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STABILITY AND SCALARIZATION FOR PERTURBED SET-VALUED OPTIMIZATION PROBLEMS WITH CONSTRAINTS VIA GENERAL ORDERING SETS

Jing-Jing Wang, Chong-Yang Shao AND Zai-Yun Peng*

Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: The aim of this paper is to discuss the Painlevé-Kuratowski convergence of minimal point sets for perturbed set-valued optimization problem (PSOP) with approximate quasi-equilibrium constraints. Firstly, we study the Painlevé-Kuratowski convergence of constraint sets for (PSOP), which are denoted by approximate solution sets for vector quasi-equilibrium problems via the free-disposal set. Then, under some types of continuity assumption, the sufficient conditions of upper (lower) Painlevé-Kuratowski convergence of E-minimal point sets, weak E-minimal point sets and Borwein E-minimal point sets for (PSOP) are obtained. Moreover, by using the oriented distance function, we establish the scalarized problem (SP) and discuss the relationships between the minimal point sets of (PSOP) and the maximum sets of (SP). Our results are new and different from the ones in the literature. Some examples are given to illustrate the results.

Key words: *perturbed set-valued optimization problems, approximate quasi-equilibrium constraints, improvement set, Painlevé-Kuratowski convergence, scalarized problem*

Mathematics Subject Classification: *49K40, 90C29, 90C31*

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1 Introduction

With the development of science and technology, optimization problems with constraints have been widely used in many fields, such as production plan problems, robot trajectory design problems, economics problems, engineering problems, air pollution control problems and so on. It is obvious that the research of optimization problems with constrains has important theoretical significance and practical value. Many authors have begun to study the various optimization problems with constraints, which include optimization problems with variational inequality constraints (see, e.g., [14, 24, 28]), bilevel programming problems (see, e.g., [2, 7, 8]) and optimization problems with equilibrium constraints (see,

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e.g., [9, 10, 18, 22, 25, 33]). Clearly, existence results, optimality conditions or characterizations of the solution set and stability of solution mappings for optimization problems with constraints have attracted widespread attention. Specially, because of the extensive practical application, the research of stability is very important and indispensable among them.

It is well known that the aim of stability analysis is to study the changes in the behaviour of the solutions under perturbation, which include Berge continuity, Hölder continuity, connectedness, Painlevé-Kuratowski convergence and so on. Among them, Painlevé-Kuratowski convergence plays an important role in stability analysis. Several researchers have studied the Painlevé-Kuratowski convergence of different optimization problems and related problems. Huang [13] discussed the convergence of solutions for vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski, when the data of the approximate problems Painlevé-Kuratowski converge to the data of the original problems. Based on the concept of continuous convergence, Lucchetti and Miglierina [21] discussed the Painlevé-Kuratowski convergence of solutions for convex vector optimization problems. Applying the same method of [13], Zeng et al. [31] obtained the sufficient conditions of Painlevé-Kuratowski convergence of efficient points for convex vector-valued optimization problem. In 2012, Lalitha and Chatterjee [16] studied the Painlevé-Kuratowski convergence of the sets of minimal, weak minimal and Henig proper minimal points for properly quasiconvex vector optimization problem. Subsequently, Li, Wang and Lin [19] discussed the Painlevé-Kuratowski convergence of solutions for set-valued optimization problems with naturally quasi-functions, which extended and improved the corresponding results of [13] and [16]. Recently, by suitable gap function, Anh et al. [1] obtained the sufficient (and necessary) conditions of upper (lower) Painlevé-Kuratowski convergence of solutions for vector quasi-equilibrium problems. Very recently, under some types of continuity assumption, Hung, Hoang and Tam [15] discussed the convergence of solutions for vector quasi-equilibrium problems in the sense of Painlevé-Kuratowski.

On the other hand, ordering relation is an important tool to study optimization problems. There are different optimization results under different order relations. Several papers have studied the Painlevé-Kuratowski convergence of solutions based on the ordering set which is a closed, convex and pointed cone with nonempty interior (e.g., [1, 13, 15, 16, 19, 21, 31]), rather than a general ordering set. However, if the ordering set is not a cone, some of the known results maybe not applicable. It is natural to raise the following question: whether the stability of solutions for these problems can be obtained under general ordering set (e.g., the improvement set)? To the best of our knowledge, up to now, there are only a few articles discussed the stability of vector optimization problems using improvement set (see [5, 6, 11, 17, 32]). Moreover, the Painlevé-Kuratowski convergence of perturbed set-valued optimization problem with approximate quasi-equilibrium constraints via general ordering set has not been discussed yet. Motivated by the literatures above, the aim of this paper is to study the Painlevé-Kuratowski convergence of minimal point sets for (PSOP) with approximate quasi-equilibrium constraints using improvement set, under functional perturbations of both objective function and feasible set. Furthermore, the relationships between minimal point sets of (PSOP) and maximum sets of (SP) are discussed. Our results improve and generalize the corresponding results in [1, 15, 17, 19, 31].

The outline of the paper is as follows. In Sect.2, we introduce the perturbed set-valued optimization problem (PSOP) with approximate quasi-equilibrium constrains, and recall some concepts which will be used in the sequel. In Sect.3, under some types of continuity assumption, we discuss the Painlevé-Kuratowski convergence of constraint sets for (PSOP). In Sect.4, we establish the sufficient conditions of upper Painlevé-Kuratowski convergence of E-minimal point sets, weak E-minimal point sets and Borwein E-minimal point sets for

(PSOP). In Sect.5, by using some assumptions of convergence, we obtain lower Painlevé-Kuratowski convergence of E-minimal point sets and Borwein E-minimal point sets for (PSOP). In Sect.6, by using the oriented distance function ($\triangle$), we establish a scalarized problem (SP) and discuss the relationships between the minimal point sets of (PSOP) and the maximum sets of (SP). Moreover, some interesting examples are also given to illustrate the results.

2 Preliminaries

Throughout the paper, unless specified otherwise, we assume that X, Y and Z are three real Banach spaces, W is also a real Banach space. Let $A \subset X$, $B \subset Z$ be nonempty compact subsets, $K_n : A \rightrightarrows A$, $T_n : A \rightrightarrows B$ and $f_n : X \rightrightarrows W$ be set-valued mappings and $g_n : A \times B \times A \longrightarrow Y$ be a single-valued mapping. $C_Y \subset Y$ ($C_W \subset W$) is a closed, convex and pointed cone with nonempty interior, i.e., $intC_Y \neq \emptyset$ ($intC_W \neq \emptyset$). Let the nonempty proper set D be a free-disposal set in Y with respect to C_Y, i.e., $D \subset Y$ satisfies the free-disposal assumption $D + C_Y = D$.

We first recall the notion of an improvement set.

Definition 2.1 ([11,32]). A nonempty set $E \subset W$ is said to be an improvement set iff

(i) $0_W \notin E$;

(ii) $E + C_W = E$.

Remark 2.2. Compared with [4,29], it is obvious that improvement set is always free-disposal set, but the converse may not be true. If E is an improvement set and C_W is a closed convex point cone with nonempty interior, then $E + intC_W = intE \neq \emptyset$ and $(W \setminus E) - C_W \subseteq W \setminus E$.

Definition 2.3 ([17]). Let E be an improvement set in W. An element $y \in A \subset W$ is said to be

(i) an E-minimal point of A iff
$$(A - y) \cap (-E) = \emptyset.$$

(ii) a weak E-minimal point of A iff
$$(A - y) \cap (-intE) = \emptyset.$$

(iii) a Borwein proper E-minimal point of A iff
$$clcone(A - y) \cap (-E) = \emptyset.$$

In this paper, consider the following perturbed set-valued optimization problem, given as
$$(PSOP): \quad min_E f_n(x) \quad s.t. \ x \in S_n$$
and
$$S_n = \{x \in K_n(x) : g_n(x, t, y) + \varepsilon_n e \in D, \quad \forall t \in T_n(x), \ \forall y \in K_n(x)\},$$
where $e \in intC_Y$, $\varepsilon_n (\neq \varepsilon_0) \geq 0$. For simplicity's sake, we denote $K_0 := K, T_0 := T, f_0 := f, g_0 := g, S_0 := S$ and $\varepsilon_0 := \varepsilon$.

Based on the above notions of minimality for a set, the set of E-minimal point, weak E-minimal point and Borwein proper E-minimal point of $(PSOP)$ are denoted by $EMinf_n(S_n), EWMinf_n(S_n)$ and $BEMinf_n(S_n)$, respectively. Clearly, $BEMinf_n(S_n) \subseteq EMinf_n(S_n) \subseteq EWMinf_n(S_n)$.

Because of the aim of this paper is to study the Painlevé-Kuratowski convergence, we give some basic definitions as follows.

Definition 2.4 ([1, 27])**.** *A sequence of sets $\{A_n \subset X : n \in N\}$ is said to converge in the sense of Painlevé-Kuratowski ($P.K.$) to A (denoted as $A_n \xrightarrow{P.K.} A$) if $limsup_{n\to\infty} A_n \subset A \subset liminf_{n\to\infty} A_n$ with*

$$\liminf_{n\to\infty} A_n := \{x \in X | \exists (x_n), x_n \in A_n, \forall n \in N, x_n \to x\},$$

$$\limsup_{n\to\infty} A_n := \{x \in X | \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in N, x_{n_k} \to x\}.$$

When $limsup_{n\to\infty} A_n \subset A$ holds, the relation is referred as upper Painlevé-Kuratowski convergence ($u.P.K$, for short), denoted as $A_n \xrightarrow{u.P.K.} A$; When $A \subset liminf_{n\to\infty} A_n$ holds, the relation is referred as lower Painlevé-Kuratowski convergence ($l.P.K$, for short), denoted as $A_n \xrightarrow{l.P.K.} A$.

Definition 2.5 ([13])**.** A sequence of vector-valued mappings $g_n : A \times B \times A \to Y$ $(n \in N)$ Painlevé-Kuratowski (P.K. for short) converges to a vector-valued mapping $g : A \times B \times A \to Y$ (written as $g_n \xrightarrow{P.K.} g$), if $epig_n \xrightarrow{P.K.} epig$, where $epig_n = \{(x,t,y,z) : z \in g_n(x,t,y) + C_Y\}$ and $epig = \{(x,t,y,z) : z \in g(x,t,y) + C_Y\}$.

Remark 2.6. By Definition 2.5, for any $x_n \to x_0$, $g_n(x_n,\cdot,\cdot) \xrightarrow{P.K.} g(x_0,\cdot,\cdot)$, if $epi(g_n)_{x_n} \xrightarrow{P.K.} epi(g)_{x_0}$, where $epi(g_n)_{x_n} = \{(x_n,t,y,z) : z \in g_n(x_n,t,y)+C_Y\}$ and $epi(g)_{x_0} = \{(x_0,t,y,z) : z \in g(x_0,t,y) + C_Y\}$.

Definition 2.7 ([23])**.** Let $g_n, g : X \times Z \times X \to Y$ and let $U(x,t,y)$ be the family of neighborhoods of $(x,t,y) \in X \times Z \times X$. We say that g_n $\Gamma-$converges to g, if for every $(x,t,y) \in X \times Z \times X$:

(i) $\forall U \in U(x,t,y)$, $\forall e \in intC_Y$, $\exists n_{e,U} \in N$ such that $\forall n \geq n_{e,U}$, $\exists (x_n,t_n,y_n) \in U$, $g_n(x_n,t_n,y_n) \in g(x,t,y) + e - C_Y$;

(ii) $\forall e \in intC_Y$, $\exists U_e \in U(x,t,y), k_e \in N$ such that $\forall (x_n,t_n,y_n) \in U_e$, $\forall n \geq k_e$, $g_n(x_n,t_n,y_n) \in g(x,t,y) - e + C_Y$.

Remark 2.8. The Painlevé-Kuratowski convergence is strictly larger than the $\Gamma-$convergence and the continuous convergence, respectively. The following example is given to illustrate the result.

Example 2.1. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C_Y = \{(x,y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq x\}$ and let $A = B = [0,+\infty)$.

Suppose that $g, g_n : A \times B \times A \to Y$ are given as

$$g(x,t,y) = g_n(x,t,y) = \begin{cases} (0,0), & x+t+y=0, \\ (x+t+y, \frac{1}{x+t+y}), & x+t+y>0. \end{cases}$$

We can easily verify that g_n P.K. converges to g. But g_n is neither continuously convergent nor $\Gamma-$convergent. In fact, take $x = t = y = 0$, $x_n = y_n = t_n = \frac{1}{2n}$ and $\varepsilon = (1, \frac{1}{2})$. For all $n \geq 2$,

$$g_n(x_n, t_n, y_n) + \varepsilon = (\frac{3}{2n}, \frac{2n}{3}) + (1, \frac{1}{2}) \notin C_Y.$$

By Definition 2.7, g_n is not $\Gamma-$convergent. At the same time, $(x_n, t_n, y_n) \to (x, t, y)$, but $g_n(x_n, t_n, y_n) \nrightarrow g(x, t, y)$. It is obvious that g_n is not continuously convergent.

$\boxed{3}$ Painlevé-Kuratowski Convergence of the Constraint Sets for $(PSOP)$

In this section, under some types of continuity assumption, we investigate the Painlevé-Kuratowski convergence of the approximate solution sets for vector quasi-equilibrium problems which are referred as the constraint sets for $(PSOP)$. We always assume that all approximate solution sets considered in this section are not equal empty sets.

Next, we give the definition related to convergence of mapping sequences which will be used in the rest of this paper.

Definition 3.1 ([1, 27])**.** Let $G_n : X \rightrightarrows Y$ be a sequence of set-valued mappings and $G : X \rightrightarrows Y$ be a set-valued mapping. $\{G_n\}$ is said to outer converge continuously (resp. inner converge continuously) to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subseteq G(x_0)$ (resp. $G(x_0) \subseteq \liminf_{n\to\infty} G_n(x_n)$), $\forall x_n \to x_0$. $\{G_n\}$ is said to converge continuously to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subseteq G(x_0) \subseteq \liminf_{n\to\infty} G_n(x_n)$, $\forall x_n \to x_0$. If $\{G_n\}$ converges continuously to G at every $x_0 \in X$, then it is said that $\{G_n\}$ converges continiously to G on X.

A sequence of mappings $\{h_n\}$, $h_n : X \to Y$, is said to continuous convergence to a mapping $h : X \to Y$ at x_0 if $\lim_{n\to\infty} h_n(x_n) = h(x_0)$ for any $x_n \to x_0$.

By virtue of the notion of Painlevé-Kuratowski convergence, the following lemma is given.

Lemma 3.2. *Let $G_n : X \rightrightarrows Y$ be a sequence of set-valued mappings and $G : X \rightrightarrows Y$ be a set-valued mapping.*

(i) *$\{G_n\}$ inner converges continuously to G at x_0, thus for any net $\{x_n\} \subset X$ with $x_n \to x_0$ and any $y_0 \in G(x_0)$, there exists $y_n \in G_n(x_n)$ such that $y_n \to y_0$.*

(ii) *$\{G_n\}$ outer converges continuously to G at x_0 with compact values, then for any net $\{x_n\} \subset X$ with $x_n \to x_0$ and for any $y_n \in G_n(x_n)$, there exist $y_0 \in G(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_n\}$ such that $y_\beta \to y_0$.*

In [3, 20], the following definitions of upper/lower semicontinuity and $C-$upper/lower semicontinuity were given.

Definition 3.3. Let X and Y be topological vector spaces, $F : X \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be upper semicontinuous (u.s.c, for short) at $x_0 \in X$, if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 in X such that $F(x) \subset V$ for all $x \in U$;

(ii) F is said to be lower semicontinuous (l.s.c, for short) at $x_0 \in X$, if for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

(iii) F is said to be continuous at $x_0 \in X$, if it is both l.s.c and u.s.c at $x_0 \in X$. F is said to be l.s.c (resp. u.s.c) on X, iff it is l.s.c (resp. u.s.c) at each $x \in X$;

(iv) F is closed, if Graph(F) is a closed set in $X \times Y$. F has compact (resp. closed) values, if $F(x)$ is a compact (resp. closed) set for each $x \in X$.

Definition 3.4. Let E be a nonempty subset of X, and let g be a mapping from E to Y. g is said to be $C-$lower semicontinuous (resp. $C-$upper semicontinuous) at $x_0 \in E$ if for any neighborhood W of 0_Y in Y, there exists a neighborhood $U(x_0)$ of x_0 such that for each $x \in U(x_0) \cap E$,

$$g(x) \in g(x_0) + W + C \ (resp.\ g(x) \in g(x_0) + W - C).$$

g is said to be $C-$lower semicontinuous (resp. $C-$upper semicontinuous) on E iff g is $C-$lower semicontinuous (resp. $C-$upper semicontinuous) at every point of E.

g is said to be $C-$semicontinuous at every point of E, if it is both $C-$upper semicontinuous and $C-$lower semicontinuous at every point of E.

Remark 3.5. g is said to be $C-$lower semicontinuious at $x \in E$ if it satisfies one of the following two equivalent conditions:

(i) For any neighborhood $V_{g(x)} \subset Z$ of $g(x)$, there exists a neighborhood $U_x \subset E$ of x such that $g(u) \in V_{g(x)} + C$ for all $u \in U_x$.

(ii) For any $k \in intC$, there exists a neighborhood $U_x \subset E$ of x such that $g(u) \in g(x) - k + intC$ for all $u \in U_x$.

Lemma 3.6 ([3]). *Let X and Y be topological vector spaces, $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) *F is lower semicontinuous at $x_0 \in X$ if and only if for any net $\{x_n\} \subset X$ with $x_n \to x_0$ and any $y_0 \in F(x_0)$, there exists $y_n \in F(x_n)$ such that $y_n \to y_0$.*

(ii) *If F has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then F is upper semicontinuous at x_0 if and only if for any net $\{x_n\} \subset X$ with $x_n \to x_0$ and for any $y_n \in F(x_n)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

Theorem 3.7. *Let $n \in N$, D be a closed free-disposal set and $\varepsilon_n \to \varepsilon_0$. Suppose that*

(i) *K_n converges continuously to K with compact values on A;*

(ii) *T_n inner converges continuously to T on A;*

(iii) *for $\overline{x}_n \in K_n(\overline{x}_n), \overline{y}_n \in K_n(\overline{x}_n), \overline{t}_n \in T_n(\overline{x}_n)$, satisfying $(\overline{x}_n, \overline{t}_n, \overline{y}_n) \to (x, t, y) \in A \times B \times A$, $\lim_{n\to\infty} g_n(\overline{x}_n, \overline{t}_n, \overline{y}_n)$ exists;*

(iv) $-g_n \xrightarrow{P.K.} -g$.

Then, $\limsup_{n\to\infty} S_n \subseteq S$.

Proof. Suppose to the contrary, $\limsup_{n\to\infty} S_n \nsubseteq S$, i.e., there exists $x_0 \in \limsup_{n\to\infty} S_n$, but $x_0 \notin S$. Since $x_0 \in \limsup_{n\to\infty} S_n$, there exists $x_{n_k} \in S_{n_k}$ such that $x_{n_k} \to x_0$, as $k \to \infty$. Then, for any $y \in K_{n_k}(x_{n_k})$, $t \in T_{n_k}(x_{n_k})$, one has

$$g_{n_k}(x_{n_k}, t, y) + \varepsilon_{n_k} e \in D. \tag{3.1}$$

As K_n outer converges continuously to K, then we have $x_0 \in K(x_0)$. As $x_0 \notin S$, there exist $y_0 \in K(x_0)$, $t_0 \in T(x_0)$ such that

$$g(x_0, t_0, y_0) + \varepsilon_0 e \notin D. \tag{3.2}$$

Since K_n, T_n inner converges continuously to K, T, respectively, for all $y_0 \in K(x_0)$, $t_0 \in T(x_0)$, there exist $y_{n_k} \in K_{n_k}(x_{n_k})$, $t_{n_k} \in T_{n_k}(x_{n_k})$ such that $y_{n_k} \to y_0$, $t_{n_k} \to t_0$, as $k \to \infty$. From (3.1), we have

$$g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) + \varepsilon_{n_k} e \in D. \tag{3.3}$$

By the assumption (iii), (iv) and $(x_{n_k}, t_{n_k}, y_{n_k}, -g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k})) \in epi(-g_n)$, there exists $z_0 \in Y$ such that

$$(x_{n_k}, t_{n_k}, y_{n_k}, -g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k})) \longrightarrow (x_0, t_0, y_0, z_0) \in epi(-g) \quad as\ k \to \infty.$$

Thus,

$$-g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) \to z_0 \in -g(x_0, t_0, y_0) + C_Y \quad as\ k \to \infty.$$

And so, there exists $n_0 \in N$ for all $n_k \geq n_0$, such that

$$-g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) \in -g(x_0, t_0, y_0) + C_Y - |\varepsilon_{n_k} - \varepsilon_0| e + C_Y.$$

Then, from (3.3) we have

$$\begin{aligned} g(x_0, t_0, y_0) + \varepsilon_{n_k} e + |\varepsilon_{n_k} - \varepsilon_0| e &\in g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) + \varepsilon_{n_k} e + C_Y \\ &\subseteq D + C_Y \\ &\subseteq D. \end{aligned}$$

By the closedness of D and $\varepsilon_{n_k} \to \varepsilon_0$, we have

$$g(x_0, t_0, y_0) + \varepsilon_0 e \in D, \tag{3.4}$$

which contradicts (3.2). Thus, $\limsup_{n\to\infty} S_n \subseteq S$. This completes the proof. $\square$

Next, we give Example 3.1 to illustrate this result.

Example 3.1. Suppose that $X = Z = \mathbb{R}$, $Y = l^\infty = \{x = (x_1, x_2, \cdots, x_i, \cdots) : \sup_{i\geq 1} |x_i| < \infty\}$, and $C_Y = \{x = (x_1, x_2, \cdots, x_i, \cdots) \in l^\infty : x_i \geq 0, i = 1, 2, \cdots\}$, $e \in intC_Y$. Let $D = C_Y \cup \{x = (x_1, x_2, \cdots, x_i, \cdots) \in l^\infty : -1 \leq x_1 \leq 0, 1 \leq x_i, i = 2, 3, \cdots\}$, $A = [0, 3]$, $B = [-5, 1]$, $\varepsilon_n = \frac{1}{n}$, $\varepsilon = 0$, and let $K(x) = [1, 2]$, $T(x) = T_n(x) = (-x, 1]$, $K_n(x) = [1 - \frac{1}{2n}, 2 + \frac{1}{2n}]$.

We consider $g, g_n : A \times B \times A \to Y$ given as

$$g(x, t, y) = t^2 y^2 (x, \frac{x}{2} \cdots, \frac{x}{i} \cdots), \quad \forall (x, t, y) \in A \times B \times A,$$

and

$$g_n(x, t, y) = (t - \frac{1}{n})^2 (y + \frac{1}{n})^2 (x, \frac{x}{2} \cdots, \frac{x}{i} \cdots), \quad \forall (x, t, y) \in A \times B \times A.$$

It is clear that all conditions of Theorem 3.7 are satisfied. By a simple computation, we can get

$$S = [1, 2],$$

and

$$S_n = [1 - \frac{1}{2n}, 2 + \frac{1}{2n}].$$

From Definition 2.4, we can obtain $\limsup_{n\to\infty} S_n \subseteq S$. Theorem 3.7 is applicable.

Theorem 3.8. *Let $n \in N$, $\varepsilon_n \searrow \varepsilon_0$, i.e., $\varepsilon_n > \varepsilon_0$ for all n and $\varepsilon_n \to \varepsilon_0$, and let D be a closed free-disposal set. Suppose that*

- (i) *K_n converges continuously to K with compact values on A, and for any $x_0 \in K(x_0)$, there exists $x_n \in K_n(x_n)$ such that $x_n \to x_0$;*
- (ii) *T_n outer converges continuously to T with compact values on A;*
- (iii) *$-g_n(x, \cdot, \cdot)$ are continuous on $B \times A$;*
- (iv) *for any $x_n \in A$, with $x_n \to x_0 \in K(x_0)$, $-g_n(x_n, \cdot, \cdot) \xrightarrow{P.K.} -g(x_0, \cdot, \cdot)$.*

Then, $S \subseteq \liminf_{n\to\infty} S_n$.

Proof. For any $x_0 \in S$, such that

$$g(x_0, t, y) + \varepsilon_0 e \in D, \quad \forall y \in K(x_0), \forall t \in T(x_0). \tag{3.5}$$

Since $x_0 \in K(x_0)$, there exists $x_n \in K_n(x_n)$ such that $x_n \to x_0$. In order to prove that $x_0 \in \liminf_{n\to\infty} S_n$, we only need to prove that $x_n \in S_n$ for n large enough. On the contrary, suppose that for all $n_0 \in N$, there exists $k \geq n_0$ such that $x_{n_k} \notin S_{n_k}$. Then, there exist $y_{n_k} \in K_{n_k}(x_{n_k})$, $t_{n_k} \in T_{n_k}(x_{n_k})$ such that

$$g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) + \varepsilon_{n_k} e \notin D. \tag{3.6}$$

As K_n, T_n outer converges continuously to K, T with compact values, respectively, without loss of generality, we can assume that $y_{n_k} \to y_0 \in K(x_0)$, $t_{n_k} \to t_0 \in T(x_0)$, respectively.

From Remark 2.6 and the assumption (iv), for $(x_0, t_0, y_0, -g(x_0, t_0, y_0)) \in epi(-g)$, there exists $(x_{n_k}, \bar{t}_{n_k}, \bar{y}_{n_k}, \bar{z}_{n_k}) \in epi(-g_{n_k})$ such that

$$(x_{n_k}, \bar{t}_{n_k}, \bar{y}_{n_k}, \bar{z}_{n_k}) \to (x_0, t_0, y_0, -g(x_0, t_0, y_0)),$$

then, there exists $n_1 \in N$, for all $k \geq n_1$ we have

$$-g(x_0, t_0, y_0) \in -g_{n_k}(x_{n_k}, \bar{t}_{n_k}, \bar{y}_{n_k}) - \frac{1}{2}(\varepsilon_{n_k} - \varepsilon_0)e + C_Y. \tag{3.7}$$

Since $-g_n(x_{n_k}, \cdot, \cdot)$ are continuous on $B \times A$, $(t_{n_k}, y_{n_k}) \to (t_0, y_0)$ and $(\bar{t}_{n_k}, \bar{y}_{n_k}) \to (t_0, y_0)$, there exists $n_2 \in N$, for all $k \geq n_2$, one has

$$-g_{n_k}(x_{n_k}, \bar{t}_{n_k}, \bar{y}_{n_k}) \in -g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) - \frac{1}{2}(\varepsilon_{n_k} - \varepsilon_0)e + C_Y.$$

From (3.7), for all $k \geq max\{n_1, n_2\}$, it follows that

$$-g(x_0, t_0, y_0) \in -g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) - (\varepsilon_{n_k} - \varepsilon_0)e + C_Y.$$

thus, combining (3.6), for all $k \geq max\{n_1, n_2\}$,

$$\begin{aligned} g(x_0, t_0, y_0) + \varepsilon_0 e &\in g_{n_k}(x_{n_k}, t_{n_k}, y_{n_k}) + \varepsilon_{n_k} e - C_Y \\ &\subseteq Y \backslash D - C_Y \\ &\subseteq Y \backslash D. \end{aligned} \tag{3.8}$$

Which contradicts (3.5) and so completed the proof. □

Remark 3.9. Theorems 3.7 and 3.8 improve and extend the latest corresponding ones of [1, 15] in the following two aspects:

(i) Theorems 3.7 and 3.8 extend the ordering relation for generalized vector quasi-equilibrium models in [1, 15] from the ordering cone case to the free-disposal set case.

(ii) The assumption of continuous convergence in [1] and the assumption of $\Gamma-$convergence in [15] are weakened to P.K. convergence.

The following examples are given to illustrate Theorem 3.8 and Remark 3.9.

Example 3.2. Let $Y = Z = \mathbb{R}^2$, $X = l^1 = \{x = (x_1, x_2, \cdots, x_i, \cdots) : \sum_{i=1}^{\infty} |x_i| < \infty\}$, and $C_Y = \mathbb{R}^2_+$, $e \in intC_Y$. And let $D = C_Y \cup \{x = (x_1, x_2) : -1 \leq x_1 \leq 0, 1 \leq x_2\}$, $\varepsilon_n = \frac{1}{2n}$, $\varepsilon = 0$, and $K(x) = K_n(x) = clcone\{\{\frac{e_i}{i}\}_{i=1}^{\infty} \cup \{0_Y\}\}$, $T(x) = T_n(x) = [0, 1]$.

Suppose that

$$g(x, t, y) + \varepsilon e = t(\sum_{i=1}^{\infty}(y_i + x_i) + 3, \sum_{i=1}^{\infty} x_i + 1), \quad \forall (x, t, y) \in X \times Z \times X,$$

and

$$g_n(x, t, y) + \varepsilon_n e = t(\sum_{i=1}^{\infty}(y_i + x_i) + 3, \sum_{i=1}^{\infty} x_i + 1) + \frac{1}{2n}e, \quad \forall (x, t, y) \in X \times Z \times X.$$

Obviously, all assumptions of Theorem 3.8 are satisfied. It is easy to obtain that

$$S = S_n = K = K_n.$$

Thus, $S \subseteq \liminf_{n\to\infty} S_n$. Therefore, Theorem 3.8 is applicable.

We give Example 3.3 to illustrate that Theorems 3.7 and 3.8 hold, even if the assumptions of continuous convergence and $\Gamma-$convergence are not satisfied.

Example 3.3. Let $Y = \mathbb{R}^2$, $X = Z = \mathbb{R}_+$, $D = C_Y = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1, 0 \leq x_2 \leq x_1\}$ and $A = [0, 8]$, $B = [0, 4]$. And let $e = (1, 1) \in intC_Y$, $\varepsilon_n = \frac{1}{2n}$, $\varepsilon = 0$, and $K_n(x) = K(x) = [\frac{1}{5}, 5]$, $T(x) = T_n(x) = [0, 2]$.

Suppose that

$$g(x, t, y) = g_n(x, t, y) = \begin{cases} (ty, ty), & x = 0 \\ (x + ty, \frac{1}{x} + ty), & x > 0 \end{cases}, \quad \forall (x, t, y) \in A \times B \times A.$$

From Definition 2.5 and Remark 2.6, it is easy to check that $-g_n \xrightarrow{P.K.} -g$ and for $x_n \to x_0 \in K(x_0)$, $-g_n(x_n, \cdot, \cdot) \xrightarrow{P.K.} -g(x_0, \cdot, \cdot)$. However, g_n is neither continuously convergent nor $\Gamma-$convergent. Indeed, take $x = t = y = 0$, $x_n = \frac{1}{n}$, $y_n = t_n = 0$ and $\varepsilon = (1, \frac{1}{2})$. For all $n \geq 1$,

$$-g_n(x_n, t_n, y_n) + \varepsilon = (-\frac{1}{n}, -n) + (1, \frac{1}{2}) \notin C_Y.$$

By Definition 2.7, g_n is not $\Gamma-$convergent. In the meanwhile, $(x_n, t_n, y_n) \to (x, t, y)$, but $g_n(x_n, t_n, y_n) \nrightarrow g(x, t, y)$. Thus g_n is not continuously convergent.

For simple computation, one can obtain

$$S = S_n = [1, 5].$$

Thus, $\limsup_{n\to\infty} S_n \subseteq S \subseteq \liminf_{n\to\infty} S_n$. Therefore, Theorems 3.7 and 3.8 are useable, but the corresponding ones in [1, 15] (e.g., Theorems 3.1 and 3.2 in [1] and Theorems 3.1 and 3.3 in [15]) are not useable here.

Theorem 3.10. *Let $n \in N$ be fixed and D be a closed free-disposal set. Assume that*

(i) *K_n is continuous with compact values on A;*

(ii) *T_n is lower semicontinuous on A;*

(iii) *g_n is C_Y-upper semicontinuous on $A \times B \times A$.*

Then, S_n is closed.

Proof. Taking $x_m \in S_n$ with $x_m \to x_0$, we only need to prove $x_0 \in S_n$. In the light of Lemma 3.6, as K_n is upper semicontinuous with compact value, for $x_m \in K_n(x_m)$, $x_m \to x_0 \in K_n(x_0)$. Because of $x_m \in S_n$, one has

$$g_n(x_m, t, y) + \varepsilon_n e \in D, \quad \forall y \in K_n(x_m), \forall t \in T_n(x_m).$$

By the lower semicontinuity of K_n, T_n, for any $y_0 \in K_n(x_0)$, $t_0 \in T_n(x_0)$, there exist $y_m \in K_n(x_m)$, $t_m \in T_n(x_m)$ such that $y_m \to y_0$, $t_m \to t_0$, respectively. Since g_n is C_Y-upper semicontinuous, for any neighborhood V of the origin 0_Y in Y, such that

$$g_n(x_0, t_0, y_0) + \varepsilon_n e \in g_n(x_m, t_m, y_m) + \varepsilon_n e + V + C_Y,$$

for m large enough, and so

$$g_n(x_0, t_0, y_0) + \varepsilon_n e \in D + V.$$

Since V is arbitrary and D is closed, we conclude that

$$g_n(x_0, t_0, y_0) + \varepsilon_n e \in D, \ \forall y_0 \in K_n(x_0), \forall t_0 \in T_n(x_0),$$

i.e., $x_0 \in S_n$, S_n is a closed set. This completes the proof. □

4 Upper Painlevé-Kuratowski Convergence of Solutions for (PSOP)

In this section, we establish the sufficient conditions of upper Painlevé-Kuratowski convergence of E-minimal point sets, weak E-minimal point sets and Borwein E-minimal point sets for perturbed set-valued optimization problem via improvement set. For simplicity's sake, in the rest of this section, we suppose that E is an improvement set, and S_n is uniformly bounded for sufficiently large n

Definition 4.1 ([13])**.** Let $f, f_n : X \rightrightarrows W (n \in \mathbb{N})$ be set-valued mappings. $A, A_n (n \in \mathbb{N})$ be sets in X and $\{(A_n, f_n) : n \in \mathbb{N}\}$ be corresponding sequence pair. We say that $(A_n, f_n) \to (A, f)$ in the sense of $P.K.$ convergence, denoted by $(A_n, f_n) \xrightarrow{P.K.} (A, f)$, iff $\overline{f_n} \xrightarrow{P.K.} \overline{f}$, where

$$\overline{f_n}(x) = \begin{cases} f_n(x), & x \in A, \\ \{+\infty\}, & x \in X \setminus A, \end{cases} \quad \text{and} \quad \overline{f}(x) = \begin{cases} f(x), & x \in A, \\ \{+\infty\}, & x \in X \setminus A. \end{cases}$$

The following example shows that there exists $(A_n, f_n) \longrightarrow (A, f)$ in the sense of P.K., but f_n doesn't converge continuously to f on X.

Example 4.1. Let $f, f_n : \mathbb{R} \rightrightarrows \mathbb{R}(n \in N)$ be, respectively, defined as

$$f(x) = [-2, x^2 - 2] \ \ and \ \ f_n(x) = [-2, x^2 + nx + \frac{1}{n}].$$

Let $A = [0, +\infty)$, $A_n = [-\frac{2}{n}, +\infty)$ and $C_W = \mathbb{R}_+$. From Definition 4.1, it is easy to check that $(A_n, f_n) \xrightarrow{P.K.} (A, f)$. However, f_n doesn't converge continuously to f on X. Taking $x_n = -\frac{1}{n}$, $x_0 = 0$, thus $f_n(x_n) = [-2, \frac{1}{n^2} - 1 + \frac{1}{n}]$ and $f(x_0) = \{-2\}$, $\limsup_{n\to\infty} f_n(x_n) \nsubseteq f(x_0)$. Obviously, f_n doesn't converge continuously to f on X.

Theorem 4.2. *Assume that $S_n \xrightarrow{P.K.} S$, f_n converges continuously to f with compact values on X, then*

$$\limsup_{n\to\infty} EWMinf_n(S_n) \subseteq EWMinf(S).$$

Proof. For any $y \in \limsup_{n\to\infty} EWMinf_n(S_n)$, there exists a subsequence $\{y_{n_k}\} \subset f_{n_k}(x_{n_k})$ in $EWMinf_n(S_n)$ such that $y_{n_k} \to y$, where $x_{n_k} \in S_n$. Thus,

$$(f_{n_k}(S_{n_k}) - y_{n_k}) \cap (-intE) = \emptyset. \tag{4.1}$$

Since $S_n \xrightarrow{P.K.} S$, there exists $x \in S$ such that $x_{n_k} \to x$. As f_n outer converges continuously to f, we have $y \in f(S)$.

Next, we prove that $y \in EWMinf(S)$. Suppose to the contrary, $y \notin EWMinf(S)$, i.e.,

$$(f(S) - y) \cap (-intE) \neq \emptyset.$$

There exists $\overline{y} \in f(\overline{x})$, where $\overline{x} \in S$, s.t.,

$$\overline{y} - y \in (f(S) - y) \cap (-intE).$$

As $S_n \xrightarrow{P.K.} S$, then there exists $\overline{x}_{n_k} \in S_n$ such that $\overline{x}_{n_k} \to \overline{x}$. Since f_n inner converges continuously to f, by the Lemma 3.2, for $\overline{y} \in f(\overline{x})$, there exists $\overline{y}_{n_k} \in f_{n_k}(\overline{x}_{n_k})$ such that $\overline{y}_{n_k} \to \overline{y}$. Hence, we have

$$\overline{y}_{n_k} - y_{n_k} \to \overline{y} - y.$$

By the openness of $-intE$, it follows that

$$\overline{y}_{n_k} - y_{n_k} \in -intE, \ for \ k \ large \ enough,$$

which contradicts (4.1). Therefore $y \in EWMinf(S)$ and $\limsup_{n\to\infty} EWMinf_n(S_n) \subseteq EWMinf(S)$. The proof is complete. □

Using the same proof method of Theorem 4.2, with suitable modification, we can obtain the upper Painlevé-Kuratowski convergence of E-minimal point sets as follows.

Theorem 4.3. *Let E be an open improvement set. If $S_n \xrightarrow{P.K.} S$ and f_n converges continuously to f, then*

$$\limsup_{n\to\infty} EMinf_n(S_n) \subseteq EMinf(S).$$

The following example is given to illustrate Theorems 4.2 and 4.3.

Example 4.2. Let $X = W = \mathbb{R}^2$, $S = [1,5]\times[1,5]$ and $S_n = [1-\frac{1}{2n}, 5-\frac{1}{2n}]\times[1-\frac{1}{2n}, 5-\frac{1}{2n}]$. And let $E = \{(x_1, x_2) : 0 < x_1, 0 < x_2\}$, $C_W = \mathbb{R}^2_+$. Suppose that

$$f(x) = [5-4x_1-x_1^2, 0]\times[5-4x_2-x_2^2, 0], \quad \forall x = (x_1, x_2) \in S,$$

and

$$f_n(x) = [5-4(x_1+\frac{1}{2n})-(x_1+\frac{1}{2n})^2, 0]\times[5-4(x_2+\frac{1}{2n})-(x_2+\frac{1}{2n})^2, 0], \ \forall x = (x_1, x_2) \in S_n.$$

It is easy to check that all assumptions of Theorems 4.2 and 4.3 are satisfied. From the simply computation, we have

$$EWMinf(S) = EMinf(S) = \{(-40, [-40, 0])\} \cup \{([-40, 0], -40)\},$$

and

$$EWMinf_n(S_n) = EMinf_n(S_n) = \{(-40, [-40, 0])\} \cup \{([-40, 0], -40)\}.$$

Thus,

$$\limsup_{n\to\infty} EMinf_n(S_n) \subseteq EMinf(S) \ \mathit{and} \ \limsup_{n\to\infty} EWMinf_n(S_n) \subseteq EWMinf(S).$$

When the convergence continuously of objective mapping sequence is replaced by $(S_n, f_n) \xrightarrow{P.K.} (S, f)$, we can obtain the following result.

Theorem 4.4. *Suppose that $S_n \xrightarrow{P.K.} S$. If $(S_n, f_n) \xrightarrow{P.K.} (S, f)$, then*

$$\limsup_{n\to\infty} EWMin(f_n(S_n) + C) \subseteq EWMin(f(S) + C).$$

Proof. For any $y \in \limsup_{n\to\infty} EWMin(f_n(S_n)+C)$, there exists $y_{n_k} \in EWMin(f_n(S_n)+C)$ such that $y_{n_k} \to y$, and also $y_{n_k} \in f_{n_k}(x_{n_k}) + C$ where $x_{n_k} \in S_{n_k}$. Thus, we have

$$(f_{n_k}(S_{n_k}) + C - y_{n_k}) \cap (-intE) = \emptyset. \tag{4.2}$$

Since $S_n \xrightarrow{P.K.} S$, there exists $x \in S$ such that $x_{n_k} \to x$. As $(S_n, f_n) \xrightarrow{P.K.} (S, f)$ and $(x_{n_k}, y_{n_k}) \in epif_{n_k}$, then $(x_{n_k}, y_{n_k}) \to (x, y) \in epif$. Therefore $y \in f(S) + C$.

Now, we prove that $y \in EWMin(f(S)+C)$. Suppose to the contrary, $y \notin EWMin(f(S)+C)$, then

$$(f(S) + C - y) \cap (-intE) \neq \emptyset.$$

There exists $\overline{y} \in f(\overline{x}) + C \subseteq f(S) + C$ such that

$$\overline{y} - y \in (f(S) + C - y) \cap (-intE).$$

Since $(S_n, f_n) \xrightarrow{P.K.} (S, f)$ and $(\overline{x}, \overline{y}) \in epif$, there exists $(\overline{x}_{n_k}, \overline{y}_{n_k}) \in epif_{n_k}$ such that $(\overline{x}_{n_k}, \overline{y}_{n_k}) \to (\overline{x}, \overline{y})$. Hence, we have

$$\overline{y}_{n_k} - y_{n_k} \to \overline{y} - y.$$

Since $\overline{y} - y \in -intE$, it follows that

$$\overline{y}_{n_k} - y_{n_k} \in (f_{n_k}(S_{n_k}) + C - y_{n_k}) \cap (-intE),$$

for k large enough, which contradicts (4.2). Therefore, $\limsup_{n\to\infty} EWMin(f_n(S_n)+C) \subseteq EWMin(f(S) + C)$. This completes the proof. □

Next, we give the following sufficient conditions of the upper Painlevé-Kuratowski convergence of Borwein E-minimal point sets for $(PSOP)$.

Theorem 4.5. *If $S_n \xrightarrow{P.K.} S$ and E is an open improvement set. Assume that f_n converges continuously to f with compact values on X, then*

$$\limsup_{n\to\infty} BEMinf_n(S_n) \subseteq BEMinf(S).$$

Proof. For any $y \in \limsup_{n\to\infty} BEMinf_n(S_n)$, there exists a subsequence $\{y_{n_k}\}$ in $BEMinf_n(S_n)$ such that $y_{n_k} \to y$. Thus,

$$clcone(f_{n_k}(S_{n_k}) - y_{n_k}) \cap (-E) = \emptyset. \tag{4.3}$$

It is obvious that $y_{n_k} \in f_{n_k}(x_{n_k})$, where $x_{n_k} \in S_{n_k}$. And since $S_n \xrightarrow{P.K.} S$, there exists $x \in S$ such that $x_{n_k} \to x$. As f_n outer converges continuously to f and $y_{n_k} \to y$, we have $y \in f(S)$.

Suppose that $y \notin BEMinf(S)$, i.e.,

$$clcone(f(S) - y) \cap (-E) \neq \emptyset.$$

So there exists $t \in clcone(f(S) - y) \cap (-E)$. We can assume that

$$t = \lambda(\overline{y} - y) \in clcone(f(\overline{x}) - y) \subseteq clcone(f(S) - y), \tag{4.4}$$

where $\lambda > 0$ and $\overline{y} \in f(\overline{x})$. As $S_n \xrightarrow{P.K.} S$, then there exists $\overline{x}_n \in S_n$ such that $\overline{x}_n \to \overline{x}$. Since f_n inner converges continuously to f, it follows that there exists $\overline{y}_{n_k} \in f_{n_k}(\overline{x}_{n_k})$ such that $\overline{y}_{n_k} \to \overline{y}$. Hence, we have

$$\lambda(\overline{y}_{n_k} - y_{n_k}) \to t.$$

By the openness of $-E$ and $\lambda(\overline{y} - y) \in -E$, there exists an open neighborhood U of the origin in Y, such that

$$\lambda(\overline{y}_{n_k} - y_{n_k}) \in \lambda(\overline{y} - y) + U \subseteq -E, for\ k\ large\ enough, \tag{4.5}$$

which contradicts (4.3). Therefore, $\limsup_{n\to\infty} BEMinf_n(S_n) \subseteq BEMinf(S)$. The proof is complete. □

Example 4.3. Continuing with Example 4.2 above, all assumptions of Theorem 4.5 are satisfied. It follows from a direct computation that

$$BEMinf(S) = \{(-40, [-40, 0])\} \cup \{([-40, 0], -40)\}$$

and

$$BEMinf_n(S_n) = \{(-40, [-40, 0])\} \cup \{([-40, 0], -40)\}.$$

Hence, by virtue of Theorem 4.5, $\limsup_{n\to\infty} BEMinf_n(S_n) \subseteq BEMinf(S)$.

5 Lower Painlevé-Kuratowski Convergence of Solutions for (PSOP)

In this section, under some assumptions of convergence, we establish the sufficient conditions of lower Painlevé-Kuratowski convergence of E-minimal point sets and Borwein E-minimal point sets for (PSOP).

Theorem 5.1. *If $S_n \xrightarrow{P.K.} S$ and S_n is uniformly bounded for sufficiently large n. Let E be a closed improvement set. Suppose that f_n converges continuously to f with compact values on X, then*

$$EMinf(S) \subseteq \liminf_{n\to\infty} EMinf_n(S_n).$$

Proof. For any $y \in EMinf(S)$, it follows that there exists $x \in S$ such that $y \in f(x)$ and

$$(f(S) - y) \cap (-E) = \emptyset. \tag{5.1}$$

By the virtue of $S_n \xrightarrow{P.K.} S$, there exists $x_n \in S_n$ such that $x_n \to x$. Since f_n inner converges continuously to f, for above $y \in f(x)$, there exists $y_n \in f_n(x_n)$ such that $y_n \to y$. Now, we only need to prove that $y_n \in EMinf_n(S_n)$.

Suppose to the contrary, assume that $y_n \notin EMinf_n(S_n)$, then there exists $\overline{x}_n \in S_n$ such that

$$(f_n(\overline{x}_n) - y_n) \cap (-E) \neq \emptyset.$$

Thus, there exists $a_n \in (f_n(\overline{x}_n) - y_n) \cap (-E)$. Without loss of generality, we can assume that $a_n = \overline{y}_n - y_n$ where $\overline{y}_n \in f_n(\overline{x}_n)$. Since $S_n \xrightarrow{P.K.} S$, there exist $\overline{x} \in S$ such that $\overline{x}_n \to \overline{x}$. As f_n outer converges continuously to f, it follows that there exist $\overline{y} \in f(\overline{x})$ and a subsequence $\{\overline{y}_{n_k}\}$ of $\{\overline{y}_n\}$ such that $\overline{y}_{n_k} \to \overline{y}$. Hence, we have

$$\overline{y}_{n_k} - y_{n_k} \to \overline{y} - y.$$

By the closedness of $-E$, one has

$$\overline{y} - y \in -E, \quad as\ k \to \infty,$$

which contradicts the fact $y \in EMinf(S)$ and so completed the proof. □

We give the following example to illustrate the assumption of the closedness of E is essential.

Example 5.1. Let $W = \mathbb{R}^2$, $X = \mathbb{R}$ and $C_W = \mathbb{R}^2_+$, $S = [0,1]$, $S_n = (0, 1 - \frac{1}{n})$. Suppose that

$$f(x) = [x, 1] \times [x, 1], \forall x \in X,$$

and

$$f_n(x) = [x, 1] \times [x, 1], \forall x \in X,$$

where $E = \{(w_1, w_2) : 2 \leq w_1, 1 \leq w_2\}$.

By simple calculation, we get that

$$EMinf(S) = [0,1] \times [0,1] \quad and \quad EMinf_n(S_n) = (0, 1 - \frac{1}{n}] \times (0, 1 - \frac{1}{n}].$$

Thus $EMinf(S) \subseteq \liminf_{n\to\infty} EMinf_n(S_n)$. However, if E is not closed, the conclusion of Theorem 5.1 may not be hold. Suppose that $E = \mathbb{R}^2_+ \setminus \{(0,0)\}$, we can obtain that

$$EMinf(S) = \{0\} \quad and \quad EMinf_n(S_n) = \emptyset,$$

thus $EMinf(S) \nsubseteq \liminf_{n\to\infty} EMinf_n(S_n)$. Therefore the assumption of the closedness of E is essential.

The sufficient conditions of lower Painlevé-Kuratowski convergence of Borwein E-minimal solution are given as follows.

Theorem 5.2. *For each $n \in N$, $BEMinf_n(S_n)$ is nonempty. Assume that*

(i) $S_n \xrightarrow{P.K.} S$ *and S_n is uniformly bounded for sufficiently large n;*

(ii) f_n *converges continuously to f with compact values on X;*

(iii) E *is a closed improvement set satisfied $\forall e_1 \in E$, there exist $M < \infty$ and $E^* = \{e^* \in \partial E | d(e^*, 0_W) \leq M\}$, such that $e_1 \in \lambda E^* \subseteq E$ where $\lambda \geq 1$.*

Then

$$BEMinf(S) \subseteq \liminf_{n\to\infty} BEMinf_n(S_n).$$

Proof. For any $y \in BEMinf(S)$, there exists $x \in S$, s.t., $y \in f(x)$ and

$$clcone(f(S) - y) \cap (-E) = \emptyset. \tag{5.2}$$

As $S_n \xrightarrow{P.K.} S$, then there exists $x_n \in S_n$ such that $x_n \to x$. And since f_n inner converges continuously to f, for $y \in f(x)$, there exists $y_n \in f_n(x_n)$ such that $y_n \to y$. Therefore, we only need to prove that $y_n \in BEMinf_n(S_n)$. If $y_n \notin BEMinf_n(S_n)$, we have

$$clcone(f_n(S_n) - y_n) \cap (-E) \neq \emptyset.$$

Then, there exists $t_n \in clcone(f_n(S_n) - y_n) \cap (-E)$. We can also assume that $t_n \in cone(\overline{y}_n - y_n) \subseteq clcone(f_n(\overline{x}_n) - y_n) \subseteq clcone(f_n(S_n) - y_n)$, where $\overline{y}_n \in f_n(\overline{x}_n) \subset BEMinf_n(S_n)$, $\overline{x}_n \in S_n$. As $S_n \xrightarrow{P.K.} S$ and S_n is uniformly bounded for sufficiently large n, then we can assume that $\overline{x}_n \to \overline{x} \in S$. Since f_n outer converges continuously to f and $\overline{x}_n \to \overline{x}$, for $\overline{y}_n \in f_n(\overline{x}_n)$, without loss of generality, there exists $\overline{y} \in f(\overline{x})$ such that $\overline{y}_n \to \overline{y}$. Next, the proof is divided to the following two cases:

Case 1. If $\overline{y} = y$, it is easy to obtain that $y \in \liminf_{n\to\infty} BEMinf_n(S_n)$.

Case 2. On the other hand, $\overline{y} \neq y$. Then there exists $\delta > 0$ such that $d(\overline{y}_n, y_n) > \delta$ for n large enough. From condition (iii), there exist $M < \infty$ and $E^* = \{e^* \in \partial E | d(e^*, 0_W) \leq M\}$, such that $e_1 \in \lambda E^* \subseteq E$ where $\lambda \geq 1$. Thus we can assume that $t_n = \lambda_0(\overline{y}_n - y_n)$ for n large enough and

$$t_n \to \lambda_0(\overline{y} - y) \in clcone(f(S) - y), \tag{5.3}$$

where $\lambda_0 = \frac{M}{\delta}$. As the closedness of E and (5.3), one has

$$\lambda_0(\overline{y} - y) \in clcone(f(S) - y) \cap (-E),$$

which contradicts (5.2). Therefore, $BEMinf(S) \subseteq \liminf_{n\to\infty} BEMinf_n(S_n)$. The proof is complete. □

Example 5.2. Suppose that $W = X = l^\infty = \{x = (x_1, x_2, \cdots, x_i, \cdots) : \sup_{i\geq 1} |x_i| < \infty\}$, and $C_W = \{x = (x_1, x_2, \cdots, x_i, \cdots) \in l^\infty : x_i \geq 0, i = 1, 2, \cdots\}$. Let $E = C \backslash \{x = (x_1, x_2, \cdots, x_i, \cdots) \in l^\infty : 0 \leq x_i < 1, i = 1, 2, \cdots\}$, $S = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times \cdots \times [0, \frac{1}{2}] \times \cdots$ and $S_n = [0, \frac{1}{2} - \frac{1}{n}] \times [0, \frac{1}{2} - \frac{1}{n}] \times \cdots \times [0, \frac{1}{2} - \frac{1}{n}] \times \cdots$.

We consider $f : S \rightrightarrows W$ and $f_n : S_n \rightrightarrows W$ given as

$$f(x) = \{(z_1, z_2, \cdots, z_i, \cdots) | z_i \in [x_i^2 - 1, -\frac{1}{2}]\}, \ \forall x \in S$$

and

$$f_n(x) = \{(z_1, z_2, \cdots, z_i, \cdots) | z_i \in [x_i^2 - 1, -\frac{1}{2} - \frac{1}{n}]\},\ \forall x \in S_n.$$

It is clear that all conditions of Theorem 5.2 are satisfied. A short calculation reveals that

$$BEMinf(S) = \{(x_1, x_2, \cdots, x_i, \cdots) | x_i = -1,\ i \in N\},$$

and

$$BEMinf_n(S_n) = \{(x_1, x_2, \cdots, x_i, \cdots) |\ x_i = -1,\ i \in N\}.$$

By virtue of Definition 2.3 and Theorem 5.2, one has $BEMinf(S) \subseteq \liminf_{n\to\infty} BEMinf_n(S_n)$. Thus, Theorem 5.2 is applicable.

6 Scalarization

In this section, by using the oriented distance function ($\triangle$), we establish a scalarized problem and discuss the relationships between the minimal point sets of $(PSOP)$ and the maximum sets for the corresponding nonlinear scalarized problem (SP).

Now, we give the following nonlinear scalarization function which was introduced by Hiriart-Urruty [12], the oriented distance function ($\triangle$).

Definition 6.1. For a set $Q \subset W$. Let the function $\triangle_Q : W \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$\triangle_Q(y) = d_Q(y) - d_{W\setminus Q}(y),$$

with $d_\emptyset(y) = +\infty$.

Some properties of the oriented distance function Δ are given by Zaffaroni [30] as follows.

Lemma 6.2. *If the set $Q \subset W$ is a nonempty set and $Q \neq W$, then*

(i) $\triangle_Q$ *is real valued;*

(ii) $\triangle_Q(y) < 0$ *for every* $y \in intQ$, $\triangle_Q(y) = 0$ *for every* $y \in \partial Q$, *and* $\triangle_Q(y) > 0$ *for every* $y \in int\ Q^c$*;*

(iii) *if* Q *is closed, then* $Q = \{y : \triangle_Q(y) \leq 0\}$.

Let E is an improvement set, for $\widehat{y} \in f(S)$, consider the following nonlinear scalarized problem $(SP)_{\widehat{y}}$:

$$(SP)_{\widehat{y}} : \quad max \triangle_{-E} (\widehat{y} - y) \quad s.t.\ y \in f(S),$$

where

$$S = \{x \in K(x) : g(x, t, y) + \varepsilon e \in D, \quad \forall t \in T(x),\ \forall y \in K(x)\}.$$

The set of global maximum $(GMax)$ and strict global maximum $(SGMax)$ of this problem, are defined as

$$GMax_{\widehat{y}} f(S) := \{y \in f(S) : \triangle_{-E}(\widehat{y} - y_0) \leq \triangle_{-E}(\widehat{y} - y),\ \forall y_0 \in f(S)\}. \tag{6.1}$$

$$SGMax_{\widehat{y}} f(S) := \{y \in f(S) : \triangle_{-E}(\widehat{y} - y_0) < \triangle_{-E}(\widehat{y} - y),\ \forall y_0 \in f(S)\ with\ y_0 \neq y\}. \tag{6.2}$$

Definition 6.3. For a set $Q \subset W$, the function $\triangle_{-Q}(\widehat{y} - y)$ is said to be

(i) order preserving on f(S), iff $\forall y_1, y_2 \in f(S)$

$$y_1 - y_2 \in -Q \implies \triangle_{-Q}(\widehat{y} - y_1) \geq \triangle_{-Q}(\widehat{y} - y_2).$$

(ii) strictly order preserving on f(S), iff $\forall y_1, y_2 \in f(S)$ and $y_1 \neq y_2$

$$y_1 - y_2 \in -intQ \implies \triangle_{-Q}(\widehat{y} - y_1) > \triangle_{-Q}(\widehat{y} - y_2).$$

Definition 6.4. For a set $Q \subset W$, the function $\triangle_{-Q}(\widehat{y} - y)$ is said to be

(i) order representing on f(S), iff

$$\{y \in f(S) : \triangle_{-Q}(\widehat{y} - y) \geq 0\} \subseteq \widehat{y} - Q \cup \{0\}.$$

(ii) strictly order representing on f(S), iff

$$\{y \in f(S) : \triangle_{-Q}(\widehat{y} - y) > 0\} \subseteq \widehat{y} - intQ \cup \{0\}.$$

Theorem 6.5. *Assume that E is a closed improvement set, then $\overline{y} \in EMinf(S)$ if and only if $\triangle_{-E}(y - \overline{y}) > 0$, $\forall y \in f(S)$.*

Proof. "$\Longrightarrow$" Let $\overline{y} \in EMinf(S)$, then we have

$$(f(S) - \overline{y}) \cap (-E) = \emptyset.$$

Thus for all $y \in f(S)$, $(y - \overline{y}) \cap (-E) = \emptyset$, i.e., $y - \overline{y} \notin -E$. In the light of Lemma 6.2 and the closedness of E, this is equivalent to the fact

$$\triangle_{-E}(y - \overline{y}) = d_{-E}(y - \overline{y}) - d_{W \setminus -E}(y - \overline{y}) > 0, \ \forall y \in f(S).$$

"$\Longleftarrow$" If for any $y \in f(S)$, $\triangle_{-E}(y - \overline{y}) > 0$, thus we can get

$$\triangle_{-E}(y - \overline{y}) = d_{-E}(y - \overline{y}) - d_{W \setminus -E}(y - \overline{y}) > 0.$$

Therefore $(y - \overline{y}) \cap (-E) = \emptyset$, i.e., $\overline{y} \in EMinf(S)$. This completes the proof. □

Similarly, we can get the following theorem.

Theorem 6.6. *Let E is an improvement set. $\overline{y} \in EWMinf(S)$ if and only if $\triangle_{-E}(y - \overline{y}) \geq 0$, $\forall y \in f(S)$.*

Proof. Using the similar proof method of Theorem 6.5, we can get the conclusion. □

Inspired by [30], the following theorems are given to illustrate the relationships between the minimal point sets of $(PSOP)$ and maximum sets of $(SP)_{\widehat{y}}$.

Theorem 6.7. *Let E is an improvement set. If $\triangle_{-E}(\widehat{y} - y)$ is a*

(i) *order preserving function on f(S), then*

$$SGMax_{\widehat{y}} f(S) \subseteq EMinf(S);$$

(ii) *strictly order preserving function on f(S), then*

$$GMax_{\widehat{y}}f(S) \subseteq EWMinf(S).$$

Proof. Because the proof methods of (i), (ii) are similar, we only give the proof of (i). To the contrary, suppose that there exists $y_0 \in SGMax_{\widehat{y}}f(S)$ but $y_0 \notin EMinf(S)$. For $y_0 \in SGMax_{\widehat{y}}f(S)$, there exists

$$\triangle_{-E}(\widehat{y} - y) < \triangle_{-E}(\widehat{y} - y_0),\ \forall y \in f(S) \text{ with } y \neq y_0. \tag{6.3}$$

By virtue of $y_0 \notin EMinf(S)$, from Definition 2.3, one has

$$(f(S) - y_0) \cap (-E) \neq \emptyset.$$

Thus there exists $y_1 \in f(S)$ such that

$$y_1 - y_0 \in -E.$$

Since $\triangle_{-E}(\widehat{y} - y)$ is a order preserving function on $f(S)$, we have

$$\triangle_{-E}(\widehat{y} - y_1) \geq \triangle_{-E}(\widehat{y} - y_0),$$

which contradicts (6.3). Therefore $y_0 \in EMinf(S)$. The proof is complete. □

Theorem 6.8. *Let E is an improvement set and $y \in f(S)$, for any $y_0 \in f(S)$, assume that $\triangle_{-E}(y - y_0)$ is a*

(i) *order representing function on f(S) and $y \in EMinf(S)$, then*

$$y \in SGMax_y f(S);$$

(ii) *strictly order representing function on f(S) and $y \in EWMinf(S)$, then*

$$y \in GMax_y f(S).$$

Proof. (i) By virtue of $y \in EMinf(S)$, for any $y_0 \in f(S)$ one has

$$y_0 - y \notin -E.$$

On the contrary, suppose that $y \notin SGMax_y f(S)$, there exists $y_1 \in f(S)$ with $y_1 \neq y$ such that

$$0 \leq \triangle_{-E}(y - y) \leq \triangle_{-E}(y - y_1).$$

Since $\triangle_{-E}(y - y_0)$ is a order representing function on $f(S)$, we have

$$y_1 \in \{y_0 \in f(S) : \triangle_{-E}(y - y_0) \geq 0\} \subseteq y - E \cup \{0\}.$$

Thus, $y_1 - y \in -E \cup \{0\}$. As $y_1 \neq y$, it follows that

$$y_1 - y \in -E,$$

which contradicts the fact $y \in EMinf(S)$. Therefore $y \in SGMax_y f(S)$.

(ii) Using the similar proof method, with suitable modifications, we can get that if $\triangle_{-E}(y - y_0)$ is a strictly order representing function on $f(S)$ and $y \in EWMinf(S)$, then

$$y \in GMax_y f(S).$$

So, the proof is complete. □

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Jing-Jing Wang
College of Mathematics and Statistics, Chongqing JiaoTong University
Chongqing 400074, P.R. China
E-mail address: wjj523166387@163.com

Chong-Yang Shao
College of Mathematics and Statistics, Chongqing JiaoTong University
Chongqing 400074, P.R. China
E-mail address: shaocyll@sina.com

Zai-Yun Peng
College of Mathematics and Statistics, Chongqing JiaoTong University
Chongqing 400074, P.R. China
E-mail address: pengzaiyun@126.com