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OPTIMALITY CONDITIONS FOR SEMI-INFINITE PROGRAMMING PROBLEMS UNDER RELAXED QUASICONVEXITY ASSUMPTIONS

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, we study the optimality condition for nonconvex semi-infinite programming problems. We introduce a new constraint qualification for this problem, namely the generalized Abadie constraint qualification. By the two star subdifferentials, we establish necessary and sufficient optimality conditions for nonconvex semi-infinite programming problems. We assume that the strict sublevel sets of the involving functions are convex only at a given point under question and hence these functions are not assumed to be quasiconvex. Our results improve the corresponding ones in the literature. Some examples are given to illustrate our main results.

Key words: *semi-infinite programming, optimality condition, subdifferential, normal cone, generalized Abadie constraint qualification*

Mathematics Subject Classification: *90C34, 90C46, 90C26* .

1 Introduction

As it is well-known, quasiconvex programs, that is problems where the objective function is quasiconvex and the feasible set is convex, play an important role due to many practical applications. A large number of results obtained by many authors for quasiconvex programs; e.g., [1, 2, 4, 10, 19–23] and the references therein. Recently, Khanh, Quyen and Yao [7] obtained the necessary optimality condition for nonconvex programming problems under the assumptions that the involved functions that need only admit a convex sublevel set at the optimal point. Their results are very important in nonconvex programs.

On the other hand, semi-infinite programming has been considered recently in several papers with various requirements on the involved functions and spaces due to its extensive applications in many fields such as reverse Chebyshev approximate, robust optimization,

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minimax problems, design centering and disjunctive programming; e.g., [3, 24, 25]. A great deal of results have appeared in the literature: e.g., [5, 8, 9, 11–18] and the references therein.

Very recently, Kanzi and Soleimani-damaneh [6] derived some necessary and sufficient optimality conditions for quasiconvex semi-infinite programming problems by the quasiconvex Slater constraint qualification.

Motivated by the results from [6, 7], our aim in this paper is to study the optimality condition for nonconvex semi-infinite programming problems. First, the Abadie constraint qualification using the star subdifferential is introduced and the relation between this new constraint qualification and the quasiconvex Slater constraint qualification is established. Second, necessary optimality conditions for nonconvex semi-infinite programming problems are derived, where the involved functions need only to admit a convex sublevel set at the optimal point. Finally, some sufficient optimality conditions are given for the considered problem. The results obtained in this paper improve the corresponding ones in [6].

2 Preliminaries

We start this section by introducing some necessary notations and concepts. Let $A \subset \mathbb{R}^n$. Denote by $\mathrm{int}A$, $\mathrm{cl}A$ and $\mathrm{bd}A$, respectively, the interior, closure and the boundary of A. Let A be convex and $x \in A$. The normal cone at x to A, denoted by $N(A;x)$, is defined by

$$N(A;x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle \leq 0,\ \forall\, y \in A\}.$$

If $x \notin \mathrm{cl}A$, then we adopt that $N(A;x) = \emptyset$.

A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be quasiconvex if its sublevel set $L_f(x) := \{y \in \mathbb{R}^n : f(y) \leq f(x)\}$ at x is convex for all $x \in \mathbb{R}^n$, or equivalently, if for each $r \in \mathbb{R}$ the strict sublevel set $\{y \in \mathbb{R}^n : f(y) < r\}$ is convex. Hence, f is quasiconvex if and only if the strict sublevel set $L_f^<(x) := \{y \in \mathbb{R}^n : f(y) < f(x)\}$ is convex for all $x \in \mathbb{R}^n$. Another equivalent statement, which is often met in the literature, is that f is quasiconvex if for any $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$, we have

$$f(x + \lambda(y - x)) \leq \max\{f(x), f(y)\}.$$

We now recall some definitions of the important subdifferentials. Let $x_0 \in \mathbb{R}^n$. The Greenber-Pierskalla subdifferential [4], which is akin to the normal cone, is defined by

$$\partial^{GP} f(x_0) := \{\xi \in \mathbb{R}^n : \langle \xi, x - x_0\rangle < 0,\ \forall\, x \in L_f^<(x_0)\}.$$

The star subdifferentials [19], are the following normal cone subdifferentials:

$$\partial^v f(x_0) := N(L_f(x_0); x_0),$$

$$\partial^{\circledast} f(x_0) := N(L_f^<(x_0); x_0),$$

$$\partial^{\star} f(x_0) := N(L_f^<(x_0); x_0)\backslash\{0\}.$$

It is easy to see that $\partial^{GP} f(x_0) \subseteq \partial^{\circledast} f(x_0)$ and $\partial^v f(x_0) \subseteq \partial^{\circledast} f(x_0)$. If f is upper semicontinuous on $L_f^<(x_0)$ (i.e., at each point of this set), then $\partial^{\circledast} f(x_0) = \partial^{GP} f(x_0) \cup \{0\}$; see [19, Proposition 8].

The following lemmas will be used in the sequel.

Lemma 2.1 ([7, Theorem 3.1]). *Consider the minimization problem:*

(P) Minimize $f(x)$ subject to $x \in C$,

where $f : \mathbb{R}^n \to \mathbb{R}$ and $C \subseteq \mathbb{R}^n$ is a convex set. Let $L_f^<(x_0)$ be a convex set. Suppose that $x_0 \in C$ is a solution to the problem (P), which is not a local minimizer of f on $\mathbb{R}^n$. Then

$$\partial^{\circledast} f(x_0) \cap (-N(C; x_0)) \neq \{0\}.$$

Lemma 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function and $x_0 \in \mathbb{R}^n$. If there is not local minimizer of f in $f^{-1}(f(x_0))$, then*

(i) [23, Lemma 2.1] $\partial^{\circledast} f(x_0) = \partial^v f(x_0)$;

(ii) [19, Proposition 8] $\partial^\star f(x_0) = \partial^v f(x_0) \backslash \{0\}$.

From Lemma 4.9 and Corollary 5.4 in [9], we have the following result.

Lemma 2.3. *Let T be an infinite index set, and the sets D and C_t, $t \in T$ be closed convex subsets of $\mathbb{R}^n$. Assume that the following conditions are satisfied:*

(i) *T is a compact metric space;*

(ii) *the set-valued mapping $t \mapsto (spanD) \cap C_t$ is lower semi-continuous on T;*

(iii) *$D \cap (\cap_{t\in T} intC_t) \neq \emptyset$.*

Then for each $x_0 \in D \cap (\cap_{t\in T} C_t)$ we have

$$N(D \cap (\cap_{t\in T} C_t); x_0) = N(D; x_0) + \sum_{t\in T} N(C_t; x_0).$$

3 Optimality Conditions

In this section, we consider the following semi-infinite programming problem:

$$\begin{array}{lll} \text{(SIP)} & \text{Minimize} & f(x), \\ & \text{subject to} & f_t(x) \leq 0,\ t \in T, \\ & & x \in \mathbb{R}^n, \end{array}$$

where T is an arbitrary (not necessarily finite) index set, $f,\ f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are two real-valued functions.

We shall use the following notations:

$$C := \{x \in \mathbb{R}^n : f_t(x) \leq 0,\ \forall\, t \in T\},$$

$$C_t := \{x \in \mathbb{R}^n : f_t(x) \leq 0\},\ t \in T,$$

$$T(x) := \{t \in T : f_t(x) = 0\},\ x \in \mathbb{R}^n.$$

Obviously, $C = \bigcap_{t\in T} C_t$. Recently, Kanzi and Soleimani-damaneh [6] introduced the following quasiconvex Slater constraint qualification to the system $\{f_t : t \in T\}$.

Definition 3.1. [6] We say that the system $\{f_t : t \in T\}$ satisfies the quasiconvex Slater constraint qualification (QSCQ) if

(a) $T \subseteq \mathbb{R}^n$ is a compact set;

(b) for each $t \in T$, C_t is a closed set;

(c) the set-valued mapping $t \mapsto C_t$ is lower semi-continuous on T;

(d) there exists a point $\overline{x} \in \mathbb{R}^n$ such that $f_t(\overline{x}) < 0$ for all $t \in T$.

Next, we introduce a new constraint qualification to the system $\{f_t : t \in T\}$.

Definition 3.2. Let C be a convex set and $x_0 \in C$. We say that the system $\{f_t : t \in T\}$ satisfies the generalized Abadie constraint qualification (GACQ) at x_0 if

$$N(C; x_0) = \sum_{t \in T(x_0)} \partial^v f_t(x_0). \tag{3.1}$$

Remark 3.3. If $x_0 \in \text{int}C$, then it is easy to see that (3.1) holds.

Proposition 3.4. *Let $x_0 \in C$. Assume that C_t, $t \in T$ is convex and f_t, $t \in T$ is upper semi-continuous on $\mathbb{R}^n$. If the system $\{f_t : t \in T\}$ satisfies the (QSCQ), then it satisfies the (GACQ) at x_0.*

Proof. By the definition of the (GSCQ), there exists $\overline{x} \in \mathbb{R}^n$ such that $f_t(\overline{x}) < 0$ for all $t \in T$. Since f_t, $t \in T$ is upper semi-continuous on $\mathbb{R}^n$,

$$\overline{x} \in \bigcap_{t \in T} \text{int}C_t.$$

As C_t, $t \in T$ is convex, we have C is convex. Note that $x_0 \in C$. Then $x_0 \in \bigcap_{t \in T} C_t$. By Lemma 2.3,

$$N(C; x_0) = \sum_{t \in T} N(C_t; x_0) = \sum_{t \in T(x_0)} N(C_t; x_0) + \sum_{t \in T \setminus T(x_0)} N(C_t; x_0).$$

For each $t \in T(x_0)$, we have

$$C_t = \{x \in \mathbb{R}^n : f_t(x) \le 0\} = \{x \in \mathbb{R}^n : f_t(x) \le f_t(x_0)\} = L_{f_t}(x_0).$$

It follows that

$$\sum_{t \in T(x_0)} N(C_t; x_0) = \sum_{t \in T(x_0)} N(L_{f_t}(x_0); x_0) = \sum_{t \in T(x_0)} \partial^v f_t(x_0).$$

For each $t \in T \setminus T(x_0)$, we have $x_0 \in \text{int}C_t$. This implies $N(C_t; x_0) = \{0\}$ for all $t \in T \setminus T(x_0)$. Hence,

$$N(C; x_0) = \sum_{t \in T(x_0)} N(C_t; x_0) + \sum_{t \in T \setminus T(x_0)} N(C_t; x_0) = \sum_{t \in T(x_0)} \partial^v f_t(x_0).$$

The proof is complete. □

The following example shows that the system $\{f_t : t \in T\}$ satisfies the (GACQ) at x_0, but it does not satisfy the (QSCQ).

Example 3.5. Consider the system $\{f_t : t \in T\}$, where $T := [1, 2]$,

$$f_1(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ 2, & \text{if } x < 0, \end{cases}$$

$$f_t(x) = \frac{-x-1}{t},\ t \in (1, 2),$$

$$f_2(x) = -x.$$

Then $C = [0, +\infty)$. Let $x_0 = 0$. It is easy to see that (1) holds. But the system $\{f_t : t \in T\}$ does not satisfy the (QSCQ) since there is not $\overline{x} \in \mathbb{R}^n$ such that $f_t(\overline{x}) < 0$ for all $t \in T$.

We now give a necessary optimality condition of KKT type to the problem (SIP).

Theorem 3.6. *Let $x_0 \in C$, the sets $L_f^{<}(x_0)$ and C be convex. Suppose that the system $\{f_t : t \in T\}$ satisfies the (GACQ) at x_0 and x_0 is a solution of the problem (SIP), but x_0 is not a local minimizer of f on $\mathbb{R}^n$. Then*

$$\partial^v f(x_0) \cap \left(- \sum_{t \in T(x_0)} \partial^v f_t(x_0) \right) \neq \{0\}. \tag{3.2}$$

Proof. Since $L_f^{<}(x_0)$ is a convex set and x_0 is a solution of the problem (SIP) which is not a local minimizer of f on $\mathbb{R}^n$, by Lemma 2.1,

$$\partial^{\circledast} f(x_0) \cap (-N(C; x_0)) \neq \{0\}. \tag{3.3}$$

Note that the system $\{f_t : t \in T\}$ satisfies the (GACQ) at x_0. Then (1) holds. This together with (3.3) yields

$$\partial^{\circledast} f(x_0) \cap \left(- \sum_{t \in T(x_0)} \partial^v f_t(x_0) \right) \neq \{0\}.$$

From Lemma 2.2 (i), we have (3.2) holds. The proof is complete. □

We now give an example to illustrate Theorem 3.6.

Example 3.7. Consider the following problem:

$$\text{(SIP)} \qquad \begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & f_t(x) \leq 0,\ t \in T := [1, 2], \end{array}$$

where

$$f(x) = \begin{cases} x - 1, & \text{if } x < 0, \\ x, & \text{if } x \geq 0 \end{cases}$$

and $f_t(x)$, $t \in T$ considered in Example 3.5. Let $x_0 = 0$. Then $C = [0, +\infty)$, $L_f^{<}(x_0) = (-\infty, 0)$ is convex. It is easy to see that x_0 is a solution for (SIP) which is not a local minimizer of f on $\mathbb{R}$. By a simple computation,

$$\partial^v f(x_0) = [0, +\infty),\ \partial^v f_t(x_0) = (-\infty, 0],\ \forall\, t \in T(x_0) = \{1, 2\}.$$

Therefore, (3.2) holds.

By Proposition 3.4 and Theorem 3.6, we have the following corollary.

Corollary 3.8. *Let $x_0 \in C$ and $L_f^<(x_0)$ be a convex set. Suppose that the system $\{f_t : t \in T\}$ satisfies the (QSCQ) and x_0 is a solution of the problem (SIP), but x_0 is not a local minimizer of f on $\mathbb{R}^n$. If C_t, $t \in T$ is a convex set and f_t, $t \in T$ is upper semi-continuous on $\mathbb{R}^n$, then (3.2) holds.*

Remark 3.9. If f and f_t, $t \in T$ are quasiconvex on $\mathbb{R}^n$, then $L_f^<(x_0)$ and C_t, $t \in T$ are convex. But the converse does not hold in general. Therefore, Corollary 3.8 improves Theorem 3.4 of Kanzi and Soleimani-damaneh [6].

Corollary 3.10 ([6, Theorem 3.4])**.** *Let the (GSCQ) be satisfied and the functions f and f_t, $t \in T$ be quasiconvex on $\mathbb{R}^n$. Suppose that x_0 is a solution of the problem (SIP), but x_0 is not a local minimizer of f on $\mathbb{R}^n$. If f and f_t, $t \in T$ are upper semi-continuous on $\mathbb{R}^n$, then (3.2) holds.*

Remark 3.11. The example below illustrates that Corollary 3.8 holds but Corollary 3.10 does not apply.

Example 3.12. Consider the following problem:

$$\begin{array}{lll} \text{(SIP)} & \text{Minimize} & f(x), \\ & \text{subject to} & f_t(x) \leq 0,\ t \in T := [1,2], \end{array}$$

where

$$f(x) = \begin{cases} -2, & \text{if } x \leq -2, \\ -1, & \text{if } -2 < x < -1, \\ -2, & \text{if } -1 \leq x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

$$f_1(x) = \begin{cases} 1, & \text{if } x < -1, \\ 2, & \text{if } x = -1, \\ -x-1, & \text{if } x > -1 \end{cases}$$

and $f_t(x) = -tx$, for all $t \in (1,2]$. Consider a point $x_0 = 0$. Then $C = [0, +\infty)$, $L_f^<(x_0) = (-\infty, 0)$ is convex, f_t, $t \in [1,2]$ is upper semi-continuous and x_0 is a solution for (SIP) which is not a local minimizer of f on $\mathbb{R}$. All assumptions of Corollary 3.8 are satisfied. By a simple computation,

$$\partial^v f(x_0) = [0, +\infty),\ \partial^v f_t(x_0) = (-\infty, 0],\ \forall\, t \in T(x_0) = (1,2].$$

Obviously, (3.2) holds. But Theorem 3.4 of Kanzi and Soleimani-damaneh [6], i.e., Corollary 3.10, is not applicable since f and f_t, $t \in T$ are not quasiconvex on $\mathbb{R}^n$, and f is not upper semi-continuous on $\mathbb{R}^n$.

From Corollary 3.8 and Lemma 2.2 (ii), we have the following corollary.

Corollary 3.13. *Let $L_f^<(x_0)$ be a convex set and the (GSCQ) be satisfied. Suppose that x_0 is a solution of the problem (SIP), but x_0 is not a local minimizer of f on $\mathbb{R}^n$. If C_t, $t \in T$ is a convex set and f_t, $t \in T$ is upper semi-continuous on $\mathbb{R}^n$, then*

$$\partial^\star f(x_0) \cap \left(- \sum_{t \in T(x_0)} \partial^\star f_t(x_0) \right) \neq \{0\}. \tag{3.4}$$

Remark 3.4 In [6], Kanzi and Soleimani-damaneh gave a sufficient condition guaranteeing (3.4) to be hold. The assumption that f and f_t, $t \in T$ are quasiconvex on $\mathbb{R}^n$ and f is upper semi-continuous on $\mathbb{R}^n$ are required in [6]. However, Corollary 3.12 does not require this assumption. Therefore, Corollary 3.12 improves Theorem 3.5 of Kanzi and Soleimani-damaneh [6].

The following theorem gives a sufficient condition assuming that x_0 is a solution of the problem (SIP).

Theorem 3.14. *Let $x_0 \in C$ and C be a convex set. Assume that*

$$\partial^{GP} f(x_0) \cap \left(- \sum_{t \in T(x_0)} \partial^v f_t(x_0) \right) \neq \{0\}. \tag{3.5}$$

Then x_0 is a solution of the problem (SIP).

Proof. Suppose by contradiction that there exists $x' \in C$ such that $f(x') < f(x_0)$. By the definition of $\partial^{GP} f(x_0)$, for any $\mu \in \partial^{GP} f(x_0)$,

$$\langle \mu, x' - x_0 \rangle < 0.$$

By (3.5), there exists $\xi \in \partial^{GP} f(x_0)$ and $\xi_t \in \partial^v f_t(x_0)$, $t \in T(x_0)$ such that

$$\xi + \sum_{t \in T(x_0)} \xi_t = 0.$$

It follows that

$$\langle \xi + \sum_{t \in T(x_0)} \xi_t, x' - x_0 \rangle = 0. \tag{3.6}$$

Note that $\xi \in \partial^{GP} f(x_0)$. Then we have $\xi \neq 0$ and

$$\langle \xi, x' - x_0 \rangle < 0. \tag{3.7}$$

Since $x' \in C$, $f_t(x') \leq 0 = f_t(x_0)$ for all $t \in T(x_0)$. Then $x' \in L_{f_t}(x_0)$. This fact together with $\xi_t \in \partial^v f_t(x_0)$ yields

$$\langle \xi_t, x' - x_0 \rangle \leq 0, \ \forall\, t \in T(x_0). \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\langle \xi + \sum_{t \in T(x_0)} \xi_t, x' - x_0 \rangle < 0,$$

which contradicts (3.6). The proof is complete. □

We now give an example to illustrate Theorem 3.14.

Example 3.15. Consider the following problem:

$$\begin{array}{lll} \text{(SIP)} & \text{Minimize} & f(x), \\ & \text{subject to} & f_t(x) \leq 0, \ t \in T := 1, 2, 3, \cdots, \end{array}$$

where

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0, \end{cases}$$

$$f_t(x) = \begin{cases} -x, & \text{if } t = 1, \\ -t|x| - 1, & \text{if } t = 2, 3, \cdots. \end{cases}$$

Then $C = [0, +\infty)$ is convex. Let $x_0 = 0$. By a simple computation,

$$\partial^{GP} f(x_0) = (0, +\infty),\ \partial^v f_t(x_0) = (-\infty, 0],\ \forall\, t \in T(x_0) = \{1\}.$$

It is easy to see (3.5) holds and x_0 is a solution to (SIP).

Corollary 3.16. *Let $x_0 \in C$ and C be a convex set. Assume that f is upper semi-continuous on $L_f^{<}(x_0)$ and*

$$\partial^{\circledast} f(x_0) \cap \left(- \sum_{t \in T(x_0)} \partial^v f_t(x_0) \right) \neq \{0\}. \tag{3.9}$$

Then x_0 is a solution of the problem (SIP).

Proof. Since f is upper semi-continuous on $L_f^{<}(x_0)$, $\partial^{\circledast} f(x_0) = \partial^{GP} f(x_0) \cup \{0\}$. This fact together with (3.9) implies (3.5) holds. It follows that x_0 is a solution of the problem (SIP). □

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References

[1] D. Aussel and N. Hadjisavvas, Adjusted sublevel sets, normal operators, and quasiconvex programming, *SIAM J. Optim.* 16 (2005) 358–367.

[2] A. Daniilidis and N. Hadjisavvas and J.E. Martmez-Legaz, An appropriate subdifferential for quasiconvex functions, *SIAM J. Optim.* 12 (2001) 407–420.

[3] M.A. Goberna and M.A. López, *Linear Semi-Infinite Optimization*, J. Wiley Sons, Chichester, 1998.

[4] H.P. Greenberg and W.P. Pierskalla, Quasiconjugate function and surrogate duality, *Cahiers du Centre d'Etude de Recherche Oper.* 15 (1973) 437–448.

[5] N. Kanzi, Constraint qualifications in semi-infinite systems and their applications in nonsmooth semi-infinite programs with mixed constraints, *SIAM J. Optim.* 24 (2014) 559–572.

[6] N. Kanzi and M. Soleimani-damaneh, Slater CQ, optimality and duality for quasiconvex semi-infinite optimization problems, *J. Math. Anal. Appl.* 434 (2016) 638–651.

[7] P.Q. Khanh, H.T. Quyen and J.C. Yao, Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials, *European J. Oper. Res.* 212 (2011) 235–241.

[8] C. Li, X.P. Zhao and Y.H. Hu, Quasi-slater and Farkas-Minkowski qualifications for semi-infinite programming with applications, *SIAM J. Optim.* 23 (2013) 2208–2230.

[9] C. Li, K.F. Ng and K. Pong, The SECQ, linear regularity and the strong CHIP for infinite system of closed convex sets in normed linear spaces, *SIAM J. Optim.* 18 (2007) 643–665.

[10] N.T.H. Linh and J.P. Penot, Optimality conditions for quasiconvex programs, *SIAM J. Optim.* 17 (2006) 500–510.

[11] X.J. Long, Z.Y. Peng, X.K. Sun and X.B. Li, Generic stability of the solution mapping for semi-infinite vector optimization problems in Banach spaces, *Pacific J. Optim.* 13 (2017) 593–605.

[12] X.J. Long, Z.Y. Peng, and X.F. Wang, Characterizations of the solution set for nonconvex semi-infinite programming problems, *J. Nonlinear Convex Anal.* 17 (2016) 251–265.

[13] X.J. Long, Y.B. Xiao and N.J. Huang, Optimality conditions of approximate solutions for nonsmooth semi-infinite programming problems, *J. Oper. Res. Soc. China* 6 (2018) 289–299.

[14] X.J. Long, Z.Y. Peng and X.F. Wang, Stable Farkas lemmas and duality for nonconvex composite semi-infinite programming problems, *Pacific J. Optim.* 15 (2019) 295–315.

[15] S.K. Mishra, M. Jaiswal and H.A.Le Thi, Nonsmooth semi-infinite programming problem using limiting subdifferentials, *J. Global Optim.* 53 (2012) 285–296.

[16] B.S. Mordukhovich and T.T.A. Nghia, Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with Lipschitzian date, *SIAM J. Optim.* 23 (2013) 406–431.

[17] Z.Y. Peng, X.B. Li, X.J. Long and X.D. Fan, Painlevé-Kuratowski stability of approximate efficient solutions for perturbed semi-infinite vector optimization problems, *Optim. Lett.* 12 (2018) 1339–1356.

[18] Z.Y. Peng, J.W. Peng, X.J. Long, J.C. Yao, On the stability of solutions for semi-infinite vector optimization problems, *J. Global Optim.* 70 (2018) 55–69.

[19] J. P. Penot, Are generalized derivatives useful for generalized convex functions?, in: J.P. Crouzeix, in: *Generalized Convexity, Generalized Monotonicity: Recent Results*, J.E. Martinez-Legaz, M. Volle (eds.), Kluwer, Dordrecht, 1998, pp. 3–59.

[20] J.P. Penot, What is Quasiconvex Analysis?, *Optimization* 47 (2000) 35–110.

[21] J.P. Penot, Characterization of solution sets of quasiconvex programs, *J. Optim. Theory Appl.* 117 (2003) 627–636.

[22] J.P. Penot and C. Zalinescu, Harmonic sum and duality, *J. Convex Anal.* 7 (2000) 95–113.

[23] J.P. Penot and C. Zalinescu, Elements of quasiconvex subdifferential calculus, *J. Convex Anal.* 7 (2000) 243–269.

[24] R. Reemtsen and J.J. Ruckmann (eds.), *Semi-Infinite Programming*, Kluwer, Boston 1998.

[25] O. Stein, How to solve a semi-infinite optimization problem, *European J. Oper. Res.* 223 (2012) 312–320.

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