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INTEGRATED STRATEGY OF PRODUCTION, TRANSPORTATION AND PRICING FOR MULTI-PRODUCT MULTI-MARKET UNDER FUZZY ENVIRONMENT AND ITS APPLICATION*

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, an integrated fuzzy production, transportation and pricing model for multi-product multi-market is presented by considering three factors, production planning, transportation strategy and pricing decision as a system. We discuss the properties of the fuzzy objective function, and propose a solution method for the integrated fuzzy production, transportation and pricing model for multi-product multi-market model based on a generalized convex theoretical foundation. An equivalence relationship between the proposed model and a fuzzy variational-like inequality problem is developed, and some relevant theorems of solution for the fuzzy variational-like inequality are proved. Finally, An numerical example is provided to illustrate the validity of the method. Compared with the traditional method, the results show that the proposed method can effectively reduce transportation costs and increase total profits.

Key words: *production planning, transportation, pricing, fuzzy variational-like inequality, generalized concave*

Mathematics Subject Classification: *90C70*

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1 Introduction

In recent years, there has been a growing body of research focusing on production planning, pricing decisions and transportation strategies under uncertainty. There are many uncertain factors when seeking to determine production, transportation and pricing, such as manufacturing costs, customer demand, and transportation costs. As these uncertain factors can affect revenue management, they can not be neglected. Much of the research in this area has tended to concentrate on the randomness aspect of uncertainty and several stochastic modeling techniques based on certain probability assumptions have been developed in which the uncertain parameters are typically modeled using probability distributions

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(see [11, 15, 31, 41]). However, in practice, probability distributions may not be available or may be difficult to estimate exactly from limited data points, e.g. when deciding upon prices, the decision-makers only has some vague ideas about the levels of demand associated with each feasible price. Under this scenario, fuzzy set theory, which was introduced by Zadeh in 1965, is considered an appropriate modeling tool when uncertain parameters cannot be adequately described in distributions. Uncertainty parameters are able to be estimated approximately using the managers' judgments, intuition and experience, and can be characterized as fuzzy variables [58].

We consider the following problem. Monopoly firms have many factories which produce multiple products with a varying demand. These products are sent to multiple sub-markets through distribution centers. However, as the different sub-markets have different fuzzy demands, the product prices are different in these sub-markets and can be represented as fuzzy functions of the demands. At the same time, the different products have different fuzzy transportation costs. How many products to be produced in the factories, how to price in the sub-markets, how to allocate these products efficiently and how to transport these products to each sub-market are problems that decision-makers have to solve to guarantee maximum profit, which is the primary focus of this paper.

There exist researches on production planning, transportation and pricing decision problems have typically focused on considering individually production/pricing problems [6, 9, 14, 33, 55], or on considering individually transportation problems. Very few researches have jointly considered the production decisions, the transportation strategies and pricing decisions in one optimization model. In terms of the production and pricing problem for multi-product multi-market in a deterministic or random environment, extensive researches have been conducted on third-degree price discrimination and its social welfare effects (see [18, 23, 25]). In a fuzzy environment, a novel two-stage fuzzy optimization method for solving the multi-product multi-period production planning problem was proposed by Yuan [56]. Vasant [52] presented new methods for solving fuzzy production planning problems with vagueness parameters alpha and fuzzy objective coefficients. Sun [51] proposed a bilevel model to describe the pricing and production decisions with fuzzy demand and fuzzy cost parameters. The upper level is to determine the optimal price and production quantity with capacity constraints. The lower level problem tries to structure a distribution pattern of customers or markets that will satisfy his demand at minimum cost. Zhao et al [57] analyzed the pricing problems of substitutable products in a fuzzy supply chain using game theory. An optimal pricing decision problem for a fuzzy closed-loop supply chain with retail competition was considered by Wei and Zhao [53]. In Ref. [1], the optimal decision problem is analyzed based on a fuzzy price and sales effort-dependent demand to evaluate how members decide wholesale price, collection rate, retail price, and sales effort under different decision-making structures. Six game theory models are established and optimal solutions are extracted and compared by applying game and fuzzy theories. Ghomi-Avili et al [19] presents a fuzzy bi-objective bi-level model with a price-dependent demand for the network design of a closed-loop supply chain in the presences of random disruptions at suppliers.

There have been significant researches on transportation problem, such as [10, 12, 36, 48, 49]. However, most of these considered a transportation equilibrium model under known conditions of supply and demand between each origin-destination pair under deterministic or random cost. In a fuzzy environment, Basirzadeh [2] proposed a systematic procedure for solving all types of fuzzy transportation problem whether maximize or minimize objective function. Chakraborty et al [7] proposed a new approach for solving a fully fuzzy transportation problem, which is an extension of popular approach of solving transportation problem (i.e., North West corner method, least cost method, Vogel's approximation method, modified distribution method). Maity and Roy [32] explored the study of fuzzy transportation problem (FTP) using multi-choice goal-programming approach. Kundu et al [27] presented type-2 fuzzy variables for solving fixed charge transportation problem where total transportation cost is minimized. Sinha et al [50] proposed a fuzzy transportation problem with two objective functions, one is to maximize the profit while the other is to minimize the transportation time. Baykasoglu et al [3] presented a novel method based on the constrained fuzzy arithmetic concept to solve fully fuzzy balanced/unbalanced transportation problems in which all of the parameters (source capacities, demands of destinations, transportation costs etc.) as well as the decision variables (transportation quantities) are considered as fuzzy numbers.

About integrated production and transportation problem, there also exist significant researches. Under certain condition, Philippe et al [39] proposed an extension of the integrated production and transportation scheduling problem by considering multiple vehicles for optimisation of supply chains. Jalil et al [24] proposed a de-centralized bi-level multi-objective model for integrated production and transportation problems in closed-loop supply chain network with positive and reverse product flow. Under fuzzy condition, Sakawa et al [45] consider a production and transportation fuzzy planning under the assumption that the manufacturer makes multiple products at factories in multiple regions and the products are in demand in each of the regions. A fuzzy linear programming to describe production and transportation plan that minimize the total sum of the production cost and the transportation cost was presented by Rommelfanger in [43]]. Liang [29] presented a novel fuzzy mutiobjective linear programming model to solve integrated production-transportation planing decision problems in supply chains in a fuzzy environment.

In this paper, we jointly consider above three problems: production planning, transportation strategy and pricing decision for multi-product multi-market in a fuzzy environment. For this complex problem, extensive research has been conducted and has produced many different fuzzy production-pricing models and fuzzy transportation strategies, or fuzzy production-transportation models and fuzzy pricing strategies. But very few have jointly considered the production planning, the transportation strategy and pricing decision for multi-product multi-market in one fuzzy optimization model. Note that most fuzzy production planning, transportation and pricing for multi-product multi-market (FPTPMM for short) problem cannot be obtained through a mere addition of a single fuzzy production-pricing strategy and single fuzzy transportation strategy, as the different sub-markets have different demands, and the different transportation paths have different transportation costs associated with the production quantities. The optimal production quantity determined in fuzzy production-pricing strategy has limited the traffic flow in fuzzy transportation planning, which means that the total profit is not necessarily maximized. Instead, it should be considered as a system which has three influence factors, namely, fuzzy market demand, fuzzy production cost and fuzzy transportation cost. It's necessary to propose an integrated FPTPMM model. Xu et al [54] have simultaneously considered a production, transportation and pricing optimization problem for multi-product multi-market in one optimization under the certain conditions. In this paper, we consider the FPTPMM problem as a complicated transportation network refer to Xu et al [54], where the supply and demand are fuzzy decision variables, and transportation is also fuzzy decision variables. The objective is to pursuit the maximum profit of the whole network, and an integrated FPTPMM model would be proposed. In order to solve the FPTPMM model, we will discuss the relationship between the integrated FPTPMM model and vector variational inequality.

The remainder of this paper is organized as follows. Section 2 gives the key problem description and some notations which will be used in the sequel. Section 3 establishes the integrated FPTPMM model based on transportation network graph. In Section 4, according to giving some preliminaries and the analysis for generalized concavity property, such as the incave and preincave, equivalence relationships between the integrated FPTPMM model and a fuzzy variational-like inequality problem are developed. We also propose a specific solution method for the integrated FPTPMM model in this section. Section 5 proposes a numerical example and its compare results with traditional model in which the production planning, transportation and pricing problems are solved separately under fuzzy environment. Section 6 presents some conclusions and offers ideas for further research.

2 Key Problem Statement

A monopolistic firm has n_1 factories which produce homogeneous products. These products are sent to n_3 sub-markets through n_2 distribution centers. Different factories have different production costs, different transportation arcs have different transportation costs and different sub-markets have different demands and product prices. For each pair of factories and sub-markets, there may be many connecting paths. We consider a production and transport network in which m products traverse the network, with a typical product denoted by j. Consider a general network $G = (N, A)$ where N denotes the set of nodes representing n_1 factories $(O_1, O_2, \cdots, O_{n_1})$ and n_2 distribution centers $(M_1, M_2, \cdots, M_{n_2})$, as well as n_3 sub-markets $(D_1, D_2, \cdots, D_{n_3})$, i.e. $|N| = n_1 + n_2 + n_3$, as depicted in Fig. 1(a). For each pair of factories (named the origin) and sub-markets (named the destination), there may be many possible paths (see Fig. 1(b), left). Price of each product at each sub-market is not an constant and dependent on its demand quantity. Because different sub-markets have different demand functions and different arcs have different transportation costs, to obtain maximum profits the decision makers need to choose production outputs for every factory and transportation flows and paths from the point of origin to the point of destination. That is to say, for each product, a monopolistic firm must decide how many products need to be produced in each factory, the most efficient way to transport these products from the factories to the sub-markets, and the ideal price for each product in each sub-market. Decision-makers need to decide the production output of each factory, the transportation flow for each arc and the price in each sub-market. The decision variable in our model is the network production flow matrix q. If we know the production flow matrix q, then we also know the factory production quantities, the price of each product in the sub-markets, and the transport methods for these products.

Motivation for employing fuzzy functions for PTPMM problem are given as follows. In real production, transportation and pricing problem, the available data are not always exact or precise. Various types of uncertainties appear in those data due to various reason such as insufficient information, lack of evidence, linguistic information, imperfect statistical analysis, etc. In order to describe and extract the useful information hidden in uncertain data and to use this data properly in practical problems, many researchers have proposed a number of improved theories such as fuzzy set, random set, rough set etc. When some of or all the system parameters associated with a decision making problem are not exact or precisely defined, moreover those are represented by fuzzy, random or rough sets(/variables), etc., then it is called that the problem is defined in those uncertain environment respectively. As different people have different feelings for uncertain demand and cost caused by the uncertain environment and there is no clear definition of this change. So in this situation, it can be characterized by uncertainty of fuzziness, and stochastic models may not be the best choice. Fuzzy set theory may provide an alternative approach for dealing with these uncertainty. It can be used to represents the uncertain production cost, unreliable transportation cost,

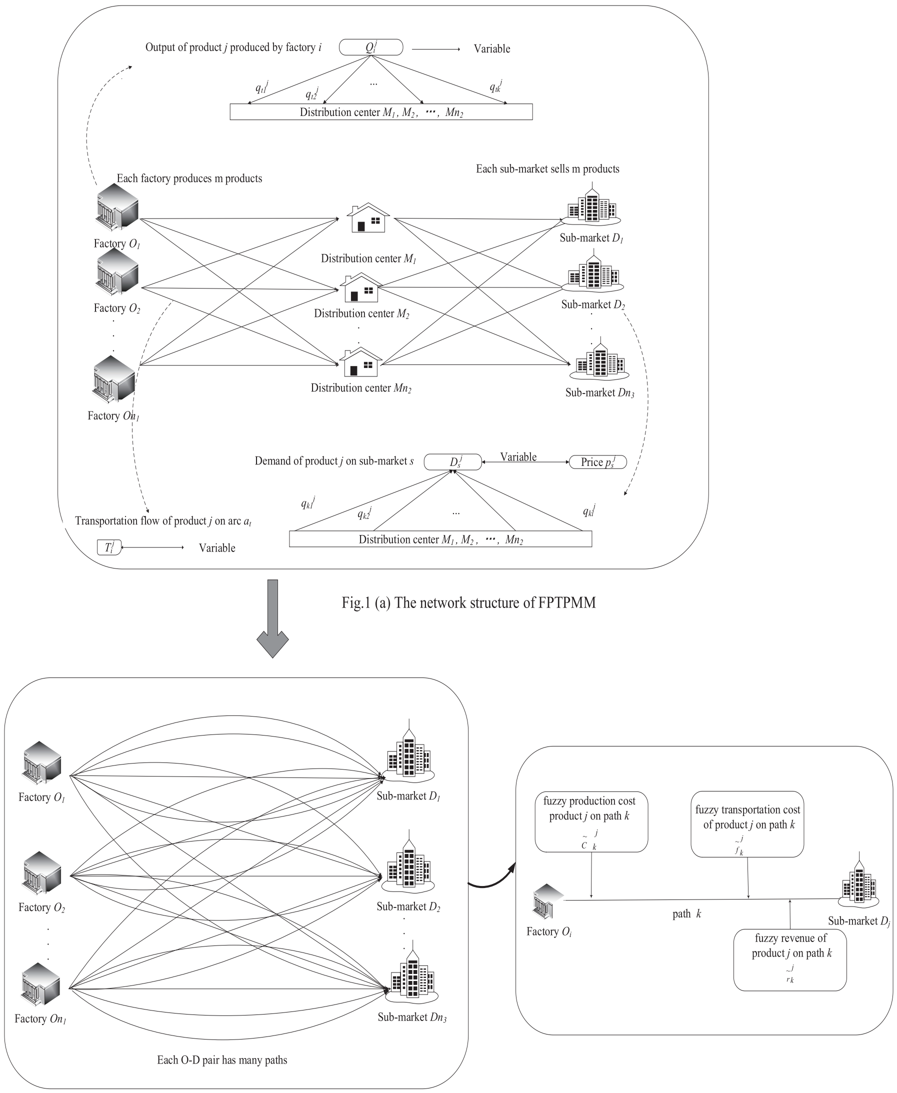

Figure 1: The network structure and path graph of FPTPMM

and the fluctuated customers demand. Therefore, fuzzy variables that can take into account fuzziness are favored by decision-makers to describe the uncertainty and vague information.

The greatest uncertainties in this study are caused by the modelling technique-related parameters. Take transportation cost of transporting one unit of the product as an example. In practice, in order to collect the transportation cost of transporting one unit of the product, investigations and surveys were made to different experts, and the experts usually can not give an exact expression for it. So in this situation, triangular fuzzy numbers are more suitable to explain uncertainty in these parameters such as transportation cost of transporting one unit of the product is about d_m, but definitely not less than d_L and grater than d_R. Therefor, triangular fuzzy numbers (d_L, d_m, d_R) is applied to express parameter for transportation cost of transporting one unit of the product. The situations are similar with the production costs and the customer demands. These can be estimated by experts and professional engineers using fuzzy variables to interpret parameters for production cost function and market demand function.

The list of symbols used for the development of the model is given in Table 1.

Notations	*Explanation*
$\mathbb{A}$	set of directed arcs, let $a_t \in \mathbb{A}$ denote an arc connecting a pair of nodes
I	set of all the O-D pairs associated with each pair of factor and sub-market, and $\lvert I\rvert = l$
P_i	set of paths that connect an O-D pair $i \in I$
$k \in P_i$	denote the path k, consisting of a sequence of arcs connecting an O-D pair of nodes i
q_k^j	transportation flow (named production flow in following) on path k of product j
q^j	production vector of product j, and $q^j = (q_1^j, q_2^j, \cdots, q_M^j)^T$, where $M = \sum_{i\in I} \lvert P_i\rvert$
q_k	production vector on path k, and $q_k = (q_k^1, q_k^2, \cdots, q_k^m)$
q	an production flow matrix of the network, $q = (q_k^j)_{M\times m}$
Q_n^j	production quantity of product j produced by factory O_n
$\tilde{c}_t^j$	fuzzy transportation cost of product j on arc a_t
$\tilde{c}_t$	total fuzzy transportation cost on arc a_t
$\tilde{C}_k^j$	fuzzy transportation cost of product j on path k
d_i^j	demand quantity of product j for the O-D pair i
D_s^j	demand quantity of product j in sub-market D_s
B_s	set of paths which the destination of the path is sub-market D_s
$\tilde{g}_n^j$	fuzzy production cost of product j produced by factory O_n
$\tilde{g}_n$	total fuzzy production cost in factory O_n
$\tilde{f}_k^j$	fuzzy production cost of product j allocated on path k
$(\tilde{C}_{total})_k^j$	total fuzzy cost for product j on path k, and $(\tilde{C}_{total})_k^j = \tilde{C}_k^j \oplus \tilde{f}_k^j$
$\tilde{p}_s^j$	fuzzy price of product j in sub-market D_s
$\tilde{r}_k^j$	fuzzy revenue of product j on path k
$\tilde{\pi}_k^j$	fuzzy profit of product j on path k
$\bigoplus$	abbreviation for fuzzy sum, for example, $\bigoplus_{i=1}^3 x_i = x_1 \oplus x_2 \oplus x_3$

Table 1: Notations used for the development of the model

3 Modelling

3.1 Assumption

Considering the complexity of the problem and referring to the existing literature [30, 40], we do some assumptions as follows:

(i) The transportation cost functions are separable, the total transportation cost on each arc is equal to the sum of the transportation costs of all products.

If the unit transportation cost for each product on each arc is constant. then the assumption (i) is reasonable. For the case that different products are delivered using the same mode of transportation (e.g., truck), we assume that the total transportation cost can be distributed to each product according to some means (e.g., the ratio of each product transportation flow).

(ii) The production cost functions are separable, i.e., the total production cost in each factory is equal to the sum of the production costs of all products.

(iii) There are no inventories in the side of factories, distribution centers and demand markets.

(iv) The transportation cost function, production cost and market demand function are fuzzy functions, and the parameters of them are triangular fuzzy numbers.

3.2 Integrated FPTPMM model

Our goal is to find the production flow matrix at which profit is maximized. To obtain the optimal production flow matrix q, we need to know the objective function for this problem, that is, the total profit function. In the following, we first allocate production cost and revenue for each product to each path and study the cost function and revenue function of each product on each path (see Fig. 1(b), right), and then calculate the profit on each path.

Because d_i^j is said to be the demand for the O-D pair i for product j, then $\sum_{k\in P_i} q_k^j = d_i^j$, which states that the demand for the O-D pair i for product j is equal to the sum of the production flows on all paths of the O-D pair i for product j. So the set

$$Q = \{q : q_k^j \geq 0, \sum_{k\in P_i} q_k^j = d_i^j, i \in I, j = 1, 2, \cdots, m\}$$

is the feasible set. Q is clearly a convex set. In many production and transportation problems, the demand on sub-market were considered as known constant (See [26, 34, 49] or fuzzy numbers with known membership function (See [13, 20, 37, 38]). In this paper, d_i^j is considered to be a decision variable which is decided by the production flow of product j on path k, q_k^j. So, the feasible set can be rewritten as

$$Q = \{q : q_k^j \geq 0, i \in I, k \in P_i, j = 1, 2, \cdots, m\}.$$

Firstly, we find the fuzzy transportation cost function for product j on path k. The fuzzy cost of product j on path k should be the fuzzy cost of all products j on arcs a_t which belong to the sequence of arcs on path k (See [?, 28]). So, according to assumption (i), the fuzzy transportation cost function for product j on path k, $\tilde{C}_k^j : R^{M\times m} \to E(j = 1, 2, \cdots, m)$,

$$\tilde{C}_k^j(q_k^j) = \bigoplus_{a_t\in\mathbb{A}} \delta_{tk}\tilde{c}_{a_t}^j(q_k^j),$$

where $\delta_{tk} = 1$, if arc a_t is contained in path k, and 0, otherwise. If unit cost of transportation of unit product on arc a_t is a fuzzy number like references [16, 37, 38], then $\tilde{c}_{a_t}^j(q_k^j)$ is a fuzzy linear function: $\tilde{c}_{a_t}^j(q_k^j) = \tilde{c}_t^j \cdot q_k^j$, which is a special case.

Remark 3.1. In this formula, we have used the fuzzy Minkowski addition $\tilde{u} \oplus \tilde{v}$ and multiplication $\lambda\tilde{u}$. About their operations we use the α-level cut set to represent. More detail about fuzzy Minkowski addition and multiplication please see Section 4.

Secondly, we need to determine the fuzzy production cost function for product j on path k. If the origin of path k is factory O_n, then the production cost for product j on path k is $g_n^j(q_k^j)$. If, for example, path 1 is composed of arc $O_1 \to M_1$ and arc $M_1 \to D_1$ in Fig. 1(a), i.e., the origin of path 1 is factory O_1, according to assumption (ii), then the fuzzy production cost function for product j on path 1 is $\tilde{f}_1^j(q_1^j) = \tilde{g}_1^j(q_1^j)$, path 5 is composed of arc $O_2 \to M_1$ and arc $M_1 \to D_1$, i.e., the origin of path 5 is factory O_2, then the fuzzy production cost for product j on path 5 is $\tilde{f}_5^j(q_5^j) = \tilde{g}_2^j(q_5^j)$. So, the fuzzy production cost for product j on path k can be written as

$$\tilde{f}_k^j(q_k^j) = \bigoplus_{n=1}^{n_1} \alpha_{kn}\tilde{g}_n^j(q_k^j),$$

where $\alpha_{kn} = 1$, if the origin of path k is factory O_n, and 0, otherwise.

Because the total cost includes production cost and transportation cost, then the total fuzzy cost for product j on path k can be written as

$$(\tilde{C}_{total})_k^j(q_k^j) = \tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j).$$

Next, we need to determine the fuzzy revenue function for product j on path k. Obviously, revenue is the simple multiplication of quantity by price: $Revenue = Quantity * Price$, where Quantity is the total number of units we sell in market, and Price is the amount we charged for each unit, which is a fuzzy function. The only variable we need to control is the quantity. But the quantity we choose will influence the price through the demand function. The fuzzy price of product j at sub-market D_s is related to the demand for all products in sub-markets D_s, i.e., the fuzzy demand function

$$\tilde{p}_s^j(q) = \tilde{p}_s^j(D_s^1, D_s^2, \cdots, D_s^m),$$

where $D_s^j = \sum_{k \in B_s} q_k^j$.

If the destination of path k is D_s, the revenue function of product j on the path k should be equal to the transport volume of product j on path k multiply by the price of product j on sub-market D_s [54]. For example, in Fig. 1(a), path 1 is composed of arc $O_1 \to M_1$ and arc $M_1 \to D_1$, i.e., the destination of path 1 is sub-market D_1, the fuzzy revenue function for product j on path 1 is $\tilde{r}_1^j(q) = \tilde{p}_1^j(q)q_1^j$, path 5 is composed of arc $O_2 \to M_1$ and arc $M_1 \to D_1$, i.e., the destination of path 2 is also sub-market D_1, then the fuzzy revenue function for product j on path 5 is $\tilde{r}_5^j(q) = \tilde{p}_1^j(q)q_5^j$, path 9 is composed of arc $O_2 \to M_1$ and arc $M_1 \to D_3$, i.e., the destination of path 9 is sub-market D_3, then the fuzzy revenue function for product j on path 9 is $\tilde{r}_9^j(q) = \tilde{p}_3^j(q)q_9^j$. So it holds that

$$\tilde{r}_k^j(q) = \bigoplus_{s=1}^{n_3} \sigma_{ks}\tilde{p}_s^j(q)q_k^j,$$

where $\sigma_{ks} = 1$, if the destination of path k is sub-market D_s, and 0, otherwise.

Finally, we need to determine the profit for product j on path k. According to the above discuss, the fuzzy profits for product j on path k can be written as

$$\tilde{\pi}_k^j(q) = \tilde{r}_k^j(q) \ominus_g (\tilde{C}_{total})_k^j(q_k^j). \tag{3.1}$$

Remark 3.2. Here, symbol $\ominus_g$ represents generalized difference (g-difference for short). In fact, fuzzy difference have three forms, Hukuhara difference (H-difference for short), generalized Hukuhara difference (gH-difference for short) and generalized difference [21, 47].

The reason we use the g-difference to express the fuzzy revenue will be given later in Section 4.

So, the fuzzy profits function of product j is

$$\tilde{F}^j(q) = \bigoplus_{i\in I}\bigoplus_{k\in P_i} \tilde{\pi}_k^j(q), j = 1, 2, \cdots, m.$$

and the total fuzzy profits function of monopolistic firm is

$$\tilde{F}(q) = \bigoplus_{j=1}^{m} \tilde{F}^j(q) = \bigoplus_{j=1}^{m}\bigoplus_{i\in I}\bigoplus_{k\in P_i} \tilde{\pi}_k^j(q). \tag{3.2}$$

The monopolistic firm seeks to maximize profits across the whole network, so we have the following integrated FPTPMM model

$$\max_{q\in Q} \tilde{F}(q) = \bigoplus_{j=1}^{m}\bigoplus_{i\in I}\bigoplus_{k\in P_i} \tilde{\pi}_k^j(q) \tag{3.3}$$

$$\begin{cases} \tilde{\pi}_k^j(q) = \tilde{r}_k^j(q) \ominus_g (\tilde{C}_{total})_k^j(q_k^j) \\ \tilde{r}_k^j(q) = \bigoplus_{s=1}^{n_3} \sigma_{ks}\tilde{p}_s^j(q)q_k^j \\ \tilde{p}_s^j(q) = \tilde{p}_s^j(D_s^1, D_s^2, \cdots, D_s^m) \\ D_s^j = \sum_{k\in B_s} q_k^j \\ (\tilde{C}_{total})_k^j(q_k^j) = \tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j) \\ \tilde{C}_k^j(q_k^j) = \bigoplus_{a_t\in \mathbb{A}} \delta_{tk}\tilde{c}_{a_t}^j(q_k^j) \\ \tilde{f}_k^j(q_k^j) = \bigoplus_{n=1}^{n_1} \alpha_{kn}\tilde{g}_n^j(q_k^j) \\ Q = \{q : q_k^j \geq 0\} \\ i \in I, k \in P_i, j = 1, 2, \ldots, m. \end{cases}$$

where $\sigma_{ks} = 1$, if the destination of path k is sub-market D_s, and 0, otherwise. $\delta_{tk} = 1$, if arc a_t is contained in path k, and 0, otherwise. $\alpha_{kn} = 1$, if the origin of path k is factory O_n, and 0, otherwise.

The proposed integrated FPTPMM model (3.3) consider the three factors affecting the total profits, fuzzy production cost, fuzzy transportation cost and fuzzy demand as a system, which does not simply add the existing production quantity decision model, distribution and transportation problem and pricing model together. This research method, allocate production cost and revenue for each product to each path and then calculate the profit for each product to each path, is not only very simple, but also can overcome the difficulties of comprehensive consideration.

4 Solution Method

In order to determine the solution methods for the above fuzzy programming problem, we must first analyze the properties of fuzzy-valued function $\tilde{F}(q)$. The notion of convexity plays an important role in economic theory and modeling, but the conditions for convex and concave fuzzy-valued function are very strict. In the following, we discuss the production, transportation and pricing problems under generalized convex and generalized concave fuzzy-valued function.

$\boxed{4.1}$ Theoretical basis

We first give some preliminary definitions and results which will be needed in the sequel.

Let E denotes the family of fuzzy numbers. Obviously, the α-level set $\tilde{a}_\alpha$ is a closed, bounded and convex subset of R for each $\alpha \in [0,1]$. i.e., a closed interval in R. Therefore, we can denote it using $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^R_\alpha]$. Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers. We have the following useful results for standard Minkowski addition, H-difference and multiplication

$$(\tilde{u} \oplus \tilde{v})_\alpha = [\tilde{u}^L_\alpha + \tilde{v}^L_\alpha, \tilde{u}^R_\alpha + \tilde{v}^R_\alpha], (\tilde{u} \ominus_H \tilde{v})_\alpha = [\tilde{u}^L_\alpha - \tilde{v}^L_\alpha, \tilde{u}^R_\alpha - \tilde{v}^R_\alpha].$$

$$\lambda \tilde{u}_\alpha = \begin{cases} [\lambda \tilde{u}^L_\alpha, \lambda \tilde{u}^R_\alpha], \lambda > 0; \\ [\lambda \tilde{u}^R_\alpha, \lambda \tilde{u}^L_\alpha], \lambda < 0. \end{cases} \tag{4.1}$$

However, the H-difference of two fuzzy numbers does not always exist for arbitrary pairs of fuzzy numbers. It only exists for fuzzy numbers $\tilde{u}$ and $\tilde{v}$ for which the widths are such that $len(\tilde{u}) \geq len(\tilde{v})$, where $len(\tilde{u}) = \tilde{u}^R_\alpha - \tilde{u}^L_\alpha$ is the length of fuzzy number $\tilde{u}$. Recently, Stefanini introduced the concept of generalized H-difference for the two fuzzy numbers $\tilde{u}, \tilde{v}$ (gH-difference for short) and is defined as follows.

Definition 4.1 ([47])**.** Given $\tilde{u}, \tilde{v} \in E$, the gH-difference is the fuzzy quantity $\tilde{w} \in E$, if it exists, such that $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Longleftrightarrow (i)\ \tilde{u} = \tilde{v} \oplus \tilde{w}$ *or* $(ii)\ \tilde{v} = \tilde{u} \oplus (-1)\tilde{w}$.

In terms of α-cuts, if the gH-difference $\tilde{u} \ominus_{gH} \tilde{v}$ exists, then

$$(\tilde{u} \ominus_{gH} \tilde{v})_\alpha = [\min\{\tilde{u}^L_\alpha - \tilde{v}^L_\alpha, \tilde{u}^R_\alpha - \tilde{v}^R_\alpha\}, \max\{\tilde{u}^L_\alpha - \tilde{v}^L_\alpha, \tilde{u}^R_\alpha - \tilde{v}^R_\alpha\}].$$

Similarly, it is possible that the gH-difference of two fuzzy numbers does not exist. So we introduce another concept of difference proposed by Stefanini, a difference that always exists.

Definition 4.2 ([47])**.** The generalized difference (g-difference for short) of two fuzzy numbers $\tilde{u}, \tilde{v}$ is given by its level sets as

$$(\tilde{u} \ominus_g \tilde{v})_\alpha = cl \bigcup_{\beta \geq \alpha} (\tilde{u}_\beta \ominus_{gH} \tilde{v}_\beta),$$

where the gH-difference $\ominus_{gH}$ is with interval operands $\tilde{u}_\beta, \tilde{v}_\beta$.

The following lemma give simplified notation for $\tilde{u} \ominus_g \tilde{v}$.

Lemma 4.3 ([5])**.** *For any two fuzzy numbers $\tilde{u}, \tilde{v} \in E$, the g-difference $\tilde{u} \ominus_g \tilde{v}$ exists and, for $\forall \alpha \in [0,1]$, we have*

$$(\tilde{u} \ominus_g \tilde{v})_\alpha = [\min\{\inf_{\beta \geq \alpha}(\tilde{u}^L_\beta - \tilde{v}^L_\beta), \inf_{\beta \geq \alpha}(\tilde{u}^R_\beta - \tilde{v}^R_\beta)\}, \max\{\sup_{\beta \geq \alpha}(\tilde{u}^L_\beta - \tilde{v}^L_\beta), \sup_{\beta \geq \alpha}(\tilde{u}^R_\beta - \tilde{v}^R_\beta)\}].$$

Let $\tilde{f} : K(\subset R^n) \to E$ is said to be a fuzzy-valued function. For any $\alpha \in (0,1]$, denote $\tilde{f}(x)_\alpha = [\tilde{f}(x)^L_\alpha, \tilde{f}(x)^R_\alpha]$. Here the endpoint functions $\tilde{f}(x)^L_\alpha, \tilde{f}(x)^R_\alpha : K \to R$ are called upper and lower functions of $\tilde{f}$, respectively.

About fuzzy derivative, both definitions for the derivative, the H-derivative (initially introduced by Puri and Ralescu [42]) and the S-derivative (initially introduced by Seikkala [46]), are very restrictive. So here we use the more general definition of the derivative for the fuzzy-valued function introduced by Bede and Gal [4] by enlarging the class of differentiable fuzzy-valued function. For more details about the H-derivative, S-derivative and G-derivative see [8, 44]. In this paper, we use the following g-differentiability concept proposed by Bede and Stefanini, that further extends the differentiability of fuzzy-valued function.

Definition 4.4 ([5]). Let $x_0 \in [a, b]$ and h be such that $x_0 + h \in [a, b]$, then the g-derivative of a function $\tilde{f} : [a, b] \to E$ at x_0 is defined as

$$\tilde{f}'(x_0) = \lim_{h \to 0} \frac{\tilde{f}(x_0 + h) \ominus_g \tilde{f}(x_0)}{h}. \tag{4.2}$$

If $\tilde{f}'(x_0) \in E$ satisfying (4.2) exists, we say that $\tilde{f}$ is generalized differentiable (g-differentiable for short) at x_0.

Lemma 4.5 ([5]). *Let $\tilde{f} : [a, b] \to E$ be such that $\tilde{f}(x)_\alpha = [\tilde{f}(x)^L_\alpha, \tilde{f}(x)^R_\alpha]$. If $\tilde{f}(x)^L_\alpha$ and $\tilde{f}(x)^R_\alpha$ are differentiable real-valued functions with respect to x, uniformly for $\alpha \in [0, 1]$, then $\tilde{f}(x)$ is g-differentiable and we have*

$$\tilde{f}'(x)_\alpha = [\inf_{\beta \geq \alpha} (\min\{\tilde{f}'(x)^L_\beta, \tilde{f}'(x)^R_\beta\}), \sup_{\beta \geq \alpha} (\max\{\tilde{f}'(x)^L_\beta, \tilde{f}'(x)^R_\beta\})].$$

Definition 4.6 ([5]). Let $\tilde{H}(x)$ be a fuzzy-valued function defined on $K \subset R^n$ and let $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ be an element of K fixed. We consider the fuzzy-valued function $\tilde{f}_i(x_i) = \tilde{H}(x_1^{(0)}, x_2^{(0)}, \ldots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \ldots, x_n^{(0)})$. If $\tilde{f}_i(x_i)$ is g-differentiable at $x_i^{(0)}$, then we say that $\tilde{H}(x)$ has the ith partial g-derivative at x_0, denoted by $\tilde{H}'_i(x_0)$, and $\tilde{H}'_i(x_0) = (\tilde{f}_i)'(x_i^{(0)})$.

Definition 4.7 ([5]). Let $\tilde{H}(x)$ be a fuzzy-valued function defined on $K \subset R^n$ and let $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ be fixed. We say that $\tilde{H}(x)$ is g-differentiable at x_0 if all the partial g-derivatives $\tilde{H}'_1(x_0), \tilde{H}'_2(x_0), \ldots, \tilde{H}'_n(x_0)$ exist on some neighborhoods of x_0 and are continuous at x_0. Given a fuzzy-valued function $\tilde{H}$, the generalized gradient of $\tilde{H}$ at x_0, denoted by $\tilde{\nabla}\tilde{H}(x_0)$ is defined by

$$\tilde{\nabla}\tilde{H}(x_0) = (\tilde{H}'_1(x_0), \tilde{H}'_2(x_0), \ldots, \tilde{H}'_n(x_0)).$$

Let $X = (x_{ij})_{m\times n}$ be a matrix of $R_{m\times n}$, $X^0 = (x_{ij}^{(0)})_{m\times n}$ be fixed, and X_{ij} denote a matrix whose the ijth element is variable x_{ij}, others are fixed constants $x_{ij}^{(0)}$. If $\tilde{H} : M(\subset R^{m\times n}) \to E$ be a fuzzy-valued function, then $\tilde{H}(X_{ij}) : T \to E$ is a fuzzy-valued function of one variable.

Next, we introduce some special notations, definitions and results which will be used in FPTPMM problem.

Definition 4.8. Let $\tilde{H} : M(\subset R_{m\times n}) \to E$ be a fuzzy-valued function and let matrix X^0 be fixed. We consider the fuzzy-valued function $\tilde{f}_{ij}(x_{ij}) = \tilde{H}(X_{ij})$. If $\tilde{f}_{ij}(x_{ij})$ is g-differentiable at $x_{ij}^{(0)}$, then we say that $\tilde{H}$ has the ijth partial g-derivative at X^0, denoted by $\tilde{H}'_{ij}(X^0)$, and $\tilde{H}'_{ij}(X^0) = (\tilde{f}_{ij})'(x_{ij}^{(0)})$. We say that $\tilde{H}$ is g-differentiable at X^0 if all the partial g-derivatives $\tilde{H}'_{ij}(X^0)$ exist on some neighborhoods of X_0 and are continuous at X_0. And if $\tilde{F}$ to be g-differentiable at X, we call $[\tilde{H}'_{ij}(X^0)]_{m\times n}$ the gradient of $\tilde{f}$ at X, also denote $\tilde{\nabla}\tilde{H}(x) = [\tilde{H}'_{ij}(X^0)]_{m\times n}$.

For example, $q = (q_k^j)_{M\times m} \in Q$, $\tilde{F} : K(\subset R_{M\times m}) \to E$ be a fuzzy-valued function, $\tilde{F}'_{kj}(q)$ denote the partially derivative of $\tilde{F}(q)$ w.r.t. the component q_k^j, then

$$\tilde{\nabla}\tilde{F}(q) = \begin{pmatrix} \tilde{F}'_{11}(q) & \tilde{F}'_{12}(q) & \cdots & \cdots \tilde{F}'_{1m}(q) \\ \tilde{F}'_{21}(q) & \tilde{F}'_{22}(q) & \cdots & \cdots \tilde{F}'_{2m}(q) \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{F}'_{M1}(q) & \tilde{F}'_{M2}(q) & \cdots & \cdots \tilde{F}'_{Mm}(q) \end{pmatrix}$$

We also need to introduce a special multiplication of the two matrices which is needed for the sequel. In order to distinguish it from a traditional matrix multiplication, we define the special matrix multiplication $q^T \odot \tilde{\nabla}\tilde{F}(q)$ as follows

$$q^T \odot \tilde{\nabla}\tilde{F}(q) = \bigoplus_{s=1}^{M}\bigoplus_{t=1}^{m} q_s^t \cdot \tilde{F}'_{st}(q).$$

Let fuzzy-valued function $\tilde{f} : K(\subset R_{M\times m}) \to E$ and $x, y \in K$. We say $\tilde{f}(x) \preceq \tilde{f}(y)$ if for every $\alpha \in (0,1]$, $\tilde{f}(x)_\alpha^L \leq \tilde{f}(y)_\alpha^L$ and $\tilde{f}(x)_\alpha^R \leq \tilde{f}(y)_\alpha^R$. We say $\tilde{f}(x) \prec \tilde{f}(y)$ if and only if $\tilde{f}(x) \preceq \tilde{f}(y)$ and $\tilde{f}(x) \neq \tilde{f}(y)$. We also write $\tilde{f}(x) \preceq \tilde{f}(y)$ for $\tilde{f}(y) \succeq \tilde{f}(x)$, and $\tilde{f}(x) \prec \tilde{f}(y)$ for $\tilde{f}(y) \succ \tilde{f}(x)$.

Next we present other concepts of invex and incave fuzzy mappings. These are variation of RufiSn-Lizana et al in [44].

Definition 4.9. Let $y \in K \subset R_{M\times m}$, we say K is invex at y w.r.t. $\eta : K \times K \to R_{M\times m}$, if for each $x \in K, \lambda \in [0,1], y + \lambda\eta(x,y) \in K$. K is said to be an invex set w.r.t. η, if K is invex at each $x \in K$.

Definition 4.10. A g-differentiable fuzzy-valued function $\tilde{f} : K(\subset R_{M\times m}) \to E$ is called fuzzy invex w.r.t. a function $\eta : K \times K \to R_{M\times m}$, if for all $x, y \in K$ $\tilde{f}(x) \succeq \eta(x,y)^T \odot \tilde{\nabla}\tilde{f}(y) \oplus \tilde{f}(y)$. $\tilde{f}$ is called fuzzy incave w.r.t. a function η, if for all $x, y \in K$, $\tilde{f}(x) \preceq \eta(x,y)^T \odot \tilde{\nabla}\tilde{f}(y) \oplus \tilde{f}(y)$.

Definition 4.11 ([44])**.** A fuzzy-valued function $\tilde{f} : K \subset R_{M\times m} \to E$ said to be preincave on invex set K w.r.t. η, if for any $x \in K, \lambda \in [0,1], \tilde{f}[y + \lambda\eta(x,y)] \succeq \lambda\tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y)$.

Definition 4.12. Let $\tilde{f} : K \subset R_{M\times m} \to E$ be a fuzzy-valued function defined on an invex set $K \subset R_{M\times m}, K \neq \varnothing$ w.r.t. $\eta : K \times K \to R^n$. If for any $x, y \in K$, there exists $\delta > 0$ for any real number $h \in (0,\delta)$, such that

$$\lim_{h\to 0^+} \frac{\tilde{f}(y + h\eta(x,y)) \ominus_g \tilde{f}(y)}{h} = \eta(x,y)^T \odot \tilde{\nabla}\tilde{f}_\eta(y),$$

then $\tilde{f}$ is called the fuzzy η-extended directional at y,$\tilde{\nabla}\tilde{f}_\eta(y)$ is called the fuzzy η-extended directional derivative at y in the direction $\eta(x,y)$.

Definition 4.13 ((Condition C) [35])**.** We say that the function $\eta : K\times K \to R_{M\times m}$ satisfies Condition C if for any $x, y \in K$, the following relations are satisfied for any $t \in [0,1]$.

(i) $\eta(y, y + t\eta(x,y)) = -t\eta(x,y)$;

(ii) $\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y)$.

The following two Theorems explain the relationship between the fuzzy incave function and fuzzy preincave function.

Theorem 4.14. *Let K is a nonempty invex set w.r.t. η, and η satisfies Condition C, if a fuzzy-valued function $\tilde{f} : K \subset R_{M\times m} \to E$ is an fuzzy incave function, then $\tilde{f}$ is a fuzzy preincave function.*

Proof. Since K is a nonempty invex set w.r.t. η, for all $x, y \in K, \lambda \in [0,1]$, we have $y^* = y + \lambda\eta(x,y) \in K$. By the incavity of $\tilde{f}$, we have

$$\tilde{f}(y) \preceq \eta(y, y^*)^T \odot \tilde{\nabla}\tilde{f}_\eta(y^*) \oplus \tilde{f}(y^*).$$

Similarly, the incavity condition applied to the pair x, y^* yields

$$\tilde{f}(x) \preceq \eta(x, y^*)^T \odot \nabla \tilde{f}_\eta(y^*) \oplus \tilde{f}(y^*).$$

Therefore, for every $\alpha \in [0,1]$, we have

$$\tilde{f}(y)^L_\alpha \leq [\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + \tilde{f}(y^*)^L_\alpha, \tag{4.3a}$$

$$\tilde{f}(y)^R_\alpha \leq [\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + \tilde{f}(y^*)^R_\alpha, \tag{4.3b}$$

$$\tilde{f}(x)^L_\alpha \leq [\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + \tilde{f}(y^*)^L_\alpha, \tag{4.3c}$$

$$\tilde{f}(x)^R_\alpha \leq [\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + \tilde{f}(y^*)^R_\alpha. \tag{4.3d}$$

Now, multiplying (4.3c) by λ and multiplying (4.3a) by $(1-\lambda)$ and adding, we note that

$$\lambda \tilde{f}(x)^L_\alpha + (1-\lambda)\tilde{f}(y)^L_\alpha \leq \lambda[\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + (1-\lambda)[\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + \tilde{f}(y^*)^L_\alpha. \tag{4.4a}$$

Similarly, multiplying (4.3d) by λ and multiplying (4.3b) by $(1-\lambda)$ and adding, we note that

$$\lambda \tilde{f}(x)^R_\alpha + (1-\lambda)\tilde{f}(y)^R_\alpha \leq \lambda[\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + (1-\lambda)[\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + \tilde{f}(y^*)^R_\alpha. \tag{4.4b}$$

From (i) and (ii) of Condition C, we have

$$\eta(y, y^*) = \eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$

$$\eta(x, y^*) = \eta(x, y + \lambda\eta(x, y)) = (1-\lambda)\eta(x, y).$$

Thus

$$\begin{aligned}
&\lambda[\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + (1-\lambda)[\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + \tilde{f}(y^*)^L_\alpha \\
&= \lambda(1-\lambda)[\eta(x, y)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^L_\alpha + (-\lambda)(1-\lambda)[\eta(x, y)^T \odot \nabla \tilde{f}_\eta(y^*)]^L_\alpha + \tilde{f}(y^*)^L_\alpha \\
&= \tilde{f}(y^*)^L_\alpha.
\end{aligned}$$

Similarly,

$$\lambda[\eta(x, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + (1-\lambda)[\eta(y, y^*)^T \odot \tilde{\nabla} \tilde{f}_\eta(y^*)]^R_\alpha + \tilde{f}(y^*)^R_\alpha = \tilde{f}_(y^*)^R_\alpha.$$

By (4.4a) and (4.4b), for every $\alpha \in [0,1]$, we have

$$\lambda \tilde{f}(x)^L_\alpha + (1-\lambda)\tilde{f}(y)^L_\alpha \leq \tilde{f}(y^*)^L_\alpha, \lambda \tilde{f}(x)^R_\alpha + (1-\lambda)\tilde{f}(y)^R_\alpha \leq \tilde{f}(y^*)^R_\alpha,$$

i.e.,

$$\lambda \tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y) \preceq \tilde{f}(y^*) = \tilde{f}(y + \lambda\eta(x, y)).$$

Therefore, $\tilde{f}$ is preincave. $\square$

Lemma 4.15. *Let fuzzy-valued function $\tilde{f} : K(\subset R_{M\times m}) \to E$ and $x, y, z \in K$. If $\tilde{f}(x) \succeq \tilde{f}(y)$, then*

$$\tilde{f}(x) \ominus_g \tilde{f}(z) \succeq \tilde{f}(y) \ominus_g \tilde{f}(z).$$

Proof. By Lemma 4.3, for all $\alpha \in [0,1]$, we have

$$\begin{aligned}(\tilde{f}(x) \ominus_g \tilde{f}(z))_\alpha &= [(\tilde{f}(x) \ominus_g \tilde{f}(z))^L_\alpha, (\tilde{f}(x) \ominus_g \tilde{f}(z))^R_\alpha] \\ &= [\min\{\inf_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta), \inf_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\}, \\ &\qquad \max\{\sup_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta), \sup_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\}],\end{aligned}$$

$$\begin{aligned}(\tilde{f}(y) \ominus_g \tilde{f}(z))_\alpha &= [(\tilde{f}(y) \ominus_g \tilde{f}(z))^L_\alpha, (\tilde{f}(y) \ominus_g \tilde{f}(z))^R_\alpha] \\ &= [\min\{\inf_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta), \inf_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\}, \\ &\qquad \max\{\sup_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta), \sup_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\}].\end{aligned}$$

Since $\tilde{f}(x) \succeq \tilde{f}(y)$, so for all $\beta \in [0,1]$, it holds that $\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta \geq \tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta$, $\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta \geq \tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta$, i.e.,

$$\inf_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta)\} \geq \inf_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta)\}, \inf_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\} \geq \inf_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\},$$

$$\sup_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta)\} \geq \sup_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta)\}, \sup_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\} \geq \sup_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\}.$$

Thus,

$$\begin{aligned}&\min\{\inf_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta), \inf_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\} \\ &\qquad \geq \min\{\inf_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta), \inf_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\},\end{aligned}$$

$$\begin{aligned}&\max\{\sup_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(z)^L_\beta), \sup_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(z)^R_\beta)\} \\ &\qquad \geq \max\{\sup_{\beta\geq\alpha}(\tilde{f}(y)^L_\beta - \tilde{f}(z)^L_\beta), \sup_{\beta\geq\alpha}(\tilde{f}(y)^R_\beta - \tilde{f}(z)^R_\beta)\}.\end{aligned}$$

Therefore, for all $\alpha \in [0,1]$, we have

$$(\tilde{f}(x) \ominus_g \tilde{f}(z))^L_\alpha \geq (\tilde{f}(y) \ominus_g \tilde{f}(z))^L_\alpha, (\tilde{f}(x) \ominus_g \tilde{f}(z))^R_\alpha \geq (\tilde{f}(y) \ominus_g \tilde{f}(z))^R_\alpha,$$

i.e., $\tilde{f}(x) \ominus_g \tilde{f}(z) \succeq \tilde{f}(y) \ominus_g \tilde{f}(z)$. □

Theorem 4.16. *Let K is a nonempty invex set w.r.t. η and $\tilde{f} : K \subset R_{M\times m} \to E$ is an fuzzy preincave w.r.t. η and η-extended directional at y, then $\tilde{f}$ is a fuzzy incave w.r.t. η.*

Proof. Since K is a nonempty invex set w.r.t. η, for all $x, y \in K, \lambda \in (0,1]$, from the definition of fuzzy preincave, we can get

$$\tilde{f}(y + \lambda\eta(x,y)) \succeq \lambda\tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y).$$

By Lemma 4.15,

$$\tilde{f}(y + \lambda\eta(x,y)) \ominus_g \tilde{f}(y) \succeq \lambda\tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y) \ominus_g \tilde{f}(y).$$

By Lemma 4.3, for all $\alpha \in [0,1]$, we have

$$\begin{aligned}
&[\lambda\tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y) \ominus_g \tilde{f}(y)]_\alpha \\
&= [\min\{\inf_{\beta\geq\alpha}(\lambda\tilde{f}(x)^L_\beta + (1-\lambda)\tilde{f}(y)^L_\beta - \tilde{f}(y)^L_\beta), \inf_{\beta\geq\alpha}(\lambda\tilde{f}(x)^R_\beta + (1-\lambda)\tilde{f}(y)^R_\beta - \tilde{f}(y)^R_\beta)\}, \\
&\quad \max\{\sup_{\beta\geq\alpha}(\lambda\tilde{f}(x)^L_\beta + (1-\lambda)\tilde{f}(y)^L_\beta - \tilde{f}(y)^L_\beta), \sup_{\beta\geq\alpha}(\lambda\tilde{f}(x)^R_\beta + (1-\lambda)\tilde{f}(y)^R_\beta - \tilde{f}(y)^R_\beta)\}] \\
&= [\min\{\inf_{\beta\geq\alpha}(\lambda\tilde{f}(x)^L_\beta - \lambda\tilde{f}(y)^L_\beta), \inf_{\beta\geq\alpha}(\lambda\tilde{f}(x)^R_\beta - \lambda\tilde{f}(y)^R_\beta)\}, \\
&\quad \max\{\sup_{\beta\geq\alpha}(\lambda\tilde{f}(x)^L_\beta - \lambda\tilde{f}(y)^L_\beta), \sup_{\beta\geq\alpha}(\lambda\tilde{f}(x)^R_\beta - \lambda\tilde{f}(y)^R_\beta)\}] \\
&= \lambda[\min\{\inf_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(y)^L_\beta), \inf_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(y)^R_\beta)\}, \\
&\quad \max\{\sup_{\beta\geq\alpha}(\tilde{f}(x)^L_\beta - \tilde{f}(y)^L_\beta), \sup_{\beta\geq\alpha}(\tilde{f}(x)^R_\beta - \tilde{f}(y)^R_\beta)\}] \\
&= [\lambda(\tilde{f}(x) \ominus_g \tilde{f}(y))]_\alpha.
\end{aligned}$$

Therefore, it follows that for all $\lambda \in (0,1]$,

$$\lambda\tilde{f}(x) \oplus (1-\lambda)\tilde{f}(y) \ominus_g \tilde{f}(y) = \lambda(\tilde{f}(x) \ominus_g \tilde{f}(y)).$$

So, we have

$$\tilde{f}(y + \lambda\eta(x,y)) \ominus_g \tilde{f}(y) \succeq \lambda(\tilde{f}(x) \ominus_g \tilde{f}(y)).$$

This implies that for all $\lambda \in (0,1]$,

$$\frac{\tilde{f}(y + \lambda\eta(x,y)) \ominus_g \tilde{f}(y)}{\lambda} \succeq \tilde{f}(x) \ominus_g \tilde{f}(y).$$

Since $\tilde{f}$ is the fuzzy η-extended directional at y, and taking $\lambda \to 0^+$, we have

$$\lim_{\lambda\to 0^+} \frac{\tilde{f}(y + \lambda\eta(x,y)) \ominus_g \tilde{f}(y)}{\lambda} \succeq \tilde{f}(x) \ominus_g \tilde{f}(y),$$

i.e.

$$\eta(x,y)^T \odot \tilde{\nabla}\tilde{f}(y) \succeq \tilde{f}(x) \ominus_g \tilde{f}(y).$$

Thus, $\tilde{f}$ is a fuzzy incave w.r.t. η. □

A well-known fact in mathematical programming is that the variational inequality problem is closely related to the optimization problem. Similarly, the fuzzy variational inequality problem is also closely related to the fuzzy optimization problem. In this section, we introduce a fuzzy variational-like inequality problems (FVLIP): to find $x^* \in K \subset R_{M\times m}$, a fuzzy matrix $\tilde{\nabla}\tilde{f}(x^*) = [\tilde{\nabla}\tilde{f}(x^*)_{ij}]_{M\times m}$ such that for any $x \in K$, such that

$$\langle\tilde{\nabla}\tilde{f}(x^*), \eta(x,x^*)\rangle = \eta(x,x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*) \preceq 0.$$

Theorem 4.17. *Let set K be an open invex set w.r.t. η, $x^* \in K$, and $\tilde{f} : K \to E$ be a g-differentiable fuzzy incave w.r.t. η. If x^* is a solution of FVLIP, then x^* is a local optimal solution of FP:* $\max_{x\in K} \tilde{f}(x)$.

Proof. Suppose that x^* is not a local optimal solution of FP, then there exists an $\bar{x} \in K \bigcap \aleph_\delta(x^*)$ such that $\tilde{f}(\bar{x}) \succ \tilde{f}(x^*)$, and for all $\alpha \in [0,1]$,

$$\tilde{f}(\bar{x})^L_\alpha > \tilde{f}(x^*)^L_\alpha, \tilde{f}(\bar{x})^R_\alpha > \tilde{f}(x^*)^R_\alpha. \tag{4.5a}$$

Since $\tilde{f}$ is a g-differentiable incave fuzzy function, it holds that

$$\tilde{f}(\bar{x}) \preceq \eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*) \oplus \tilde{f}(x^*),$$

then for all $\alpha \in [0,1]$

$$\begin{aligned} \tilde{f}(\bar{x})^L_\alpha &\le [\eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*)]^L_\alpha + \tilde{f}(x^*)^L_\alpha, \\ \tilde{f}(\bar{x})^R_\alpha &\le [\eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*)]^R_\alpha + \tilde{f}(x^*)^R_\alpha. \end{aligned} \tag{4.5b}$$

By (4.5a) and (4.5b), it holds that

$$[\eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*)]^L_\alpha > 0,\ [\eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*)]^R_\alpha > 0,$$

then

$$\eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*) = \langle \tilde{\nabla}\tilde{f}(x^*), \eta(x, x^*) \rangle \succ 0.$$

This contradicts the fact that x^* is a solution of FVLIP. □

Theorem 4.18. *Let the set K be an open invex set w.r.t. η, $x^* \in K$, and $\tilde{f} : K \to E$ be a g-differentiable invex fuzzy w.r.t. η. If x^* is a local optimal solution of FP, then x^* is a solution of FVLIP.*

Proof. Suppose that x^* is not a solution of FVLIP, then there exists an $\bar{x} \in K$, such that

$$\langle \tilde{\nabla}\tilde{f}(x^*), \eta(\bar{x}, x^*) \rangle = \eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*) \succ 0.$$

Since $\tilde{f}$ is a g-differentiable fuzzy invex function, then

$$\tilde{f}(\bar{x}) \succeq \eta(\bar{x}, x^*)^T \odot \tilde{\nabla}\tilde{f}(x^*) \oplus \tilde{f}(x^*).$$

Thus, we have

$$\tilde{f}(\bar{x}) \succ \tilde{f}(x^*).$$

This contradicts the fact that x^* is a local optimal solution of FP. □

Theorems 4.17 and 4.18 show that the fuzzy variational-like inequality problem is not completely equivalent to the fuzzy optimization problem and depends on the characteristics of the fuzzy-valued function $\tilde{f}$.

4.2 Specific solution for integrated FPTPMM model

For FPTPMM problem, the fuzzy variational-like inequality problems (FVLIP) should be: to find $q^* \in Q$, a fuzzy matrix $\tilde{\nabla}\tilde{F}(q^*) = [\tilde{\nabla}\tilde{F}(q^*)_{ij}]_{M\times m}$ such that for any $q \in Q$, such that

$$\langle \tilde{\nabla}\tilde{F}(q^*), \eta(q, q^*) \rangle = \eta(q, q^*)^T \odot \tilde{\nabla}\tilde{F}(q^*) \preceq 0. \tag{4.6}$$

The following theorem is an immediate consequence of Theorems 4.17 and 4.18.

Theorem 4.19. *Let the feasible set Q be an open invex set w.r.t. η, $q^* \in Q$, we have*

(i) *If* $\tilde{F}: Q \to E$ *be a g-differentiable fuzzy incave w.r.t.* η, q^* *is a solution of* $\langle \tilde{\nabla} \tilde{F}(q^*), \eta(q, q^*) \rangle \preceq 0$, *then* q^* *is a local optimal solution of the integrated FPTPMM model.*

(ii) *If* $\tilde{F}: Q \to E$ *be a g-differentiable fuzzy invex w.r.t.* η, q^* *is a local optimal solution of the integrated FPTPMM model, then* q^* *is a solution of* $\langle \tilde{\nabla} \tilde{F}(q^*), \eta(q, q^*) \rangle \preceq 0$.

Remark 4.20. By (3.2), we can obtain $\nabla \tilde{F}(q)$ is a $M \times m$ matrix whose components are

$$\tilde{F}'_{st}(q) = \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{M} [\tilde{\pi}_k^j(q)]'_{st}, (s = 1, 2, \cdots, M, t = 1, 2, \cdots, m),$$

where $[\tilde{\pi}_k^j(q)]'_{st}$ represents the partially derivative of $\tilde{\pi}_k^j(q)$ at q w.r.t. the component q_s^t.

Remark 4.21. Theorem 4.19 shows that the solution of the integrated FPTPMM model can be obtained by solving the FVLIP when $\tilde{F}(q)$ is a g-differentiable fuzzy incave. In fact, according to the relationship between the fuzzy incave and fuzzy preincave (Theorem 4.14 and Theorem 4.16), if $\tilde{F}(q)$ is a g-differentiable fuzzy preincave w.r.t. η and η-extended directional at q, then the solution of the integrated FPTPMM model can also be obtained by solving the FVLIP (8).

If $\eta(q, q^*) = q - q^*$, then the economic interpretation of FVLIP (8) is that for all production flow matrix $q \in Q$, if the combination increment of fuzzy total profit on all paths at production flow matrix q^* is non-positive, i.e., for all production flow matrix $q \in Q$, the fuzzy total profit on all paths at production flow matrix q is not more than that at production flow matrix q^*. then the production flow matrix q^* is a point where the fuzzy function $\tilde{F}(q)$ achieves its maximum.

According to Theorem 4.19, to obtain the solution to the integrated FPTPMM model (3.3), we only need to solve the FVLIP (8). However, it is very difficult to solve FVLIP (8) for the general $\eta(x, y)$. But we are able to transform it equivalently into a generalized nonlinear complementarity problem if $\eta(x, y)$ have the following special form: $\eta(x, y) = G(x) - G(y)$. So we present the solution methods for when $\eta(x, y) = G(x) - G(y)$.

The generalized nonlinear complementarity problem, denoted by GNCP, is to find $q^* \geq 0$ such that

$$\tilde{\nabla} \tilde{F}(q^*) \succeq 0, G(q^*)^T \odot \tilde{\nabla} \tilde{F}(q^*) = 0. \tag{4.7}$$

Using Fischer-Burmeister NCP-function [17]

$$\varphi(a, b) = \frac{1}{2}[\sqrt{a^2 + b^2} - (a + b)]^2.$$

(4.7) equivalent to finding a global minimum for the unconstrained minimization problem

$$\min_{q \in Q} \bigoplus_{t=1}^{m} \bigoplus_{s=1}^{M} \varphi[G(q_s^t), \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{M} [\tilde{\pi}_k^j(q)]'_{st}]. \tag{4.8}$$

(4.8) is a fuzzy unconstrained minimization problem, we transform it to crisp unconstrained minimization problem by defuzzification.

As we all know, it is very difficult to handle a optimistic problem when it involves uncertain information, so it is necessary to transform the fuzzy numbers into a determinate form. In this paper, we use a ranking fuzzy numbers approach to defuzzification. There have been many different methods proposed for ranking fuzzy intervals, most of which suggest mapping each fuzzy interval into a real line to define a ranking function. The ranking function, called the Average Value (AV), was introduced by Ibáñez and Muñoz [22] and was defined as a dependent on several parameters, allowing for flexibility in the final classification. The following definition for the ranking function is a particular case of an AV that considers a mean optimism degree.

Definition 4.22. Let $\alpha \in [0,1]$, then the defuzzified value of $\tilde{f}(x)$ is given by

$$\overline{f(x)} = \int_0^1 \alpha[\tilde{f}(x)_\alpha^L + \tilde{f}(x)_\alpha^R]d\alpha.$$

In the following, we give the solution method for the integrated FPTPMM model under the condition of general convex and general concave fuzzy-valued function.

Step 1: For all $k \in P_i$ and all j calculate the profits for product j on path k, $\tilde{\pi}_k^j(q)$, using formula (3.1).

Step 2: For all $k \in P_i$ and all j calculate the partially derivative of $\tilde{\pi}_k^j(q)$ at q w.r.t. the component q_s^t, $[\tilde{\pi}_k^j(q)]'_{st}$.

Step 3: Defuzzification, we denote $\overline{[\tilde{\pi}_k^j(q)]'_{st}}$ as the defuzzified value of $[\tilde{\pi}_k^j(q)]'_{st}$, and

$$\overline{[\tilde{\pi}_k^j(q)]'_{st}} = \int_0^1 \alpha\{[(\tilde{\pi}_k^j(q))'_{st}]_\alpha^L + [(\tilde{\pi}_k^j(q))'_{st}]_\alpha^R\}d\alpha. \tag{4.9}$$

Step 4: Using (3.2), obtain the function $\tilde{F}(q)$ for the integrated FPTPMM model.

Step 5: Obtain the function $\eta(x,y)$ which make the function $\tilde{F}(q)$ is a G-differentiable incave fuzzy-valued function w.r.t. $\eta(x,y)$, and rewritten $\eta(x,y)$ as $\eta(x,y) = G(x) - G(y)$.

Step 6: Finding a global minimum for the crisp unconstrained minimization problem

$$\min_{q\in Q} \sum_{t=1}^m \sum_{s=1}^M \varphi\{G(q_s^t), \sum_{j=1}^m \sum_{k=1}^M \overline{[\pi_k^j(q)]'_{st}}\}.$$

Remark 4.23. Note that a fuzzy concave function is a incave fuzzy by taking $\eta(x,y) = x-y$. The above solution method is contained in the solution method under a concavity condition. Similarly, if $\tilde{F}(q)$ is a g-differentiable fuzzy preincave w.r.t. η and η-extended directional at q, then the above solution method is also can be used.

5 An Illustrative Example

In this section, a practical example is used to demonstrate the practicality of the proposed optimization methodology and to guide similar real-world applications. In this part, the case problem is first presented. Following that, we use the above solution method to solve this problem. Finally, computations are done and the results demonstrate .

5.1 Case presentation

In this section, we give an example to demonstrate the application of the model. We assume that a milk products firm has 3 factories which have two types of production, pure milk and yogurt (m=2). These products are sent through 4 distribution centers to 4 sub-markets. These factories, distribution centers and sub-markets are located in eleven different cities.

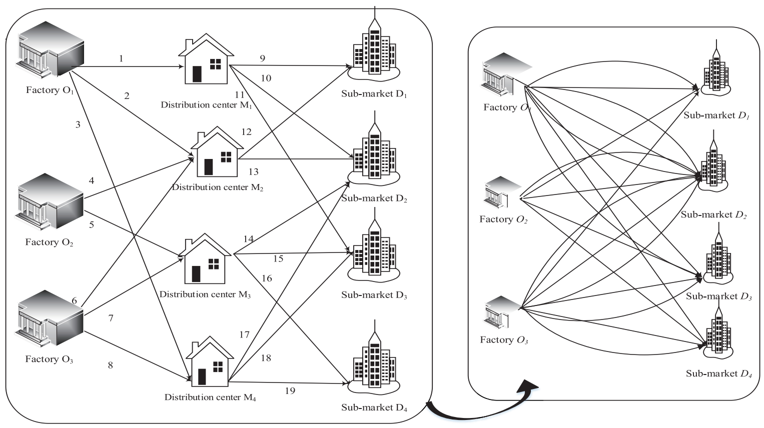

Figure 2: Fig.2 Network topology and path graph for illustrative example

Because these cities all have different distance and transport conditions, and the different sub-markets have different consumers, decision makers need to decide the production quantities for pure milk and yogurt, the transportation methods for these products and the sub-market pricing. We use the above model to solve these problems. The example has a topology as depicted in the left of Fig. 2. Table 2 summarizes the constituent paths of each $O-D$ pair and the path graph ia depicted in the right of Fig. 2. Thus $|I| = 12, |\mathbb{A}| = 19, M = 21$.

i	OD_i	P_i	sequence of arcs	q_k^j	i	OD_i	P_i	sequence of arcs	q_k^j
1	$O_1 \to D_1$	k_1	$\{a_1, a_9\}$	q_1^j	7	$O_2 \to D_3$	k_{12}	$\{a_5, a_{15}\}$	q_{12}^j
		k_2	$\{a_2, a_{12}\}$	q_2^j	8	$O_2 \to D_4$	k_{13}	$\{a_5, a_{16}\}$	q_{13}^j
2	$O_1 \to D_2$	k_3	$\{a_1, a_{10}\}$	q_3^j	9	$O_3 \to D_1$	k_{14}	$\{a_6, a_{12}\}$	q_{14}^j
		k_4	$\{a_2, a_{13}\}$	q_4^j	10	$O_3 \to D_2$	k_{15}	$\{a_6, a_{13}\}$	q_{15}^j
		k_5	$\{a_3, a_{17}\}$	q_5^j			k_{16}	$\{a_7, a_{14}\}$	q_{16}^j
3	$O_1 \to D_3$	k_6	$\{a_1, a_{11}\}$	q_6^j			k_{17}	$\{a_8, a_{17}\}$	q_{17}^j
		k_7	$\{a_3, a_{18}\}$	q_7^j	11	$O_3 \to D_3$	k_{18}	$\{a_7, a_{15}\}$	q_{18}^j
4	$O_1 \to D_4$	k_8	$\{a_3, a_{19}\}$	q_8^j			k_{19}	$\{a_8, a_{18}\}$	q_{19}^j
5	$O_2 \to D_1$	k_9	$\{a_4, a_{12}\}$	q_9^j	12	$O_3 \to D_4$	k_{20}	$\{a_7, a_{16}\}$	q_{20}^j
6	$O_2 \to D_2$	k_{10}	$\{a_4, a_{13}\}$	q_{10}^j			k_{21}	$\{a_8, a_{19}\}$	q_{21}^j
		k_{11}	$\{a_5, a_{14}\}$	q_{11}^j					

Table 2: OD pairs and their constituent paths

The fuzzy parameters for cost and demand functions have triangular membership functions. The fuzzy transportation cost functions for product j on the arc a_t are given by

$$\tilde{c}_t^j(q) = \tilde{m}_t^j q, (j = 1, 2, t = 1, 2, \cdots, 19). \tag{5.1}$$

where

$$\begin{pmatrix} \tilde{m}_1^1 \cdots & \tilde{m}_{19}^1 \\ \tilde{m}_1^2 \cdots & \tilde{m}_{19}^2 \end{pmatrix} = \begin{pmatrix} \tilde{7.5} & \tilde{7.5} & \tilde{10.5} & \tilde{9} & \tilde{10.5} & \tilde{6} & \tilde{7.5} & \tilde{7.5} \\ \tilde{4.5} & \tilde{10.5} & \tilde{6} & \tilde{7.5} & \tilde{9} & \tilde{7.5} & \tilde{9} & \tilde{3} \end{pmatrix.}$$
$$\left. \begin{matrix} \tilde{9} & \tilde{4.5} & \tilde{7.5} & \tilde{7.5} & \tilde{6} & \tilde{9} & \tilde{12} & \tilde{6} & \tilde{7.5} & \tilde{7.5} & \tilde{12} \\ \tilde{6} & \tilde{6} & \tilde{6} & \tilde{4.5} & \tilde{9} & \tilde{6} & \tilde{7.5} & \tilde{6} & \tilde{7.5} & \tilde{4.5} & \tilde{6} \end{matrix} \right)$$

Obviously, the transportation cost functions is convex.

The membership functions of the fuzzy parameters are triangular membership functions as follows

$$\mu_{\tilde{m}_k^j}(x) = \begin{cases} (x - m_k^j + 3)/2, & m_k^j - 3 < x \le m_k^j; \\ (-x + m_k^j + 3)/2, & m_k^j < x < m_k^j + 3; \\ 0, & otherwise. \end{cases}$$

The fuzzy production cost functions for factories, $\tilde{g}_i^j(q)$, are given by

$$\tilde{g}_1^j(q) = \tilde{h}_1^j q, \tilde{g}_2^j(q) = \tilde{h}_2^j q, \tilde{g}_3^j(q) = \tilde{h}_3^j q, (j = 1, 2). \tag{5.2}$$

where

$$\begin{pmatrix} \tilde{h}_1^1 & \tilde{h}_2^1 & \tilde{h}_3^1 \\ \tilde{h}_1^2 & \tilde{h}_2^2 & \tilde{h}_3^2 \end{pmatrix} = \begin{pmatrix} \tilde{7.5} & \tilde{6} & \tilde{7.5} \\ \tilde{6} & \tilde{7.5} & \tilde{7.5} \end{pmatrix}$$

The membership functions of the fuzzy parameters are triangular membership functions as follows

$$\mu_{\tilde{h}_k^j}(x) = \begin{cases} (x - h_k^j + 1)/2, & h_k^j - 1 < x \le h_k^j; \\ (-x + h_k^j + 1)/2, & h_k^j < x < h_k^j + 1; \\ 0, & otherwise. \end{cases}$$

The demand functions,$\tilde{p}_i^j$, at the sub-markets are

$$\tilde{p}_1^1 = 750 - \tilde{3}d_1^1 - \tilde{3}d_1^2, \tilde{p}_1^2 = 750 - \tilde{6}d_1^2 - \tilde{4.5}d_1^1,$$
$$\tilde{p}_2^1 = 750 - \tilde{6}d_2^1 - \tilde{3}d_2^2, \tilde{p}_2^2 = 750 - \tilde{4.5}d_2^2 - \tilde{6}d_2^1,$$
$$\tilde{p}_3^1 = 750 - \tilde{3}d_3^1 - \tilde{6}d_3^2, \tilde{p}_3^2 = 750 - \tilde{4.5}d_3^2 - \tilde{4.5}d_3^1,$$
$$\tilde{p}_4^1 = 750 - \tilde{4.5}d_4^1 - \tilde{6}d_4^2, \tilde{p}_4^2 = 750 - \tilde{6}d_4^2 - \tilde{3}d_4^1. \tag{5.3}$$

where d_i^j is the demand in sub-market i for product j, and the membership functions of the fuzzy parameters for p_i^j are also triangular membership functions with an interval length 2.

5.2 Case solution

In this illustrative example, the fuzzy price for product j at destination D_s: $\tilde{p}_s^j(q)$ is concave, the fuzzy cost function for product j on the path k and the fuzzy production cost functions for the factories are both convex, then integrated FPTPMM model (3.3) is a concave programming. So, we use the above solution method to solve this problem.

Step 1: For all $k \in P_i$ and all j calculate the profits for product j on path $k \in P_i$ using (3.1).

Let A_1, A_2, A_3, A_4 denote the set of paths k for which the destination paths are D_1, D_2, D_3, D_4, respectively. Then $A_1 = \{1, 2, 9, 14\}, A_2 = \{3, 4, 5, 10, 11, 15, 16, 17\}, A_3 = \{6, 7, 12, 18, 19\}, A_4 = \{8, 13, 20, 21\}$. So

$$d_1^j = \sum_{k \in A_1} q_k^j, d_2^j = \sum_{k \in A_2} q_k^j, d_3^j = \sum_{k \in A_3} q_k^j, d_4^j = \sum_{k \in A_4} q_k^j.$$

Using formula (3.1), the fuzzy profits for product j on path $k \in P_i$ are given by

$$\tilde{\pi}_k^j(q) = \begin{cases} \tilde{p}_1^j q_k^j \ominus_g [\tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j)], k \in A_1; \\ \tilde{p}_2^j q_k^j \ominus_g [\tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j)], k \in A_2; \\ \tilde{p}_3^j q_k^j \ominus_g [\tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j)], k \in A_3; \\ \tilde{p}_4^j q_k^j \ominus_g [\tilde{C}_k^j(q_k^j) \oplus \tilde{f}_k^j(q_k^j)], k \in A_4. \end{cases} \tag{5.4}$$

where $\tilde{C}_k^j(q_k^j) = \tilde{n}_k^j q_k^j$, and

$$\begin{aligned}(\tilde{n}_1^1, \tilde{n}_2^1, \cdots, \tilde{n}_{21}^1) = (&\tilde{16.5}, \tilde{15}, \tilde{12}, \tilde{13.5}, \tilde{18}, \tilde{15}, \tilde{18}, \tilde{22.5}, \tilde{16.5}, \tilde{15}, \tilde{18}, \tilde{22.5}, \tilde{16.5}, \tilde{13.5}, \\ &\tilde{12}, \tilde{16.5}, \tilde{15}, \tilde{19.5}, \tilde{15}, \tilde{15}, \tilde{19.5}); \\ (\tilde{n}_1^2, \tilde{n}_2^2, \cdots, \tilde{n}_{21}^2) = (&\tilde{10.5}, \tilde{18}, \tilde{10.5}, \tilde{19.5}, \tilde{13.5}, \tilde{10.5}, \tilde{10.5}, \tilde{12}, \tilde{15}, \tilde{16.5}, \tilde{15}, \tilde{16.5}, \tilde{15}, \tilde{15}, \\ &\tilde{16.5}, \tilde{15}, \tilde{10.5}, \tilde{16.5}, \tilde{7.5}, \tilde{16.5}, \tilde{9})\end{aligned}$$

$$\tilde{f}_k^j(q_k^j) = \begin{cases} \tilde{h}_1^j q_k^j, k = 1, 2, \ldots, 8; \\ \tilde{h}_2^j q_k^j, k = 9, 10, \ldots, 13; \\ \tilde{h}_3^j q_k^j, k = 14, 15, \ldots, 21. \end{cases} \tag{5.5}$$

Step 2: For all $k \in P_i$ and all j calculate the partial derivative of $\tilde{\pi}_k^j(q)$ w.r.t. the component q_s^t, $[\tilde{\pi}_k^j(q)]'_{st}$.

Step 3: Defuzzification. Using formula (4.9), determine the defuzzified value for $[\tilde{\pi}_k^j(q)]'_{st}$, $\overline{[\tilde{\pi}_k^j(q)]'_{st}}$ as follows

$$\overline{[\tilde{\pi}_k^j(q)']_{st}} = \int_0^1 \alpha\{[(\tilde{\pi}_k^j(q))'_{st}]_\alpha^L + [(\tilde{\pi}_k^j(q))'_{st}]_\alpha^R\} d\alpha.$$

Step 4: Using (3.2), obtain the function $\tilde{F}(q)$ for the integrated FPTPMM model:

$$\tilde{F}(q) = \bigoplus_{j=1}^{2} \bigoplus_{k=1}^{21} \tilde{\pi}_k^j(q).$$

Step 5: From formulae (5.1)-(5.3), we know that the objective function $\tilde{F}(q)$ is a concave fuzzy function and also is an incave fuzzy function, where $\eta(x, y) = x - y$.

Step 6: Finding a global minimum for the crisp unconstrained minimization problem:

$$\min_{q \in Q} \sum_{t=1}^{2} \sum_{s=1}^{21} \varphi\{q_s^t, \sum_{j=1}^{2} \sum_{k=1}^{21} \overline{[\tilde{\pi}_k^j(q)]'_{st}}\}.$$

This problem can be solved using the optimization software 1stOpt1.5. Here, we only give the following optimal seven solutions to reference for decision makers, which are shown in Table 3. Although the solution to the original problem is a set, the defuzzified total profits for all solutions are 147684.69.

From the optimal production flow matrix, we derive more information for the decision makers. The production quantity for product j $(j = 1, 2)$ at factory O_n $(n = 1, 2, 3)$ can be given as follows

$$Q_1^1 = \sum_{k=1}^{8} (q_k^1)^*, Q_2^1 = \sum_{k=9}^{13} (q_k^1)^*, Q_3^1 = \sum_{k=14}^{21} (q_k^1)^*,$$

Table 3: Numerical results for integrated FPTPMM model

	solution 1		solution 2		solution 3		solution 4		solution 5		solution 6		solution 7	
q^*	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$
q_1^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_2^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_3^j	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5
q_4^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_5^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_6^j	67.72	0	60.6	0	68.03	0	60	0	120.17	0	30.02	0	97.46	0
q_7^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_8^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_9^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{10}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{11}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{12}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{13}^j	39.62	0	38.85	0	70.5	0	79.72	0	80.38	0	23.71	0	34.9	0
q_{14}^j	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0
q_{15}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{16}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{17}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{18}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q_{19}^j	53.53	0	60.66	0	53.22	0	72.25	0	1.08	0	91.23	0	23.8	0
q_{20}^j	41.22	0	42	0	10.36	0	1.12	0	0.5	0	57.13	0	45.93	0
q_{21}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$Q_1^2 = \sum_{k=1}^{8}(q_k^2)^*, Q_2^2 = \sum_{k=9}^{13}(q_k^2)^*, Q_3^2 = \sum_{k=14}^{21}(q_k^2)^*.$$

The optimal transportation flow for product j on arc a_t can be described as follows

$$T_t^j = \bigoplus_{k=1}^{21} \delta_{tk} q_k^j, (j = 1, 2, t = 1, 2, \cdots, 19)$$

where $\delta_{tk} = 1$, if arc a_t is contained in path k, and 0, otherwise.

Table 4: The production quantity of products at factories for integrated FPTPMM model

	solution 1	solution 2	solution 3	solution 4	solution 5	solution 6	solution 7
Q_i^j	$(j=1, j=2)$	$(j=1, j=2)$	$(j=1, j=2)$	$(j=1, j=2)$	$(j=1, j=2)$	$(j=1, j=2)$	$(j=1, j=2)$
Q_1^j	(66.72, 81.5)	(60.6, 81.5)	(68.03, 81.5)	(60, 81.5)	(120.17, 81.5)	(30.02, 81.5)	(97.46, 81.5)
Q_2^j	(39.62, 0)	(38.85, 0)	(70.5, 0)	(79.72, 0)	(80.38, 0)	(23.71, 0)	(34.9, 0)
Q_3^j	(216.25, 0)	(224.16, 0)	(185.08, 0)	(194.87, 0)	(123.08, 0)	(268.86, 0)	(191.23, 0)

Table 5: Demand and price of products at sub-markets for integrated FPTPMM model

		sub-market D_1		sub-market D_2		sub-market D_3		sub-market D_4	
		$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$
solution 1	demand	121.5	0	0	81.5	121.25	0	80.84	0
	price	383.25	-	-	386.25	386.25	-	386.22	-
solution 2	demand	121.5	0	0	81.5	121.26	0	80.85	0
	price	383.25	-	-	386.25	386.22	-	386.175	-
solution 3	demand	121.5	0	0	81.5	121.25	0	80.86	0
	price	383.25	-	-	386.25	386.25	-	386.13	-
solution 4	demand	121.5	0	0	81.5	132.25	0	80.84	0
	price	383.25	-	-	386.25	353.25	-	386.22	-
solution 5	demand	121.5	0	0	81.5	121.25	0	80.88	0
	price	383.25	-	-	386.25	386.25	-	386.04	-
solution 6	demand	121.5	0	0	81.5	121.25	0	80.84	0
	price	383.25	-	-	386.25	386.25	-	386.25	-
solution 7	demand	121.5	0	0	81.5	121.26	0	80.83	0
	price	383.25	-	-	386.25	386.22	-	386.22	-

Table 6: Transportation flow of products on arc a_t for integrated FPTPMM model

	solution 1		solution 2		solution 3		solution 4		solution 5		solution 6		solution 7	
T_t^j	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$
T_1^j	67.72	81.5	60.6	81.5	68.03	81.5	60	81.5	120.17	81.5	30.02	81.5	97.46	81.5
T_2^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_3^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_4^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_5^j	39.62	0	38.85	0	70.5	0	79.72	0	80.38	0	23.71	0	34.9	0
T_6^j	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0
T_7^j	41.22	0	42	0	10.36	0	1.12	0	0.5	0	57.13	0	45.93	0
T_8^j	53.53	0	60.66	0	53.22	0	72.25	0	1.08	0	91.23	0	23.8	0
T_9^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_{10}^j	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5	0	81.5
T_{11}^j	67.72	0	60.6	0	68.03	0	60	0	120.17	0	30.02	0	97.46	0
T_{12}^j	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0	121.5	0
T_{13}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_{14}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_{15}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_{16}^j	80.84	0	80.85	0	80.86	0	80.84	0	80.88	0	80.84	0	80.83	0
T_{17}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_{18}^j	53.53	0	60.66	0	53.22	0	72.25	0	1.08	0	91.23	0	23.8	0
T_{19}^j	0	0	0	0	0	0	0	0	0	0	0	0	0	0

More information is given in Tables 4-6, where Table 4 shows the production quantity of products at factories for integrated FPTPMM model, Table 5 shows the demand and price for products at the sub-market for integrated FPTPMM model and Table 6 shows the transportation flow for products on arc $a_t \in \mathbb{A}$ for integrated FPTPMM model. The results for seven solutions show that only paths $6, 13, 14, 19, 20$ allocated product 1, while, for product 2, only factory O_1 produce product 2 and these products are transported sub-markets through unique path 3. arcs $a_2, a_3, a_4, a_9, a_{13}, a_{14}, a_{15}, a_{17}$ and a_{19} do not transport any production. Although the production quantities of products at factories and transportation flow on arcs are different, the demands and prices at sub-markets are almost the same.

$\boxed{5.3}$ Comparison of results

To demonstrate the advantages of the proposed integrated FPTPMM model, we compare it with traditional approach in which the production planning, transportation and pricing problems are solved separately.

The traditional approach considers the FPTPMM problem in two separate phases. First, the optimal outputs and price is determined based on the most profitable conditions. And then, according to the known supplies and demands obtained in first phase, the optimal transportation means are determined. So we first consider individually production and pricing problems. The fuzzy production and pricing problems for multi-product multi-market (FPPMM) optimization model is formulated as

$$\max \tilde{F}(q) = [\bigoplus_{j=1}^{2}\bigoplus_{i=1}^{4}\tilde{p}_i^j \cdot d_i^j] \ominus_g [\bigoplus_{j=1}^{2}\bigoplus_{n=1}^{3}\tilde{g}_n^j(Q_n^j)]$$

subject to

$$\begin{cases} Q_1^1 + Q_2^1 + Q_3^1 = d_1^1 + d_2^1 + d_3^1 + d_4^1 \\ Q_1^2 + Q_2^2 + Q_3^2 = d_1^2 + d_2^2 + d_3^2 + d_4^2 \\ Q_n^j \geq 0, d_i^j \geq 0, n = 1,2,3, j = 1,2, i = 1,2,3,4. \end{cases}$$

After defuzzifying the objective function, this problem also can be solved using the optimization software 1stOpt1.5. A approximate solution can be given as follows

$$d_1^1 = 124, d_2^1 = 10.88, d_3^1 = 124, d_4^1 = 78.67, d_1^2 = 0, d_2^2 = 81.5, d_3^2 = 0, d_4^2 = 0,$$

$$Q_1^1 = 0, Q_2^1 = 337.55, Q_3^1 = 0, Q_1^2 = 81.5, Q_2^2 = 0, Q_3^2 = 0.$$

Secondly, according to the known supplies from the factories and the known demand in the sub-markets, the Fuzzy transportation problems for multi-product multi-market (FTMM) can be modeled as a fuzzy linear programming problem. Using a simplified approach for solving fuzzy transportation problems proposed by Ebrahimnejad [16], we obtain the following optimal solution

$$q^* =$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 124 & 10.88 & 0 & 124 & 78.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 81.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

About the demand and price for products at the sub-market and the transportation flow for products on arc $a_t \in \mathbb{A}$ for traditional model are given in Tables 7-8.

	sub-market D_1		sub-market D_2		sub-market D_3		sub-market D_4	
	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$	$j=1$	$j=2$
demand	124	0	10.88	81.5	124	0	78.67	0
price	378	-	440.22	377.97	317.25	-	395.99	-

Table 7: Demand and price for products at sub-market for traditional FPPMM model

Table 8: Transportation flow of products on arc a_t for traditional FTMM method

T_t^j	$j=1$	$j=2$	T_i^j	$j=1$	$j=2$	T_i^j	$j=1$	$j=2$	T_i^j	$j=1$	$j=2$
T_1^j	0	81.5	T_6^j	0	0	T_{11}^j	0	0	T_{16}^j	78.67	0
T_2^j	0	0	T_7^j	0	0	T_{12}^j	124	0	T_{17}^j	0	0
T_3^j	0	0	T_8^j	0	0	T_{13}^j	10.88	0	T_{18}^j	0	0
T_4^j	134.88	0	T_9^j	0	0	T_{14}^j	0	0	T_{19}^j	0	0
T_5^j	202.67	0	T_{10}^j	0	81.5	T_{15}^j	124	0			

The results for solution q^* show that factories O_1 and O_3 do not produce production 1, factory O_3 does not produce production 2, arcs $a_2, a_3, a_6, a_7, a_8, a_9, a_{11}, a_{14}, a_{17}, a_{18}, a_{19}$ do not transport any production.

Table 9: comparison of two methods

	total output		revenue	production cost	transportation cost	total profit
	$j=1$	$j=2$				
Traditional method	337.55	81.5	152957.295	2514	7153.005	143289.99
Integrated method	324.89	81.5	156012.84	2845.935	5482.215	147684.69
Increase from Integrated method	-12.66	0	3055.545	331.935	-1670.79	4394.685
Increased percentage compared with traditional method	-3.75	0	1.998	13.2	-23.358	3.067

Table 9 provides the comparison of results for the integrated method and traditional method. The row labelled 'Percentage of increase from Integrated method' give the value from integrated method minus the value from traditional method divided by the value from traditional method and expressed as a percentage. The results shows that the revenues have not changed much in the integrated model, it only increased by 1.998%, but the transportation cost have changed much, which can be reduced by a surprising number 23.358%. Compared with traditional method, the percentage of increase in total profit is 3.067%.

6 Conclusion

In this paper, we have simultaneously considered a production, transportation and pricing optimization problem for multi-product multi-market under uncertain demand and uncertain cost, i.e. the demand and cost are fuzzy-valued functions. We first developed the integrated fuzzy production planning, transportation and pricing for multi-product multi-market (FPTPMM for short) model. Then, we discussed the solution methods for the integrated FPTPMM model based on the fact that $\tilde{F}$ is a special generalized concave function and derived an equivalence relation between the integrated FPTPMM model and the fuzzy variational-like inequality. Finally, an illustrative example was given to demonstrate an application of the theoretical results. The main contributions of this thesis can be summarized as follows:

(1) This is the first attempt in fuzzy environment to bridge these two streams of study: the fuzzy production-pricing problem and the fuzzy transportation problem, or the fuzzy production transportation problem and the fuzzy pricing problem. This study not only provides a solution for production, pricing and transportation decision problems in supply

chain under the fuzzy condition, but also provides theoretical support for collaborative production, transportation, pricing strategy in supply chain under the fuzzy condition.

(2) The proposed integrated FPTPMM model consider the three factors affecting the fuzzy total profits, fuzzy production cost, fuzzy transportation cost and fuzzy demand as a system, which does not simply add the existing fuzzy production quantity decision model, fuzzy distribution and transportation problem and fuzzy pricing model together. This research method, allocate production cost and revenue for each product to each path and then calculate the profit for each product to each path, is not only very simple, but also can overcome the difficulties of comprehensive consideration.

(3) An equivalence relationship between the FPTPMM model and a fuzzy variational-like inequality problem is developed. Some theorem solutions are proved for the fuzzy variational-like inequality, which extends the application fields of fuzzy variational-like inequalities.

However, a general solution method for the integrated FPTPMM model cannot be given when $\eta(x, y)$ cannot be rewrite as $G(x) - G(y)$. With this in mind, one direction for future research may be a study of the methods for the case when $\eta(x, y)$ cannot be rewritten as $G(x) - G(y)$, and further studies as to how the incave function $\tilde{F}$ can be converted into a concave fuzzy-valued function using an invertible transformation.

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