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SOME APPLICATIONS OF NEAR (C,ε)-SUBCONVEXLIKENESS IN VECTOR OPTIMIZATION PROBLEMS WITH SET-VALUED MAPS

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: The main aim of this work is to study (C,ε)-Benson proper efficient elements of a vector optimization problem with nearly (C,ε)-subconvexlike set-valued maps. First, some properties of near (C,ε)-subconvexlikeness are discussed. Then, Lagrangian multiplier theorem for (C,ε)-Benson proper efficient elements of a constrained vector optimization problem with set-valued maps is obtained by using the technique of linear scalarization. Our results generalize and improve other similar results in the literature.

Key words: *(C,ε)-Benson proper efficient elements, near (C,ε)-subconvexlikeness, Lagrangian multiplier theorem*

Mathematics Subject Classification: *90C29*

1 Introduction

Convexity and generalized convexity play a significant role in many aspects of vector optimization, especially in studying various types of solutions (including optimality conditions, characterizations of solutions, the existence of solutions) and duality theory of vector optimization problems. Therefore, the investigation of generalized convexity greatly promotes the development of vector optimization theory and methods. The most common generalized convexities of vector functions with set-valued data include cone-convexity [1], cone-convexlikeness [8], cone-subconvexlikeness [9], generalized cone-subconvexlikeness [12] and near cone-subconvexlikeness [13], etc. Near cone-subconvexlikeness is still one of the weakest convexity. Further research on near cone-subconvexlikeness has been done by Sach [11], and some important property of near cone-subconvexlikeness is given as follows:

$$\begin{gathered} clcone(B+K) \text{is convex} \quad \Leftrightarrow \\ \forall \alpha \in (0,1), \forall b_i \in B, i=1,2,\ \alpha b_1 + (1-\alpha) b_2 \in clcone(B+K), \end{gathered}$$

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that is,

$$clcone(B+K)\text{is convex} \Leftrightarrow co(B) \subseteq clcone(B+K),$$

where B is a nonempty subset of Banach space Y, $K \subseteq Y$ is a convex cone. By using this characterization of near cone-subconvexlikeness, Sach established saddle-point criterion for Benson proper efficient solutions of a vector optimization problem with nearly subconvexlike objectives and constraints.

It is worth noting that co-radiant set is a useful tool for studying vector optimization problems. Its role in vector optimization is similar to that of a convex cone, for example, see [3–5]. Some interesting questions then arise naturally: Whether the above property holds when the convex cone K is replaced by a co-radiant set $C(\varepsilon)$ $(\varepsilon \geq 0)$? Could this characterization be used to investigate (C,ε)-Benson proper efficient elements of vector optimization problem with set-valued maps?

Based on the above considerations, we have three objectives in this work. First, to explore some properties of near (C,ε)-subconvexlikeness; Second, to establish linear scalarization result for (C,ε)-Benson proper efficient elements of a vector optimization problem with set-valued maps; Third, to obtain Lagrangian multiplier theorem for (C,ε)-Benson proper efficient elements of a vector optimization problem with set-valued maps via linear scalarization method.

2 Preliminaries

Throughout the paper, X will be a real linear space, Y and Z be two real locally convex Hausdorff topological vector spaces. The topological dual spaces of Y and Z are denoted by Y^* and Z^*, respectively. Let $\mathbb{R}^n$ be the usual $n-$dimensional Euclidean space with the nonnegative orthant $\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i \geq 0, i=1,\dots,n\}$, and denote $\mathbb{R}^n_{++} := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_i > 0, i=1,\dots,n\}$.

For any nonempty set $B \subseteq Y$, $intB$ and clB denote its topological interior and closure, respectively. The cone hull, positive cone hull, convex hull of B are defined as, respectively,

$$coneB = \{\lambda b : b \in B, \lambda \geq 0\},$$
$$cone_+B = \{\lambda b : b \in B, \lambda > 0\},$$
$$coB = \{\sum_{i=1}^{k} \lambda_i b_i : \sum_{i=1}^{k} \lambda_i = 1, b_i \in B, \lambda_i > 0, \forall i \in 1,\dots,k, k \in N\}.$$

A set $B \subseteq Y$ is said to be proper if $\emptyset \neq B \neq Y$, and be pointed if $B \cap (-B) \subseteq \{0\}$. We denote the positive polar cone and the strict positive polar cone of B by B^+ and B^{+i}, respectively, i.e.,

$$B^+ = \{y^* \in Y^* : y^*(b) \geq 0, \forall b \in B\},$$
$$B^{+i} = \{y^* \in Y^* : y^*(b) > 0, \forall b \in B \backslash \{0\}\}.$$

Recall that a base of a cone $K \subseteq Y$ is a convex subset B of K such that $0 \notin B$ and $K = coneB$.

Definition 2.1 ([5]). A set $B \subseteq Y$ is called a co-radiant set if $\alpha B \subseteq B$ for each scalar $\alpha > 1$.

Let $C \subseteq Y$ be a nonempty co-radiant set, $\varepsilon \in \mathbb{R}$ and denote

$$C(\varepsilon) := \varepsilon C, \forall \varepsilon > 0; \qquad C(0) := \bigcup_{\varepsilon > 0} C(\varepsilon) = cone_+ C.$$

By definition, it is easy to see that $coneC = C(0) \bigcup \{\mathbf{0}\}$, and $C(\varepsilon)^{+i} = C^{+i}$ for each $\varepsilon \geq 0$.

Some elementary properties of co-radiant sets to be used later are collected in the following two lemmas.

Lemma 2.2. *Suppose $C \subseteq Y$ is a proper pointed co-radiant set. Then*

(i) $\forall \varepsilon \geq 0$, *$C(\varepsilon)$ is a proper pointed co-radiant.*

(ii) $C(0) \cup \{\mathbf{0}\}$ *is a pointed cone.*

(iii) $\forall \varepsilon_1, \varepsilon_2 \geq 0$*:* $\varepsilon_1 \leq \varepsilon_2$*,* $C(\varepsilon_2) \subseteq C(\varepsilon_1)$.

(iv) *For any nonempty set $B \subseteq Y$,*

$$C(\varepsilon) \subseteq clcone(B + C(\varepsilon)), \forall \varepsilon \geq 0, \tag{2.1}$$

$$clcone(B + C(0)) = clcone(B + C(0) \cup \{0\}). \tag{2.2}$$

Proof. The straightforward proof of (i)-(iii) can be found in [5].

(iv): Let $\varepsilon \geq 0$. Pick arbitrarily $q \in C(\varepsilon)$, then

$$q = \lim_{\lambda \to +\infty} \frac{1}{\lambda}(b + \lambda q), \forall b \in B.$$

According to (i), we have that $\lambda q \in C(\varepsilon)$ for $\lambda > 1$, which along with the above equality implies $q \in clcone(B + C(\varepsilon))$. Hence, (2.1) holds.

Next, we show (2.2). Since

$$\begin{aligned} B + C(0) &\subseteq B + C(0) \cup \{0\} \\ &\subseteq clconeB + clC(0) \\ &= clcone_+B + clC(0) \\ &\subseteq clcone_+(B + C(0)) \\ &= clcone(B + C(0)), \end{aligned}$$

we get

$$clcone(B + C(0)) \subseteq clcone(B + C(0) \cup \{0\}) \subseteq clcone(B + C(0)),$$

i.e., (2.2) holds. □

Remark 2.3. It should be noted that even if C is a proper pointed convex co-radiant set, and B is a nonempty convex set, the following equation may be not true

$$clcone(B + C(\varepsilon)) = clcone(B + C(\varepsilon) \cup \{0\}),$$

where $\varepsilon > 0$. For example, let $B = \{(-1, -1)\} \subseteq \mathbb{R}^2$, $C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 1, y_2 \geq 1\}$, $\varepsilon = 1$. It's clear that $clcone(B + C(\varepsilon)) = \mathbb{R}^2_+$, but $clcone(B + C(\varepsilon) \cup \{0\}) = \mathbb{R}^2_+ \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = y_1, y_1 \leq 0\}$.

Lemma 2.4. *Suppose that $C \subseteq Y$ is a convex co-radiant set. Then*

(i) $\forall \varepsilon \geq 0$, *$C(\varepsilon)$ is convex.*

(ii) $\forall \varepsilon_1, \varepsilon_2 \geq 0$, $C(\varepsilon_2) + C(\varepsilon_1) \subseteq C(\varepsilon_i),\ i = 1, 2$.

(iii) $\forall \varepsilon_1, \varepsilon_2 \geq 0$, $intC(\varepsilon_2) + C(\varepsilon_1) \subseteq intC(\varepsilon_i),\ i = 1, 2$.

(iv) $\forall \varepsilon \geq 0$, $C(\varepsilon) + intC(0) = intC(\varepsilon)$.

(v) *For any nonempty set $B \subseteq Y$,*

$$clcone(B + C(\varepsilon_1)) + C(\varepsilon_2) \subseteq clcone(B + C(\varepsilon_1)), \forall \varepsilon_1, \varepsilon_2 \geq 0.$$

Proof. The proof of (i)-(ii) can be seen in [6], and (iii) is a direct consequence of (ii).

(iv): According to (iii) we see that $C(\varepsilon) + intC(0) \subseteq intC(\varepsilon)$. Conversely, if $d \in intC(\varepsilon)$, then for each $p \in intC(0)$, there exists $\lambda > 0$ such that $d - \lambda p \in C(\varepsilon)$. Notice that $intC(0) = cone_+ intC(0)$, it follows that $\lambda p \in intC(0)$, and $d \in \lambda p + C(\varepsilon) \subseteq intC(0) + C(\varepsilon)$. By the arbitrariness of d, it follows that $intC(\varepsilon) \subseteq C(\varepsilon) + intC(0)$.

(v): Let $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ be fixed but arbitrary. It follows from Lemma 2.2(iii) that

$$\begin{aligned} clcone(B + C(\varepsilon_1)) + C(\varepsilon_2) &\subseteq clcone(B + C(\varepsilon_1)) + C(0) \\ &= clcone_+(B + C(\varepsilon_1)) + C(0) \\ &\subseteq clcone_+(B + C(\varepsilon_1) + C(0)). \end{aligned} \tag{2.3}$$

Since C is convex co-radiant, by (ii), we have

$$C(\varepsilon_1) + C(0) \subseteq C(\varepsilon_1).$$

which combined with (2.3) yields that

$$\begin{aligned} clcone(B + C(\varepsilon_1)) + C(\varepsilon_2) &\subseteq clcone_+(B + C(\varepsilon_1) + C(0)) \\ &\subseteq clcone_+(B + C(\varepsilon_1)) \\ &= clcone(B + C(\varepsilon_1)). \end{aligned}$$

□

3 Charaterizations of near (C, ε)–subconvexlikeness

Let $B \subseteq Y$ be a nonempty set, $C \subseteq Y$ be a nonempty co-radiant set, and $\varepsilon \geq 0$.

Definition 3.1. The set B is said to be nearly (C, ε)–sunconvexlike if $clcone(B + C(\varepsilon))$ is a convex set.

In general, the set $clcone(B + C(\varepsilon))$ $(\varepsilon > 0)$ is different from the set $clcone(B + C(0))$. Therefore, when the set B is nearly (C, ε_0)–subconvexlike for some $\varepsilon_0 > 0$, B may be not nearly $(C, 0)$–subconvexlike. For example, let $B = \{(-1, 0), (0, -1)\}$, $C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 1\} \cap \mathbb{R}^2_+$, $\varepsilon_0 = 1$. Obviously, $clcone(B + C(1)) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$ is convex, but $clcone(B + C(0)) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0\} \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$ is nonconvex. However, we have the following result.

Theorem 3.2. *If the set B is nearly (C, ε)–subconvexlike for each $\varepsilon > 0$, then B is nearly $(C, 0)$–subconvexlike.*

Proof. As $clcone(B+C(0))$ is a cone, it follows that B is nearly $(C,0)$–subconvexlike if and only if

$$clcone(B+C(0))+clcone(B+C(0))=clcone(B+C(0)). \tag{3.1}$$

In the sequence, we show (3.1) holds.

We first prove

$$cone_+(B+C(0))+cone_+(B+C(0))\subseteq clcone(B+C(0)). \tag{3.2}$$

For any $y_1,y_2\in cone_+(B+C(0))$, there exist $\lambda_1,\lambda_2>0$, $b_1,b_2\in B$, $c_1,c_2\in C$, $\varepsilon_1,\varepsilon_2>0$ such that $y_i=\lambda_i(b_i+\varepsilon_i c_i)$, $i=1,2$ and

$$\begin{aligned}&\lambda_1(b_1+\varepsilon_1 c_1)+\lambda_2(b_2+\varepsilon_2 c_2)\\ \in\ &clcone(B+C(\varepsilon_1))+clcone(B+C(\varepsilon_1)).\end{aligned} \tag{3.3}$$

Now, consider two possible cases:

Case 1. $\varepsilon_1=\varepsilon_2$. Since $clcone(B+C(\varepsilon_1))$ is convex,

$$clcone(B+C(\varepsilon_1))+clcone(B+C(\varepsilon_1))=clcone(B+C(\varepsilon_1)).$$

By (3.3), we have

$$\begin{aligned}&\lambda_1(b_1+\varepsilon_1 c_1)+\lambda_2(b_2+\varepsilon_2 c_2)\\ \in\ &clcone(B+C(\varepsilon_1))\subseteq clcone(B+C(0)).\end{aligned}$$

Case 2. $\varepsilon_1\neq\varepsilon_2$. Without loss of generality we assume that $\varepsilon_1<\varepsilon_2$. It's clear from Lemma 2.2(iii) that

$$clcone(B+C(\varepsilon_2))\subseteq\ clcone(B+C(\varepsilon_1)).$$

As $clcone(B+C(\varepsilon_1))$ is convex,

$$clcone(B+C(\varepsilon_1))+clcone(B+C(\varepsilon_1))=clcone(B+C(\varepsilon_1)).$$

Then, it follows from (3.3) that

$$\begin{aligned}&\lambda_1(b_1+\varepsilon_1 c_1)+\lambda_2(b_2+\varepsilon_2 c_2)\\ \in\ &clcone(B+C(\varepsilon_1))+clcone(B+C(\varepsilon_2))\\ \subseteq\ &clcone(B+C(\varepsilon_1))+clcone(B+C(\varepsilon_1))\\ =\ &clcone(B+C(\varepsilon_1))\\ \subseteq\ &clcone(B+C(0)).\end{aligned}$$

Combining two cases, we conclude that (3.2) holds.

Next, we show that

$$clcone(B+C(0))+clcone(B+C(0))=clcone(B+C(0)). \tag{3.4}$$

Since

$$\begin{aligned}clcone(B+C(0))&\subseteq clcone(B+C(0))+clcone(B+C(0))\\ &=clcone_+(B+C(0))+clcone_+(B+C(0))\\ &\subseteq cl[cone_+(B+C(0))+cone_+(B+C(0))],\end{aligned}$$

it follows from (3.2) that

$$\begin{aligned} clcone(B+C(0)) &\subseteq clcone(B+C(0)) + clcone(B+C(0)) \\ &\subseteq clcone(B+C(0)), \end{aligned}$$

which means (3.1) holds. This completes the proof. □

Remark 3.3. Theorem 3.2 provides a sufficient condition for near $(C, 0)$–subconvexlikeness without the assumption of closedness. Therefore, Theorem 3.2 improves Lemma 3.2 in [4].

Subsequently, we present some sufficient conditions for near (C, ε)–subconvexlikeness of the set B.

Theorem 3.4. *Let C be a convex co-radiant set, and $\varepsilon \geq 0$. If $co(B) \subseteq clcone(B + C(\varepsilon))$, then B is a nearly (C, ε)–subconvexlike set.*

Proof. Since C is convex, it follows from Lemma 2.4 (i) that $C(\varepsilon)$ is convex, and

$$co(B + C(\varepsilon)) = co(B) + C(\varepsilon).$$

Under the assumption of $co(B) \subseteq clcone(B + C(\varepsilon))$, we have

$$\begin{aligned} co(B + C(\varepsilon)) = co(B) + C(\varepsilon) &\subseteq co(B) + C(0) \\ &\subseteq clcone(B + C(\varepsilon)) + C(0). \end{aligned}$$

By Lemma 2.4 (v), we get

$$clcone(B + C(\varepsilon)) + C(0) \subseteq clcone(B + C(\varepsilon)).$$

Therefore $co(B+C(\varepsilon)) \subseteq clcone(B+C(\varepsilon))$, which means that B is nearly (C, ε)–subconvexlike. □

Remark 3.5. It's worth noting that there are two situations in Theorem 3.4:
Case 1. $\varepsilon = 0$. If C is convex co-radiant, then

$$B \text{ is nearly } (C, \varepsilon)\text{–subconvexlike } \Leftrightarrow co(B) \subseteq clcone(B + C(\varepsilon)).$$

In fact, $clcone(B + C(0)) = clcone(B + C(0) \cup \{\mathbf{0}\})$ is convex if and only if $co(B) \subseteq clcone(B + C(0) \cup \{\mathbf{0}\}) = clcone(B + C(0))$.
Case 2. $\varepsilon > 0$. If C is convex co-radiant, then

$$B \text{ is nearly } (C, \varepsilon)\text{–subconvexlike} \not\Leftrightarrow co(B) \subseteq clcone(B + C(\varepsilon)),$$

which means that the convexity of $clcone(B+C(\varepsilon))$ does not guarantee the inclusion relation $co(B) \subseteq clcone(B + C(\varepsilon))$ in general. For example, let $B = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = -1, -1 \leq y_1 \leq 0\}$, $C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 1\} \cap \mathbb{R}^2_+$, $\varepsilon = 1$. It's clear that $clcone(B + C(\varepsilon)) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$ is convex, however $co(B) = B \nsubseteq clcone(B + C(\varepsilon))$.

In the following, we study some conditions under which the converse statement of Theorem 3.4 is true when $\varepsilon > 0$.

Theorem 3.6. *Let C be a convex co-radiant set, and $\varepsilon > 0$. Assume that $\mathbf{0} \in cl(C(\varepsilon))$, then*

$$B \textit{ is nearly } (C,\varepsilon)\textit{–subconvexlike} \Longrightarrow co(B) \subseteq clcone(B + C(\varepsilon)).$$

Proof. Since $\mathbf{0} \in cl(C(\varepsilon))$,

$$co(B) \subseteq co(B) + clC(\varepsilon) \subseteq cl[co(B) + C(\varepsilon)]. \tag{3.5}$$

Note that C and $clcone(B + C(\varepsilon))$ are convex, we have

$$co(B) + C(\varepsilon) = co(B + C(\varepsilon)) \subseteq clcone(B + C(\varepsilon)). \tag{3.6}$$

Combining (3.5) and (3.6), we get $co(B) \subseteq clcone(B + C(\varepsilon))$. □

Remark 3.7. When C is convex and co-radiant, the condition $intC(0) \subseteq C(\varepsilon)$ can ensure the condition $\mathbf{0} \in cl(C(\varepsilon))$. In fact, $\mathbf{0} \in cl(C(\varepsilon)) \Leftrightarrow intC(0) \subseteq C(\varepsilon)$.

In view of Theorem 3.4, Remark 3.5 (i) and Theorem 3.6, the following result is an immediate consequence.

Corollary 3.8. *Let C be a convex co-radiant set, and $\varepsilon \geq 0$. Assume that $\mathbf{0} \in cl(C(\varepsilon))$, then*

$$B \textit{ is nearly } (C,\varepsilon)\textit{–subconvexlike} \Leftrightarrow co(B) \subseteq clcone(B + C(\varepsilon)).$$

Gutiérrez et al. [7] introduceed the notion of near (C,ε)–subconvexlikeness for set-valued map.

Definition 3.9. The set-valued map $F : X \rightrightarrows Y$ is nearly (C,ε)–subconvexlike on a nonempty set $A \subseteq X$ if $clcone(imF|_A + C(\varepsilon))$ is a conex set, where $imF|_A$ is the image of A under F.

In view of Definition 3.1 and Definition 3.9, the map F is nearly (C,ε)–subconvexlike on A if and only if $clcone(F(A) + C(\varepsilon))$ is convex. Consequently, the characterizations of near (C,ε)–subconvexlikeness of a set-valued map can be easily derived by using the property of near (C,ε)–subconvexlikeness of a set.

4 Lagrangian Multiplier Theorem for (C,ε)–Benson Proper Efficient Elements of A Vector Optimization Problems with Set-Valued Maps

Let $A \subseteq X$ be a nonempty set, $D \subseteq Y$ be a closed proper convex cone with nonempty interior, C be a co-radiant set in Y, and $\varepsilon \geq 0$.

Consider the following vector optimization problem with set-valued maps:

$$(VP) \quad \min_{x \in V} F(x),$$

where $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, $V = \{x \in A : G(x) \cap (-D) \neq \emptyset\}$.

Definition 4.1 ([4])**.** Let $\bar{x} \in V$, $\bar{y} \in F(\bar{x})$. $(\bar{x}, \bar{y})$ is a (C,ε)–Benson proper efficient element of (VP), if

$$clcone(F(V) - \bar{y} + C(\varepsilon)) \cap -C(\varepsilon) \subset \{\mathbf{0}\}. \tag{4.1}$$

Remark 4.2. Since $clcone(F(V)-\bar{y}+C(\varepsilon))$ is a cone, $(\bar{x},\bar{y})$ is a (C,ε)–Benson proper efficient element of (VP) if and only if

$$clcone(F(V)-\bar{y}+C(\varepsilon))\cap -(C(0)\cup\{\mathbf{0}\})=\{\mathbf{0}\}. \tag{4.2}$$

Theorem 4.3. *Let $\varepsilon>0$. Assume that*

(i) *$C\subseteq Y$ is a convex co-radiant set and $\mathbf{0}\in cl(C(\varepsilon))$;*

(ii) *$F\times G$ is nearly $((C,\varepsilon)\times D)$–subconvexlike on A;*

(iii) *$-intD\neq\emptyset$ and $V'=\{x\in A: G(x)\cap(-intD)\neq\emptyset\}\neq\emptyset$.*

Then F is nearly (C,ε)–subconvexlike on V'.

Proof. In view of Corollary 3.8, it is sufficient to check that

$$coF(V')\subseteq clcone(F(V')+C(\varepsilon)). \tag{4.3}$$

Take arbitrary $y_1,y_2\in F(V')$, there exist $x_1,x_2\in V'$ with $y_1,y_2\in F(x_i)$ and $z_1,z_2\in G(x_i)\cap(-intD)$. For any $\alpha\in(0,1)$,

$$\alpha y_1+(1-\alpha)y_2\in coF(V'),$$
$$\alpha z_1+(1-\alpha)z_2\in coG(A).$$

Consider two cases:
Case 1. $\alpha y_1+(1-\alpha)y_2=\mathbf{0}$. It's clear that

$$\alpha y_1+(1-\alpha)y_2\in clcone(F(V')+C(\varepsilon)). \tag{4.4}$$

Case 2. $\alpha y_1+(1-\alpha)y_2\neq\mathbf{0}$. Since $F\times G$ is nearly $((C,\varepsilon)\times D)$–subconvexlike on A, by Corollary 3.8, it follows that

$$co(F\times G)(A)\subseteq clcone[(F\times G)(A)+(C(\varepsilon)\times D)].$$

Since $V'\subseteq A$, we have

$$co(F\times G)(V')\subseteq clcone[(F\times G)(A)+(C(\varepsilon)\times D)].$$

Consequently, there exist $\{\lambda_n\}\subseteq\mathbb{R}_{++}$, $\{x_n\}\subseteq A$, $\{y_n\}\subseteq F(x_n)$, $\{z_n\}\subseteq G(x_n)$, $\{q_n\}\subseteq C(\varepsilon)$ and $\{d_n\}\subseteq D$ such that

$$\alpha y_1+(1-\alpha)y_2=\lim_n\lambda_n(y_n+q_n), \tag{4.5}$$

$$\alpha z_1+(1-\alpha)z_2=\lim_n\lambda_n(z_n+d_n). \tag{4.6}$$

As $intD$ is convex and $z_1,z_2\in -intD$, it follows that $\alpha z_1+(1-\alpha)z_2\in -intD$. By (4.6), there exists a number N such that $\lambda_n(z_n+d_n)\in -intD$ for all $n>N$. Note that $intD$ is a cone and $d_n\in D$, we get that $z_n\in -intD$, $\forall n>N$, i.e.,

$$x_n\in V',\ \forall n>N,$$

which along with (4.5) yields that (4.4) holds.

Due to the arbitrariness of $y_1,y_2\in F(V')$ and $\alpha\in(0,1)$, it follows from (4.4) that (4.3) holds. $\square$

Remark 4.4. Similar results of Theorem 4.3 can be obtained if $\varepsilon = 0$, as the condition of $\mathbf{0} \in clC(0)$ holds automatically. In this case, Theorem 4.3 reduces to Lemma 4.4 in [11].

Dauer and Saleh in [2] introduced an important separation theorem for two cones.

Lemma 4.5. *[2] Let M and N be cones in Y and $M \cap N = \{\mathbf{0}\}$. Suppose that one of the following assumptions holds:*

(i) *M is closed and N has a compact base;*

(ii) *M is weakly closed and N has a weakly compact base.*

Then there is a pointed convex cone S in Y such that $N\backslash\{\mathbf{0}\} \subseteq intS$ and $M \cap S = \{\mathbf{0}\}$.

According to Theorem 4.3, Remark 4.4(i) and Lemma 4.5, we can obtain scalarization results for (VP).

Theorem 4.6. *Suppose that*

(I) *$\varepsilon \geq 0$ and $C \subseteq Y$ is a nonempty convex co-radiant set;*

(II) *$\bar{x} \in V$, $\bar{y} \in F(\bar{x})$ and $(F - \bar{y}) \times G$ is nearly $((C,\varepsilon) \times D)$–subconvexlike on the set A;*

(III) *One of the following assumptions holds:*

  (i) *$C(0) \cup \{\mathbf{0}\}$ has a weakly compact base and $\mathbf{0} \in cl(C(\varepsilon))$,*

  (ii) *$C(0) \cup \{\mathbf{0}\}$ has a compact base;*

(IV) *$-intD \neq \emptyset$ and $V' = \{x \in A : G(x) \cap (-intD) \neq \emptyset\} \neq \emptyset$.*

If $(\bar{x},\bar{y})$ is a (C,ε)-Benson proper efficient element of (VP), then there exists $(y^*,z^*) \in C^{+i} \times D^+$ such that

$$\langle y^*, y + \varepsilon q\rangle + \langle z^*, z\rangle \geq \langle y^*, \bar{y}\rangle, \quad \forall x \in A, \forall (y,z) \in F(x) \times G(x), \forall q \in C, \tag{4.7}$$

$$0 \geq \langle z^*, \bar{z}\rangle \geq \varepsilon\sigma_{-C}(y^*), \forall \bar{z} \in G(\bar{x}) \cap (-D), \tag{4.8}$$

where $\sigma_{-C}(y^*) = \sup_{y\in -C}\langle y^*, y\rangle$.

Proof. When condition (i) in (III) is satisfied. It follows from Theorem 4.3 that $F-\bar{y}$ is nearly (C,ε)–subconvexlike on V', i.e., $clcone(F(V') - \bar{y} + C(\varepsilon))$ is convex, then $clcone(F(V') - \bar{y} + C(\varepsilon))$ is weakly closed.
Since $(\bar{x},\bar{y})$ is a (C,ε)-Benson proper efficient element of (VP), it follows from Remark 4.2 that

$$clcone(F(V) - \bar{y} + C(\varepsilon)) \cap -(C(0) \cup \{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Note that $V' \subseteq V$, we have

$$clcone(F(V') - \bar{y} + C(\varepsilon)) \cap -(C(0) \cup \{\mathbf{0}\}) = \{\mathbf{0}\}.$$

In view of Lemma 4.5, there exists a pointed convex cone S such that $C(0)\backslash\{\mathbf{0}\} \subseteq intS$ and

$$clcone(F(V') - \bar{y} + C(\varepsilon)) \cap -S = \{\mathbf{0}\}. \tag{4.9}$$

Now, we justify

$$clcone[((F-\bar{y})\times G)(A)+C(\varepsilon)\times D]\cap -(intS\times intD)=\emptyset. \tag{4.10}$$

Otherwise, there exist $x_0\in A$, $y_0\in F(x_0)$, $z_0\in G(x_0)$, $q_0\in C$ and $d_0\in D$ such that

$$(y_0-\bar{y}+\varepsilon q_0, z_0+d_0)\in -(intS\times intD).$$

So, $x_0\in V'$ and $clcone((F(V')-\bar{y})+C(\varepsilon)\cap -S\neq\{\mathbf{0}\}$, contradicting (4.9). Hence (4.10) holds.

When condition (ii) in (III) is satisfied. (4.10) can be obtained similarly to the above.

Applying the separation theorem to (4.10), there exists $(y^*,z^*)\in Y^*\times Z^*\backslash\{(\mathbf{0},\mathbf{0})\}$ such that

$$\langle y^*, y-\bar{y}+\varepsilon q\rangle+\langle z^*, z+d\rangle\geq 0,\quad \forall x\in A, \forall (y,z)\in F(x)\times G(x), \forall q\in C, \forall d\in D, \tag{4.11}$$

$$\langle y^*, s\rangle+\langle z^*, d\rangle>0, \forall s\in intS, \forall d\in intD. \tag{4.12}$$

Due to (4.12), it is obvious that

$$y^*\in S^+, z^*\in D^+.$$

Next, we show $y^*\neq\mathbf{0}$. Otherwise the inequalities (4.11) and (4.12) reduce to, respectively,

$$\langle z^*, z\rangle\geq 0, \forall x\in A, z\in G(x), \tag{4.13}$$

$$\langle z^*, d\rangle>0, \forall d\in intD. \tag{4.14}$$

Under the assumption of (IV), there exist $\tilde{x}\in A$ and $\tilde{z}\in G(\tilde{x})\cap(-intD)$. Substituting $\tilde{z}$ into (4.13) and (4.14), we arrive at

$$\langle z^*, \tilde{z}\rangle\geq 0>\langle z^*, \tilde{z}\rangle.$$

This is a contradiction. Hence $y^*\neq\mathbf{0}$.

Note that $y^*\in S^+\backslash\{\mathbf{0}\}$ and $C(\varepsilon)\backslash\{\mathbf{0}\}\subseteq C(0)\backslash\{\mathbf{0}\}\subseteq intS$, therefore $y\in C^{+i}$, which together with (4.11) yields the desired results. □

Remark 4.7. (i) Although the conditions of the above theorem and Theorem 3.8 in [7] are different, the same scalarization results are obtained.
(ii) When the co-radiant set C is replaced by a closed convex cone K or $K\backslash\{\mathbf{0}\}$, Theorem 4.6 reduces to Corollary 4.1 in [11].

Based on Theorem 4.6, we derive Lagrangian multiplier theorem for (C,ε)-Benson proper efficient elements of (VP). For convenience, denote the set of all continuous linear operators from Z into Y by $L(Z,Y)$.

Theorem 4.8. *Assume that*

(I) $\varepsilon\geq 0$ *and* $C\subseteq Y$ *is a convex co-radiant set with nonempty interior;*

(II) $\bar{x}\in V$, $\bar{y}\in F(\bar{x})$ *and* $(F-\bar{y})\times G$ *is nearly* $(C,\varepsilon)\times D$*-suconvexlike on the set* A*;*

(III) *One of the following conditions holds:*

(i) $C(0)\cup\{\mathbf{0}\}$ *has a weakly compact base and* $\mathbf{0}\in cl(C(\varepsilon))$,

(ii) $C(0)\cup\{\mathbf{0}\}$ *has a compact base;*

(IV) $-intD \neq \emptyset$ *and* $V' = \{x \in A : G(x)\cap(-intD) \neq \emptyset\}$.

If $(\bar{x},\bar{y})$ *is a* (C,ε)*-Benson proper efficient element of* (VP) *and* $\mathbf{0} \in G(\bar{x})$*, there exist* $T \in L_+(Z,Y) = \{T \in L(Z,Y) : T(D) \subseteq C(0)\cup\{\mathbf{0}\}\}$*,* $\bar{p} \in int(C(0))$ *and* $y^* \in C^{+i}$ *such that*

$$T(G(\bar{x})\cap(-D)) \subseteq [\varepsilon\sigma_{-C}(y^*),0]\bar{p}\backslash(-intC(\varepsilon)), \tag{4.15}$$

and $(\bar{x},\bar{y})$ *is a* (C,ε)*-Benson proper efficient element of the following problem*

$$(UVP) \quad \min_{x\in A} F(x) + T(G(x)).$$

Proof. From the proof of Theorem 4.6, it follows that there exists $(y^*,z^*) \in C^{+i}\times D^+ = C(0)^{+i}\times D^+$ such that

$$\langle y^*, y+\varepsilon q\rangle + \langle z^*, z\rangle \geq \langle y^*,\bar{y}\rangle, \quad \forall x \in A, \forall (y,z) \in F(x)\times G(x), \forall q \in C, \tag{4.16}$$

$$0 \geq \langle z^*,\bar{z}\rangle \geq \varepsilon\sigma_{-C}(y^*), \quad \forall \bar{z} \in G(\bar{x})\cap(-D). \tag{4.17}$$

As $y^* \in C(0)^{+i}$, we get

$$\langle y^*, p\rangle > 0, \forall p \in intC(0).$$

Consequently, there exits $\bar{p} \in intC(0)$ such that $\langle y^*,\bar{p}\rangle = 1$.

Let

$$T := \langle z^*, z\rangle\bar{p},\ z \in Z.$$

It's easy to see that $T \in L_+(Z,Y)$. From (4.17), it follows that

$$T(\bar{z}) = \langle z^*,\bar{z}\rangle\bar{p} \in [\varepsilon\sigma_{-C}(y^*),0]\bar{p}, \forall \bar{z} \in G(\bar{x})\cap(-D).$$

Now, we claim $T(\bar{z}) \notin -int(C(\varepsilon)), \forall \bar{z} \in G(\bar{x})\cap(-D)$. Otherwise, there exists $z_0 \in G(\bar{x})\cap(-D)$ such that $T(z_0) \in -intC(\varepsilon)$. In view of Lemma 2.4(iv), we get $C(\varepsilon)+intC(0) = intC(\varepsilon)$, and

$$T(z_0) \in -C(\varepsilon) - intC(0),$$

i.e., there exists $q_0 \in C$ such that $T(z_0)+\varepsilon q_0 \in -intC(0)$. Since $y^* \in C(0)^{+i}$, we have that

$$\langle z^*, z_0\rangle + \varepsilon\langle y^*, q_0\rangle = \langle y^*, T(z_0)+\varepsilon q_0\rangle < 0,$$

which contradicts to (4.17). Therefore,

$$T(\bar{z}) \in [\varepsilon\sigma_{-C}(y^*),0]\bar{p}\backslash(-intC(\varepsilon)), \forall \bar{z} \in G(\bar{x})\cap(-D),$$

i.e., (4.15) holds

Notice that $y^* \in C(0)^{+i}$. By (4.16), we can conclude that

$$clcone(F(A)+T(G(A))-\bar{y}+C(\varepsilon))\cap -(C(0)\cup\{\mathbf{0}\}) = \{\mathbf{0}\}.$$

Under the assumption of $\mathbf{0} \in G(\bar{x})$, it's obvious that $\bar{y} \in F(\bar{x}) \subseteq F(\bar{x})+T(G(\bar{x}))$. Hence, $(\bar{x},\bar{y})$ is a (C,ε)-Benson proper efficient element of (UVP). □

Remark 4.9. (i) If the co-radiant set C in Theorem 4.8 is replaced by a closed pointed convex cone K or $K\backslash\{\mathbf{0}\}$, then (4.15) becomes $T(G(\bar{x})\cap(-D)) = \{\mathbf{0}\}$. In this case, Theorem 4.8 reduces to Theorem 5.1 in [10].

(ii) The near $((C,\varepsilon)\times D)$–subconvexlikeness of $(F-\bar{y})\times G$ on A does not imply the near (C,ε)–subconvexlikeness of $(F-\bar{y})$ on V. We illustrate this issue in the example below.

Example 4.1. Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \{(y_1,y_2)\in\mathbb{R}^2 : y_1 \geq 0, y_2 \geq 1\}\bigcup\{(y_1,y_2)\in\mathbb{R}^2 : y_1 \geq 1, y_2 \geq 0\}$, $D = \mathbb{R}_+ \subseteq Z$ and $A = \{(0,0),(0,1),(1,0)\} \subseteq X$. Consider the maps $F : A \rightrightarrows Y$, $G : A \rightrightarrows Z$ defined as follows:

$$\begin{aligned}
F((0,0)) &= [-2,+\infty)\times\{0\},\\
F((0,1)) &= \{0\}\times[-2,+\infty),\\
F((1,0)) &= \{(-2,-1)\},\\
G((0,0)) &= \{0\},\\
G((0,1)) &= \{0\},\\
G((1,0)) &= (1,+\infty).
\end{aligned}$$

Take $\bar{x} = (0,0) \in A$, $\bar{y} = (0,0) \in F(\bar{x})$. Clearly $V = \{(0,0),(0,1)\}$, and it's easy to check that $clcone[((F-\bar{y})\times G)(A)+C(1)\times D] = \mathbb{R}^2\times\mathbb{R}_+$ is convex, but $clcone(F(V)-\bar{y}+C(1)) = \{(y_1,y_2)\in\mathbb{R}^2 : y_1 \geq 0\}\bigcup\{(y_1,y_2)\in\mathbb{R}^2 : y_2 \geq 0\}$ is nonconvex. Therefore, Theorem 4.8 improves and extends Theorem 5.2 in [4].

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