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OPTIMALITY CONDITIONS AND DUALITY OF THE SET-VALUED FRACTIONAL PROGRAMMING PROBLEM

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, firstly, a property of generalized cone convex set-valued map is obtained via the generalized second-order contingent epiderivative. Secondly, as applications of this property, we establish a sufficient optimality condition of set-valued fractional programming problem in the sense of ϵ-weak minimizer. Finally, a new type dual, called second-order parametric type dual involving the generalized second-order contingent epiderivative, is introduced. We also derive the weak, strong and converse duality results.

Key words: *set-valued fractional programming, generalized second-order contingent epiderivative, optimality condition, duality*

Mathematics Subject Classification: *90C26, 90C29, 90C30*

1 Introduction

It is well-known that the concept of convexity and its various generalizations play a significant role in the establishment of optimality conditions and duality theory in vector optimization (see [1–9] and the references therein). Recently, Aubin [10] has introduced the notion of the contingent derivative of set-valued maps. Since then, many scholars have used various generalized derivatives of set-valued maps to study the optimality conditions and duality of set-valued optimization problems under the assumptions of convexity and generalized convexity. In [11], Corley used the contingent derivative of set-valued maps to establish optimality conditions for set-valued optimization problems. For other notions and applications of epiderivatives of set-valued maps, one can refer to [12–16]. The second-order optimality conditions of set-valued optimization problems have been extensively studied (see [17–21] and the references therein). The concept of second-order contingent set introduced by Aubin and Frankowska [17] plays an important role in the establishment of second-order optimality conditions. In [18], Durea used different kinds of derivatives of set-valued maps to establish some second-order optimality conditions. Jahn et al. [19] proposed a generalized second-order contingent epiderivative of set-valued maps and gave an existence theorem for the

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generalized second-order epiderivative. Zhu et al. [20] established some second-order optimality conditions for set-valued optimization via the second-order composed contingent derivative of set-valued maps. Peng and Xu [21] applied second-order tangent epiderivative of set-valued maps to obtain the second-order Fritz John and Kuhn-Tucker necessary optimality conditions under the assumption of near cone-subconvexlikeness.

On the other hand, many authors have paid attentions to set-valued fractional programming problems. For instance, Bhatia and Mehra [22] defined the concept of cone preinvexity of set-valued maps and derived the Lagrangian type dualiy of set-valued fractional programming problem. In [23], the duality results associated with Geoffrion efficient solution of set-valued fractional programming problem were proved. Gadhi and Jawhar [24] established the necessary optimality conditions of set-valued fractional programming problem by using the extremal principle (an approach initiated by Mordukhovich, which does not involve any convex approximations and convex separation arguments). Das and Nahak [25] introduced the notion of the ρ-cone convexity of set-valued maps and applied it to establish the sufficient Karush-Kuhn-Tuker optimality conditions of set-valued fractional programming problem. Furthermore, by applying the contingent epiderivative of set-valued maps, the authors in [25] introduced various types of dual models associated with the set-valued fractional programming problem, such as parametric type dual, Mond-Weir type dual, Wolfe type dual and mixed type dual. Motivated by the work reported in [25], we can use the generalized second-order contingent epiderivative introduced in [19] to propose a second-order parametric type dual associated with the set-valued fractional programming problem.

The aim of this paper is to study the optimality and duality of set-valued fractional programming problem. Firstly, we obtain a property of the generalized cone convex set-valued map based on the generalized second-order contingent epiderivative and introduce the notion of ϵ-weak minimal solution for a set-valued fractional programming problem. Then, by virtue of this property and the generalized second-order contingent epiderivative, we derive a sufficient optimality condition in the sense of ϵ-weak minimizer of set-valued fractional programming problem, which is different from the corresponding result in [25]. Finally, we introduce a second-order parametric type dual associated with the set-valued fractional programming problem. The weak duality, strong duality and converse duality are investigated under some suitable conditions.

This paper is organized as follows. In Section 2, we recall some basic concepts and give a property of the generalized cone convex set-valued map. In Section 3, as applications of this property, we establish the sufficient optimality condition of set-valued fractional programming problem in the sense of ϵ-weak minimizer. In Section 4, we introduce a new kind of second-order parametric type dual and study the weak duality, strong duality and converse duality.

2 Preliminaries

In this paper, let X and Y be real normed spaces and Ω be a nonempty subset of Y. The generated cone of Ω is defined $\text{cone}(\Omega) := \{\lambda a : a \in \Omega, \lambda \geq 0\}$. We denote by $\text{int}(\Omega)$ and $\text{cl}(\Omega)$ the interior and the closure of Ω, respectively. Let $C \subseteq Y$ be nontrivial pointed closed convex cone with $\text{int}(C) \neq \emptyset$. Let Y be partially ordered by C. The topological dual space of Y is denoted by Y^*. Let 0 denote the zero element for every space. The topological dual cone C^+ of C is defined as

$$C^+ := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\}.$$

Let $F : X \rightrightarrows Y$ be a set-valued map. The domain, graph and epigraph of F are, respectively, defined as:

$$\mathrm{dom}(F) := \{x \in X : F(x) \neq \emptyset\},$$
$$\mathrm{gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

and

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y : y \in F(x) + C\}.$$

The nonnegative orthant $\mathbb{R}^m_+$ of m-dimensional Euclidean space $\mathbb{R}^m$ is defined by

$$\mathbb{R}^m_+ = \{y = (y_1, \ldots, y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, \ldots, m\}.$$

Now, we recall the concept of the second-order contingent set.

Definition 2.1 ([17])**.** Let M be a nonempty subset of X, $w \in X$ and $\bar{x} \in \mathrm{cl}(M)$. The second-order contingent set of M at $\bar{x}$ in the direction w is

$$T^2(M, \bar{x}, w) := \left\{v \in X : \exists t_n \downarrow 0, \exists v_n \to v, \text{such that } \bar{x} + t_n w + \frac{1}{2}t_n^2 v_n \in M, \forall n \in \mathbb{N}\right\}.$$

Definition 2.2 ([26])**.** Let S be a nonempty subset of Y. A points $\bar{y} \in S$ is called a minimal element of S iff $(S - \bar{y}) \cap (-C) = \{0\}$. The set of all minimal elements of S with respect to C is denoted by $\mathrm{Min}\, S$.

Definition 2.3 ([26])**.** (a) The cone C is called Daniell iff any decreasing sequence in Y having a lower bound converges to its infimum.
(b) A subset A of Y is said to be minorized iff there is a $y \in Y$ so that $A \subseteq \{y\} + C$.
(c) The domination property holds for a subset A of Y iff $A \subseteq \mathrm{Min}\, A + C$.

Definition 2.4 ([19])**.** Let $F : X \rightrightarrows Y$ be a set-valued map, $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ and $(\bar{u}, \bar{v}) \in X \times Y$. A set-valued map $D^2_g F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$, defined by

$$D^2_g F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) := \mathrm{Min}\{y \in Y : (x, y) \in T^2(\mathrm{epi}(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v}))\}, \quad \forall x \in X,$$

is called generalized second-order contingent epiderivative of F at $(\bar{x}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$.

Definition 2.5 ([16])**.** Let Γ be a nonempty convex subset in X. A set-valued map $F : \Gamma \rightrightarrows Y$ is called generalized C-convex on Γ iff $\exists\, \theta \in \mathrm{int}(C)$ such that, $\forall x, y \in \Gamma$, $\forall \lambda \in [0, 1]$, $\forall \varepsilon > 0$,

$$\varepsilon\theta + (1 - \lambda)F(x) + \lambda F(y) \subseteq F((1 - \lambda)x + \lambda y) + C.$$

Remark 2.6. Note that the generalized C-convexity of the set-valued map is a proper generalization of the C-convexity of the set-valued map (see Example 2.1 in [16]).

In the following proposition, we will use the generalized second-order contingent epiderivative of set-valued map to present a property of the generalized C-convex set-valued map.

Proposition 2.7. *Let Γ be a nonempty convex subset of X. Let C be a closed convex pointed cone being Daniell and the set-valued map $F : \Gamma \rightrightarrows Y$ be generalized C-convex on Γ. Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$, $(\bar{u}, \bar{v}) \in \mathrm{epi}(F)$. For any $x \in \Gamma$, write $\Phi(x - \bar{x}) := \{y : (x - \bar{x}, y) \in T^2(\mathrm{epi}(F), (\bar{x}, \bar{y}), (\bar{u} - \bar{x}, \bar{v} - \bar{y}))\}$. If $\Phi(x - \bar{x})$ is minorized and fulfills the domination property for any $x \in \Gamma$, then*

$$F(x) - \{\bar{y}\} \subseteq D^2_g F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) + C, \quad \forall x \in \Gamma.$$

Proof. Take any $x \in \Gamma$ and $y \in F(x)$. Let $\{\lambda_n\}$ be a sequence in $\mathbb{R}$ such that $\lambda_n \in (0,1)$ with $\lim_{n\to\infty} \lambda_n = 0$. Since F is generalized C-convex on Γ, there exists $\theta \in \operatorname{int}(C)$, for $x, \bar{x} \in \Gamma$,

$$\begin{aligned}\bar{y} + \frac{1}{4}\lambda_n^2(y - \bar{y} + \lambda_n\theta) &= \frac{1}{4}\lambda_n^3\theta + \left(1 - \frac{1}{4}\lambda_n^2\right)\bar{y} + \frac{1}{4}\lambda_n^2 y \\ &\in \frac{1}{4}\lambda_n^3\theta + \left(1 - \frac{1}{4}\lambda_n^2\right)F(\bar{x}) + \frac{1}{4}\lambda_n^2 F(x) \\ &\subseteq F\left(\bar{x} + \frac{1}{4}\lambda_n^2(x - \bar{x})\right) + C \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}\bar{y} + \lambda_n\left(\bar{v} - \bar{y} + \frac{1}{4}\lambda_n^2\theta\right) &= \frac{1}{4}\lambda_n^3\theta + (1-\lambda_n)\bar{y} + \lambda_n\bar{v} \\ &\in \frac{1}{4}\lambda_n^3\theta + (1-\lambda_n)F(\bar{x}) + \lambda_n F(\bar{u}) + C \\ &\subseteq F(\bar{x} + \lambda_n(\bar{u} - \bar{x})) + C. \qquad (2.2)\end{aligned}$$

Since F is generalized C-convex on Γ, it follows from (2.1) and (2.2) that

$$\begin{aligned}\bar{y} + \frac{1}{2}\lambda_n(\bar{v} - \bar{y}) + \frac{1}{8}\lambda_n^2(y - \bar{y} + 3\lambda_n\theta) &= \frac{1}{8}\lambda_n^3\theta + \frac{1}{2}\left(\bar{y} + \lambda_n(\bar{v} - \bar{y} + \frac{1}{4}\lambda_n^2\theta)\right) \\ &\quad + \frac{1}{2}\left(\bar{y} + \frac{1}{4}\lambda_n^2(y - \bar{y} + \lambda_n\theta)\right) \\ &\in \frac{1}{8}\lambda_n^3\theta + \frac{1}{2}F(\bar{x} + \lambda_n(\bar{u} - \bar{x})) \\ &\quad + \frac{1}{2}F(\bar{x} + \frac{1}{4}\lambda_n^2(x - \bar{x})) + C \\ &\subseteq F\left(\bar{x} + \frac{1}{2}\lambda_n(\bar{u} - \bar{x}) + \frac{1}{8}\lambda_n^2(x - \bar{x})\right) + C. \qquad (2.3)\end{aligned}$$

Now, we define two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_n := \bar{x} + \frac{1}{2}\lambda_n(\bar{u} - \bar{x}) + \frac{1}{8}\lambda_n^2(x - \bar{x}), \quad y_n := \bar{y} + \frac{1}{2}\lambda_n(\bar{v} - \bar{y}) + \frac{1}{8}\lambda_n^2(y - \bar{y} + 3\lambda_n\theta). \quad (2.4)$$

Notice that $x_n \in \Gamma$. In fact,

$$x_n = \frac{1}{2}\left(\frac{1}{4}\lambda_n^2 x + \left(1 - \frac{1}{4}\lambda_n^2\right)\bar{x}\right) + \frac{1}{2}(\lambda_n\bar{u} + (1-\lambda_n)\bar{x}) \in \Gamma. \qquad (2.5)$$

According to (2.3)-(2.5), $(x_n, y_n) \in \operatorname{epi}(F)$. On the other hand, we obtain

$$(\bar{x}, \bar{y}) + \frac{1}{2}\lambda_n(\bar{u} - \bar{x}, \bar{v} - \bar{y}) + \frac{1}{2}\left(\frac{1}{2}\lambda_n\right)^2(x - \bar{x}, y - \bar{y} + 3\lambda_n\theta) = (x_n, y_n) \in \operatorname{epi}(F) \qquad (2.6)$$

and

$$\lim_{n\to\infty}(x - \bar{x}, y - \bar{y} + 3\lambda_n\theta) = (x - \bar{x}, y - \bar{y}). \qquad (2.7)$$

By (2.6), (2.7) and Definition 2.1, we have

$$(x - \bar{x}, y - \bar{y}) \in T^2(\operatorname{epi}(F), (\bar{x}, \bar{y}), (\bar{u} - \bar{x}, \bar{v} - \bar{y})),$$

i.e.,

$$y - \bar{y} \in \Phi(x - \bar{x}) = \{y \in Y : (x - \bar{x}, y) \in T^2(\mathrm{epi}(F), (\bar{x}, \bar{y}), (\bar{u} - \bar{x}, \bar{v} - \bar{y}))\}.$$

Since $\Phi(x - \bar{x})$ satisfies the domination property, we have

$$\Phi(x - \bar{x}) \subseteq D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) + C.$$

Therefore,

$$F(x) - \{\bar{y}\} \subseteq D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) + C.$$

□

Remark 2.8. In Proposition 2.7, the conditions that C is Daniell and $\Phi(x - \bar{x})$ is minorized guarantee the existence of $D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x})$.

Now, the following example is used to illustrate Proposition 2.7.

Example 2.9. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $\Gamma = \{x \in \mathbb{R} : x \in [0, 2]\}$. The set-valued map $F : \Gamma \rightrightarrows Y$ is defined as follows:

$$F(x) = \begin{cases} (0, \frac{1}{2}], & \text{if} \quad x = 0 \\ (0, x), & \text{if} \quad x \in (0, 1] \backslash \{\frac{1}{2}\} \\ [0, \frac{1}{2}), & \text{if} \quad x = \frac{1}{2} \\ [0, x - 1), & \text{if} \quad x \in (1, 2] \end{cases}.$$

It is easy to check that F is generalized C-convex on Γ. Take $\hat{x} = \frac{1}{2}$, $\hat{y} = 2$ and $\hat{\lambda} = \frac{1}{3}$. Obviously, $0 \in (1 - \hat{\lambda})F(\hat{x}) + \hat{\lambda}F(\hat{y})$. However, $0 \notin F((1 - \hat{\lambda})\hat{x} + \hat{\lambda}\hat{y}) + C$. Hence, the set-valued map F is not C-convex on Γ. On the other hand, if we take $(\bar{x}, \bar{y}) = (\frac{3}{2}, 0)$ and $(\bar{u}, \bar{v}) = (2, 0)$, then $T^2(\mathrm{epi}(F), (\bar{x}, \bar{y}), (\bar{u} - \bar{x}, \bar{v} - \bar{y})) = \mathbb{R} \times \mathbb{R}_+$. Moreover, for any $x \in \Gamma$, we have $\Phi(x - \bar{x}) = \mathbb{R}_+$. Clearly, $\Phi(x - \bar{x})$ is minorized and fulfills the domination property. It follows from Definition 2.4 that $D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) = 0$. Thus,

$$F(x) - \{\bar{y}\} \subseteq D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) + C.$$

3 Optimality Conditions

In this section, we will apply the generalized second-order contingent epiderivative of set-valued maps to characterize sufficient optimality condition for a set-valued fractional programming involving the generalized C-convexity.

Let $A, B \subseteq \mathbb{R}^m$, $D \subseteq \mathrm{int}(\mathbb{R}^m_+)$, $\lambda \in \mathbb{R}^m$. We define

$$A + B := \{a + b : a \in A, b \in B\}, \quad \lambda A := \{\lambda a = (\lambda_1 a_1, \ldots, \lambda_m a_m) : a \in A\}$$

and

$$\frac{A}{D} := \left\{\frac{a}{d} = (\frac{a_1}{d_1}, \ldots, \frac{a_m}{d_m}) : a = (a_1, \ldots, a_m) \in A, d = (d_1, \ldots, d_m) \in D\right\}.$$

Let $F = (F_1, \ldots, F_m) : X \rightrightarrows \underbrace{2^{\mathbb{R}} \times \cdots \times 2^{\mathbb{R}}}_{m}$, $G = (G_1, \ldots, G_m) : X \rightrightarrows \underbrace{2^{\mathbb{R}} \times \cdots \times 2^{\mathbb{R}}}_{m}$ and $H = (H_1, \ldots, H_p) : X \rightrightarrows \underbrace{2^{\mathbb{R}} \times \cdots \times 2^{\mathbb{R}}}_{p}$ be set-valued maps, where the set-valued maps $F_i : X \rightrightarrows 2^{\mathbb{R}}$, $G_i : X \rightrightarrows 2^{\mathbb{R}}$, $i = 1, \ldots, m$ and $H_j : X \rightrightarrows 2^{\mathbb{R}}$, $j = 1, \ldots, p$.

Let $x \in \Gamma \subseteq X$,

$$y = (y_1, \dots, y_m) \in F(x) \Leftrightarrow y_i \in F_i(x), \quad \forall i = 1, \dots, m,$$

$$z = (z_1, \dots, z_m) \in G(x) \Leftrightarrow z_i \in G_i(x), \quad \forall i = 1, \dots, m$$

and

$$w = (w_1, \dots, w_p) \in H(x) \Leftrightarrow w_j \in H_j(x), \quad \forall j = 1, \dots, p.$$

We assume that, for every $x \in \Gamma$ and $i = 1, \dots, m$, $F_i(x) \subseteq \mathbb{R}_+$ and $G_i(x) \subseteq \mathrm{int}(\mathbb{R}_+)$.

Now, we consider the following set-valued fractional programming problem:

$$\text{(FP)} \qquad \text{Minimize} \quad \frac{F(x)}{G(x)} = \left(\frac{F_1(x)}{G_1(x)}, \dots, \frac{F_m(x)}{G_m(x)} \right), \\ \text{subject to} \quad H(x) \cap (-\mathbb{R}^p_+) \neq \emptyset, x \in \Gamma.$$

Let $E := \{x \in \Gamma : H(x) \cap (-\mathbb{R}^p_+) \neq \emptyset\}$ be the set of feasible solution of (FP).

Definition 3.1. Let $\epsilon \in \mathbb{R}^m_+$. $\bar{x} \in E$ is called an ϵ-weak minimal solution of (FP) with respect to $\mathbb{R}^m_+$ (denoted by $\bar{x} \in \epsilon\text{-WMin}(\frac{F(E)}{G(E)}, \mathbb{R}^m_+)$) iff there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

$$\left(\frac{F(E)}{G(E)} - \frac{\bar{y}}{\bar{z}} + \epsilon \right) \cap (-\mathrm{int}(\mathbb{R}^m_+)) = \emptyset.$$

The point pair $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ is called ϵ-weak minimizer of (FP).

Remark 3.2. When $\epsilon = 0$, Definition 3.1 reduces to Definition 2.6 in [25].

It is well-known that the problem (FP) is equivalent to the following non-fractional parametric problem:

$$\text{(P)}_\alpha \qquad \text{Minimize} \quad (F_1(x) - \alpha_1 G_1(x), \dots, F_m(x) - \alpha_m G_m(x)) \\ \text{subject to} \quad x \in E,$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$.

Definition 3.3. Let $\tilde{\epsilon} \in \mathbb{R}^m_+$, $\bar{x} \in E$ and $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_m) \in \mathbb{R}^m$. The point $(\bar{x}, \bar{y} - \bar{\alpha}\bar{z}) \in X \times \mathbb{R}^m$ is called an $\tilde{\epsilon}$-weak minimizer of $\text{(P)}_{\bar{\alpha}}$ with respect to $\mathbb{R}^m_+$ (denoted by $\bar{x} \in \tilde{\epsilon}\text{-WMin}(F(E) - \bar{\alpha}G(E), \mathbb{R}^m_+)$) iff there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

$$((F(E) - \bar{\alpha}G(E)) - (\bar{y} - \bar{\alpha}\bar{z}) + \tilde{\epsilon}) \cap (-\mathrm{int}(\mathbb{R}^m_+)) = \emptyset.$$

Remark 3.4. When $\epsilon = 0 \in \mathbb{R}^m_+$, Definition 3.3 reduces to Definition 2.8 in [25].

The following lemma presents the relations between (FP) and $\text{(P)}_{\bar{\alpha}}$.

Proposition 3.5. *Let $\epsilon \in \mathbb{R}^m_+$ and $(\bar{x}, \frac{\bar{y}}{\bar{z}}) \in X \times \mathbb{R}^m$. Then, $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ is an ϵ-weak minimizer of (FP) iff $(\bar{x}, \bar{y} - \bar{\alpha}\bar{z})$ is an $\tilde{\epsilon}$-weak minimizer of $\text{(P)}_{\bar{\alpha}}$, where $\bar{\alpha} = \frac{\bar{y}}{\bar{z}} - \epsilon$ and $\tilde{\epsilon} = \epsilon\bar{z}$.*

Proof. Necessity. Suppose that $(\bar{x}, \bar{y} - \bar{\alpha}\bar{z})$ is not a $\tilde{\epsilon}$-weak minimizer of $\text{(P)}_{\bar{\alpha}}$. Then there exist $x \in E$, $y \in F(x)$ and $z \in G(x)$ such that $(y - \bar{\alpha}z) - (\bar{y} - \bar{\alpha}\bar{z}) + \tilde{\epsilon} \in -\mathrm{int}(\mathbb{R}^m_+)$. Therefore, $\frac{y}{z} - \frac{\bar{y}}{\bar{z}} + \epsilon \in -\mathrm{int}(\mathbb{R}^m_+)$, which contradicts $\bar{x}$ is an ϵ-weak minimal solution of (FP).

Sufficiency. It is similar to the necessity. □

Remark 3.6. If $\epsilon = 0$, then Proposition 3.5 reduces to Lemma 3.1 in [22].

Theorem 3.7. *Let $\epsilon \in \mathbb{R}^m_+$. Let $\Gamma \subseteq X$ be a nonempty convex set, $\bar{x}$ be an element of the feasible set E of (FP), $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, $\bar{\alpha} = \frac{\bar{y}}{\bar{z}} - \epsilon$ and $\bar{w} \in H(\bar{x}) \cap (-\mathbb{R}^p_+)$. Let $(\bar{u}, \bar{v}) \in \text{epi}(F)$, $(\bar{u}, \bar{\pi}) \in \text{epi}(-\bar{\alpha}G)$ and $(\bar{u}, \bar{\tau}) \in \text{epi}(H)$. Suppose that the following conditions hold:*

(i) *F, $-\bar{\alpha}G$ are generalized $\mathbb{R}^m_+$-convex on Γ and H is generalized $\mathbb{R}^p_+$-convex on Γ;*

(ii) *For any $x \in \Gamma$,*

$$\Phi_F(x - \bar{x}) := \{y : (x - \bar{x}, y) \in T^2(\text{epi}(F), (\bar{x}, \bar{y}), (\bar{u} - \bar{x}, \bar{v} - \bar{y}))\},$$

$$\Phi_{-\bar{\alpha}G}(x - \bar{x}) := \{z : (x - \bar{x}, z) \in T^2(\text{epi}(-\bar{\alpha}G), (\bar{x}, -\bar{\alpha}z), (\bar{u} - \bar{x}, \bar{\pi} - (-\bar{\alpha}\bar{z})))\}$$

and

$$\Phi_H(x - \bar{x}) := \{w : (x - \bar{x}, w) \in T^2(\text{epi}(H), (\bar{x}, \bar{w}), (\bar{u} - \bar{x}, \bar{\tau} - \bar{w}))\}$$

are minorized and fullfil the domination property;

(iii) *There exists $(y^*, z^*) \in \mathbb{R}^m_+ \backslash \{0\} \times \mathbb{R}^p_+$ such that*

$$\langle y^*, D^2_g F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(x - \bar{x}) + D^2_g(-\bar{\alpha}G)(\bar{x}, -\bar{\alpha}\bar{z}, \bar{u} - \bar{x}, \bar{\pi} - (-\bar{\alpha}\bar{z}))(x - \bar{x})\rangle + \langle z^*, D^2_g H(\bar{x}, \bar{w}, \bar{u} - \bar{x}, \bar{\tau} - \bar{w})(x - \bar{x})\rangle \geq 0, \quad \forall x \in \Gamma \tag{3.1}$$

and

$$\langle z^*, \bar{w}\rangle = 0. \tag{3.2}$$

Then, $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ is an ϵ-weak minimizer of (FP).

Proof. Suppose that $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ is not an ϵ-weak minimizer of (FP), then there exists $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$ such that

$$\frac{\hat{y}}{\hat{z}} - \frac{\bar{y}}{\bar{z}} + \epsilon \in -\text{int}(\mathbb{R}^m_+).$$

It follows from $y^* \in \mathbb{R}^m_+ \backslash \{0\}$ that

$$\langle y^*, \frac{\hat{y}}{\hat{z}} - \frac{\bar{y}}{\bar{z}} + \epsilon\rangle < 0,$$

i.e.,

$$\langle y^*, \hat{y} - \bar{\alpha}\hat{z}\rangle < 0. \tag{3.3}$$

As $\bar{\alpha} = \frac{\bar{y}}{\bar{z}} - \epsilon$, we have

$$\langle y^*, \bar{y} - \bar{\alpha}\bar{z} - \epsilon\bar{z}\rangle = 0. \tag{3.4}$$

Since $\hat{x} \in E$, there exists $\hat{w} \in H(\hat{x}) \cap (-\mathbb{R}^p_+)$. By $z^* \in \mathbb{R}^p_+$, we get

$$\langle z^*, \hat{w}\rangle \leq 0. \tag{3.5}$$

According to (3.2) and (3.5), we obtain

$$\langle z^*, \hat{w} - \bar{w}\rangle \leq 0. \tag{3.6}$$

By (3.3), (3.4) and (3.6), we have

$$\langle y^*, \hat{y} - \bar{\alpha}\hat{z} - (\bar{y} - \bar{\alpha}\bar{z} - \epsilon\bar{z})\rangle + \langle z^*, \hat{w} - \bar{w}\rangle < 0. \tag{3.7}$$

From conditions (i), (ii) and Proposition 2.7, we have

$$F(\hat{x}) - \{\bar{y}\} \subseteq D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(\hat{x} - \bar{x}) + \mathbb{R}_+^m,$$

$$(-\bar{\alpha}G)(\hat{x}) - \{-\bar{\alpha}\bar{z}\} \subseteq D_g^2(-\bar{\alpha}G)(\bar{x}, -\bar{\alpha}\bar{z}, \bar{u} - \bar{x}, \bar{\pi} - (-\bar{\alpha}\bar{z}))(\hat{x} - \bar{x}) + \mathbb{R}_+^m$$

and

$$H(\hat{x}) - \{\bar{w}\} \subseteq D_g^2 H(\bar{x}, \bar{w}, \bar{u} - \bar{x}, \bar{\tau} - \bar{w})(\hat{x} - \bar{x}) + \mathbb{R}_+^p.$$

Hence,

$$\hat{y} - \bar{y} \in D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(\hat{x} - \bar{x}) + \mathbb{R}_+^m, \tag{3.8}$$

$$-\bar{\alpha}\hat{z} + \bar{\alpha}\bar{z} \in D_g^2(-\bar{\alpha}G)(\bar{x}, -\bar{\alpha}\bar{z}, \bar{u} - \bar{x}, \bar{\pi} - (-\bar{\alpha}\bar{z}))(\hat{x} - \bar{x}) + \mathbb{R}_+^m \tag{3.9}$$

and

$$\hat{w} - \bar{w} \in D_g^2 H(\bar{x}, \bar{w}, \bar{u} - \bar{x}, \bar{\tau} - \bar{w})(\hat{x} - \bar{x}) + \mathbb{R}_+^p. \tag{3.10}$$

Clearly, $\epsilon\bar{z} \in \mathbb{R}_+^m$. Therefore, we can prove from (3.8) that

$$\hat{y} - \bar{y} + \epsilon\bar{z} \in D_g^2 F(\bar{x}, \bar{y}, \bar{u} - \bar{x}, \bar{v} - \bar{y})(\hat{x} - \bar{x}) + \mathbb{R}_+^m. \tag{3.11}$$

By (3.1), (3.9)-(3.11), we have

$$\langle y^*, \hat{y} - \bar{\alpha}\hat{z} - (\bar{y} - \bar{\alpha}\bar{z} - \epsilon\bar{z})\rangle + \langle z^*, \hat{w} - \bar{w}\rangle \geq 0,$$

which contradicts (3.7). Consequently, $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ is an ϵ-weak minimizer of (FP). □

Remark 3.8. Notice that Theorem 3.7 is different from Theorem 3.3 in [25]: (i) The generalized C-convexity is different from the ρ-cone convexity; (ii) The contingent epiderivative in [25] is replaced by the generalized second-order contingent epiderivative in Theorem 3.7; (iii) The weak minimizer in Theorem 3.3 in [25] is replaced by the ϵ-weak minimizer in Theorem 3.7.

4 Second-Order Parametric Type Duality

In this section, we introduce a second-order parametric type dual problem for a set-valued fractional programming problem characterized by the generalized second-order contingent epiderivative and investigate the weak duality, strong duality and converse duality results.

The second-order parametric type dual problem associated with (FP) is as follows:

(PD) Minimize α

subject to

$$\begin{aligned}
&\langle y^*, D_g^2 F(x', y', u' - x', v' - y')(x - x') \\
&+ D_g^2(-\alpha G)(x', -\alpha z', u' - x', \pi' - (-\alpha z'))(x - x')\rangle \\
&+ \langle z^*, D_g^2 H(x', w', u' - x', \tau' - w')(x - x')\rangle \geq 0, \quad \forall x \in \Gamma,
\end{aligned} \tag{4.1}$$

$$y' - \alpha z' \in \mathbb{R}_+^m, \tag{4.2}$$

$$\langle z^*, w'\rangle \geq 0, \tag{4.3}$$

$$y^* \in \mathbb{R}_+^m \backslash \{0\}, z^* \in \mathbb{R}_+^p, \tag{4.4}$$

$$x' \in \Gamma, y' \in F(x'), z' \in G(x'), \alpha \in \frac{F(x)}{G(x)}, w' \in H(x') \tag{4.5}$$

$$(u', v') \in \operatorname{epi}(F), (u', \pi') \in \operatorname{epi}(-\alpha G), (u', \tau') \in \operatorname{epi}(H). \tag{4.6}$$

A point $(x', y', z', \alpha', w', y^*, z^*)$ satisfying all the constraints of (PD) is called a feasible point of (PD). Let $K := \{\alpha : (x', y', z', \alpha, w', y^*, z^*)$ satsifies conditions (4.1)-(4.6)$\}$.

Definition 4.1. Let $\epsilon \in \mathbb{R}^m_+$. A feasible point $(x', y', z', \alpha', w', y^*, z^*)$ of (PD) is called an ϵ-weak maximizer of (PD) iff $(K - \alpha' - \epsilon) \cap \text{int}(\mathbb{R}^m_+) = \emptyset$.

Theorem 4.2 (**Weak duality**). *Let $\epsilon \in \mathbb{R}^m_+$ and Γ be a nonempty convex subset of X, and let $\bar{x}$ and $(x', y', z', \alpha', w', y^*, z^*)$ be feasible points for (FP) and (PD), respectively. Suppose that the following conditions are satisfied:*

(i) *F and $-\alpha' G$ are generalized $\mathbb{R}^m_+$-convex on Γ and H is generalized $\mathbb{R}^p_+$-convex on Γ;*

(ii) *The sets*

$$\Phi_F(\bar{x} - x') := \{y : (\bar{x} - x', y) \in T^2(\text{epi}(F), (x', y'), (u' - x', v' - y'))\},$$

$$\Phi_{-\alpha' G}(\bar{x} - x') := \{z : (\bar{x} - x', z) \in T^2(\text{epi}(-\alpha' G), (x', -\alpha' z'), (u' - x', \pi' - (-\alpha' z')))\}$$

and

$$\Phi_H(\bar{x} - x') := \{w : (\bar{x} - x', w) \in T^2(\text{epi}(H), (x', w'), (u' - x', \tau' - w'))\}$$

are minorized and fulfill the domination property.

Then

$$\frac{\bar{y}}{\bar{z}} - \alpha' + \epsilon \notin -\text{int}(\mathbb{R}^m_+).$$

Proof. We proceed by contracdiction. Suppose that $\frac{\bar{y}}{\bar{z}} - \alpha' + \epsilon \in -\text{int}(\mathbb{R}^m_+)$. Since $y^* \in \mathbb{R}^m_+ \backslash \{0\}$, $\langle y^*, \frac{\bar{y}}{\bar{z}} - \alpha' + \epsilon \rangle < 0$, i.e.,

$$\langle y^*, \bar{y} - \alpha' \bar{z} + \epsilon \bar{z} \rangle < 0. \tag{4.7}$$

From (4.2), we have

$$\langle y^*, y' - \alpha' z' \rangle \geq 0. \tag{4.8}$$

As $\bar{x}$ is feasible point for (FP), there exists $\bar{w} \in H(\bar{x}) \cap (-\mathbb{R}^p_+)$ such that $\langle z^*, \bar{w} \rangle \leq 0$. On the other hand, it follows from (4.3) that $\langle z^*, -w' \rangle \leq 0$. Thus,

$$\langle z^*, \bar{w} - w' \rangle \leq 0. \tag{4.9}$$

From (4.7)-(4.9), we get

$$\langle y^*, \bar{y} - \alpha' \bar{z} + \epsilon \bar{z} - (y' - \alpha' z') \rangle + \langle z^*, \bar{w} - w' \rangle < 0. \tag{4.10}$$

By Proposition 2.7, we obtain

$$F(\bar{x}) - \{y'\} \subseteq D^2_g(F)(x', y', u' - x', v' - y')(\bar{x} - x') + \mathbb{R}^m_+,$$

$$-\alpha' G(\bar{x}) + \{\alpha' z'\} \subseteq D^2_g(-\alpha' G)(x', -\alpha' z', u' - x', \pi' - (-\alpha' z'))(\bar{x} - x') + \mathbb{R}^m_+$$

and

$$H(\bar{x}) - \{w'\} \subseteq D^2_g H(x', y', u' - x', \tau' - w')(\bar{x} - x') + \mathbb{R}^p_+.$$

Hence,

$$\bar{y} - y' \in D^2_g F(x', y', u' - x', v' - y')(\bar{x} - x') + \mathbb{R}^m_+, \tag{4.11}$$

$$-\alpha' \bar{z} + \alpha' z' \in D^2_g(-\alpha' G)(x', -\alpha' z', u' - x', \pi' - (-\alpha' z'))(\bar{x} - x') + \mathbb{R}^m_+, \tag{4.12}$$

and

$$\bar{w} - w' \in D_g^2 H(x', y', u' - x', \tau' - w')(\bar{x} - x') + \mathbb{R}_+^p, \tag{4.13}$$

Obviously, $\epsilon\bar{z} \in \mathbb{R}_+^m$. Therefore, we can derive from (4.11) that

$$\bar{y} - y' + \epsilon\bar{z} \in D_g^2 F(x', y', u' - x', v' - y')(\bar{x} - x') + \mathbb{R}_+^m. \tag{4.14}$$

By (4.1), (4.12)-(4.14), we have $\langle y^*, \bar{y} - \alpha'\bar{z} + \epsilon\bar{z} - y' + \alpha' z'\rangle + \langle z^*, \bar{w} - w'\rangle \geq 0$, which contradicts (4.10). □

Theorem 4.3 (**Strong Duality**). *Let $\epsilon \in \mathbb{R}_+^m$, $\bar{\alpha} \in \mathbb{R}^m$, $(\bar{x}, \frac{\bar{y}}{\bar{z}})$ be an ϵ-weak minimizer of (FP), $\bar{w} \in H(\bar{x}) \cap (-\mathbb{R}_+^p)$ and $\bar{\alpha} = \frac{\bar{y}}{\bar{z}} - \epsilon$. Let $(\bar{u}, \bar{v}) \in \text{epi}(F)$, $(\bar{u}, \bar{\pi}) \in \text{epi}(-\bar{\alpha}G)$ and $(\bar{u}, \bar{\tau}) \in \text{epi}(H)$. Suppose that, for some $(y^*, z^*) \in \mathbb{R}_+^m\backslash\{0\} \times \mathbb{R}_+^p$, (3.1) and (3.2) are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, y^*, z^*)$. Then $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, y^*, z^*)$ is a feasible solution of (PD). Furthermore, if the conditions of Theorem 4.2 hold, then $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, y^*, z^*)$ is a 2ϵ-weak maximizer of (PD).*

Proof. Since (3.1) and (3.2) hold, it is obvious that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, y^*, z^*)$ is feasible solution of (PD). Now, we will show that

$$(K - \bar{\alpha} - 2\epsilon) \cap \text{int}(\mathbb{R}_+^m) = \emptyset. \tag{4.15}$$

Suppose that (4.15) does not hold, then there exists $\alpha' \in K$ such that

$$\alpha' - \bar{\alpha} - 2\epsilon \in \text{int}(\mathbb{R}_+^m),$$

i.e.,

$$\frac{\bar{y}}{\bar{z}} - \alpha' + \epsilon \in -\text{int}(\mathbb{R}_+^m).$$

This contradicts Theorem 4.2. Hence, $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, y^*, z^*)$ is a 2ϵ-weak maximizer of (PD). □

Theorem 4.4 (**Converse Duality**). *Let $\epsilon \in \mathbb{R}_+^m$, and let Γ be a nonempty convex subset of X and $(x', y', z', w', \alpha', y^*, z^*)$ be a feasible point of (PD), where $\alpha' = \frac{y'}{z'} - \epsilon$. Suppose that the following conditions are satisfied:*

(i) *F, $-\alpha' G$ are generalized $\mathbb{R}_+^m$-convex on Γ and H is generalized $\mathbb{R}_+^p$-convex on Γ;*

(ii) *The sets $\Phi_F(x - x')$, $\Phi_{-\alpha' G}(x - x')$ and $\Phi_H(x - x')$ given by Theorem 4.2 are minorized and fulfill the domination property;*

(iii) *x' is an element of the feasible set E of (FP).*

Then, $(x', \frac{y'}{z'})$ is an ϵ-weak minimizer of (FP).

Proof. By contradiction. Suppose that $(x', \frac{y'}{z'})$ is not an ϵ-weak minimizer of (FP), then there exist $\hat{x} \in \Gamma$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$ such that $\frac{\hat{y}}{\hat{z}} - \frac{y'}{z'} + \epsilon \in -\text{int}(\mathbb{R}_+^m)$. Noticing that $y^* \in \mathbb{R}_+^m\backslash\{0\}$, we have $\langle y^*, \frac{\hat{y}}{\hat{z}} - \frac{y'}{z'} + \epsilon\rangle < 0$, i.e.,

$$\langle y^*, \hat{y} - \alpha'\hat{z}\rangle < 0. \tag{4.16}$$

From (4.2), we obtain

$$\langle y^*, y' - \alpha' z'\rangle \geq 0. \tag{4.17}$$

Since $\hat{x} \in E$, there exists $\hat{w} \in H(\hat{x}) \cap (-\mathbb{R}^p_+)$ such that $\langle z^*, \hat{w} \rangle \leq 0$. We derive from (4.3) that

$$\langle z^*, \hat{w} - w' \rangle \leq 0. \tag{4.18}$$

By (4.16)-(4.18), we get

$$\langle y^*, \hat{y} - \alpha' \hat{z} - (y' - \alpha' z') \rangle + \langle z^*, \hat{w} - w' \rangle < 0. \tag{4.19}$$

On the other hand, from Proposition 2.7, we have

$$F(\hat{x}) - \{y'\} \subseteq D^2_g F(x', y', u' - x', v' - y')(\hat{x} - x') + \mathbb{R}^m_+,$$

$$(-\alpha' G)(\hat{x}) - \{-\alpha' z'\} \subseteq D^2_g(-\alpha' G)(x', -\alpha' z', u' - x', \pi' - y')(\hat{x} - x') + \mathbb{R}^m_+$$

and

$$H(\hat{x}) - \{w'\} \subseteq D^2_g H(x', w', u' - x', \tau' - w')(\hat{x} - x') + \mathbb{R}^p_+.$$

Hence,

$$\hat{y} - y' \in D^2_g F(x', y', u' - x', v' - y')(\hat{x} - x') + \mathbb{R}^m_+, \tag{4.20}$$

$$-\alpha' \hat{z} + \alpha' z' \in D^2_g(-\alpha' G)(x', -\alpha' z', u' - x', \pi' - y')(\hat{x} - x') + \mathbb{R}^m_+ \tag{4.21}$$

and

$$\hat{w} - w' \in D^2_g H(x', w', u' - x', \tau' - w')(\hat{x} - x') + \mathbb{R}^p_+. \tag{4.22}$$

According to (4.1), (4.20)-(4.22), we have

$$\langle y^*, \hat{y} - \alpha' \hat{z} - (y' - \alpha' z') \rangle + \langle z^*, \hat{w} - w' \rangle \geq 0,$$

which contradicts (4.19). Hence, $(x', \frac{y'}{z'})$ is an ϵ-weak minimizer of (FP). □

5 Conclusions

In this paper, a sufficient optimality condition of set-valued fractional programming problem is established for ϵ-weak minimizer under the assumption of generalized cone convexity. We also introduce a second-order parametic type dual characterized by the generalized second-order contingent epiderivative. Moreover, the weak, strong and converse dual results are derived. Recently, Li and Chen [27] have introduced the higher order generalized contingent epiderivative and the higher order generalized adjacent epiderivative of set-valued maps. Moreover, they investigated the higher order necessary and sufficient optimality conditions for Henig properly efficiency of the set-valued optimization problem with constraints. Whether the results obtained in this paper can be characterized by the higher order generalized contingent epiderivative or the higher order generalized adjacent epiderivative of set-valued maps are worth investigating.

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