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REMARKS ON QUASI INTERIOR IN VECTOR OPTIMIZATION*

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, by using the notion of quasi interior, we first give some properties of improvement set which is an important tool to deal with vector optimization problems in a unified framework. Furthermore, we establish linear scalarization of weakly efficient solutions via quasi interior for a class of vector optimization problems with set-valued maps. In particular, we present some examples to show that the corresponding generalized convexity can not be weaken to the classical near-subconvexlikeness.

Key words: *vector optimization, quasi interior, improvement set, scalarization, near-subconvexlikeness*

Mathematics Subject Classification: *90C26, 90C29, 90C30*

1 Introduction

As is known to all, the theory and methods of vector optimization have been playing an important role in many research fields and have been widely used to solve various real-life problems such as economic management and engineering design. So far, fundamental and important results have been obtained by many scholars and some related research works can refer to [3, 6] and the references therein. Study on properties and characterizations of various kinds of solutions is one of the most meaningful research topics in vector optimization. Aiming at this research topic, linear scalarization with some suitable generalized convexity of objective functions is one of the most important methods, see [12, 13] and the references therein. Especially, Yang, Li and Wang proposed the near-subconvexlikeness which is one of the most general generalized convexity until now and established linear scalarization results of weakly efficient solutions for vector optimization problems in [12].

In general, there may not exist any exact solutions when the compactness conditions are removed and so various kinds of approximate solutions have been introduced by some scholars. The earlier works can be seen in [7, 9] and the references therein. Rong and Wu introduced ε-weakly efficient solutions of vector optimization problems with set-valued

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maps and derived scalarization results under the cone-subconvexlikeness in [10]. Some other related works of approximate solutions can be seen in [5, 11]. Recently, Chicco et al. [4] introduced the concept of improvement set and presented E-efficient solutions via improvement set. Improvement set is an important tool to study vector optimization problems in a unified framework and study properties and applications of improvement set in vector optimization, see [17–19].

Borwein and Lewis pointed out there exist many convex sets which often has empty interior, and whence, they proposed a kind of generalized interior notion named as quasi relative interior in [2]. Soon afterwards, Limber and Goodrich introduced the quasi interior in [8]. It is well-known that the classic weakly efficient solutions depend on the nonemptiness of the interior of ordering cone in vector optimization. When the ordering cone has empty interior, naturally some generalized interior tools will be employed. Zălinescu obtained some properties of quasi interior and quasi relative interior and applications in optimization problems in [15,16]. Bao and Mordukhovich considered a class of multiobjective optimization problems and established some existence results of weak efficiency defined by several kinds of generalized interiors by means of variational analysis tool in [1]. Xia, Zhang and Zhao gave some properties of improvement set, introduced weak E-efficient solutions and established the corresponding alternative theorem and linear scalarization result in the sense of quasi interior in [14].

Motivated by the works of [12,14–16], in this paper, we mainly focus on some properties of improvement set and applications in vector optimization problems. By using the notion of quasi interior, we obtain some new properties of improvement set and establish linear scalarization of weakly efficient solutions defined by quasi interior for vector optimization problems. Especially, we also present some examples to show that the generalized convexity can not be weaken to the classical near-subconvexlikeness.

2 Preliminaries

Let X be a real linear space, Y be a real nontrivial separated locally convex topological vector space and Y^* be the topological dual space of Y. We denote the n-dimensional Euclidean space, the nonnegative orthant and the positive orthant by $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. For a nonempty subset A in Y, A is said to be proper if $A \neq \emptyset$ and $A \neq Y$, the generated cone and the positive dual cone of A are respectively defined as

$$\text{cone}A = \{\alpha a | \alpha \geq 0, a \in A\}, \quad A^+ = \{\mu \in Y^* | \langle \mu, y \rangle \geq 0, \forall y \in A\}, \tag{2.1}$$

and the support functional of the set A be defined as $\sigma_A(\mu) = \sup\limits_{a\in A}\{\langle \mu, a \rangle\}, \quad \forall \mu \in Y^*$. Moreover, we denote the topological interior and topological closure of the set A by int$A$ and cl$A$, respectively. And for a convex subset A, the quasi interior and quasi relative interior denoted by qi$A$ and qri$A$ are respectively defined as

$$\begin{aligned} &\text{qi}A = \{y \in A | \text{cl}(\text{cone}(A - y)) = Y\}; \\ &\text{qri}A = \{y \in A | \text{cl}(\text{cone}(A - y)) \text{ is a linear subspace in } Y\}. \end{aligned} \tag{2.2}$$

It is well known that int$A \subset$ qi$A \subset$ qri$A$; If int$A \neq \emptyset$, then int$A$ = qi$A$ = qri$A$; If qi$A \neq \emptyset$, then qi$A$ = qri$A$. The important equivalent definitions of the quasi interior and quasi relative interior are introduced by Zălinescu [16] as follows:

$$\text{qi}A = A \cap \text{qi}(\text{cl}A), \qquad \text{qri}A = A \cap \text{qri}(\text{cl}A).$$

Consider the following vector optimization problem:

$$\text{(VOP)} \quad \min F(x) \text{ subject to } \; x \in S, \tag{2.3}$$

where $F : S \rightrightarrows Y$, $S \subset X$ and $S \neq \emptyset$.

Definition 2.1. [4,19] Let K be a proper convex cone in Y and E be a nonempty subset in Y. E is said to be an improvement set with respect to K if $0 \notin E$ and $E + K = E$.

Definition 2.2. [14] Let K be a proper convex cone with nonempty quasi interior in Y and E be a convex improvement set with respect to K in Y. A point pair $(\overline{x}, \overline{y})$ is called a weak E-efficient point of (VOP) if $\overline{x} \in S, \overline{y} \in F(\overline{x})$ such that

$$(\overline{y} - \mathrm{qi}E) \cap F(S) = \emptyset.$$

Consider the following scalar optimization problem:

$$\text{(VOP)}_\mu \quad \min_{x \in S} \langle \mu, F(x) \rangle, \quad \mu \in Y^* \backslash \{0_{Y^*}\},$$

where $\langle \mu, F(x) \rangle = \{\langle \mu, y \rangle | y \in F(x)\}$. Let E be an improvement set with respect to K in Y. A point $\overline{x} \in S$ is called an optimal solution of (VOP)_μ with respect to E if there exists $\overline{y} \in F(\overline{x})$ such that

$$\langle \mu, y - \overline{y} \rangle \geq \sigma_{-E}(\mu), \forall x \in S, \forall y \in F(x).$$

From the definition of $\sigma_A(\mu)$, we have $\sigma_{-E}(\mu) = \sup_{e \in E} \langle \mu, -e \rangle$. Moreover, the point pair $(\overline{x}, \overline{y})$ is called an optimal point of (VOP)_μ with respect to E.

3 Properties of improvement sets via quasi interior and applications in (VOP)

In this section, we first give some properties of improvement sets by means of the notion of quasi interior and improve some corresponding results established by Xia, Zhang and Zhao in [14]. As applications, we obtain a scalarization result of weakly efficient solutions defined by improvement set and quasi interior for (VOP). In particular, we also point out that the corresponding generalized convexity of objective function could not be weaken to the near-subconvexlikeness even if for the case of the exact weakly efficient solutions established by Yang, Li and Wang in [12].

Lemma 3.1. *Let K be a proper convex cone with nonempty quasi interior in Y. If K is closed, then* $0 \notin \mathrm{qi}K$.

Proof. On the contrary, assume that $0 \in \mathrm{qi}K$. Then by using the definition of quasi interior, we have

$$\mathrm{cl}(\mathrm{cone}(K - 0)) = Y.$$

Hence from the closedness of K, it follows that $K = Y$, which contradicts to the fact that K is proper. □

Remark 3.1. Noting that the converse of Lemma 3.1 may not be true. In fact, if we take $Y = l^2$, $K = l^1_+$, then it is clear that K is a proper convex cone with $0 \notin \mathrm{qi}K$, and K is not closed.

Remark 3.2. Lemma 3.1 shows that the closedness of K implies $0 \notin \text{qi}K$ when K is a proper convex cone. Naturally, if K is a proper convex cone, it remains one open question that whether the pointedness of K implies $0 \notin \text{qi}K$. However, we would like to point out that this result may not be valid when Y is a locally convex topological vector space and is not necessarily separated.

Example 3.1. Let Y be the real two-dimensional linear space and

$$K = \{(y_1, y_2) \in Y | y_1 - y_2 \le 0\} \setminus \{(y_1, y_2) \in Y | y_1 - y_2 = 0, y_1 < 0\}.$$

It is clear that K is a proper pointed convex cone. Define a new topology on Y as follows:

$$\mathscr{T} = \{A \times B \subset Y \mid A = (\alpha, \beta), \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha < \beta, B = (-\infty, +\infty)\} \cup \{\emptyset, Y\}.$$

Clearly, $\mathscr{T}$ is a topology and it has a convex neighborhood basis of zero $\mathscr{T}_0$ defined as follows:

$$\mathscr{T}_0 = \{A \times B \subset Y \mid A = (-\alpha, \alpha), \alpha \in \mathbb{R}_{++}, B = (-\infty, +\infty)\}.$$

And on this basis, we can verify that the map $TV_1 : Y \times Y \to Y$ defined as $(y_1, y_2) \mapsto y_1 + y_2$ and the map $TV_2 : \mathbb{R} \times Y \to Y$ defined as $(\lambda, y) \mapsto \lambda y$ are continuous. Hence, $(Y, \mathscr{T})$ is a real locally convex topological vector space. Moreover, we can obtain that K is not closed since $K \subsetneq Y = \text{cl}K$. However, from the definition of quasi interior and $\text{cl}K = Y$, it follows that $0 \in \text{qi}K$. Therefore, in this case, the pointedness of K does not necessarily imply $0 \notin \text{qi}K$.

Based on Lemma 3.1 and by using the notion of quasi interior, we can obtain some properties of improvement set which improve some corresponding results given by Xia, Zhang and Zhao in [14].

Theorem 3.2. *Let A be a nonempty subset in Y, K be a proper convex cone with nonempty quasi interior and $0 \notin \text{qi}K$ in Y and E be a convex improvement set with respect to K in Y. If $A \cap (-\text{qi}E) = \emptyset$, then $\text{cone}(A + E) \cap (-\text{qi}K) = \emptyset$.*

Proof. The proof is similar to Theorem 3.2 in [14]. □

Theorem 3.3. *Let K be a proper convex cone with nonempty quasi interior and $0 \notin \text{qi}K$ in Y and E be a convex improvement set with respect to K in Y. If E satisfies Assumption (Q), i.e., $\text{qi}E \subset E + \text{qi}K$ and $\text{cone}(F(S) + E)$ is a closed convex set, then one and only one of the following statements is true:*

(i) $\exists x \in S$, $F(x) \cap (-\text{qi}E) \neq \emptyset$;

(ii) $\exists \mu \in K^+ \setminus \{0_{Y^*}\}$, $\langle \mu, y \rangle \ge \sigma_{-E}(\mu)$, $\forall y \in F(S)$.

Proof. The proof is similar to Theorem 4.1 in [14]. □

Remark 3.3. Assume that K is a proper convex cone with nonempty topological interior. Yang, Li and Wang proposed the near-subconvexlikeness, i.e., $\text{cl cone}(F(S) + K)$ is a convex set and established alternative theorem and linear scalarization result of weakly efficient solutions of (VOP) in [12]. Near-subconvexlikeness is one of the most general generalized convexity to derive the linear scalarization results of (VOP). Furthermore, Zhao, Yang and Peng generalizes near-subconvexlikeness to the case of improvement set in [19], i.e., $\text{cl cone}(F(S) + E)$ is a convex set. More generally speaking, if K is a proper convex cone with nonempty quasi interior, then it is worth noting that the assumption condition "$\text{cone}(F(S) + E)$ is a closed convex set" in Theorem 3.3 could not be weaken to the case that $\text{cl cone}(F(S) + E)$ is a convex set.

In the following, we let $\mathbb{N}^+$ be the set of all positive integers and

$$l^p = \left\{ y = (y_n)_{n\in\mathbb{N}^+} \Big| \sum_{n\in\mathbb{N}^+} |y_n|^p < +\infty \right\}, \quad 1 \le p < +\infty$$

endowed with its usual norm. The positive cone of l^p, denoted by l^p_+, is

$$l^p_+ = \left\{ y = (y_n)_{n\in\mathbb{N}^+} \in l^p \Big| y_n \ge 0, n \in \mathbb{N}^+ \right\}, \quad 1 \le p < +\infty.$$

It is well-known that

$$\text{qi} l^p_+ = l^p_{++} = \left\{ y = (y_n)_{n\in\mathbb{N}^+} \in l^p \Big| y_n > 0, n \in \mathbb{N}^+ \right\}, \quad 1 \le p < +\infty.$$

Example 3.2. Let

$$\begin{aligned} &X = Y = l^2, \quad S = -l^2_+ \setminus -l^1_+, \qquad F(x) = [1,2]x, \\ &K = l^1_+, \quad E = \{y = (y_1, y_2, \cdots) \in l^1_+ | y_1 \ge 1, y_2 \ge 1\} = l^1_+ + (1,1,0,\cdots). \end{aligned}$$

Clearly, $F(S) = -l^2_+ \setminus -l^1_+$, K is a proper convex cone with $0 \notin \text{qi}K = l^1_{++}$, E is a convex improvement set with respect to K and $\text{qi}E = l^1_{++} + (1,1,0,\cdots)$. So E satisfies Assumption (Q). We first verify that $F(S)$ is a convex set. In fact, for any $x = (x_1, x_2, \cdots) \in F(S)$, $y = (y_1, y_2, \cdots) \in F(S)$ and $\lambda \in (0,1)$, we have $\lambda x + (1-\lambda)y \in -l^2_+$ and

$$\begin{aligned} \sum_{n=1}^{\infty} |\lambda x_n + (1-\lambda) y_n| &= -\sum_{n=1}^{\infty} (\lambda x_n + (1-\lambda) y_n) \\ &\ge -\lambda \sum_{n=1}^{\infty} x_n = \lambda \sum_{n=1}^{\infty} |x_n| = +\infty, \end{aligned}$$

which implies $\lambda x + (1-\lambda)y \notin l^1$. Hence, by the fact that $-l^1_+ \subset l^1$, we get $\lambda x + (1-\lambda)y \in F(S)$ and so $F(S)$ is a convex set. Therefore, $\text{cone}(F(S)+E)$ is a convex set and thus it follows that $\text{cl cone}(F(S)+E)$ is a convex set. However, $\text{cone}(F(S)+E)$ is not closed. In fact, we can verify that

$$\begin{aligned} &(0,0,1,0,0,\cdots) \notin \text{cone}(F(S)+E), \\ &(0,0,1,0,0,\cdots) \in \text{cl cone}(F(S)+E). \end{aligned}$$

Since for any $x = (x_1, x_2, \cdots) \in F(S)$, $y = (y_1, y_2, \cdots) \in E$, then it is clear that $x + y \in l^2$ and

$$\sum_{n=1}^{\infty} |x_n + y_n| \ge \sum_{n=1}^{\infty} |x_n| - \sum_{n=1}^{\infty} |y_n| \ge +\infty,$$

which implies $x + y \notin l^1$. Hence $x + y \in l^2 \setminus l^1$, that is $F(S) + E \subset l^2 \setminus l^1$. Therefore, $\text{cone}(F(S)+E) \subset l^2 \setminus l^1$ and from $(0,0,1,0,0,\cdots) \in l^1$, it follows that $(0,0,1,0,0,\cdots) \notin \text{cone}(F(S)+E)$. Furthermore, we prove $(0,0,1,0,0,\cdots) \in \text{cl cone}(F(S)+E)$. Since

$$-\left(1, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right) \in F(S) + (1,1,0,0,\cdots),$$

then by virtue of the fact that $\left(1, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right) \in \mathrm{qi} l_+^2$, we have

$$\begin{aligned} Y &= \mathrm{cl\ cone}\left(l_+^2 - \left(1, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right)\right) \\ &= \mathrm{cl\ cone\ cl}\left(l_+^1 - \left(1, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right)\right) \\ &= \mathrm{cl\ cone}\left(l_+^1 - \left(1, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right)\right) \\ &\subset \mathrm{cl\ cone}(l_+^1 + F(S) + (1, 1, 0, 0, \cdots)) = \mathrm{cl\ cone}(F(S) + E) \subset Y, \end{aligned}$$

which implies $\mathrm{cl\ cone}(F(S)+E) = Y$ and so $(0, 0, 1, 0, 0, \cdots) \in \mathrm{cl\ cone}(F(S)+E)$. Therefore, it follows that $\mathrm{cone}(F(S) + E)$ is not closed.

In the following, we can verify that Theorem 3.3 does not hold. Clearly, $F(S) \cap (-\mathrm{qi}E) = \emptyset$. And we only need to verify that for any $\mu \in K^+ \setminus \{0_{Y^*}\}$, there exists $y \in F(S)$ such that

$$\langle \mu, y \rangle < \sigma_{-E}(\mu) = \sup_{e \in -E} \langle \mu, e \rangle.$$

For any given $\mu = (\mu_1, \mu_2, \cdots) \in K^+ \setminus \{0_{Y^*}\}$, it is clear that there exists a component μ_{k_0} such that $\mu_{k_0} > 0$. We can take $y = (y_1, y_2, \cdots)$ satisfying

$$y_n = \begin{cases} -\dfrac{1}{n}, & n \neq k_0, \\ -\dfrac{\mu_1 + \mu_2 + \mu_n}{\mu_n}, & n = k_0, \end{cases}$$

then it is clear that $y \in F(S)$ and

$$\langle \mu, y \rangle = \sum_{n=1}^{\infty} \mu_n y_n \leq \mu_{k_0} y_{k_0} = -\mu_1 - \mu_2 - \mu_{k_0} < -\mu_1 - \mu_2 = \sigma_{-E}(\mu).$$

Based on Theorem 3.3, we have the following scalarization result of weakly efficient solutions defined by improvement set and quasi interior for (VOP).

Theorem 3.4. *Let $\overline{x} \in S, \overline{y} \in F(\overline{x})$, K be a proper convex cone with nonempty quasi interior and $0 \notin \mathrm{qi}K$ in Y and E be a convex improvement set with respect to K in Y. If E satisfies Assumption (Q) and $\mathrm{cone}(F(S) - \overline{y} + E)$ is a closed convex set, then $(\overline{x}, \overline{y})$ is a weak E-efficient point of (VOP) if and only if there exists $\mu \in K^+ \setminus \{0_{Y^*}\}$ such that $(\overline{x}, \overline{y})$ is an optimal point of (VOP)$_\mu$ with respect to E.*

Proof. The proof is similar to Theorem 4.2 in [14]. □

Remark 3.4. The assumption condition "$\mathrm{cone}(F(S) - \overline{y} + E)$ is a closed convex set" could not be weaken to the near E-subconvexlikeness proposed by Zhao, Yang and Peng in [19].

Example 3.3. Let

$$\begin{aligned} &X = Y = l^2, K = l_+^1, E = l_+^1 \setminus \{0\}, \\ &S = (-l_+^2 \setminus -l_+^1) \cup \{(-\alpha, 0, 0, \cdots) | \alpha \in [1, 2]\}, F(x) = [1, 2]x. \end{aligned}$$

Clearly, K is a proper convex cone with $0 \notin \mathrm{qi}K = l^1_{++}$, E is a convex improvement set with respect to K, $\mathrm{qi}E = l^1_{++}$, E satisfies Assumption (Q) and

$$F(S) = (-l^2_+ \setminus -l^1_+) \cup \{(-\alpha, 0, 0, \cdots)|\alpha \in [1, 4]\}.$$

Let $\overline{x} = \overline{y} = (-1, 0, 0, \cdots)$. It is clear that $(\overline{x}, \overline{y})$ is a weak E-efficient point of (VOP). Similar to the analysis of Example 3.2, we can verify that $\mathrm{cone}(F(S)-\overline{y}+E)$ is convex but not closed and $\mathrm{cl\ cone}(F(S) - \overline{y} + E)$ is convex. Furthermore, for any $\mu = (\mu_1, \mu_2, \cdots) \in K^+ \setminus \{0_{Y^*}\}$, there exists a component μ_{k_0} satisfying $\mu_{k_0} > 0$. We can take $y = (y_1, y_2, \cdots)$ satisfying $y_{k_0} = -3$ and $y_n = -\dfrac{1}{n}, n \neq k_0$, then it is clear that $y \in F(S)$ and

$$\langle \mu, y - \overline{y} \rangle \leq \mu_{k_0}(y_{k_0} - \overline{y}_{k_0}) \leq -\mu_{k_0} < 0 = \sigma_{-E}(\mu).$$

This implies that in this case, Theorem 3.4 is not valid.

Remark 3.5. In the case of $\mathrm{int}K \neq \emptyset$, Yang, Li and Wang obtain the following linear scalarization result of weakly efficient solutions for (VOP) under the near-subconvexlikeness in [12].

Let $\overline{x} \in S, \overline{y} \in F(\overline{x})$, K be a proper convex cone with nonempty interior and $\mathrm{cl\ cone}(F(S)-\overline{y} + K)$ is a convex set. Then $(\overline{x}, \overline{y})$ is a weakly efficient point of (VOP) if and only if there exists $\mu \in K^+ \setminus \{0_{Y^*}\}$ such that $(\overline{x}, \overline{y})$ is an optimal point of $(\mathrm{VOP})_\mu$.

It is worth noting that the above result also could not be generalized to the case of quasi interior. For example, let

$$X = Y = l^2, K = l^1_+, S = (-l^2_+ \setminus -l^1_+) \cup \{(-\alpha, 0, 0, \cdots)|\alpha \in [1, 2]\}, F(x) = [1, 2]x.$$

and $\overline{x} = \overline{y} = (-1, 0, 0, \cdots)$. We can verify that K be a proper convex cone with nonempty quasi interior, $\mathrm{cl\ cone}(F(S) - \overline{y} + K)$ is a convex set and $(\overline{x}, \overline{y})$ is a weakly efficient point of (VOP). However, for any given $\mu \in K^+ \setminus \{0_{Y^*}\}$, $(\overline{x}, \overline{y})$ is not an optimal point of $(\mathrm{VOP})_\mu$.

4 Concluding Remarks

In this paper, by using the notion of quasi interior, we give some properties of improvement set and applications in vector optimization. The main results improve some corresponding results established by Xia, Zhang and Zhao in [14]. In particular, we also present some examples to show that the generalized convexity of objective function could not be weaken to the near-subconvexlikeness even if for the classical case of the exact weakly efficient solutions.

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