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OPTIMALITY CONDITIONS OF THE APPROXIMATE QUASI-WEAK ROBUST EFFICIENCY FOR UNCERTAIN MULTI-OBJECTIVE CONVEX OPTIMIZATION*

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: This paper aims to study the optimality conditions and saddle point theorems for approximate quasi-weak robust efficient solutions in an Uncertain Multi-objective Convex Optimization Problem (UMCOP). Firstly, the approximate quasi-weak robust efficient solution to problem (UMCOP) is defined and an example is given to illustrate its existence. Secondly, a scalarization theorem to problem (UMCOP) in sense of approximate quasi-weak robust efficiency is proposed by using an alternative theorem. Thirdly, the approximate optimality conditions are established under a kind of constrained qualifications, termed robust closed convex cone constrained qualification. Finally, the approximate quasi-weak saddle points to problem (UMCOP) are defined, and are also applied to characterize approximate quasi-weak robust efficient solutions.

Key words: *robust multi-objective optimization, robust efficient solution, approximate optimality condition, approximately quasi-weak saddle point*

Mathematics Subject Classification: *90C25, 90C46, 90C30*

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1 Introduction

Multi-objective Optimization is an interdisciplinary subject of operation research and decision science, its theories and methods are widely used in the fields of financial decision-making, economic planning, engineering design and military science. In practical problems, the optimization model is often affected by various factors, thus many problems contain uncertain data, such as: financial investment, engineering design and transportation network optimization, and so on. Therefore, the uncertain multi-objective optimization problem has been widely concerned by scholars [7,8]. Robust optimization method is an effective method to deal with uncertain optimization problems, and this method aims at finding a worst-case solution which is immunized against the data uncertainty to optimization problems. In this paper, the robust optimization method is used to study an Uncertain Multi-objective Convex Optimization Problem (UMCOP). It is well known that the (weak) efficient solution of

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multi-objective optimization problems usually do not exist in the case of non-compactness, however the approximate solution exists under weaker conditions. In addition, most of solutions obtained by numerical algorithm in practical application are approximate solution. Therefore, it is of great theoretical value and practical significance to investigate the approximate solution of multi-objective optimization problem.

Optimality conditions and saddle point theorems are two important contents in the study of optimization theory. Recently, much attention have been paid on these two subjects in robust optimization problems. In the case of single objective optimization: Lee and Jiao [9] and Lee and Lee [10] focused on robust quasi-approximate solution and approximate solution, respectively, and Sun et al [15] paid attention to the robust optimal solution. For multi-objective optimization problems: Kuroiwa and Lee [8] discussed the weakly efficient solution by using an alternative theorem; Sun et al [14] studied robust approximate weak efficient solutions; Kim [7] defined robust weak vector saddle point and proposed the saddle point characterization for weak efficient solution. To the best of our knowledge, there are few literatures involving the quasi-approximate efficient solution and saddle point for multi-objective robust optimization problems. This paper is devoted to the study of scalarization, optimality conditions and saddle point for approximate quasi-weak robust efficient solutions to an uncertain multi-objective convex optimization problem. The constrained qualification adopted in present paper is termed as robust closed convex cone constrained qualification, which was given in [10].

The content of this paper is arranged as follows: In Section 2, the symbols used in this paper are presented, some concepts and lemmas used in subsequent sections are given. In Section 3, the concept of approximate quasi-weak robust efficient solution to problem (UMCOP) is introduced, and its scalarization theorem and the optimality conditions are proved by using an alternative theorem. In Section 4, the definition of approximate quasi-weak saddle point for problem (UMCOP) is defined, and the corresponding saddle point theorems are established to characterize approximate quasi-weak robust efficient solutions.

2 Notations and Preliminaries

Firstly, let's present some of the notations and definitions which will be used in this paper. Let $\mathbb{R}^n$ be the n-dimensional Euclid space, and $\mathbb{B}$ be closed unit ball in $\mathbb{R}^n$. The inner product in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$. We set

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geq 0, \ i = 1, ..., n\}, \ \ \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \ x_i > 0, \ i = 1, ..., n\}.$$

This paper adopts the following order relations:

$$x < y \Leftrightarrow y - x \in \mathbb{R}^n_{++}; \ \ x \leqq y \Leftrightarrow y - x \in \mathbb{R}^n_+.$$

Let $f : \ \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a extended real value function. The effective domain and epigraph of f are defined by

$$\text{dom} f := \{x \in \mathbb{R}^n : \ f(x) < +\infty\},$$

$$\text{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leq r\}.$$

It is said that f is proper, if for any $x \in \mathbb{R}^n$, $f(x) > -\infty$ and $\text{dom} f \neq \emptyset$. f is called to be a lower semicontinuous function if for any $x \in \mathbb{R}^n$,

$$\liminf_{y \to x} f(y) \geq f(x),$$

and f is said to be convex, if epi$f$ is a convex set, or equivalently,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in \mathbb{R}^n,\ \lambda \in [0,1].$$

Moreover, if $-f$ is convex, then f is termed to be a concave function. As usual, for any proper convex function f on $\mathbb{R}^n$, its conjugate function $f^*:\ \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) :\ x \in \mathbb{R}^n\}, \qquad \forall x^* \in \mathbb{R}^n,$$

clearly, f^* is a proper, lower semicontinuous and convex function (see [15]).

Let $A \in \mathbb{R}^n$ be a subset. The closure and convex hull generated by A are represented by cl$A$, co$A$ respectively. Let $\bar{x} \in A,\ \varepsilon \geq 0$. The normal cone and $\varepsilon-$normal cone to A at $\bar{x}$ are defined by (see [3, 4, 10]).

$$N(\bar{x}, A) = \{x^* \in \mathbb{R}^n :\ \langle x^*, x - \bar{x}\rangle\} \leq 0,\ \forall x \in A\},$$

$$N_\varepsilon(\bar{x}, A) = \{x^* \in \mathbb{R}^n :\ \langle x^*, x - \bar{x}\rangle\} \leq \varepsilon,\ \forall x \in A\}.$$

The indicator function $\delta_A:\ \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of A is given by

$$\delta_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A, \end{cases}$$

and the support function $\sigma_A:\ \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of A is defined by

$$\sigma_A(x^*) = \sup_{x \in \mathbb{R}^n} \langle x^*, x\rangle, \quad \forall x^* \in \mathbb{R}^n.$$

Remark 2.1. Let $A \subset \mathbb{R}^n$. the indicator function and support function of A satisfy the following properties (see [1]),

$$\sigma_A(x^*) = \sup_{x \in A}\langle x^*, x\rangle = \delta_A^*(x^*), \qquad \forall x^* \in \mathbb{R}^n. \tag{2.1}$$

Now, let us give the concept of subdifferential of convex function and its related properties. For any proper convex function $f:\ \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of f at $\bar{x} \in$ dom$f$ is defined by

$$\partial f(\bar{x}) = \{x^* \in \mathbb{R}^n :\ \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}),\ \forall x \in \mathbb{R}^n\}.$$

More generally, for any $\varepsilon \geq 0$, the $\varepsilon-$subdifferent of f at $\bar{x} \in$ dom$f$ is given by

$$\partial_\varepsilon f(\bar{x}) = \{x^* \in \mathbb{R}^n :\ \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \varepsilon,\ \forall x \in \mathbb{R}^n\}.$$

Remark 2.2. Let $C \subset \mathbb{R}^n$ be nonempty closed convex set, $\bar{x} \in$ dom$f$. It is pointed out in [2] that the following properties hold:

$$\partial \delta_C(\bar{x}) = N(\bar{x}, C), \tag{2.2}$$

$$\partial_\varepsilon \delta_C(\bar{x}) = N_\varepsilon(\bar{x}, C). \tag{2.3}$$

The next Lemma 2.3 and Lemma 2.4 give some basic properties of conjugate function and subdifferential, which play an important role in proving our main results.

Lemma 2.3 (see [6,10,15])**.** *If $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions and $dom f_1 \cap dom f_2 \neq \emptyset$, then*

$$epi(f_1 + f_2)^* = cl(epi f_1^* + epi f_2^*).$$

Moreover, if one of the function f_1, f_2 is continuous, then

$$epi(f_1 + f_2)^* = epi f_1^* + epi f_2^*,$$

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Lemma 2.4 (see [5])**.** *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and $\bar{x} \in dom f$, then*

$$epi f^* = \bigcup_{\varepsilon \geq 0} \{(x^*, \langle x^*, \bar{x}\rangle + \varepsilon - f(\bar{x}) : \ x^* \in \partial_\varepsilon f(\bar{x}))\}.$$

As we all know, the alternative theorem is critical to establish the theory of optimization. In this paper, we will utilize Gordan's alternative theorem to establish the scalarization theorem and optimality conditions for robust multi-objective convex optimization problems.

Lemma 2.5 (see [12,13])**.** *Let $C \subset \mathbb{R}^n$ be a convex set, $\psi_1(x), ..., \psi_m(x)$ are convex on C, if there is no solution to the following system of inequalities on C,*

$$\psi_i(x) < 0, \quad i = 1, ..., m,$$

then there exist $\lambda_1, ..., \lambda_m \geq 0$, not all zero, such that

$$\sum_{i=1}^{m} \lambda_i \psi_i(x) \geq 0, \qquad \forall x \in C.$$

Consider the following multi-objective convex optimization problem:

$$\text{(MCOP)} \qquad \begin{cases} \min & f(x) = (f_1(x), f_2(x), ..., f_l(x)) \\ \text{s.t.} & x \in C, \ g_j(x) \leq 0, \ j = 1, ..., m, \end{cases}$$

where $C \subset \mathbb{R}^n$ is a nonempty closed convex set, $f_i, g_j : \mathbb{R}^n \to \mathbb{R}, \ i = 1, ..., l, \ j = 1, ..., m$ are continuous convex functions, the feasible set of problem (MCOP) is denoted as

$$\mathcal{F}_0 := \{x \in C : \ g_j(x) \leq 0, \ j = 1, ..., m\}.$$

The program (MCOP) in the face of uncertain data both in the objectives and constraints can be captured by the next problem:

$$\text{(UMCOP)} \qquad \begin{cases} \min & f(x, u) = (f_1(x, u_1), f_2(x, u_2), ..., f_l(x, u_l)) \\ \text{s.t.} & x \in C, \ g_j(x, v_j) \leq 0, \ j = 1, ..., m, \end{cases}$$

here, u_i, v_j are uncertain parameters which belong to the corresponding convex and compact uncertainty sets $\mathcal{U}_i \subset \mathbb{R}^p$ and $\mathcal{V}_j \subset \mathbb{R}^q$, $i = 1, ..., l, \ j = 1, ..., m$. *Throughout of the rest paper, we always assume that $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, \ i = 1, ..., l$ are continuous, and are convex related to the first variable, that is $f_i(\cdot, u_i)$ is convex for any $u_i \in \mathcal{U}_i$, and are concave to the second variable, that is $f_i(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$, $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, \ j = 1, ..., m$ are continuous, and g_j is convex related to the first variable.*

In this paper, the robust optimization approach is applied to examine the optimality of problem (UMCOP), thus, we consider its robust counterpart [8]:

$$\text{(RUMCOP)} \qquad \begin{cases} \min & \phi(x) = (\max\limits_{u_1 \in \mathcal{U}_1} f_1(x,u_1), \max\limits_{u_2 \in \mathcal{U}_2} f_2(x,u_2), ..., \max\limits_{u_l \in \mathcal{U}_l} f_l(x,u_l)) \\ \text{s.t.} & x \in C,\ g_j(x,v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, \quad j = 1,...,m. \end{cases}$$

It is worth noting that the robust counterpart, which is termed as the *robust optimization problem*. The feasible set of problem (RUMCOP) is denoted as

$$\mathcal{F} := \{x \in C:\ g_j(x,v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, \quad j = 1,...,m\},$$

obviously, $\mathcal{F}$ is a convex set.

Definition 2.6. Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_l) \in \mathbb{R}^l_+$ and $\bar{x} \in \mathcal{F}$.

(i) It is said that $\bar{x}$ is a quasi-weak robust ε-efficient solution of (UMCOP), iff $\bar{x}$ is a quasi-weak ε-efficient solution of (RUMCOP), that is there does not exist $x \in \mathcal{F}$, such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x,u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i) - \sqrt{\varepsilon_i} \parallel x - \bar{x} \parallel, \qquad i = 1,...,l.$$

(ii) It is called that $\bar{x}$ is a weak robust ε-efficient solution of (UMCOP), iff $\bar{x}$ is a weak ε-efficient solution of (RUMCOP), that is there does not exist $x \in \mathcal{F}$, such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x,u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i) - \varepsilon_i, \qquad i = 1,...,l.$$

(iii) It is said that $\bar{x}$ is a weak robust efficient solution of (UMCOP), iff $\bar{x}$ is a weak efficient solution of (RUMCOP), that is there does not exist $x \in \mathcal{F}$, such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x,u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i), \qquad i = 1,...,l.$$

Remark 2.7. When $\varepsilon = 0$, quasi-weak robust ε-efficient solutions and weak robust ε-efficient solutions degrade into weak robust efficient solution [8]. When $\varepsilon > 0$, a weak robust efficient solution must be a quasi-weak robust ε-efficient solution, conversely, it is not true, see the next Example 2.8.

Example 2.8. Let $\mathcal{U}_1 = \mathcal{U}_2 = [-1,1]$, $\mathcal{V} = [-1,1]$, $f_1, f_2, g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f_1((x_1,x_2),u_1) = x_1^2 - x_2 - u_1^2, \quad f_2((x_1,x_2),u_2) = -x_2 + u_2, \quad g((x_1,x_2),v) = x_1^2 - vx_1,$$

where $u_1 \in \mathcal{U}_1,\ u_2 \in \mathcal{U}_2,\ v \in \mathcal{V}$ are uncertain parameters. Clearly, the above functions f_1, f_2, g are convex related to the first variable, and f_1, f_2 are concave to the second variable. Considering the following robust multi-objective convex optimization problem:

$$\text{(RUMCOP)}^1 \qquad \begin{cases} \min & (\max\limits_{u_1 \in \mathcal{U}_1} (x_1^2 - x_2 - u_1^2), \ \max\limits_{u_2 \in \mathcal{U}_2} (-x_2 + u_2)) \\ \text{s.t.} & x_1^2 - vx_1 \leq 0,\ v \in \mathcal{V}. \end{cases}$$

Obviously,

$$\max_{u_1 \in \mathcal{U}_1} (x_1^2 - x_2 - u_1^2) = x_1^2 - x_2, \quad \max_{u_2 \in \mathcal{U}_2} (-x_2 + u_2) = -x_2 + 1.$$

Thus, feasible set of (RUMCOP)^1 is

$$\mathcal{F}^1 = \{(x_1,x_2) \in \mathbb{R}^2:\ x_1^2 - vx_1 \leq 0,\ v \in [-1,1]\} = \{(x_1,x_2) \in \mathbb{R}^2:\ x_1 = 0,\ x_2 \in \mathbb{R}\}.$$

Let $\varepsilon = (1,4),\ \bar{x} = (0,0)$. For any $x \in \mathcal{F}^1$, and we can easily get that $\bar{x}$ is a quasi-weak ε-efficient solution of (RUMCOP)^1. However, $\bar{x}$ is not a weak efficient solution.

3 Approximate Optimal Conditions

From now on, we always suppose $\varepsilon = (\varepsilon_1, ..., \varepsilon_l) \in \mathbb{R}^l_+$. Firstly, we use an alternative theorem to establish the scalarization theorem to problem (UMCOP) in the sense of quasi-weak robust ε-efficiency.

Theorem 3.1. *In problem (UMCOP), let $\bar{x} \in \mathcal{F}$. Then, $\bar{x}$ is a quasi-weak robust ε-efficient solution of (UMCOP) if and only if there exist $\bar{\mu}_i \geq 0$, and $\sum_{i=1}^l \bar{\mu}_i = 1,\ \bar{u}_i \in \mathcal{U}_i,\ i = 1, ..., l$, such that for any $x \in \mathcal{F}$,*

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(x, \bar{u}_i) \geq \sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|, \tag{3.1}$$

and

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i). \tag{3.2}$$

Proof. Firstly, we claim that: $\bar{x}$ is a quasi-weak robust ε-efficient solution of problem (UMCOP) if and only if there exist $\bar{\mu}_i \geq 0,\ i = 1, ..., l$, and $\sum_{i=1}^l \bar{\mu}_i = 1$, such that

$$\sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|,\ \forall x \in \mathcal{F}. \tag{3.3}$$

In fact, assume that $\bar{x}$ is a quasi-weak robust ε-efficient solution of problem (UMCOP), then the following system of inequalities have no solution on $\mathcal{F}$,

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) + \sqrt{\varepsilon_i} \parallel x - \bar{x} \parallel < 0,\ i = 1, ..., l.$$

Moreover, for any $i = 1, ..., l$, obviously,

$$\psi_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) + \sqrt{\varepsilon_i} \parallel x - \bar{x} \parallel$$

is convex function on convex set $\mathcal{F}$. Thus, according to Lemma 2.5, there exist $\mu_i^{'} \geq 0,\ i = 1, ..., l$, not all zero, such that

$$\sum_{i=1}^{l} \mu_i^{'} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i=1}^{l} \mu_i^{'} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) + \sum_{i=1}^{l} \mu_i^{'} \sqrt{\varepsilon_i} \|x - \bar{x}\| \geq 0,\ \forall x \in \mathcal{F}.$$

Both sides of the above equation is divided by $\sum_{i=1}^l \mu_i^{'}$, and let $\bar{\mu}_i = \frac{\mu_i^{'}}{\sum_{i=1}^l \mu_i^{'}}$, therefore, there are real numbers $\bar{\mu}_i \geq 0,\ i = 1, ..., l$ and $\sum_{i=1}^l \bar{\mu}_i = 1$ such that

$$\sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|,\ \forall x \in \mathcal{F},$$

which shows that (3.3) is satisfied. Conversely, suppose that (3.3) holds, however $\bar{x}$ is not a quasi-weak robust ε-efficient solution of program (UMCOP), then there exist $\hat{x} \in \mathcal{F}$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sqrt{\varepsilon_i} \parallel \hat{x} - \bar{x} \parallel,\ i = 1, ..., l.$$

Therefore, for not all zero real value $\bar{\mu}_i \geq 0,\ i = 1, ..., l$, with$\sum_{i=1}^{l} \bar{\mu}_i = 1$, we get

$$\sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \sum_{i=1}^{l} \bar{\mu}_i (\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sqrt{\varepsilon_i} \parallel \hat{x} - \bar{x} \parallel),$$

which contradicts to (3.3) .

Secondly, noticing that the equation (3.3) is equivalent to

$$\inf_{x \in \mathcal{F}} \max_{(u_1,...,u_l) \in \prod_{i=1}^{l} \mathcal{U}_i} \sum_{i=1}^{l} \bar{\mu}_i (f_i(x, u_i) + \sqrt{\varepsilon_i} \|x - \bar{x}\|) \geq \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

By min-max theorem (see [13]), we arrive at

$$\max_{(u_1,...,u_l) \in \prod_{i=1}^{l} \mathcal{U}_i} \inf_{x \in \mathcal{F}} \sum_{i=1}^{l} \bar{\mu}_i (f_i(x, u_i) + \sqrt{\varepsilon_i} \|x - \bar{x}\|) \geq \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

So, there exists $\bar{u}_i \in \mathcal{U}_i,\ i = 1, ..., l$, such that

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(x, \bar{u}_i) \geq \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|,\ \forall x \in \mathcal{F}.$$

That is

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(x, \bar{u}_i) \geq \sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|,\ \forall x \in \mathcal{F},$$

where

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

This completes the proof. □

Remark 3.2. In literature [11], the scalarization result for weak robust ε-efficient solution is obtained by using a separation theorem. Our approach is different from that of [11], it is examined by utilizing an alternative theorem in the sense of quasi-weak robust ε-efficiency .

Before we present the optimality conditions for the quasi-weak robust ε-efficient solution to problem (UMCOP) , we need to introduce the following robust closed convex cone constrained qualification.

Definition 3.3. (see [10, 14]) In problem (RUMCOP), if

$$\bigcup_{\lambda_j \geq 0, v_j \in \mathcal{V}_j} \mathrm{epi}(\sum_{j=1}^{m} \lambda_j g_j(., v_j))^* + \mathrm{epi}\sigma_C.$$

is a closed convex cone, then, it is called that the problem (RUMCOP) satisfies robust closed convex cone constrained qualification.

Lemma 3.4. *(see [10, 14]) Let $\varphi :\ \mathbb{R}^n \to \mathbb{R}$ be a convex function, and $\mathcal{F}$ be the feasible set of problem (RUMCOP). Then the following conclusions are equivalent:*

$$(i)\ \mathcal{F} = \{x \in C :\ g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j,\ j = 1, ..., m\} \subseteq \{x \in \mathbb{R}^n :\ \varphi(x) \geq 0\};$$

$$(ii)\ (0, 0) \in epi\varphi^* + cl(co(\bigcup_{\lambda_j \geq 0, v_j \in \mathcal{V}_j} epi(\sum_{j=1}^{m} \lambda_j g_j(., v_j))^* + epi\sigma_C)).$$

Theorem 3.5. *In problem (RUMCOP), let $\bar{x} \in \mathcal{F}$. If the problem (RUMCOP) satisfies robust closed convex cone constrained qualification, then $\bar{x}$ is a quasi-weak ε-efficient solution of program (RUMCOP) if and only if there exist $\bar{\mu}_i \geq 0$, and $\sum_{i=1}^{l} \bar{\mu}_i = 1$, $\bar{u}_i \in \mathcal{U}_i$, $i = 1, ..., l$, $\bar{v}_j \in \mathcal{V}_j$, $\bar{\lambda}_j \geq 0$, $j = 1, ..., m$, such that*

$$0 \in \sum_{i=1}^{l} \partial \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \partial \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) + \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \mathbb{B} + N(\bar{x}, C), \tag{3.4}$$

$$\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, ..., m, \tag{3.5}$$

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \tag{3.6}$$

where $\partial \bar{\mu}_i f_i(\bar{x}, \bar{u}_i)$ and $\partial \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j)$ are subdifferential of functions $\bar{\mu}_i f_i(\cdot, \bar{u}_i)(x)$ and $\bar{\lambda}_j g_j(\cdot, \bar{v}_j)(x)$, $i = 1, ..., l$, $j = 1, ..., m$, related to the first variable x at $\bar{x}$.

Proof. By Theorem 3.1, we know that $\bar{x}$ is a quasi-weak ε-efficient solution of problem (RUMCOP) if and only if there exist $\bar{\mu}_i \geq 0$, and $\sum_{i=1}^{l} \bar{\mu}_i = 1$, $\bar{u}_i \in \mathcal{U}_i$, $i = 1, ..., l$, such that

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(x, \bar{u}_i) \geq \sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) - \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|, \ \forall \ x \in \mathcal{F}, \tag{3.7}$$

and (3.6) holds. Let

$$\varphi(x) = \sum_{i=1}^{l} \bar{\mu}_i f_i(x, \bar{u}_i) - \sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) + \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|x - \bar{x}\|, \ \forall \ x \in \mathcal{F},$$

From (3.7), we arrive at

$$\varphi(x) \geq 0, \quad \forall \ x \in \mathcal{F}.$$

It follows that

$$x \in \mathcal{F} = \{x \in C : \ g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, ..., m\} \ \Rightarrow \ \varphi(x) \geq 0. \tag{3.8}$$

Again because the program (RUMCOP) satisfies robust closed convex cone constrained qualification, we get

$$\text{cl}(\text{co}(\bigcup_{\lambda_j \geq 0, v_j \in \mathcal{V}_j} \text{epi}(\sum_{j=1}^{m} \lambda_j g_j(., v_j))^* + \text{epi}\sigma_C)) = \bigcup_{\lambda_j \geq 0, v_j \in \mathcal{V}_j} \text{epi}(\sum_{j=1}^{m} \lambda_j g_j(., v_j))^* + \text{epi}\sigma_C.$$

It follows from (3.8) and Lemma 3.4 that

$$(0, 0) \in \text{epi}\varphi^* + \bigcup_{\lambda_j \geq 0, v_j \in \mathcal{V}_j} \text{epi}(\sum_{j=1}^{m} \lambda_j g_j(., v_j))^* + \text{epi}\sigma_C. \tag{3.9}$$

Since φ is continuous for any $x \in \mathcal{F}$, by Lemma 2.3, it leads to

$$\text{epi}\varphi^* = \text{epi}(\sum_{i=1}^{l} \bar{\mu}_i f_i(\cdot, \bar{u}_i))^* + \text{epi}(\sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \| \cdot - \bar{x}\| \ - \sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i))^*.$$

According to the definition of conjugate function, we know that

$$(\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\cdot-\bar{x}\| \ -\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i))^*(x^*)=$$

$$\begin{cases}\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\| \ +\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i), \text{if } \|x^*\|\le\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i},\\ +\infty, \qquad \text{else.}\end{cases}$$

So

$$\text{epi}(\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\cdot-\bar{x}\| \ -\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i))^*=\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\mathbb{B}\times[\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\| \ +\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i),+\infty).$$

Again, it follows from (3.9) that

$$(0,-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\|-\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i))\in\text{epi}(\sum_{i=1}^{l}\bar{\mu}_i f_i(\cdot,\bar{u}_i))^*$$
$$+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\mathbb{B}\times\mathbb{R}_+ +\bigcup_{\lambda_j\ge 0, v_j\in\mathcal{V}_j}\text{epi}(\sum_{j=1}^{m}\lambda_j g_j(.,v_j))^*+\text{epi}\sigma_C.$$

Therefore, there exist $\bar{\lambda}_j\ge 0,\ \bar{v}_j\in\mathcal{V}_j,\ j=1,...,m$, such that

$$(0,-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\|-\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i))\in\text{epi}(\sum_{i=1}^{l}\bar{\mu}_i f_i(\cdot,\bar{u}_i))^*$$
$$+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\mathbb{B}\times\mathbb{R}_+ +\text{epi}(\sum_{j=1}^{m}\bar{\lambda}_j g_j(.,\bar{v}_j))^*+\text{epi}\sigma_C.$$

On the other hand, noticing that $f_i,\ i=1,...,l,\ g_j,\ j=1,...,m$ are continuous functions related to the first variable, we get from Lemma 2.3 that

$$(0,-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\|-\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i))\in\sum_{i=1}^{l}\text{epi}(\bar{\mu}_i f_i(\cdot,\bar{u}_i))^*$$
$$+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\mathbb{B}\times\mathbb{R}_+ +\sum_{j=1}^{m}\text{epi}(\bar{\lambda}_j g_j(.,\bar{v}_j))^*+\text{epi}\sigma_C. \tag{3.10}$$

Furthermore, it follows from Lemma 2.4 that there exist $\eta_i\in\mathbb{R}^n,\ \alpha_i\ge 0,\ i=1,...,n,\ \xi_j\in\mathbb{R}^n,\ \beta_j\ge 0,\ j=1,...,m$, such that

$$\sum_{i=1}^{l}\text{epi}(\bar{\mu}_i f_i(\cdot,\bar{u}_i))^*=\sum_{i=1}^{l}\bigcup_{\alpha_i\ge 0}\{(\eta_i,\ \langle\eta_i,\ \bar{x}\rangle+\alpha_i-\bar{\mu}_i f_i(\bar{x},\ \bar{u}_i))\mid\eta_i\in\partial_{\alpha_i}\bar{\mu}_i f_i(\cdot,\bar{u}_i)(\bar{x})\}, \tag{3.11}$$

$$\sum_{j=1}^{m}\text{epi}(\bar{\lambda}_j g_j(.,\bar{v}_j))^*=\sum_{j=1}^{m}\bigcup_{\beta_j\ge 0}\{(\xi_j,\langle\xi_j,\bar{x}\rangle+\beta_j-\bar{\lambda}_j g_j(\bar{x},\bar{v}_j))\mid\xi_j\in\partial_{\beta_j}\bar{\lambda}_j g_j(\cdot,\bar{v}_j)(\bar{x})\}. \tag{3.12}$$

In addition, It yields from Remark 2.1 that there exist $\gamma \geq 0$ and $\zeta \in \mathbb{R}^n$ such that

$$\text{epi}\sigma_C = \text{epi}\delta_C^* = \bigcup_{\gamma\geq 0}\{\zeta,\ \langle\zeta,\ \bar{x}\rangle + \gamma - \delta_C(\bar{x}) \mid \zeta \in \partial_\gamma\delta_C(\bar{x})\}. \tag{3.13}$$

Since

$$N_\gamma(\bar{x}, C) = \partial_\gamma\delta_C(\bar{x}),$$

we get from (3.10)-(3.13) that there exist $\bar{\alpha}_i \geq 0,\ \bar{\eta}_i \in \partial_{\bar{\alpha}_i}\bar{\mu}_i f_i(\cdot, \bar{u}_i)(\bar{x}),\ i = 1, ..., l,\ \bar{\beta}_j \geq 0,\ \bar{\xi}_j \in \partial_{\bar{\beta}_j}\bar{\lambda}_j g_j(\cdot, \bar{v}_j)(\bar{x}),\ j = 1, ..., m,\ b \in \mathbb{B},\ \bar{r} \in \mathbb{R}_+,\ \bar{\gamma} \geq 0,\ \bar{\zeta} \in N_{\bar{\gamma}}(\bar{x}, C)$, such that

$$(0, -\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\| - \sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x}, \bar{u}_i)) = \sum_{i=1}^{l}(\bar{\eta}_i, \langle\bar{\eta}_i, \bar{x}\rangle + \bar{\alpha}_i - \bar{\mu}_i f_i(\bar{x}, \bar{u}_i))$$

$$+\sum_{j=1}^{m}(\bar{\xi}_j, \langle\bar{\xi}_j, \bar{x}\rangle + \bar{\beta}_j - \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j)) + (\bar{\zeta}, \langle\bar{\zeta}, \bar{x}\rangle + \bar{\gamma}) + (\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b, \bar{r}).$$

Hence,

$$\sum_{i=1}^{l}\bar{\eta}_i + \sum_{j=1}^{m}\bar{\xi}_j + \bar{\zeta} + \sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b = 0. \tag{3.14}$$

Since

$$\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\| - \sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\langle b, \bar{x}\rangle \geq 0,$$

we get

$$\begin{aligned}
0 \leq \sum_{j=1}^{m}\bar{\beta}_j + \sum_{i=1}^{l}\bar{\alpha}_i + \bar{\gamma} &\leq \sum_{j=1}^{m}\bar{\beta}_j + \sum_{i=1}^{l}\bar{\alpha}_i + \bar{\gamma} - \sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\langle b, \bar{x}\rangle + \sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\bar{x}\| + \bar{r}\\
&= \sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j)\\
&\leq 0.
\end{aligned}$$

Now, this leads to

$$\bar{\beta}_j = 0,\ \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, ..., m,\ \bar{\alpha}_i = 0,\ i = 1, ..., l,\ \bar{\gamma} = 0.$$

Thus, the equation (3.5) holds and

$$\bar{\eta}_i \in \partial\bar{\mu}_i f_i(\cdot, \bar{u}_i)(\bar{x}),\ i = 1, ..., l,\ \bar{\xi}_j \in \partial\bar{\lambda}_j g_j(\cdot, \bar{v}_j)(\bar{x}),\ j = 1, ..., m,\ \bar{\zeta} \in N(\bar{x}, C).$$

Together with (3.14), we obtain that

$$0 \in \sum_{i=1}^{l}\partial\bar{\mu}_i f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m}\partial\bar{\lambda}_j g_i(\bar{x}, \bar{v}_j) + \sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\mathbb{B} + N(\bar{x}, C).$$

Conversely, suppose that there exist $\bar{\mu}_i \geq 0,\ \sum_{i=1}^{l}\bar{\mu}_i = 1,\ \bar{u}_i \in \mathcal{U}_i,\ i = 1, ..., l,\ \bar{v}_j \in \mathcal{V}_j,\ \bar{\lambda}_j \geq 0,\ j = 1, ..., m$, such that (3.4) and (3.5) hold, then there exist $\bar{\eta}_i \in$

$\partial\bar{\mu}_i f_i(\cdot,\bar{u}_i)(\bar{x}),\ i=1,...,l,\ \bar{\xi}_j\in\partial\bar{\lambda}_j g_j(\cdot,\bar{v}_j)(\bar{x}),\ j=1,...,m,\ \bar{\zeta}\in N(\bar{x},C),\ b\in\mathbb{B}$, such that

$$\langle\bar{\eta}_i,x-\bar{x}\rangle\le\bar{\mu}_i f_i(x,\bar{u}_i)-\bar{\mu}_i f_i(\bar{x},\bar{u}_i),\ i=1,...,l,\ \forall x\in\mathcal{F}, \tag{3.15}$$

$$\langle\bar{\xi}_j,x-\bar{x}\rangle\le\bar{\lambda}_j g_j(x,\bar{v}_j)-\bar{\lambda}_j g_j(\bar{x},\bar{v}_j),\ j=1,...,m,\ \forall x\in\mathcal{F}, \tag{3.16}$$

$$\langle\bar{\zeta},x-\bar{x}\rangle\le\delta_C(x)-\delta_C(\bar{x}),\ \forall x\in\mathcal{F}, \tag{3.17}$$

$$\sum_{i=1}^{l}\bar{\eta}_i+\sum_{j=1}^{m}\bar{\xi}_j+\bar{\zeta}+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b=0. \tag{3.18}$$

Observing that $\bar{\lambda}_j g_j(\bar{x},\bar{v}_j)=0,\ \delta_C(\bar{x})=0$, and for any $x\in\mathcal{F},\ \bar{\lambda}_j g_j(x,\bar{v}_j)\le 0,\ \delta_C(x)=0,\ j=1,...,m$, we obtain from (3.15)- (3.17) that

$$\langle\sum_{i=1}^{l}\bar{\eta}_i+\sum_{j=1}^{m}\bar{\xi}_j+\bar{\zeta},x-\bar{x}\rangle\le\sum_{i=1}^{l}\bar{\mu}_i f_i(x,\bar{u}_i)-\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i),\ \forall\ x\in\mathcal{F}.$$

It follows from (3.18) that

$$\langle-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b,x-\bar{x}\rangle\le\sum_{i=1}^{l}\bar{\mu}_i f_i(x,\bar{u}_i)-\sum_{i=1}^{n}\bar{\mu}_i f_i(\bar{x},\bar{u}_i),\ \forall\ x\in\mathcal{F}.$$

Again, since

$$-\langle\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b,x-\bar{x}\rangle\ge-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|x-\bar{x}\|,\ \forall\ x\in\mathcal{F},$$

we obtain

$$\sum_{i=1}^{l}\bar{\mu}_i f_i(\bar{x},\bar{u}_i)-\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|x-\bar{x}\|\le\sum_{i=1}^{l}\bar{\mu}_i f_i(x,\bar{u}_i),\ \forall\ x\in\mathcal{F},$$

which shows that $\bar{x}$ is a quasi-weak ε-efficient solution for problem (RUMCOP) due to Theorem 3.1. □

Remark 3.6. The literature [14] dealt with the approximate optimality conditions for a robust optimization problem, in which the uncertain data are only involved in constraint functions. In presented paper, the uncertainty are considered for both the objective and constraint functions.

4 Approximate Weak Saddle Point Theorems

In this section, we firstly use the Lagrangian function to define the quasi ε-weak saddle point for problem (RUMCOP) , and provide an example to illustrate its existence. Then, the saddle point theorems for quasi-weak ε-efficiency are established.

Let $e=(1,...,1)\in\mathbb{R}^l,\ x\in C,\ \lambda=(\lambda_1,...,\lambda_m)\in\mathbb{R}^m_+$. Then, the Lagrangian function of problem (RUMCOP) is defined as follows:

$$L(x,v,\lambda)=\phi(x)+\sum_{j=1}^{m}\lambda_j g_j(x,v_j)\cdot e,$$

where

$$\phi(x)=(\max_{u_1\in\mathcal{U}_1}f_1(x,u_1),\max_{u_2\in\mathcal{U}_2}f_2(x,u_2),...,\max_{u_l\in\mathcal{U}_l}f_l(x,u_l)).$$

$$\phi_i(x)=\max_{u_i\in\mathcal{U}_i}f_i(x,u_i),\ i=1,...,l.$$

Definition 4.1. Let $\mathcal{V} = (\mathcal{V}_1, ..., \mathcal{V}_m),\ v = (v_1, ..., v_m) \in \mathcal{V}$. It is called that $(\bar{x}, \bar{v}, \bar{\lambda}) \in C \times \mathcal{V} \times \mathbb{R}^m_+$ is a quasi ε-weak saddle point of problem (RUMCOP), if

$$L(\bar{x}, v, \lambda) - \|\lambda - \bar{\lambda}\| \cdot \sqrt{\varepsilon} \leqq L(\bar{x}, \bar{v}, \bar{\lambda}),\ \forall v \in \mathcal{V},\ \lambda \in \mathbb{R}^m_+, \tag{4.1}$$

$$L(x, \bar{v}, \bar{\lambda}) + \|\ x - \bar{x}\| \cdot \sqrt{\varepsilon} - L(\bar{x}, \bar{v}, \bar{\lambda}) \notin -\mathbb{R}^l_{++},\ \forall x \in C, \tag{4.2}$$

where $\sqrt{\varepsilon} = (\sqrt{\varepsilon_1}, ..., \sqrt{\varepsilon_l})$.

In Definition 4.1, take $\varepsilon = (\varepsilon_1, ..., \varepsilon_l) = (0, ..., 0)$, it degrades into the concept of robust weak saddle point presented in literature [7]. Now, we give a concrete example to illustrate the existence of robust quasi ε-weak saddle point.

Example 4.2. Let $C = \mathbb{R}^2,\ \mathcal{U}_1 = \mathcal{U}_2 = [-1, 1],\ \mathcal{V} = [-1, 1]$. $f_1, f_2, g:\ \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f_1((x_1, x_2), u_1) = |x_1| - x_2 - u_1^2, \quad f_2((x_1, x_2), u_2) = -x_2 - |u_2|,$$

$$g((x_1, x_2), v) = x_1^2 - vx_1,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, and $u_1 \in \mathcal{U}_1,\ u_2 \in \mathcal{U}_2,\ v \in \mathcal{V}$ are uncertain parameters. Obviously, the functions f_1, f_2, g are convex functions related to the first variable, and f_1, f_2 are concave functions to the second variable. Consider the following robust multi-objective convex optimization problem:

$$(\text{RUMCOP})^2 \qquad \begin{cases} \min & (\max\limits_{u_1 \in \mathcal{U}_1} (|x_1| - x_2 - u_1^2), \max\limits_{u_2 \in \mathcal{U}_2} (-x_2 - |u_2|)) \\ \text{s.t.} & x_1^2 - vx_1 \leq 0,\ v \in [-1, 1]. \end{cases}$$

It is clear that

$$\max_{u_1 \in \mathcal{U}_1} (|x_1| - x_2 - u_1^2) = |x_1| - x_2, \quad \max_{u_2 \in \mathcal{U}_2} (-x_2 - |u_2|) = -x_2,$$

and the feasible set of $(\text{RUMCOP})^2$ is

$$\mathcal{F}^2 = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1 = 0,\ x_2 \in \mathbb{R}\}.$$

For any $x = (x_1,\ x_2) \in \mathbb{R}^2,\ v \in \mathcal{V} = [-1, 1],\ \lambda \in \mathbb{R}_+$, then, the Lagrangian function of the problem $(\text{RUMCOP})^2$ is given by

$$L(x, v, \lambda) = (|x_1| - x_2, -x_2) + \lambda(x_1^2 - vx_1, x_1^2 - vx_1),$$

and taking $\varepsilon = (1, 1) \in \mathbb{R}^2_+,\ \bar{x} = (0, 0),\ \bar{\lambda} = 1,\ \bar{v} = 1$, then we have

$$\begin{cases} L(\bar{x}, v, \lambda) - \|\lambda - \bar{\lambda}\| \cdot \sqrt{\varepsilon} = -(|\lambda - 1|, |\lambda - 1|) \leqq (0, 0) = L(\bar{x}, \bar{v}, \bar{\lambda}),\ \forall v \in [-1, 1],\ \lambda \in \mathbb{R}_+, \\ L(x, \bar{v}, \bar{\lambda}) + \|\ x - \bar{x}\| \cdot \sqrt{\varepsilon} - L(\bar{x}, \bar{v}, \bar{\lambda}) \\ \qquad = (|x_1| - x_2 + x_1^2 - x_1, -x_2 + x_1^2 - x_1) + (\sqrt{x_1^2 + x_2^2}, \sqrt{x_1^2 + x_2^2}) - (0, 0) \\ \qquad \notin -\mathbb{R}^2_{++},\ \forall x \in \mathbb{R}^2. \end{cases}$$

Hence, $(\bar{x}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^2 \times \mathcal{V} \times \mathbb{R}_+$ is a quasi ε-weak saddle points of problem $(\text{RUMCOP})^2$.

The next Theorem 4.3 shows that the quasi-weak ε-efficient solution of problem (RUMCOP) is quasi ε-weak saddle point under the assumption of robust closed convex cone constraint qualification.

Theorem 4.3. *Let $\bar{x} \in \mathcal{F}$ be a quasi-weak ε-efficient solution of (RUMCOP). If the problem (RUMCOP) satisfies robust closed convex cone constraint qualification, then there exist $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_m) \in \mathbb{R}^m_+,\ \bar{v} = (\bar{v}_1, ..., \bar{v}_m) \in (\mathcal{V}_1, ..., \mathcal{V}_m) = \mathcal{V}$, such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a quasi ε-weak saddle point.*

Proof. Since $\bar{x}$ is a quasi-weak ε-efficient solution of (RUMCOP), it follows from Theorem 3.5 that there exist $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_l) \in \mathbb{R}^l_+$, and $\sum_{i=1}^{l} \bar{\mu}_i = 1,\ \bar{u} = (\bar{u}_1, ..., \bar{u}_l) \in \mathcal{U},\ \bar{v} = (\bar{v}_1, ..., \bar{v}_m) \in \mathcal{V},\ \bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_m) \in \mathbb{R}^m_+$, and by Lemma 2.3, such that

$$0 \in \partial\Big(\sum_{i=1}^{l} \bar{\mu}_i f_i(\cdot, \bar{u}_i)\Big)(\bar{x}) + \sum_{j=1}^{m} \partial\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) + \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i}\mathbb{B} + N(\bar{x}, C),$$

and

$$\sum_{i=1}^{l} \bar{\mu}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

So,

$$0 \in \partial \sum_{i=1}^{l} \bar{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) + \sum_{j=1}^{m} \partial\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) + \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i}\mathbb{B} + N(\bar{x}, C).$$

Since

$$\phi_i(x) = \max_{u_i \in \mathcal{U}_i} f_i(x, u_i),\ i = 1, ..., l,$$

thus, we get that there exist $\bar{\eta} \in \partial\big(\sum_{i=1}^{l} \bar{\mu}_i \phi_i\big)(\bar{x}),\ \bar{\xi}_j \in \partial\bar{\lambda}_j g_j(\cdot, \bar{v}_j)(\bar{x}),\ j = 1, ..., m,\ \bar{\zeta} \in N(\bar{x}, C),\ b \in \mathbb{B}$, such that for any $x \in C$,

$$\langle \bar{\eta}, x - \bar{x} \rangle \leq \sum_{i=1}^{l} \bar{\mu}_i \phi_i(x) - \sum_{i=1}^{l} \bar{\mu}_i \phi_i(\bar{x}), \tag{4.3}$$

$$\sum_{j=1}^{m} \langle \bar{\xi}_j, x - \bar{x} \rangle \leq \sum_{j=1}^{m} (\bar{\lambda}_j g_j(x, \bar{v}_j) - \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j)), \tag{4.4}$$

$$\sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \langle b, x - \bar{x} \rangle \leq \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} \|\ x - \bar{x}\|, \tag{4.5}$$

$$\langle \bar{\zeta}, x - \bar{x} \rangle \leq 0, \tag{4.6}$$

$$\bar{\eta} + \sum_{j=1}^{m} \bar{\xi}_j + \bar{\zeta} + \sum_{i=1}^{l} \bar{\mu}_i \sqrt{\varepsilon_i} b = 0. \tag{4.7}$$

Now, we prove that the equation (4.1) holds. In fact, since $\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) = 0$ and for any $\lambda_j \geq 0,\ v_j \in \mathcal{V}_j,\ \lambda_j g_j(\bar{x}, v_j) \leq 0,\ j = 1, ..., m$, it holds

$$\sum_{j=1}^{m} \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) = 0, \quad \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, v_j) \leq 0.$$

Therefore,

$$\sum_{j=1}^{m} \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) - \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, v_j) \geq 0,$$

this leads to

$$-\|\lambda-\bar{\lambda}\|\cdot\sqrt{\varepsilon}\leqq(0,...,0)\leqq\phi(\bar{x})+(\sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x},\bar{v}_j))\cdot e-\phi(\bar{x})-(\sum_{j=1}^{m}\lambda_j g_j(\bar{x},v_j))\cdot e.$$

Thus, we arrive at

$$L(\bar{x},v,\lambda)-\|\lambda-\bar{\lambda}\|\cdot\sqrt{\varepsilon}\leqq L(\bar{x},\bar{v},\bar{\lambda}),\ \forall v\in\mathcal{V},\ \lambda\in\mathbb{R}^m_+.$$

Next, we examine that

$$L(x,\bar{v},\bar{\lambda})+\|\ x-\bar{x}\|\cdot\sqrt{\varepsilon}-L(\bar{x},\bar{v},\bar{\lambda})\notin-\mathbb{R}^l_{++},\ \forall x\in C.$$

We proceed it by contradiction. If the above equation is not true, then there exists $\hat{x}\in C$ such that

$$\phi(\hat{x})+\sum_{j=1}^{m}\bar{\lambda}_j g_j(\hat{x},\bar{v}_j)\cdot e+\|\ \hat{x}-\bar{x}\|\cdot\sqrt{\varepsilon}-(\phi(\bar{x})+\sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x},\bar{v}_j)\cdot e)\in-\mathbb{R}^l_{++},$$

that is,

$$\phi_i(\hat{x})+\sum_{j=1}^{m}\bar{\lambda}_j g_j(\hat{x},\bar{v}_j)+\|\hat{x}-\bar{x}\|\cdot\sqrt{\varepsilon_i}<\phi_i(\bar{x})+\sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x},\bar{v}_j),\ i=1,...,l.$$

Since $\sum_{i=1}^{l}\bar{\mu}_i=1$, one has that

$$\sum_{i=1}^{l}\bar{\mu}_i\phi_i(\hat{x})-\sum_{i=1}^{l}\bar{\mu}_i\phi_i(\bar{x})+\sum_{j=1}^{m}(\bar{\lambda}_j g_j(\hat{x},\bar{v}_j)-\bar{\lambda}_j g_j(\bar{x},\bar{v}_j))+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}\|\ \hat{x}-\bar{x}\|<0.$$

Together with (4.3)-(4.6), we get that

$$\langle\bar{\eta}+\sum_{j=1}^{m}\xi_j+\sum_{i=1}^{l}\bar{\mu}_i\sqrt{\varepsilon_i}b+\bar{\zeta},\hat{x}-\bar{x}\rangle<0,$$

which contradicts to (4.7). □

We conclude this paper by presenting the following Theorem 4.4, which discloses that a quasi ε-weak saddle point is a quasi-weak ε- efficient solution to problem (RUMCOP) under mild assumptions.

Theorem 4.4. *In problem (RUMCOP), suppose that $(\bar{x},\bar{v},\bar{\lambda})\in C\times\mathcal{V}\times\mathbb{R}^m_+$ is a quasi ε-weak saddle point of problem (RUMCOP), and $\bar{x}$ is an optimal solution of program $\max\limits_{x\in C}\sum_{j=1}^{m}\bar{\lambda}_j g_j(x,\bar{v}_j)$, then $\bar{x}$ is a quasi-weak ε-efficient solution of (RUMCOP).*

Proof. Because $(\bar{x},\bar{v},\bar{\lambda})$ is a quasi ε-weak saddle point for problem (RUMCOP), it follows from (4.2) that

$$\phi(x)-\phi(\bar{x})+(\sum_{j=1}^{m}\bar{\lambda}_j g_j(x,\bar{v}_j)-\sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x},\bar{v}_j))\cdot e+\sqrt{\varepsilon}\cdot\|\ x-\bar{x}\|\notin-\mathbb{R}^l_{++},\ \forall x\in C.\quad(4.8)$$

On the other hand, since $\bar{x}$ is an optimal solution of program $\max\limits_{x\in C}\sum_{j=1}^{m}\bar{\lambda}_j g_j(x,\bar{v}_j)$, one has that

$$\sum_{j=1}^{m}\bar{\lambda}_j g_j(x,\bar{v}_j)-\sum_{j=1}^{m}\bar{\lambda}_j g_j(\bar{x},\bar{v}_j)\leq 0,\ \forall x\in C.$$

By (4.8), we obtain that

$$\phi(x)-\phi(\bar{x})+\sqrt{\varepsilon}\cdot\|\ x-\bar{x}\|\notin-\mathbb{R}^{l}_{++},\ \forall x\in C.$$

Therefore, $\bar{x}$ is a quasi-weak ε-efficient solution of (RUMCOP). □

Remark 4.5. Kim [7] established the weak saddle point theorems for a robust multi-objective convex optimization problem. This paper proposes quasi ε-weak saddle point theorems for problem (RUMCOP). Since the concept of quasi ε-weak saddle point is the generalization of weak saddle point, so the above results extend those of literature [7].

5 Conclusions

In this paper, the optimality conditions and saddle point theorems for approximate solution of the uncertain multi-objective convex optimization problem (UMCOP) are studied by using the robust optimization method. Firstly, the scalarization theorem of quasi-weak robust ε-efficient solution of problem (UMCOP) is proved. Secondly, under the assumption that problem (UMCOP) satisfies robust closed convex cone constraint quafication, the optimality conditions for quasi-weak robust ε-efficient solution are established. Finally, the concept of robust quasi ε-weak saddle point is introduced and the corresponding saddle point theorems are provided.

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