Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 15, No. 4, pp. 565 – 582, 2019

CONVERGENCE OF THE PROPER EFFICIENCY IN VECTOR OPTIMIZATION PROBLEMS WITH EQUILIBRIUM CONSTRAINTS*

Yunan Wu†, June Dong and Ding Zhang

Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, the proper efficiency in vector optimization problems with vector equilibrium constraints is introduced and some convergence results, for instance, the convergence of the marginal map, the value map and the solution map under perturbations, are obtained. A nonlinear scalarization function is employed in the study. A special case where the constraint is a vector variational inequality problem is also considered.

Key words: *vector optimization problems with equilibrium constraints, proper efficiency, lower convergence, upper convergence, variational inequality*

Mathematics Subject Classification: *90C29, 90C35, 90B06, 90B10*

1 Introduction

Let X be a vector space and Y be a compact subset of another vector space. Let Z be a finite dimensional Euclidean space partially ordered by a closed, convex and pointed cone P with nonempty interior $intP$ and $K \subset X$ be a nonempty compact set. $C : X \rightrightarrows Y$ is a set-valued map. We consider the following vector optimization problem with vector equilibrium constraints (VPEC)

$$\text{(VPEC)} \quad \begin{cases} min & f(x,y) \\ s.t. & x \in K, \text{ and } y \in S(x), \end{cases}$$

where $f : X \times Y \to Z$, and, for each $x \in K$, $S(x)$ is the solution set of the following vector-valued equilibrium problem(VEP); that is, $y \in S(x)$ if and only if $y \in C(x)$, and y satisfies the formula:

$$\text{(VEP)} \quad g(x,y,z) \notin -intP, \quad \text{for any } z \in C(x),$$

*This work was supported by the National Natural Science Foundation of China under grant number 71071023 and 71571020. This work was also sponsored by Key Laboratory of Optimization and Control (Chongqing Normal University).

†Corresponding author

with $g : X \times Y \times Y \to Z$.

In fact, the modelling as mentioned above is a generalization of a bilevel program (BP), which was introduced in the operations research literature in the early 1970s by Bracken and McGill (Ref. 3). A bilevel program is in turn a special case of a hierarchical mathematical program where a subset of the variables is constrained to be a solution of a given optimization problem parameterized by the remaining variables. The hierarchical optimization structure appears naturally in many applications when lower level actions depend on upper level decisions. The applications of bilevel (and multilevel) programming include transportation (taxation, network design, trip demand estimation), management (coordination of multidivisional firms, network facility location, credit allocation), planning (agricultural policies, electric utility), and optimal design. Many important mathematical programs, such as minimax problems, linear integer programs, bilinear and quadratic programs, can be stated as special instances of bilevel programs. These facts illustrate the importance of bilevel problems.

One could generalize the bilevel program in different ways. It is known that the equilibrium problem that Blum and Oettli (Ref. 2) put forward unifies many interesting problems, for instance, optimization problems, complementarity problems, fixed point problems and variational inequalities. Correspondingly, vector equilibrium problems include vector optimization problems, vector complementarity problems, vector variational inequalities and so on as special cases. In Ref. 12, Wu and Cheng consider the vector optimization problems with vector equilibrium constraints.

As to the vector optimization problem, various generalizations such as weak efficient solutions, efficient solutions and proper efficient solutions have been proposed, see Refs. 6,9,10 and 13. It is known that the set of proper efficient solutions can be equivalently characterized by a scalar minimization problem while the set of efficient solutions can't. Thus we only consider the proper efficiency. In the present paper, we consider a bilevel program where the upper level is a vector optimization problem and the lower level is a vector equilibrium problem. We establish the convergence of the marginal map, the value map and the solution map under perturbations. A nonlinear scalarization function from Ref. 8 is employed.

The paper is outlined as follows. In Section 2, we introduce some definitions of convergence and prove a preliminary lemma. In Sections 3 and 4, we derive the convergence results of the marginal map, the Benson′s proper efficient points set and the proper efficient solutions set of the vector optimization problem with equilibrium constraints respectively. Finally in Section 5, the application to a vector optimization problem with vector variational inequality constraints is given.

2 Notations and Preliminaries

Firstly, we recall some convergence notations for real-valued, vector-valued and set-valued maps(Ref 11). Let $\phi_n, \phi_0 : X \to R \cup \{+\infty\}$, $\{h_n\}$ be a sequence of vector-valued maps from X to Z and $\{H_n\}$ be a sequence of set-valued maps from X to Y. Let h_0, H_0 be a vector-valued and a set-valued map similar to $\{h_n\}$ and $\{H_n\}$ respectively. We denote the epigraph of ϕ_0 and h_0 respectively by *epi* ϕ_0 and *epi* h_0, that is,

$$epi\ \phi_0 := \{(x, r) \in X \times R : \phi_0(x) \leqslant r\};$$

$$epi\ h_0 := \{(x, z) \in X \times Z : z \in h_0(x) + P\}.$$

Let $\{K_n\}$ be a sequence of nonempty subsets of X. The lower and upper limits of $\{K_n\}$ are defined respectively by

$$\liminf_{n\to+\infty} K_n := \{x \in X : \exists \{x_n\} \subset X \text{ convergent to } x \text{ such that } x_n \in K_n, \quad \text{for sufficiently large } n\};$$

$$\limsup_{n\to+\infty} K_n := \{x \in X : \exists \{x_k\} \subset X \text{ convergent to } x \text{ such that } x_k \in K_{n_k}, \text{ for a subsequence } \{n_k\}\}.$$

Obviously,

$$\liminf_{n\to+\infty} K_n \subset \limsup_{n\to+\infty} K_n.$$

Definition 2.1. $\{K_n\}$ is convergent to $K_0 \subset X$ if ,

$$\limsup_{n\to+\infty} K_n \subset K_0 \subset \liminf_{n\to+\infty} K_n.$$

We denote it by $K_n \to K_0$.

Definition 2.2. We say that the sequence $\{\phi_n\}$ epiconverges to ϕ_0 if

(i) for every $x_0 \in X$ and every $\{x_n\} \subset X$ convergent to x_0, one has

$$\phi_0(x_0) \leq \liminf_{n\to+\infty} \phi_n(x_n);$$

(ii) for every $x_0 \in X$, there exists $\{x_n\} \subset X$ convergent to x_0 such that

$$\limsup_{n\to+\infty} \phi_n(x_n) \leq \phi_0(x_0).$$

We denote it by $\phi_n \stackrel{epi}{\to} \phi_0$.

By Ref. 1, we get the following proposition.

Proposition 2.3. $\phi_n \stackrel{epi}{\to} \phi_0$ *if and only if epi* $\phi_n \to$ *epi* ϕ_0.

Definition 2.4. The sequence $\{h_n\}$ epiconverges to h_0 if

$$epi\ h_n \to epi\ h_0,$$

where the convergence of sets *epi* h_n is understood in the sense of Definition 2.1. That is,

(i) for every $x_0 \in X$ and every $z_0 \in h_0(x_0) + P$, there exist $\{x_n\} \subset X$ convergent to x_0 and $\{z_n\} \subset Z$ convergent to z_0, such that

$$z_n \in h_n(x_n) + P, \text{ for sufficiently large } n;$$

(ii) for every $x_0 \in X$ and $z_0 \in Z$ satisfying that there exist $\{x_k\} \subset X$ convergent to x_0 and $z_k \in h_{n_k}(x_k) + P$ convergent to z_0 for a subsequence $\{n_k\}$, one has

$$z_0 \in h_0(x_0) + P.$$

We denote it as $h_n \overset{epi}{\to} h_0$.

Definition 2.5. The sequence $\{h_n\}$ continuously converges to h_0 if $\{h_n\}$ epiconverges to h_0 and $\{-h_n\}$ epiconverges to $-h_0$.

Let $k^0 \in intP$. The nonlinear scalarization function $\xi : Z \to R$ is defined by

$$\xi(z) := min\{\lambda \in R : z \in \lambda k^0 - P\}, \qquad \forall z \in Z.$$

Proposition 2.6 (Ref. 8)**.** *For each $\lambda \in R$, we have*

(1) $\xi(z) < \lambda \Leftrightarrow z \in \lambda k^0 - intP$;

(2) $\xi(z) \leq \lambda \Leftrightarrow z \in \lambda k^0 - P$;

(3) $\xi(z) \geq \lambda \Leftrightarrow z \notin \lambda k^0 - intP$;

(4) $\xi(z) > \lambda \Leftrightarrow z \notin \lambda k^0 - P$;

(5) $\xi(z) = \lambda \Leftrightarrow z \in \lambda k^0 - \partial P$,

where ∂P is the topological boundary of P.

Then we obtain the following Proposition 2.7.

Proposition 2.7. $h_n \overset{epi}{\to} h_0 \Longrightarrow \xi h_n \overset{epi}{\to} \xi h_0$.

Proof. By Proposition 2.3, we only need to prove

$$epi\ \xi h_n \to epi\ \xi h_0.$$

Firstly, we prove

$$\limsup_{n\to+\infty} epi\ \xi h_n \subset epi\ \xi h_0.$$

For any $(x_0, r_0) \in \limsup_{n\to+\infty} epi\ \xi h_n$, there exists sequence $\{(x_k, r_k)\} \subset X \times R$ convergent to (x_0, r_0) such that $(x_k, r_k) \in epi\ \xi h_{n_k}$, for a subsequence $\{n_k\}$. That is, $(\xi h_{n_k})(x_k) \leq r_k$. By Proposition 2.6,

$$h_{n_k}(x_k) \in r_k k^0 - P.$$

Then

$$r_k k^0 \in h_{n_k}(x_k) + P.$$

By the definition of $epi\ h_n$, we get $(x_k, r_k k^0) \in epi\ h_{n_k}$. Thus, $\left(x_0, r_0 k^0\right) \in \limsup_{n\to+\infty} epi\ h_n$. Since $h_n \overset{epi}{\to} h_0$, by Definition 2.4, $epi\ h_n \to epi\ h_0$. So we have $\left(x_0, r_0 k^0\right) \in epi\ h_0$. That is $r_0 k^0 \in h_0(x_0) + P$, i.e. $h_0(x_0) \in r_0 k^0 - P$. By Proposition 2.6. $r_0 \geq (\xi h_0)(x_0)$. Therefore, $(x_0, r_0) \in epi\ \xi h_0$.

Conversely, we prove

$$epi\ \xi h_0 \subset \liminf_{n\to+\infty} epi\ \xi h_n.$$

For any $(x_0, r_0) \in epi\ \xi h_0$, that is, $(\xi h_0)(x_0) \leq r_0$. By Proposition 2.6, $h_0(x_0) \in r_0 k^0 - P$, i.e. $\left(x_0, r_0 k^0\right) \in epi\ h_0$. Since $h_n \overset{epi}{\to} h_0$, $epi\ h_n \to epi\ h_0$. Hence, there exists $\{\left(x_n, r_n k^0\right)\} \subset X \times Z$ convergent to $\left(x_0, r_0 k^0\right)$ such that $\left(x_n, r_n k^0\right) \in epi\ h_n$ for sufficiently large n. So we get $r_n k^0 \in h_n(x_n) + P$. It is equivalent to $h_n(x_n) \in r_n k^0 - P$. By the definition of ξ, $(\xi h_n)(x_n) \leq r_n$. That is, $(x_n, r_n) \in epi\ \xi h_n$. Therefore, there exists $\{(x_n, r_n)\} \subset X \times R$ convergent to (x_0, r_0) such that $(x_n, r_n) \in epi\ \xi h_n$. Thus, $(x_0, r_0) \in \liminf_{n\to+\infty} epi\ \xi h_n$. □

Definition 2.8 (Ref. 1)**.** The sequence $\{H_n\}$ is lower convergent to H_0 if, for any $x_0 \in X$, any $\{x_n\} \subset X$ convergent to x_0, and any $y_0 \in H_0(x_0)$, there exists $\{y_n\} \subset Y$ convergent to y_0 such that $y_n \in H_n(x_n)$, for sufficiently large n.

Definition 2.9 (Ref. 1)**.** The sequence $\{H_n\}$ is upper convergent or G$^+$ convergent to H_0 if, for any $x_0 \in X$, any $y_0 \in Y$, any $\{x_n\} \subset X$ convergent to x_0, and any $\{y_k\} \subset Y$ convergent to y_0 such that $y_k \in H_{n_k}(x_{n_k})$, for a selection of integers $\{n_k\}$, one has $y_0 \in H_0(x_0)$.

Definition 2.10 (Ref. 1)**.** The sequence $\{H_n\}$ is G$^-$ convergent to H_0 if, for any $x_0 \in X$, and any $y_0 \in H_0(x_0)$, there exist $\{x_n\} \subset X$ convergent to x_0 and $\{y_n\} \subset Y$ convergent to y_0 such that $y_n \in H_n(x_n)$ for sufficiently large n.

We say that the sequence $\{H_n\}$ is G convergent to H_0 if $\{H_n\}$ is G$^+$ and G$^-$convergent to H_0; $\{H_n\}$ is convergent to H_0 if $\{H_n\}$ is lower and upper convergent to H_0.

Now, from the definitions above, we can easily get the following propositions.

Proposition 2.11. *For any $x_0 \in X$, and any $\{x_n\} \subset X$ convergent to x_0,*

(1) *$\{H_n\}$ is upper convergent to H_0 if and only if*

$$\limsup_{n\to+\infty} H_n(x_n) \subset H_0(x_0);$$

(2) *$\{H_n\}$ is lower convergent to H_0 if and only if*

$$H_0(x_0) \subset \liminf_{n\to+\infty} H_n(x_n).$$

Analogously, we derive the following result.

Proposition 2.12. *$\{H_n\}$ is G^- convergent to H_0 if and only if there exists $\{x_n\} \subset X$ convergent to x_0 such that*

$$H_0(x_0) \subset \liminf_{n\to+\infty} H_n(x_n),$$

for any $x_0 \in X$.

From Propositions 2.11 and 2.12, it is easy to deduce the following corollary.

Corollary 2.13. *If $\{H_n\}$ is lower convergent to H_0, then $\{H_n\}$ is G^- convergent to H_0. That is, the property of G^- convergence is weaker than the lower convergence.*

Let $B \subset Z$ be a nonempty subset, $\bar{z} \in B$ is called a Benson's proper efficient point of B with respect to the ordered cone P written $\bar{z} \in BE(B,P)$ if

$$clP(B + P - \bar{z}) \cap (-P) = \{0\},$$

where clA is the closure of A and the generated cone $P(A)$ of A is defined as

$$P(A) := \{\lambda a : \lambda \geq 0,\ a \in A\} = \bigcup_{\lambda\geq 0} \lambda A.$$

We introduce several notations which will be used throughout this paper. The marginal map $W : K \rightrightarrows Z$ is denoted as

$$W(x) := BE(f(x, S(x)), P), \qquad \forall x \in K.$$

The set of all Benson's proper efficient points of (VPEC) is

$$V := BE\left(W\left(K\right), P\right).$$

The set of all Benson's proper efficient solutions to (VPEC) is

$$R := \{x \in K : W(x) \subset V\}.$$

We consider the sequence of perturbations $\{f_n\}_{n\in N}$ of f. The related perturbed (VPEC) is the following:

$$(\text{VPEC})_n \qquad \begin{cases} min & f_n(x,y) \\ s.t. & x \in K,\ and \quad y \in S_n(x), \end{cases}$$

where $f_n : X \times Y \to Z$, and $y \in S_n(x)$ if and only if $y \in C_n(x)$, and satisfies the inequality:

$$(\text{VEP})_n \qquad g_n\left(x, y, z\right) \notin -intP, \quad \text{for any } z \in C_n\left(x\right),$$

with $g_n : X \times Y \times Y \to Z,\ C_n : X \rightrightarrows Y$.

In this paper, we make the blank assumption that the feasible regions of (VPEC) and $(\text{VPEC})_n$ are nonempty. $W_n(x)$, V_n and R_n are defined similarly as $W(x)$, V and R, respectively.

3 Convergence Results for Marginal Map

Firstly, we consider the convergence of $\{S_n\}$ in $(\text{VPEC})_n$.

Lemma 3.1. *Assume that the following conditions hold:*

(i) *the sequence $\{g_n\}$ is continuously convergent to g;*

(ii) *$\{C_n\}$ is upper and lower convergent to C.*

Then, the sequence $\{S_n\}$ is lower and upper convergent to S.

Proof. For any $x_0 \in X$, any $\{x_n\} \subset X$ converging to x_0, and any $y_0 \in S(x_0)$, it holds that $y_0 \in C(x_0)$, and

$$g(x_0, y_0, z_0) \notin -intP, \quad \forall z_0 \in C(x_0). \tag{3.1}$$

Since $\{C_n\}$ is lower convergent to C, there exists $\{y_n\} \subset Y$ converging to y_0 such that $y_n \in C_n(x_n)$, for sufficiently large n. We assume that there exists $\hat{z}_n \in C_n(x_n)$ such that

$$g_n\left(x_n, y_n, \hat{z}_n\right) \in -intP. \tag{3.2}$$

By the compactness of Y, there exists $\hat{z} \in Y$ such that subsequence $\{\hat{z}_{n_k}\} \subset Y \to \hat{z}$. Since $\{C_n\}$ is upper convergent to C, $\hat{z} \in C(x_0)$. But $\{g_n\}$ is epiconvergent to g, from Proposition 2.7, $\xi g_n \overset{epi}{\to} \xi g$. By (3.2), we know

$$g_{n_k}\left(x_{n_k}, y_{n_k}, \hat{z}_{n_k}\right) \in 0k^0 - intP.$$

From Proposition 2.6, $\xi g_{n_k}\left(x_{n_k}, y_{n_k}, \hat{z}_{n_k}\right) < 0$. Then for sufficiently large k, there exists $\varepsilon_k < 0$ such that $\xi g_{n_k}\left(x_{n_k}, y_{n_k}, \hat{z}_{n_k}\right) \leqslant \varepsilon_k$. If not, we get that for any $\varepsilon_k < 0$, $\xi g_{n_k}\left(x_{n_k}, y_{n_k}, \hat{z}_{n_k}\right) > \varepsilon_k$. Thus, we obtain $\xi g_{n_k}\left(x_{n_k}, y_{n_k}, \hat{z}_{n_k}\right) \geqslant 0$. Since ξ is upper semi-continuous on Z (Ref. 5), it holds that there exists $\eta_k > 0$ such that for any $\tilde{g} \in$

$B(g_{n_k}(x_{n_k}, y_{n_k}, \hat{z}_{n_k}), \eta_k)$, $\xi\tilde{g} \leqslant \varepsilon_k$. Hence, $\xi\tilde{g} < 0$. Since $\{g_n\}$ is continuously convergent to g, from Ref. 7 we have $g_{n_k}(x_{n_k}, y_{n_k}, \hat{z}_{n_k}) \to g(x_0, y_0, \hat{z})$ when $(x_{n_k}, y_{n_k}, \hat{z}_{n_k}) \to (x_0, y_0, \hat{z})$. So $g(x_0, y_0, \hat{z}) \in B(g_{n_k}(x_{n_k}, y_{n_k}, \hat{z}_{n_k}), \eta_k)$. Thus,

$$\xi g(x_0, y_0, \hat{z}) < 0.$$

Hence,

$$g(x_0, y_0, \hat{z}) \in 0k^0 - intP = -intP.$$

This is a contradiction to (3.1). Therefore,

$$g_n(x_n, y_n, z_n) \notin -intP, \quad \forall z_n \in C_n(x_n).$$

i.e., there exists $\{y_n\} \subset Y$ converging to y_0 such that $y_n \in S_n(x_n)$, for sufficiently large n, which completes the proof of the lower convergence result of $\{S_n\}$.

For any $x_0 \in X$, any $y_0 \in Y$, any $\{x_n\} \subset X$ converging to x_0, and any $\{y_k\} \subset Y$ converging to y_0 such that $y_k \in S_{n_k}(x_{n_k})$, for a selection of integers $\{n_k\}$, we get $y_k \in C_{n_k}(x_{n_k})$, and

$$g_{n_k}(x_{n_k}, y_k, z_{n_k}) \notin -intP, \quad \forall z_{n_k} \in C_{n_k}(x_{n_k}). \tag{3.3}$$

Since $\{C_n\}$ is upper convergent to C, $y_0 \in C(x_0)$. If there exists $\bar{z}_0 \in C(x_0)$ such that

$$g(x_0, y_0, \bar{z}_0) \in -intP, \tag{3.4}$$

then by the lower convergence of $\{C_n\}$, there exists $\{\bar{z}_{n_k}\} \subset Y$ converging to $\bar{z}_0$ such that $\bar{z}_{n_k} \in C_{n_k}(x_{n_k})$, for sufficiently large k. From the epiconvergence of $\{g_n\}$, $\xi g_n \overset{epi}{\to} \xi g$. We set $t_k = \xi g_{n_k}(x_{n_k}, y_k, z_{n_k})$, $\quad t = \xi g(x_0, y_0, \bar{z}_0)$, so

$$t = \min\{\lambda \in R : g(x_0, y_0, \bar{z}_0) \in \lambda k^0 - P\}.$$

By (3.4), we know

$$g(x_0, y_0, \bar{z}_0) \in 0k^0 - intP.$$

Hence, $t < 0$. By Definition 2.4,

$$\limsup_{k\to+\infty} \xi g_{n_k}(x_{n_k}, y_k, z_{n_k}) < 0.$$

That is

$$\xi g_{n_k}(x_{n_k}, y_k, z_{n_k}) < 0, \text{ for sufficiently large } k.$$

Thus,

$$g_{n_k}(x_{n_k}, y_k, \bar{z}_{n_k}) \in 0k^0 - intP = -intP, \text{ for sufficiently large } k,$$

which contradicts (3.3). Hence, there exists $y_0 \in C(x_0)$ such that

$$g(x_0, y_0, z_0) \notin -intP, \qquad \forall z_0 \in C(x_0).$$

That is, there exists $y_0 \in S(x_0)$. This completes the whole proof. □

In the following, we consider some convergence results of the marginal map $\{W_n\}$ in $(\text{VPEC})_n$. Firstly, we give a lemma which derives the closedness property of the set $f(x, S(x))$, for any $x \in K$.

Lemma 3.2. *If $f(x, .)$ and $g(x, ., z)$ are both continuous on Y, then, for any $x \in K$, $f(x, S(x))$ is closed.*

Proof. Firstly, we show that $S(x)$ is closed. For any $y_n \in S(x)$ and $y_n \to y$, we know

$$g(x, y_n, z) \notin -intP, \quad \text{for any } z \in C(x).$$

Since $g(x, ., z)$ is continuous on Y and $intP$ is an open set, we obtain

$$g(x, y, z) \notin -intP, \quad \text{for any } z \in C(x).$$

That is, $y \in S(x)$, i.e. the set $S(x)$ is closed. For any $t_n \in f(x, S(x))$ and $t_n \to t$, we can always find $s_n \in S(x)$ such that $t_n = f(x, s_n)$. Since Y is compact, there exists $\{s_{n_k}\} \subset \{s_n\} \subset S(x)$ such that $\lim_{k\to+\infty} s_{n_k} = s$, for some $s \in Y$. Since $S(x)$ is closed, we get $s \in S(x)$. Since $f(x, .)$ is continuous on Y, we have

$$t = \lim_{k\to+\infty} t_{n_k} = \lim_{k\to+\infty} f(x, s_{n_k}) = f(x, s).$$

Hence, $t \in f(x, S(x))$. We complete the proof. □

Next, we obtain the convergence results of the marginal map $\{W_n\}$.

Proposition 3.3. *Let $f(x, .)$, $g(x, ., z)$, $f_n(x_n, .)$ and $g_n(x_n, ., z_n)$ be all continuous on Y. Assume that (i) and (ii) in Lemma 3.1, together with*

(iii) *for any $x \in K$, $y \in S(x)$, and for any $\{(x_n, y_n)\}$ convergent to (x, y),*

$$\limsup_{n\to+\infty} f_n(x_n, y_n) \in f(x, y) - intP$$

hold, where the definition of $\limsup_{n\to+\infty} f_n(x_n, y_n)$ is the upper limit of the sequence $\{f(x_n, y_n)\}$. Then $\{W_n\}$ is upper convergent to W. That is, for any $x \in K$ and any $\{x_n\}$ convergent to x,

$$\limsup_{n\to+\infty} W_n(x_n) \subset W(x).$$

Proof. For any $x \in K$, $\{x_n\}$ convergent to x and any $w \in \limsup_{n\to+\infty} W_n(x_n)$, there exists $\{w_k\}$ convergent to w such that $w_k \in W_{n_k}(x_{n_k})$, for a subsequence $\{n_k\}$. That is,

$$clP(f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k})) + P - w_k) \cap (-P) = \{0\}, \text{ for sufficiently large } k.$$

By Lemma 3.2, $f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k}))$ is closed. This fact, together with the closedness of P, gives

$$P(f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k})) + P - w_k) \cap (-P) = \{0\}.$$

If there exists $\tilde{t} \neq 0$ such that $\tilde{t} \in (f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k})) + P - w_k) \cap (-P)$, then for any $\lambda > 0$, $\lambda\tilde{t} \in P(f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k})) + P - w_k) \cap (-P)$ and $\lambda\tilde{t} \neq 0$. It is a contradiction. Thus,

$$(f_{n_k}(x_{n_k}, S_{n_k}(x_{n_k})) + P - w_k) \cap (-P) = \{0\}.$$

That is, there exists $y_k \in S_{n_k}(x_{n_k})$ such that

$$(f_{n_k}(x_{n_k}, y_k) + P - w_k) \cap (-P) = \{0\}.$$

Since Y is compact, there exists a subsequence $\{y_{k_t}\} \subset \{y_k\}$ such that $\lim_{t\to+\infty} y_{k_t} = y$, for some $y \in Y$. Thus, $y_{k_t} \in S_{n_{k_t}}\left(x_{n_{k_t}}\right)$ and

$$\left(f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w_{k_t}\right) \cap (-P) = \{0\}, \quad \text{for } t \text{ large.} \tag{3.5}$$

So,

$$0 \in f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w_{k_t}.$$

That is,

$$-f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + w_{k_t} \in P.$$

Let $t \to +\infty$, since P is closed we get

$$-\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + w \in P.$$

i.e. $0 \in -P$ and

$$-\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) \in P - w. \tag{3.6}$$

Hence,

$$0 \in (\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w) \cap (-P).$$

If there exists an $a \in (\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w) \cap (-P)$ and $a \neq 0$, then

$$\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) - w \in a - P.$$

So,

$$f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) - w_{k_t} \in a - P, \quad \text{for } t \text{ large.}$$

Thus $a \neq 0$ and

$$a \in \left(f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w_{k_t}\right) \cap (-P).$$

It contradicts (3.5). Hence,

$$(\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) + P - w) \cap (-P) = \{0\}. \tag{3.7}$$

From Lemma 3.1, $\{S_n\}$ is upper convergent to S, that is, for any $x \in K$, $y \in Y$, $\{x_n\} \to x$ and $\{y_{k_t}\}$ convergent to y such that $y_{k_t} \in S_{n_{k_t}}\left(x_{n_{k_t}}\right)$, we have $y \in S(x)$. Thus, by (iii),

$$\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) \in f(x,y) - intP. \tag{3.8}$$

(3.6) and (3.8) show that

$$0 \in f(x,y) - w + P - intP.$$

If $0 \notin f(x,y) - w + P$, then there exists $t \in f(x,y) - w + P$ and $t \in -intP$. Thus, $f(x,y) - w = 0$. By (3.8),

$$\limsup_{t\to+\infty} f_{n_{k_t}}(x_{n_{k_t}}, y_{k_t}) - w \in f(x,y) - intP - w = -intP.$$

Since $0 \in P$, we get

$$\limsup_{t\to+\infty} f_{n_{k_t}}(x_{n_{k_t}}, y_{k_t}) - w \in \limsup_{t\to+\infty} f_{n_{k_t}}(x_{n_{k_t}}, y_{k_t}) - w + P.$$

Hence

$$0 \neq \limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) - w \in \left(\limsup_{t\to+\infty} f_{n_{k_t}}\left(x_{n_{k_t}}, y_{k_t}\right) - w + P\right) \cap (-P).$$

It contradicts (3.7). So $0 \in f(x,y) - w + P \subset f(x, S(x)) - w + P$. Then

$$0 \in clP\left(f(x, S(x)) - w + P\right) \cap (-P).$$

We assume that there exists $t \in clP(f(x,S(x)) - w + P) \cap (-P)$ and $t \neq 0$. Since both $f(x, S(x))$ and P are closed, there exists $\lambda > 0$ such that

$$\frac{t}{\lambda} \in (f(x,S(x)) - w + P) \cap (-P).$$

Thus, there exists $y \in S(x)$ such that

$$\frac{t}{\lambda} \in \left(f(x,y) - w + P\right) \cap (-P). \tag{3.9}$$

From Lemma 3.1, $\{S_n\}$ is lower convergent to S, that is, for $x \in X$, $\{x_n\} \subset X$ convergent to x and any $y \in S(x)$, there exists $\{y_n\} \subset Y$ convergent to y such that $y_n \in S_n(x_n)$, for sufficiently large n. Then by (iii),

$$\limsup_{n\to+\infty} f_n(x_n, y_n) - w \in f(x,y) - w - intP.$$

That is,

$$f(x,y) - w \in \limsup_{n\to+\infty} f_n(x_n, y_n) - w + intP.$$

Thus,

$$f(x,y) - w + P \subset \limsup_{n\to+\infty} f_n(x_n, y_n) - w + P.$$

By (3.9),

$$0 \neq \frac{t}{\lambda} \in \left(\limsup_{n\to+\infty} f_n(x_n, y_n) - w + P\right) \cap (-P).$$

This contradicts (3.7). Therefore

$$clP\left(f(x, S(x)) - w + P\right) \cap (-P) = \{0\},$$

i.e. $w \in W(x)$. Hence, for any $x \in K$ and any $\{x_n\}$ convergent to x,

$$\limsup_{n\to+\infty} W_n(x_n) \subset W(x).$$

□

Proposition 3.4. *Let $f_n(x_n, .)$ and $g_n(x_n, ., z_n)$ be continuous on Y. Assume that (i), (ii) in Lemma 3.1, and the following*

(iv) *for any $x \in K$, $y \in S(x)$ and any $\{(x_n, y_n)\}$ convergent to (x,y),*

$$\liminf_{n\to+\infty} f_n(x_n, y_n) \in f(x,y) + intP$$

hold. Then, $\{W_n\}$ is lower convergent to W, i.e. for any $x \in K$ and any $\{x_n\} \to x$,

$$W(x) \subset \liminf_{n\to+\infty} W_n(x_n).$$

Proof. To the contrary, we assume that there exists $\bar{x} \in K$ and $\{\bar{x}_n\} \to \bar{x}$ such that

$$W(\bar{x}) \not\subset \liminf_{n\to+\infty} W_n(\bar{x}_n).$$

That is, there exists $\bar{w} \in W(\bar{x})$ such that $\bar{w} \notin \liminf_{n\to+\infty} W_n(\bar{x}_n)$. Hence for any $\{\bar{w}_n\}$ convergent to $\bar{w}$, $\bar{w}_n \notin W_n(\bar{x}_n)$, for sufficiently large n. Thus,

$$clP(f_n(\bar{x}_n, S_n(\bar{x}_n)) + P - \bar{w}_n) \cap (-P) \neq \{0\}.$$

We know $0 \in clP(f_n(\bar{x}_n, S_n(\bar{x}_n)) + P - \bar{w}_n) \cap (-P)$. Then there exists $t \neq 0$ such that

$$t \in clP(f_n(\bar{x}_n, S_n(\bar{x}_n)) + P - \bar{w}_n) \cap (-P).$$

So there exists $\{t_m\} \to t$ such that $t_m \in P(f_n(\bar{x}_n, S_n(\bar{x}_n)) + P - \bar{w}_n)$, for sufficiently large m. Since both $f_n(\bar{x}_n, S_n(\bar{x}_n))$ and P are closed, let $m \to +\infty$ we get

$$t \in P(f_n(\bar{x}_n, S_n(\bar{x}_n)) + P - \bar{w}_n).$$

That is, there exists $\lambda > 0$, and $\bar{y}_n \in S_n(\bar{x}_n)$ such that

$$\frac{t}{\lambda} \in f_n(\bar{x}_n, \bar{y}_n) + P - \bar{w}_n.$$

Let $n \to +\infty$, we have

$$\frac{t}{\lambda} \in \liminf_{n\to+\infty} f_n(\bar{x}_n, \bar{y}_n) + P - \bar{w}.$$

Since Y is compact, there exists $\{\bar{y}_{n_k}\} \subset \{\bar{y}_n\}$ such that $\lim_{k\to+\infty} \bar{y}_{n_k} = \bar{y}$, for some $\bar{y} \in Y$. Thus, $\bar{y}_{n_k} \in S_{n_k}\left(\bar{x}_{n_k}\right)$ and

$$\frac{t}{\lambda} \in \liminf_{k\to+\infty} f_{n_k}(\bar{x}_{n_k}, \bar{y}_{n_k}) + P - \bar{w}. \tag{3.10}$$

From Lemma 3.1, $\{S_n\}$ is upper convergent to S, that is, for $\bar{x} \in K$, $\bar{y} \in Y$, $\{\bar{x}_n\} \to \bar{x}$ and $\{\bar{y}_{n_k}\}$ convergent to $\bar{y}$ such that $\bar{y}_{n_k} \in S_{n_k}\left(\bar{x}_{n_k}\right)$ for a selection of integers $\{n_k\}$, we have $\bar{y} \in S(\bar{x})$. Thus, by (iv) and (3.10) we obtain

$$\frac{t}{\lambda} \in f(\bar{x}, \bar{y}) + P + intP - \bar{w} \subset f(\bar{x}, \bar{y}) + P - \bar{w}.$$

Hence,

$$\frac{t}{\lambda} \in clP(f(\bar{x}, \bar{y}) + P - \bar{w}).$$

Since $\lambda > 0$ and P is a cone, $\frac{t}{\lambda} \in -P$. So we get $\frac{t}{\lambda} \neq 0$ and

$$\frac{t}{\lambda} \in clP(f(\bar{x}, \bar{y}) + P - \bar{w}) \cap (-P) \subset clP(f(\bar{x}, S(\bar{x})) + P - \bar{w}) \cap (-P).$$

By $\bar{w} \in W(\bar{x})$ we know

$$clP(f(\bar{x}, S(\bar{x})) + P - \bar{w}) \cap (-P) = \{0\}.$$

It is a contradiction. Therefore, for any $x \in K$ and any $\{x_n\} \to x$,

$$W(x) \subset \liminf_{n\to+\infty} W_n(x_n).$$

□

Corollary 3.5. *On the conditions of Proposition 3.3 and (iv) in Proposition 3.4, $\{W_n\}$ is convergent to W.*

4 Convergence Results of Value Map and Solution Map

In the following part, we get the convergence results of $\{V_n\}$ and $\{R_n\}$. Firstly, we prove a closedness property of $W(K)$.

Lemma 4.1. *If $f(x,.)$ and $g(x,.,z)$ are continuous on Y, then for any $x \in K$, $W(x)$ is closed, thus $W(K)$ is also closed.*

Proof. For any $x \in K$, $w_n \in W(x)$ such that $w_n \to w$, we know

$$clP\left(f\left(x,S\left(x\right)\right)+P-w_n\right)\cap(-P)=\{0\}.$$

By Lemma 3.2, $f(x,S(x))$ is closed. Hence we get

$$P\left(f\left(x,S\left(x\right)\right)+P-w_n\right)\cap(-P)=\{0\}.$$

If there exists $\tilde{t} \neq 0$ such that $\tilde{t} \in (f(x,S(x))+P-w_n)\cap(-P)$, then for any $\lambda > 0$, $\lambda\tilde{t} \in P(f(x,S(x))+P-w_n)\cap(-P)$ and $\lambda\tilde{t} \neq 0$. It is a contradiction. Hence we have

$$\left(f\left(x,S\left(x\right)\right)+P-w_n\right)\cap(-P)=\{0\}.$$

Let $n \to +\infty$ we obtain

$$\left(f\left(x,S\left(x\right)\right)+P-w\right)\cap(-P)=\{0\}.$$

Therefore $w \in W(x)$ and $W(x)$ is closed. Since K is closed, $W(K)$ is also closed. □

Theorem 4.2. *Let $f(x,.)$, $g(x,.,z)$, $f_n(x_n,.)$ and $g_n(x_n,.,z_n)$ be all continuous on Y. Assume (i), (ii) in Lemma 3.1, (iii) in Proposition 3.3 and (iv) in Proposition 3.4 hold. Then $\{V_n\}$ is convergent to V. That is,*

$$\limsup_{n\to+\infty} V_n \subset V \subset \liminf_{n\to+\infty} V_n.$$

Proof. Firstly, we show $V \subset \liminf_{n\to+\infty} V_n$. For any $v \in V$, we have

$$clP\left(W\left(K\right)+P-v\right)\cap(-P)=\{0\}.$$

By Lemma 4.1, $W(K)$ is closed, hence we obtain

$$\left(W\left(K\right)+P-v\right)\cap(-P)=\{0\}.$$

Thus, there exists an $x \in K$ such that

$$\left(W\left(x\right)+P-v\right)\cap(-P)=\{0\}. \tag{4.1}$$

We assume that $v \notin \liminf_{n\to+\infty} V_n$, that is, for any $\{v_n\} \to v$, $v_n \notin V_n$, for sufficiently large n. Hence

$$clP\left(W_n(K)+P-v_n\right)\cap(-P)\neq\{0\}.$$

There are two cases: (i) $clP(W_n(K)+P-v_n)\cap(-P)=\emptyset$; (ii) there exists $t \in clP(W_n(K)+P-v_n)\cap(-P)$ and $t \neq 0$.

Case (i). Since $W_n(K)+P-v_n \subset clP(W_n(K)+P-v_n)$, we have

$$\left(W_n\left(K\right)+P-v_n\right)\cap(-P)=\emptyset.$$

Then for any $\{x_n\} \subset K$ convergent to x, one has

$$(W_n(x_n) + P - v_n) \cap (-P) = \emptyset. \tag{4.2}$$

For any $t \in \liminf_{n\to+\infty} W_n(x_n) + P - v$, there exists $p \in P$ such that $t \in \liminf_{n\to+\infty} W_n(x_n) + p - v$. So $t - p + v \in \liminf_{n\to+\infty} W_n(x_n)$. It shows that there exists $\{t_n\}$ convergent to $t - p + v$ such that $t_n \in W_n(x_n)$. Thus, $t_n + p - v_n \to t - p + v + p - v = t$ and $t_n + p - v_n \in W_n(x_n) + P - v_n$. Since $W_n(x_n)$ and P are closed, $t \in W_n(x_n) + P - v_n$. By (4.2), $t \notin -P$. Hence

$$(\liminf_{n\to+\infty} W_n(x_n) + P - v) \cap (-P) = \emptyset.$$

From Proposition 3.4, $W(x) \subset \liminf_{n\to+\infty} W_n(x_n)$. Thus,

$$(W(x) + P - v) \cap (-P) = \emptyset.$$

It contradicts (4.1).

Case (ii). By Lemma 4.1,

$$t \in P(W_n(K) + P - v_n) \cap (-P).$$

Thus, there exists $\lambda > 0$ and $\mu \in (W_n(K) + P - v_n) \cap (-P)$ such that $t = \lambda\mu$ and $\mu \neq 0$. By $\mu \in (W_n(K) + P - v_n)$, we know that there exists $p \in P$ such that $\mu - p + v_n \in W_n(K)$, for sufficiently large n. By $\mu - p + v_n \to \mu - p + v$, one has $\mu - p + v \in \limsup_{n\to+\infty} W_n(K)$, i.e. $\mu \in \limsup_{n\to+\infty} W_n(K) + P - v$. Since K is compact, from Proposition 3.3, we get $\limsup_{n\to+\infty} W(K) \subset W_n(K)$. Hence

$$\mu \in (W(K) + P - v) \cap (-P).$$

It is easy to see that this contradicts with $v \in V$ by $\mu \neq 0$. Therefore there exists $\{v_n\}$ convergent to v such that $v_n \in V_n$. That is, $v \in \liminf_{n\to+\infty} V_n$. So we complete the proof of

$$V \subset \liminf_{n\to+\infty} V_n.$$

Secondly, for any $v \in \limsup_{n\to+\infty} V_n$, there exists $\{v_k\}$ convergent to v such that $v_k \in V_{n_k}$, for sufficiently large k. That is

$$clP(W_{n_k}(K) + P - v_k) \cap (-P) = \{0\}.$$

By Lemma 4.1, $W_{n_k}(K)$ and P are closed for sufficiently large k. So we get

$$(W_{n_k}(K) + P - v_k) \cap (-P) = \{0\}.$$

Thus, there exists $\{x_{n_k}\} \subset K$ such that

$$(W_{n_k}(x_{n_k}) + P - v_k) \cap (-P) = \{0\}. \tag{4.3}$$

Hence there exists $p \in P$ such that $0 \in W_{n_k}(x_{n_k}) + p - v_k$. That is, $v_k - p \in W_{n_k}(x_{n_k})$, for sufficiently large k. Thus, $v - p \in \limsup_{n\to+\infty} W_n(x_n)$ by $v_k \to v$. i.e.

$$0 \in \limsup_{n\to+\infty} W_n(x_n) + P - v \subset \limsup_{n\to+\infty} W_n(K) + P - v.$$

Since K is compact, from Proposition 3.3, we get

$$0 \in W(K) + P - v.$$

If there exists $t \neq 0$ such that $t \in clP(W(K) + P - v) \cap (-P)$, it follows from the closedness of $W(K)$ and P that there exists $\lambda > 0$ such that

$$\frac{t}{\lambda} \in (W(K) + P - v) \cap (-P).$$

Thus, there exist $x \in K$ and $p \in P$ such that

$$\frac{t}{\lambda} \in (W(x) + P - v) \cap (-P).$$

That is,

$$\frac{t}{\lambda} - p + v \in W(x).$$

By Proposition 3.4, for any $\{x_n\}$ convergent to x,

$$\frac{t}{\lambda} - p + v \in \liminf_{n \to +\infty} W_n(x_n) \subset \limsup_{n \to +\infty} W_n(x_n).$$

Since $\frac{t}{\lambda} - p + v_k \to \frac{t}{\lambda} - p + v$, we obtain $\frac{t}{\lambda} - p + v_k \in W_{n_k}(x_{n_k})$, for sufficiently large k. That is, $\frac{t}{\lambda} \in (W_{n_k}(x_{n_k}) + P - v_k) \cap (-P)$. It contradicts (4.3) by $\frac{t}{\lambda} \neq 0$. Therefore

$$clP(W(K) + P - v) \cap (-P) = \{0\}.$$

That is, $v \in BE(W(K), P) = V$. We complete the proof. □

Theorem 4.3. *Let $f(x,.)$, $g(x,.,z)$, $f_n(x_n,.)$ and $g_n(x_n,.,z_n)$ be all continuous on Y. Assume (i), (ii) in Lemma 3.1, (iii) in Proposition 3.3 and (iv) in Proposition 3.4 hold. Then the sequence $\{R_n\}$ of the proper efficient solution sets for (VPEC)$_n$ is upper convergent to R, i.e.*

$$\limsup_{n \to +\infty} R_n \subset R.$$

Proof. For any $x \in \limsup_{n \to +\infty} R_n$, there exists $\{x_k\}$ convergent to x such that $x_k \in R_{n_k}$, for sufficiently large k. That is, $W_{n_k}(x_k) \in V_{n_k}$. For any $w \in W(x)$, by Proposition 3.4,

$$w \in \liminf_{k \to +\infty} W_{n_k}(x_k) \subset \limsup_{k \to +\infty} W_{n_k}(x_k).$$

So there exists $\{w_t\}$ convergent to w such that $w_t \in W_{n_{k_t}}(x_{k_t})$, for sufficiently large t. Thus, we get $w_t \in V_{n_{k_t}}$. Hence

$$clP\left(W_{n_{k_t}}(K) + P - w_t\right) \cap (-P) = \{0\}. \tag{4.4}$$

Since both $W_{n_{k_t}}(K)$ and P are closed, we have

$$(W_{n_{k_t}}(K) + P - w_t) \cap (-P) = \{0\}, \text{ for sufficiently large } t.$$

Then there exists $\{\tilde{x}_t\} \subset K$ such that

$$\left(W_{n_{k_t}}(\tilde{x}_t) + P - w_t\right) \cap (-P) = \{0\}.$$

Since K is compact, there exists $\{\tilde{x}_{t_p}\} \subset \{\tilde{x}_t\}$ such that $\lim_{p\to+\infty} \tilde{x}_{t_p} = \tilde{x}$ for some $\tilde{x} \in K$. So one gets

$$\left(W_{n_{k_{t_p}}}\left(\tilde{x}_{t_p}\right) + P - w_{t_p}\right) \cap (-P) = \{0\}, \text{ for sufficiently large } p.$$

That is, there exists $s \in P$ such that $w_{t_p} - s \in W_{n_{k_{t_p}}}\left(\tilde{x}_{t_p}\right)$. Thus, we obtain $w - s \in \limsup_{p\to+\infty} W_{n_{k_{t_p}}}\left(\tilde{x}_{t_p}\right)$ by $w_{t_p} - s \to w - s$. From Proposition 3.3, $w - s \in W\left(\tilde{x}\right)$. Hence,

$$0 \in clP(W(K) + P - w) \cap (-P).$$

We assume that there exists $t \in clP\left(W\left(K\right) + P - w\right) \cap (-P)$ and $t \neq 0$. Then there exists $\lambda > 0$ such that

$$\frac{t}{\lambda} \in \left(W\left(K\right) + P - w\right) \cap \left(-P\right).$$

Like the proof of Theorem 4.2, we get the contradiction with (4.4). Therefore,

$$clP(W(K) + P - w) \cap (-P) = \{0\}.$$

That is, $w \in BE\left(W\left(K\right), P\right) = V$. Hence, $x \in$R. The proof is completed. □

5 Special Case: Vector Variational Inequality Constraints

In this section, we investigate a special case of (VPEC), that is, a vector optimization problem with vector variational inequality constraints (VPVIC in short). Assume that the same assumptions for the maps f and C as in (VPEC) hold. The problem is

$$\text{(VPVIC)} \quad \begin{cases} min & f\left(x, y\right) \\ s.t. & x \in K, \text{ and } y \in SV\left(x\right). \end{cases}$$

For each $x \in K$, $SV(x)$ is the solution set of a vector variational inequality (VVI); that is, $y \in SV\left(x\right)$ if and only if $y \in C\left(x\right)$, and

$$\text{(VVI)} \quad \langle T(x, y), z - y\rangle \notin -intP, \quad \text{for any } z \in C(x),$$

where $T: X \times Y \to L(Y, Z)$ is a function and by $L(Y, Z)$ we denote the set of all continuous linear operators from Y to Z. We refer to Refs. 4 and 14 for the references on vector variational inequality problems.

The perturbed (VPVIC) is

$$\text{(VPVIC)}_n \quad \begin{cases} min & f_n\left(x, y\right) \\ s.t. & x \in K, \text{ and } y \in SV_n\left(x\right), \end{cases}$$

where $y \in SV_n\left(x\right)$ if and only if $y \in C_n\left(x\right)$ satisfies the inequality

$$\text{(VVI)}_n \quad \langle T_n(x, y), z - y\rangle \notin -intP, \quad \text{for any } \ z \in C_n(x),$$

with the maps $f_n : X \times Y \to Z$, $C_n : X \rightrightarrows Y$ and $T_n : X \times Y \to L(Y, Z)$.

Let $L(Y, Z)$ be partially ordered by a closed, convex and pointed cone $L^+(Y, Z)$, defined by

$$L^+(Y, Z) := \{l \in L(Y, Z) : \ l(y) \in P, \ \forall \ y \in Y\}.$$

Since the vector variational inequality is a special case of the vector equilibrium problem, we set

$$g\,(x,y,z) = \langle T(x,y), z-y\rangle$$

and

$$g_n\,(x,y,z) = \langle T_n(x,y), z-y\rangle.$$

Obviously, $T(x,\cdot)$ ($T_n(x,\cdot)$ respectively) is continuous on Y ensures that $g(x,\cdot,z)$ ($g_n(x,\cdot,z)$ respectively) is continuous on Y.

In order to obtain the similar results as those in Section 4, we will prove

Lemma 5.1. *If the sequence $\{T_n\}$ is epiconvergent to T, then the sequence $\{g_n\}$ is also epiconvergent to g.*

Proof. Firstly we prove (i) in Definition 2.4. For every $(x_0,y_0,z_0) \in X\times Y\times Y$ and every $g_0 \in g_0(x_0,y_0,z_0)+P$, we get $g_0 \in \langle T(x_0,y_0), z_0-y_0\rangle + P$. It is easy to find $l_0 \in T(x_0,y_0)+L^+(Y,Z)$ such that $g_0 = \langle l_0, z_0-y_0\rangle$. By the assumption that T_n is epiconvergent to T, there exist $\{(x_n,y_n)\}\subset X\times Y$ convergent to (x_0,y_0) and $\{l_n\}\subset L(Y,Z)$ convergent to l_0 such that

$$l_n \in T_n(x_n,y_n)+L^+(Y,Z),\ \text{for sufficiently large } n.$$

Let $\{z_n\}\subset Y$ be any a sequence convergent to z_0. Then there holds

$$\begin{aligned}\langle l_n, z_n-y_n\rangle &\in \langle T_n(x_n,y_n), z_n-y_n\rangle + P\\ &= g_n\,(x_n,y_n,z_n)+P, \qquad \text{for } n \text{ large.}\end{aligned}$$

Since $\{l_n\}$ is convergent to l_0, $\langle l_n, z_n-y_n\rangle$ is convergent to $\langle l_0, z_0-y_0\rangle$. Hence there exist $\{(x_n,y_n,z_n)\}\subset X\times Y\times Y$ convergent to (x_0,y_0,z_0) and $\{\langle l_n, z_n-y_n\rangle\}\subset Z$ convergent to $g_0=\langle l_0,z_0-y_0\rangle$, such that

$$\langle l_n, z_n-y_n\rangle \in g_n(x_n,y_n,z_n)+P,\ \text{for sufficiently large } n.$$

In (ii), for every $(x_0,y_0,z_0)\in X\times Y\times Y$ and $g_0\in Z$ satisfying that there exist $\{(x_k,y_k,z_k)\}\subset X\times Y\times Y$ convergent to (x_0,y_0,z_0) and $g_k\in g_{n_k}(x_k,y_k,z_k)+P$ convergent to g_0 for a subsequence $\{n_k\}$, we have $g_k\in\langle T_{n_k}(x_k,y_k), z_k-y_k\rangle+P$. Then there exists $l_k\in T_{n_k}(x_k,y_k)+L^+(Y,Z)$ such that $g_k=\langle l_k, z_k-y_k\rangle$. By the convergence of $\{g_k\}$ to g_0 we know $\{l_k\}$ is convergent. If we assume that the limit of $\{l_k\}$ is $l_0\in L(Y,Z)$, then $g_0=\langle l_0, z_0-y_0\rangle$. Since T_n is epiconvergent to T, one gets $l_0\in T(x_0,y_0)+L^+(Y,Z)$. Hence,

$$g_0=\langle l_0,z_0-y_0\rangle\in\langle T(x_0,y_0), z_0-y_0\rangle+P.$$

We complete the whole proof. □

Analogously, we derive

Lemma 5.2. *If the sequence $\{-T_n\}$ is epiconvergent to $-T$, then the sequence $\{-g_n\}$ is also epiconvergent to $-g$.*

By Lemma 5.1 and Lemma 5.2 we get the following corollary.

Corollary 5.3. *If the sequence $\{T_n\}$ is continuously convergent to T, then the sequence $\{g_n\}$ is also continuously convergent to g.*

Thus we can derive the parallel results to those in Section 4 when the constraint is a vector variational inequality.

Theorem 5.4. *Let $f(x,.)$, $T(x,.)$, $f_n(x_n,.)$ and $T_n(x_n,.)$ be all continuous on Y. Assume that the sequence $\{T_n\}$ is continuously convergent to T, $\{C_n\}$ is upper and lower convergent to C, and for any $x \in K$, $y \in S(x)$, and any $\{(x_n, y_n)\}$ convergent to (x, y),*

$$\limsup_{n\to+\infty} f_n(x_n, y_n) \in f(x,y) - intP,$$

and

$$\liminf_{n\to+\infty} f_n(x_n, y_n) \in f(x,y) + intP.$$

Then $\{V_n\}$ is convergent to V and $\{R_n\}$ is upper convergent to R, i.e.

$$\limsup_{n\to+\infty} V_n \subset V \subset \liminf_{n\to+\infty} V_n.$$

and

$$\limsup_{n\to+\infty} R_n \subset R,$$

where the definitions of V_n, R_n, V and R are as same as for (VPEC).

References

[1] H. Attouch, *Variational Convergence for Functions and Operators*, Appl. Math. Ser. Pitman, London, 1984.

[2] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Mathematics Student* 63 (1994) 123–145.

[3] J. Bracken and J.T. McGill, Mathematical programs with optimization programs in the constraints, *Operations Research* 21 (1973) 37–43.

[4] G.Y. Chen and X.Q. Yang, The Vector complementary problem and its equivalence with the weak minimal element in ordered space, *Journal of Mathematical Analysis and Applications* 153 (1990) 136–158.

[5] G.Y. Chen , X.Q. Yang and H. Yu, A nonlinear scalarization function and generalized quasi-vector equilibrium problems, *Journal of Global Optimization*, 32 (2005) 451–466.

[6] L. Chen , G.H. Lin and X.M. Yang, Expected residual minimization method for stochastic mixed variational inequality problems, *Pacific Jourbal of Optimization*, **14** (2018) 703–722.

[7] N.H. Chieu and G. M. Lee, Constraint qualifications for mathematical programs with equilibrium constraints and their local preservation property, *Journal of Optimization Theory and Applications* 163 (2014) 755–776.

[8] C. Gerth and P. Weidner, Nonconvex separation theorems and some applications in vector optimization, *Journal of Optimization Theory and Applications* 67 (1990) 297–320.

[9] G.X. Lin, Z.P. Wan and X.K. Zhao, Optimality conditions for bilevel optimization program with both levels problems being multiobjective, *Pacific Jourbal of Optimization* 13 (2017) 421–441.

[10] Y. Pandey and S.K. Mishra, Duality of mathematical programming problems with equilibrium constraints, *Pacific Jourbal of Optimization* 13 (2017) 105–122.

[11] F. Toyasaki, P. Daniele and T. Wakolbinger, A variational inequality formulation of equilibrium models for end-of-life products with nonlinear constraints, *European Journal of Operational Research* 236 (2014), 340–350.

[12] Y.N. Wu and T.C.E. Cheng, Convergence results of weak efficiency in vector optimization problems with equilibrium constraints, *Journal of Optimization Theory and Applications* 125 (2005) 453–472.

[13] Y.N. Wu, Y.C. Peng, L. Peng and L. Xu, Super efficiency of multi-criterion network equilibrium model and vector variational inequality, *Journal of Optimization Theory and Applications* 153 (2012) 485–496.

[14] X.Q. Yang and C.J. Goh, On vector variational inequalities: application to vector equilibria, *Journal of Optimization Theory and Applications* 95 (1997), 431–443.

Manuscript received 28 March 2019
revised 2 April 2019
accepted for publication 9 July 2019

Yunan Wu
International Business School
Beijing Foreign Studies University
Beijing 100089, China
E-mail address: wuyunan@bfsu.edu.cn

Ding Zhang
Marketing and Management Department
State University of New York at Oswego
Oswego, NY 13126, USA
E-mail address: june.dong@oswego.edu

June Dong
Marketing and Management Department
State University of New York at Oswego
Oswego, NY 13126, USA
E-mail address: ding.zhang@oswego.edu