-2)]})M^{-1}\overline{\mathcal{M}}\bar{x}^{m-2} \ge 0. \end{aligned}$$

If $M^{-1}b > 0$, then it follows from the last equality that

$$\phi'(\bar{x})\bar{x} = \bar{x} - \alpha\ \mathrm{diag}(\bar{x}^{[-(m-2)]})M^{-1}\mathcal{M}\bar{x}^{m-1} = \bar{x} - \alpha\ \mathrm{diag}(\bar{x}^{[-(m-2)]})M^{-1}b < \bar{x}. \tag{3.17}$$

Since $\phi'(\bar{x})$ is nonnegative, we claim that the spectral radius of $\phi'(\bar{x})$ satisfies $\rho(\phi'(\bar{x})) < 1$. As a result, the convergence of the sequence $\{x_k\}$ is linear. Theorem 4.5 in [17] shows that the condition $M^{-1}b > 0$ implies that the the M-TEQ has a unique positive solution.

The above arguments have shown the following theorem.

Theorem 3.3. *Suppose that $\mathcal{M}$ is a strong M-tensor and M is its majorization matrix. If $M^{-1}b > 0$, then the convergence rate of $\{x_k\}$ generated by the S-MEQM (3.5) converges to the unique positive solution of the M-tensor equation (2.1) linearly.*

Let D, $-L$ and $-U$ be the diagonal part, strict lower triangular part and strict upper triangle part of the matrix M, i.e.,

$$M = D - L - U. \tag{3.18}$$

Similar to the splitting methods for solving system of linear equations, on the basis of (3.5), we can develop splitting type methods.

The Jacobi iteration:

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k D^{-1} F(x_k),\ k = 0, 1, 2, \dots. \tag{3.19}$$

This iterative scheme is the same as the Jacobi method by Ding and Wei [6].

The Gauss-Seidel iteration:

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k (D - L)^{-1} F(x_k),\ k = 0, 1, 2, \dots. \tag{3.20}$$

Successive over-relaxation (SOR) iteration:

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k \omega (D - \omega L)^{-1} F(x_k),\ k = 0, 1, 2, \dots. \tag{3.21}$$

Similar to Theorem 3.2, it is not difficult to establish the monotone convergence of the above iterative methods. As an example, we derive the convergence of the Gauss-Seidel method below.

Theorem 3.4. *Let the sequence $\{x_k\}$ be generated by the Gauss-Seidel method (3.20) with initial point $x_0 \in \mathcal{S}$. Then $\{x_k\} \subset \mathcal{S}$ and converges to a nonnegative solution of (2.1) monotonically.*

Proof. For $k = 0$, we have $F(x_0) \le 0$ and

$$x_1^{[m-1]} = x_0^{[m-1]} - \alpha_0 (D - L)^{-1} F(x_0) \ge x_0^{[m-1]}. \tag{3.22}$$

It implies

$$\begin{aligned} F(x_1) &= \mathcal{M} x_1^{m-1} - b = M x_1^{[m-1]} + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= (D - L - U) x_1^{[m-1]} + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= (D - L) x_1^{[m-1]} - U x_1^{m-1} + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= (D - L) x_0^{[m-1]} - \alpha_0 F(x_0) - U x_1^{m-1} + \overline{\mathcal{M}} x_1^{m-1} - b \\ &= (1 - \alpha_0) F(x_0) - U(x_1^{[m-1]} - x_0^{[m-1]}) + \overline{\mathcal{M}}(x_1^{m-1} - x_0^{m-1}) \\ &\le 0, \end{aligned}$$

Suppose the inequalities $x_k \ge x_{k-1} \ge 0$ and $F(x_k) \le 0$ hold for some $k \ge 1$. We can get for all $\alpha_k \in (0, 1]$,

$$x_{k+1}^{[m-1]} = x_k^{[m-1]} - \alpha_k (D-L)^{-1} F(x_k) \geq x_k^{[m-1]} \tag{3.23}$$

and

$$\begin{aligned}
F(x_{k+1}) &= Mx_{k+1}^{[m-1]} + \overline{\mathcal{M}} x_{k+1}^{m-1} - b \\
&= (D-L)x_{k+1}^{[m-1]} - Ux_{k+1}^{[m-1]} + \overline{\mathcal{M}} x_{k+1}^{m-1} - b \\
&= (D-L)x_k^{[m-1]} - \alpha_k F(x_k) - Ux_{k+1}^{[m-1]} + \overline{\mathcal{M}} x_{k+1}^{m-1} - b \\
&= Mx_k^{[m-1]} - \alpha_k F(x_k) - U(x_{k+1}^{[m-1]} - x_k^{[m-1]}) + \overline{\mathcal{M}} x_{k+1}^{m-1} - b \\
&= (1-\alpha_k) F(x_k) - U(x_{k+1}^{[m-1]} - x_k^{[m-1]}) + \overline{\mathcal{M}} (x_{k+1}^{m-1} - x_k^{m-1}) \leq 0.
\end{aligned}$$

By induction, it is not difficult to show that if $x_0 \geq 0$ and $F(x_0) \leq 0$, then the following inequalities hold for all $k \geq 0$:

$$x_{k+1}^{[m-1]} \geq x_k^{[m-1]} \geq 0, \quad F(x_k) \leq 0. \tag{3.24}$$

Since $\{x_k\}$ is bounded from above, we claim that $\{x_k\}$ converges and hence $\{F(x_k)\}$ converges to 0. The proof is complete. □

4 An Improvement

In this section, we make some improvement to the monotone method proposed in the last section. We rewrite the Newton method as

$$0 = \mathcal{M} x_k^{m-2}(x_{k+1} - x_k) + \frac{1}{m-1} F(x_k) = Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \frac{1}{m-1} F(x_k) + r_k, \tag{4.1}$$

where

$$r_k = \mathcal{M} x_k^{m-2}(x_{k+1} - x_k) - (Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]}). \tag{4.2}$$

The S-MEQ method (3.5) developed in Section 3 neglected the term r_k. Since $r_k = O(\|x_{k+1} - x_k\|)$, it might be important for Newton's method to be quadratically convergent. As a result, the method (3.5) may not be a good approximation to Newton's method. In this section, we consider to improve the S-MEQ method by using more information of r_k.

Without loss of generality, we suppose that $\mathcal{M}$ is semi-symmetric. So we have

$$(\mathcal{M} x^{m-1})' = (m-1)\mathcal{M} x^{m-2}. \tag{4.3}$$

Denote

$$r(x) = \frac{1}{m-1}(\mathcal{M} x^{m-1} - (m-1)Mx^{[m-1]}). \tag{4.4}$$

We have

$$r'(x) = \mathcal{M} x^{m-2} - (m-1)M \mathrm{diag}(x^{[m-2]}) \tag{4.5}$$

and

$$\begin{aligned}
r_k &= \mathcal{M} x_k^{m-2}(x_{k+1} - x_k) - M(x_{k+1}^{[m-1]} - x_k^{[m-1]}) \\
&= \mathcal{M} x_k^{m-2}(x_{k+1} - x_k) - (m-1)M \mathrm{diag}(x_k^{[m-2]})(x_{k+1} - x_k) + o(\|x_{k+1} - x_k\|) \\
&= r'(x_k)(x_{k+1} - x_k) + o(\|x_{k+1} - x_k\|) \\
&= r(x_{k+1}) - r(x_k) + o(\|x_{k+1} - x_k\|)
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{m-1}(F(x_{k+1})-F(x_k)) - M(x_{k+1}^{[m-1]} - x_k^{[m-1]}) + o(\|x_{k+1}-x_k\|) \\ &\triangleq \frac{1}{m-1}y_k - Ms_k + o(\|x_{k+1}-x_k\|), \end{aligned}$$

where

$$y_k = F(x_{k+1}) - F(x_k), \quad s_k = x_{k+1}^{[m-1]} - x_k^{[m-1]}. \tag{4.6}$$

Consequently, it follows from Newton's method that

$$\begin{aligned} 0 &= \frac{1}{m-1}F(x_k) + \mathcal{M}x_k^{m-2}(x_{k+1}-x_k) \\ &= \frac{1}{m-1}F(x_k) + (Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]}) + (\mathcal{M}x_k^{m-2}(x_{k+1}-x_k) - M(x_{k+1}^{[m-1]} - x_k^{[m-1]})) \\ &= \frac{1}{m-1}F(x_k) + (Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]}) + r'(x_k)(x_{k+1}-x_k) + o(\|x_{k+1}-x_k\|) \\ &= \frac{1}{m-1}F(x_k) + (Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]}) + r(x_{k+1}) - r(x_k) + o(\|x_{k+1}-x_k\|), \\ &\qquad k = 0,1,2,\ldots. \end{aligned}$$

A reasonable approximation to Newton's method is to let x_{k+1} satisfy

$$\frac{1}{m-1}F(x_k) + (Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]}) + r(x_{k+1}) - r(x_k) = 0. \tag{4.7}$$

However, the point x_{k+1} is not known in advance. So, we consider to use $\epsilon_k \triangleq r(x_k) - r(x_{k-1})$ instead of the term $r(x_{k+1})-r(x_k)$ in the last equation. This results in the following iteration:

$$Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \delta_k = 0, \quad k = 0,1,2,\ldots, \tag{4.8}$$

where

$$\delta_k = \frac{1}{m-1}F(x_k) + \epsilon_k. \tag{4.9}$$

More generally, we propose the following iterative scheme:

$$Mx_{k+1}^{[m-1]} - Mx_k^{[m-1]} + \alpha_k F(x_k) + \epsilon_k = 0, \quad k = 0,1,2,\ldots, \tag{4.10}$$

where $\alpha_k \in (0,1]$ and ϵ_k is chosen in the way that $\epsilon_0 = 0$ and for $k \geq 1$, $\epsilon_k = r(x_k) - r(x_{k-1})$. However, such a simple choice rule for ϵ_k could not guarantee the generated sequence $\{x_k\}$ contained in $\mathcal{S}$. As a result, the convergence of the related method is doubtful. In what follows, we give some other reasonable choice for ϵ_k to ensure the monotone convergence of $\{x_k\}$ to a nonnegative solution of (2.1).

In order for the method to be monotonically convergent, we need the requirement $\alpha_k F(x_k) + \epsilon_k \leq 0$, for all $k \geq 0$ to ensure the monotone property of $\{x_k\}$. It is satisfied if we let ϵ_k satisfy

$$\epsilon_k \leq -\alpha_k F(x_k) \triangleq \epsilon_k^+. \tag{4.11}$$

On the other hand, we also need the condition $F(x_k) \leq 0$, for all $k \geq 0$ to ensure $\{x_k\} \subset \mathcal{S}$. This motives us to determine ϵ_k in the following way. At iteration k, we first let $\epsilon_k = \min\{\epsilon_k^+, r(x_k) - r(x_{k-1})\}$ and then solve the system of linear equations (4.10) to get an $\bar{x}_{k+1}$. If $F(\bar{x}_{k+1}) \leq 0$, then we let $x_{k+1} = \bar{x}_{k+1}$. Otherwise, we let $\epsilon_k = 0$ and solve (4.10) to get x_{k+1}. Here the vector operation $\min\{x, y\}$ is also defined by elements.

Based on the above arguments, we propose an approximate Newton method for solving the M-tensor equation (2.1) as follows.

Algorithm 4.1. (Approximate Newton Method)

Initial. Given positive sequence $\{\alpha_k\} \subset (0,1]$. Given constant $\eta > 0$ and initial point $x_0 \in \mathcal{S}$. Let $\epsilon_0 = 0$ and $k = 0$.

Step 1. Stop if $\|F(x_k)\| \le \eta$.

Step 2. Solve the system of linear equation (4.10) to get $\bar{x}_{k+1}$.

Step 3. If $F(\bar{x}_{k+1}) \le 0$, then go to Step 4. Otherwise, let $\epsilon_k = 0$. Go to Step 2.

Step 4. Let $x_{k+1} = \bar{x}_{k+1}$ and $\epsilon_{k+1} = \min\{\epsilon^+_{k+1}, r(x_{k+1}) - r(x_k)\}$. Let $k = k+1$. Go to Step 1.

Remark 1. Notice that if $\epsilon_k = 0$, then the method reduces to the S-MEQM. Consequently, if for some k, $F(\bar{x}_{k+1}) > 0$, then the iterate x_{k+1} is generated by the S-MEQM, which guarantees $F(x_{k+1}) \le 0$. In other words, at each iteration, the circle between Steps 2 and 3 is no more than once.

Remark 2. To implement the algorithm, the initial point must be chosen in the set S. In general, finding a point in S is not an easy task. In the special case $b \ge 0$ or the system of linear equation $Mx - b = 0$ (M is the majorization matrix of $\mathcal{M}$) has a nonnegative solution, we can simply let x_0 be the solution to the system of linear equations $Mx - b = 0$.

It is easy to show by induction that the sequence $\{x_k\}$ generated by Algorithm 4.1 satisfies $x_{k+1} \ge x_k$ and $F(x_k) \le 0$, for all k. Similar to the proof of Theorem 3.2, we have the following result.

Theorem 4.2. *Suppose that $\mathcal{M}$ is a strong M-tensor. Then the sequence $\{x_k\}$ be generated by Algorithm 4.1 satisfies $\{x_k\} \subset \mathcal{S}$ and*

$$x_{k+1} \ge x_k, \quad \forall k \ge 0. \tag{4.12}$$

Moreover, it converges to a nonnegative solution to the tensor equation (2.1).

5 Numerical Results

In this section, we do numerical experiments to test the effectiveness of the proposed methods. We implemented our methods in Matlab R2015b and ran the codes on a computer with 3.60 GHz CPU and 20.0 GB RAM. We used a tensor toolbox [1] to proceed some tensor computation.

While do numerical experiments, similar to [10, 11], we solved the tensor equation

$$\hat{F}(x) = \hat{\mathcal{M}} x^{m-1} - \hat{b} = 0 \tag{5.1}$$

instead of the tensor equation (2.1), where $\hat{\mathcal{M}} := \mathcal{M}/\omega$ and $\hat{b} := b/\omega$ with ω is the largest value among the absolute values of components of $\mathcal{M}$ and b. The stopping criterion is set to

$$\|\hat{F}(x_k)\| \le 10^{-8}. \tag{5.2}$$

or the number of iteration reaches to 2000. The latter case means that the method is failure for the problem. In all cases, we take the parameter $\alpha_k = \alpha$ to be independent of k.

The test problems are from [6, 13, 24].

Problem 1. We solve tensor equation (2.1) where $\mathcal{M}$ is a symmetric strong M-tensor of order m ($m = 3, 4, 5$) in the form $\mathcal{M} = s\mathcal{I} - \mathcal{B}$, where tensor $\mathcal{B}$ is symmetric whose entries are uniformly distributed in $(0, 1)$, and

$$s = (1 + 0.01) \cdot \max_{i=1,2,\ldots,n} (\mathcal{B}\mathbf{e}^{m-1})_i, \tag{5.3}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^T$.

Problem 2. We solve tensor equation (2.1) where $\mathcal{M}$ is a symmetric strong M-tensor of order m ($m = 3, 4, 5$) in the form $\mathcal{M} = s\mathcal{I} - \mathcal{B}$, and tensor $\mathcal{B}$ is a nonnegative tensor with

$$b_{i_1 i_2 \ldots i_m} = |\sin(i_1 + i_2 + \ldots + i_m)|, \tag{5.4}$$

and $s = n^{m-1}$.

Problem 3. We solve tensor equation (2.1) where $\mathcal{M}$ is a non-symmetric strong M-tensor of order m ($m = 3, 4, 5$) in the form $\mathcal{M} = s\mathcal{I} - \mathcal{B}$, and tensor $\mathcal{B}$ is nonnegative tensor whose entries are uniformly distributed in $(0, 1)$. The parameter s is set to

$$s = (1 + 0.01) \cdot \max_{i=1,2,\ldots,n} (\mathcal{B}\mathbf{e}^{m-1})_i, \tag{5.5}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^T$.

We first tested the performance of the sequential M-matrix linear equation method (3.5) (abbreviated as S-MEQM) and the approximate Newton method Algorithm 4.1 (abbreviated as A-Newton) with different values of α on Problem 1 with b uniformly distributed in $(0, 1)$. We set the start point $x_0 = b^{[1/(m-1)]}$.

For each α, we tested the methods on 100 problems with different sizes of tensor. The results are listed in Tables 1 and 2, where the columns 'Iter', 'Time' and 'Res' stand for the total number of iterations, the computational time (in second) used for the method and residual $\|\hat{\mathcal{A}}x_k^{(m-1)} - \hat{b}\|$. The results of the method S-MEQM with $\alpha \in (0, 0.5)$ were not listed in the table because those results are not as good as the method with $\alpha \in (0.5, 1]$. We also tested the S-MEQM with the value $\alpha > 1$. Although we could not establish the convergence of the method S-MEQM with $\alpha > 1$, the numerical results in Table 1 seem to show that the method still works for all $\alpha \in (1, 2)$. The best parameter α for the method S-MEQM seems close to 1.9 while the best α for the A-Newton seems to 1.

The results in Tables 1 and 2 show that the A-Newton method performed much better then the method S-MEQM if we choose the parameter α appropriately.

We then test the methods S-MEQM and A-Newton on Problems 1-3 where the constant term b has negative elements. Notice that for an arbitrary $\bar{x} \in R^n$ whose elements are in the interval $(0, 1)$, the vector $\mathcal{M}\bar{x}^{(m-1)}$ has negative elements and positive elements in most cases. As a result, we first choose an x_0 whose elements are uniformly distributed in $(0, 1)$. Let $e = \mathcal{M}x_0^{m-1}$ and

$$b_i = \begin{cases} 1.5e_i, & \text{if } e_i > 0, \\ 0.5e_i, & \text{if } e_i \le 0. \end{cases} \tag{5.6}$$

It is easy to see that b has negative elements and positive elements. Moreover, x_0 is feasible, i.e., $x_0 \in \mathcal{S} = \{x \in R_+^n \,|F(x) \le 0\}$.

Tables 3,4,5 list the performance of the methods S-MEQM and A-Newton with $\alpha_k \equiv 1$ on Problems 1-3 with arbitrary constant term b. They show that in all cases both methods can find a nonnegative solution to the problem. The method A-Newton performed better than the method S-MEQM did.

Table 1: Results for S-MEQM with different α_k on Problem 1.

(m,n)	(3,10)		(3,500)		(4,10)		(4,150)		(5,10)	
α	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.5	464.6	0.0056	1002.5	89.3071	980.0	0.0509	654.0	192.8567	1353.3	0.0513
0.6	372.3	0.0045	836.8	74.7805	818.1	0.0432	542.1	159.5753	1109.3	0.0416
0.7	324.7	0.0039	712.5	63.5823	701.2	0.0376	471.2	139.4450	975.7	0.0359
0.8	291.8	0.0035	626.2	55.9513	602.0	0.0320	408.2	120.7247	821.2	0.0299
0.9	256.7	0.0031	560.5	50.1437	554.5	0.0301	361.3	107.1271	743.6	0.0272
1.0	235.9	0.0028	497.6	44.1223	494.5	0.0261	326.0	97.1213	665.7	0.0245
1.1	199.5	0.0024	451.7	40.3780	457.8	0.0244	298.4	88.9745	607.3	0.0225
1.2	184.4	0.0022	417.7	37.4080	414.9	0.0221	269.6	80.1724	549.1	0.0206
1.3	158.2	0.0019	385.5	34.5943	379.2	0.0202	252.5	74.9961	508.0	0.0190
1.4	159.0	0.0019	351.4	31.3947	338.5	0.0180	232.4	68.9249	475.0	0.0179
1.5	152.1	0.0018	332.5	29.8046	338.6	0.0182	219.0	64.3919	445.6	0.0164
1.6	141.0	0.0017	310.9	27.7868	307.2	0.0167	201.0	59.4630	414.2	0.0149
1.7	132.6	0.0016	293.1	26.2350	289.7	0.0157	187.0	55.4343	379.2	0.0140
1.8	123.5	0.0015	276.8	24.8428	273.8	0.0149	187.0	56.4978	371.4	0.0138
1.9	201.1	0.0024	260.6	23.3867	268.1	0.0145	169.0	50.7594	350.2	0.0130
2.0	-	-	272.6	24.4338	-	-	162.0	49.1474	-	-

Table 2: Results for A-Newton with different α_k on Problem 1.

(m,n)	(3,10)		(3,500)		(4,10)		(4,150)		(5,10)	
α	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.1	81.3	0.0042	77.4	7.2864	102.6	0.0082	49.9	15.3825	104.4	0.0077
0.2	64.9	0.0034	56.9	5.4034	75.8	0.0055	35.0	10.7034	75.2	0.0067
0.3	52.4	0.0028	46.7	4.4692	63.7	0.0047	28.1	8.6363	66.0	0.0060
0.4	47.5	0.0025	41.1	3.9717	57.3	0.0042	24.9	7.6523	56.7	0.0054
0.5	44.3	0.0024	39.4	3.8176	53.6	0.0040	22.0	6.7758	51.5	0.0050
0.6	42.2	0.0023	35.9	3.4991	49.8	0.0037	20.0	6.1671	50.0	0.0050
0.7	40.2	0.0022	31.4	3.0693	46.4	0.0035	21.7	6.8789	46.3	0.0046
0.8	37.8	0.0021	28.3	2.7784	43.6	0.0033	17.0	5.2891	43.0	0.0043
0.9	36.8	0.0021	26.5	2.6148	42.6	0.0033	16.0	4.9991	43.0	0.0043
1	34.7	0.0020	27.8	2.7497	41.9	0.0032	16.6	5.2963	42.2	0.0042

Table 3: Comparison between S-MEQM and A-Newton on Problem 1 with arbitrary b.

	S-MEQM			A-Newton		
(m,n)	Iter	Time	Res	Iter	Time	Res
(3,10)	282.0	0.0109	9.71E-09	46.6	0.0027	6.49E-09
(3,100)	928.6	0.3770	9.91E-09	67.0	0.0321	6.39E-09
(3,200)	1161.0	6.5738	9.93E-09	72.2	0.4421	6.22E-09
(3,300)	1284.2	23.6522	9.93E-09	79.7	1.5731	7.58E-09
(3,400)	1362.3	62.6525	9.94E-09	77.3	3.7846	5.83E-09
(3,500)	1409.0	124.1550	9.94E-09	67.5	6.4069	6.57E-09
(4,10)	738.9	0.0394	9.88E-09	64.9	0.0047	6.63E-09
(4,50)	1440.6	5.6390	9.94E-09	77.9	0.3314	6.43E-09
(4,100)	1584.6	92.3761	9.95E-09	73.3	4.6216	5.00E-09
(4,150)	1632.6	475.5009	9.94E-09	73.5	22.9804	8.01E-09
(5,10)	1241.7	0.0788	9.94E-09	78.7	0.0065	7.35E-09
(5,30)	1608.6	22.3078	9.95E-09	70.4	1.0548	5.27E-09
(5,50)	1661.5	290.7334	9.94E-09	78.1	14.7327	7.40E-09

Table 4: Comparison between S-MEQM and A-Newton on Problem 2 with arbitrary b.

	S-MEQM			A-Newton		
(m,n)	Iter	Time	Res	Iter	Time	Res
(3,10)	39.2	0.0018	8.11E-09	19.7	0.0017	6.37E-09
(3,100)	42.6	0.0180	8.19E-09	22.0	0.0149	5.78E-09
(3,200)	42.8	0.2380	8.12E-09	22.0	0.1987	5.14E-09
(3,300)	42.9	0.8210	7.85E-09	22.0	0.6444	4.97E-09
(3,400)	42.9	2.0550	7.76E-09	22.0	1.5968	5.12E-09
(3,500)	42.9	3.8912	7.87E-09	22.0	3.0263	5.20E-09
(4,10)	44.7	0.0025	8.34E-09	23.1	0.0023	6.34E-09
(4,50)	46.3	0.1834	8.28E-09	23.8	0.1441	6.43E-09
(4,100)	46.2	2.7358	8.57E-09	23.9	2.1451	5.93E-09
(4,150)	46.2	13.7176	8.46E-09	23.9	10.7461	5.71E-09
(5,10)	48.1	0.0031	8.44E-09	24.6	0.0028	6.79E-09
(5,30)	48.3	0.6749	8.52E-09	23.8	0.4994	7.13E-09
(5,50)	48.4	8.5743	8.34E-09	23.7	6.3121	6.68E-09

Table 5: Comparison between S-MEQM and A-Newton on Problem 3 with arbitrary b.

	S-MEQM			A-Newton		
(m,n)	Iter	Time	Res	Iter	Time	Res
(3,10)	182.6	0.0072	9.57E-09	39.8	0.0024	6.69E-09
(3,100)	702.0	0.2893	9.89E-09	60.5	0.0300	6.55E-09
(3,200)	947.1	5.5298	9.91E-09	70.6	0.4396	6.50E-09
(3,300)	1095.1	20.0662	9.93E-09	73.4	1.4450	7.07E-09
(3,400)	1196.3	55.2341	9.93E-09	71.0	3.5047	5.33E-09
(3,500)	1252.3	107.3120	9.94E-09	72.2	6.7266	7.12E-09
(4,10)	465.9	0.0255	9.80E-09	58.6	0.0043	6.45E-09
(4,50)	1252.8	4.8686	9.93E-09	76.9	0.3261	6.50E-09
(4,100)	1486.4	86.5543	9.95E-09	84.5	5.3253	7.62E-09
(4,150)	1567.3	454.2121	9.94E-09	84.2	26.3310	4.67E-09
(5,10)	932.7	0.0622	9.91E-09	73.5	0.0063	6.84E-09
(5,30)	1502.3	20.6186	9.95E-09	84.0	1.2499	7.42E-09
(5,50)	1607.5	281.4558	9.95E-09	71.1	13.4823	5.43E-09

Table 6: Comparison on Problem 1.

	S-MEQM		A-Newton		HM		QCA		NGS		CTNN	
(m,n)	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
(3,10)	215.7	0.0087	34.1	0.0014	5.0	0.0412	11.7	0.0549	57.9	0.1149	14.7	0.0319
(3,100)	505.4	0.2103	38.1	0.0157	5.0	0.0645	15.9	0.1333	104.6	0.3378	15.3	0.0783
(3,200)	538.9	2.8621	31.6	0.1799	5.1	0.2385	16.8	0.4969	134.0	1.6543	16.3	0.3079
(3,300)	529.9	9.9090	34.8	0.6837	5.0	0.6139	17.1	1.3200	135.5	4.4780	16.6	0.8152
(3,400)	518.2	23.843	30.7	1.5187	5.0	1.3513	17.9	3.1165	142.9	10.892	15.3	1.5976
(3,500)	498.1	43.162	26.7	2.4923	5.0	2.6725	18.4	6.3641	147.7	22.798	16.0	3.1778
(4,10)	504.3	0.0308	39.9	0.0037	5.1	0.0541	12.7	0.0765	620.6	1.8773	405.2	1.0868
(4,50)	569.3	2.3466	32.2	0.1428	5.0	0.1977	14.3	0.3580	-	-	712.2	8.4887
(4,100)	430.5	24.854	21.3	1.3252	5.0	3.0205	15.3	5.5518	-	-	784.5	143.671
(4,150)	324.3	96.654	16.0	5.0916	5.0	15.338	16.0	29.923	-	-	1193.3	1075.93
(5,10)	677.5	0.0426	41.9	0.0035	5.0	0.0508	12.4	0.0759	163.7	0.4234	32.1	0.0905
(5,30)	454.5	6.1231	24.7	0.3580	5.0	0.8796	12.9	1.3142	386.9	20.3873	23.6	1.1602
(5,50)	271.8	46.4756	15.0	2.7196	5.0	10.5094	14.2	16.5825	-	-	21.5	12.2233

Table 7: Comparison on Problem 2.

	S-MEQM		A-Newton		HM		QCA		NGS		CTNN	
(m,n)	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
(3,10)	27.0	0.0011	12.0	0.0010	5.0	0.0342	8.2	0.0392	6.6	0.0140	10.7	0.0232
(3,100)	19.0	0.0081	8.3	0.0056	5.0	0.0488	11.5	0.0987	7.7	0.0269	11.8	0.0608
(3,200)	15.9	0.0857	7.9	0.0639	5.0	0.1491	12.2	0.3558	7.8	0.1000	11.4	0.2143
(3,300)	14.0	0.2686	6.0	0.1657	5.0	0.4030	12.4	0.9697	7.6	0.2686	10.9	0.5218
(3,400)	12.7	0.6224	6.0	0.4060	5.0	0.9350	13.1	2.3042	7.9	0.6405	11.2	1.1724
(3,500)	11.7	1.0947	6.0	0.7801	5.0	1.8970	13.4	4.6906	7.9	1.3052	10.8	2.1978
(4,10)	23.3	0.0051	10.2	0.0017	5.0	0.0742	7.7	0.0871	7.1	0.0215	28.1	0.1375
(4,50)	13.1	0.0529	6.0	0.0339	5.0	0.1235	9.4	0.2126	9.1	0.1392	126.4	1.4118
(4,100)	8.4	0.5346	4.0	0.3405	5.0	2.1085	9.6	3.5381	9.9	1.9866	280.0	51.571
(4,150)	6.0	2.0199	3.0	1.4520	5.0	10.7969	10.0	18.7512	9.7	9.3607	403.1	369.39
(5,10)	19.0	0.0013	8.9	0.0010	5.0	0.0293	7.5	0.0416	90.5	0.2250	28.2	0.0708
(5,30)	9.0	0.1345	4	0.0805	5.0	0.5789	7.3	0.7566	76.4	3.1008	19.4	1.0028
(5,50)	4.0	0.8641	2	0.5109	5.0	6.2314	6.5	7.9730	169.3	96.8653	18.9	11.1137

Following an anonymous referee's suggestion, we compared S-MEQM and A-Newton method with some existing algorithms that can find a nonnegative solution to the M-tensor equation. The methods to be compared together with their abbreviations are listed in the following.

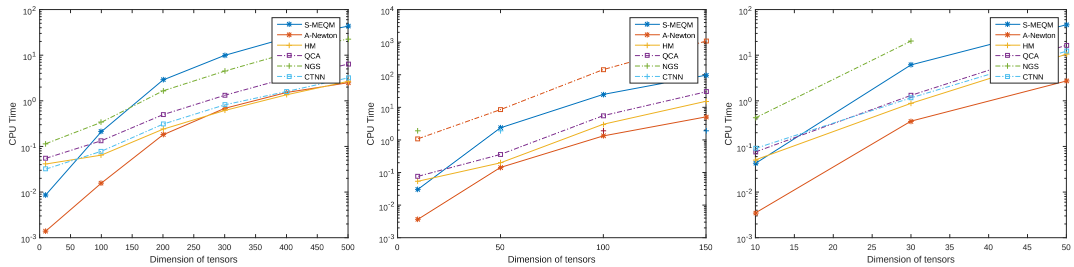
- HM: the Homotopy method [10].
- QCA: the Quadratically Convergent Algorithm [11].
- NGS: the Newton-Gauss-Seidel method [13].
- CTNN: the Continuous time neural network method [22].

We take parameters $\delta = 0.5, \gamma = 0.8, \sigma = 0.2, \bar{t} = 2/(5\gamma)$ in QCA and $\beta = 0.45$ in CTNN. The sub-problem in QCA is solved by 'pcg' method. The code of method HM by Han [10] was downloaded from Han's homopage. The codes of the methods QCA and CTNN were provided by the authors of the methods. We compared our method with the above methods on problems 1 and 2 with positive constant term b. The initial point is to $x_0 = b^{[1/(m-1)]}$. The results are listed in Tables 6 and 7.

To facilitate understanding the performance of those methods, we draw two figures to show the comparison among the methods. They are Figures 1 and 2. We can easily see from Tables 6 and 7 and Figures 1 and 2 that the method A-Newton performed best among all the methods.

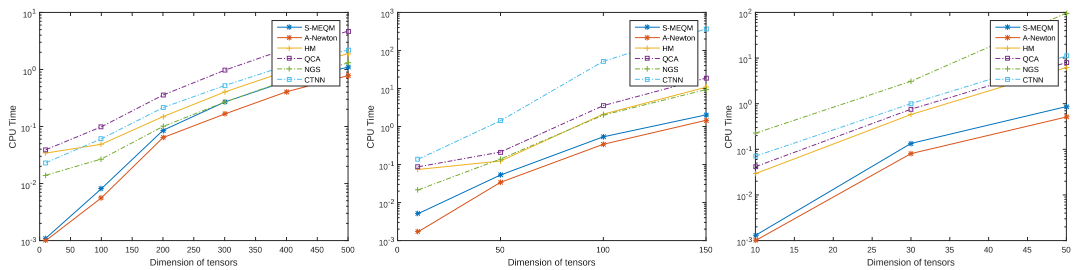
6 Conclusion

We developed a sequential M-matrix equation based method and its improvement for solving M-tensor equations. The methods can be regarded as approximate Newton methods. An



(a) (m=3) (b) (m=4) (c) (m=5)

Figure 1: Comparison among different methods on Problem 1.



(a) (m=3) (b) (m=4) (c) (m=5)

Figure 2: Comparison among different methods on Problem 2.

advantage of the method is that the subproblems of the methods are systems of linear equations with the same M-matrix as coefficient matrix. If the initial point is appropriate chosen, the methods possess monotone convergence property even if the constant term b has negative elements. Our numerical results show that when $\alpha \in (1, 2)$ the performance of the sequence M-matrix method can be better than the method with $\alpha \in (0, 1]$. However, at the moment, we could not establish the related convergence theory. We leave it as a further research topic.

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Dong-Hui Li
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, China
E-mail address: lidonghui@m.scnu.edu.cn

Hong-Bo Guan
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, China;
School of Mathematical Sciences and Energy Engineering
Hunan Institute of Technology
HengYang,421002, China
E-mail address: hongbo_guan@m.scnu.edu.cn

Xiao-Zhou Wang
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, China
E-mail address: 2014020896@m.scnu.edu.cn