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PROPER EFFICIENCY FOR NONCONVEX VECTOR OPTIMIZATION PROBLEM*

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: This paper focuses on the Benson proper efficiency for a nonconvex vector optimization problem with cone constraints. First, we give a characterization of the Benson proper efficiency in the image space, and discuss the relations among the Benson proper efficiency, image regularity condition and regular separation. Then, we analyse the generalized saddle-point and image regularity condition, and establish sufficient and necessary conditions of the generalized saddle-point criterion for the nonconvex vector optimization problems.

Key words: *Benson proper efficiency, image space, image regularity condition, nonconvex separation.*

Mathematics Subject Classification: *90C26, 90C30*

1 Introduction

Optimality notion is an important topic in vector optimization. Since the notion of efficiency contains some undesirable situations, people proposed the proper efficiency to improve it. Researchers defined various concepts of proper efficiency, and considered the relationships between these concepts and the scalar optimization problems; see [1, 2, 4, 7, 8, 11–15]. The notion of proper efficiency can be traced back to Kuhn-Tucker [15] and modified by Geoffrion [8], and later it was formulated in a more general framework (Borwein [2], Benson [1], Henig [11]). Subsequently, Jahn [12] characterized the Borwein properly minimal elements of a set based on the norm scalarization functional. Recently, Kasimbeyli [13,14] exploited a conic scalarization method to characterize the Benson proper efficiency in vector optimization problems. In general, people employed the Lagrangian functions to study the proper efficiency of constrained vector optimization problems; see [2, 4, 6–8]. Especially, Chen and Rong [4] gave some characterizations of the Benson proper efficiency under generalized convexity in terms of scalarization, Lagrangian multipliers, saddle-point criterion, and duality. In [7], Gasimov obtained a generalized saddle-point type sufficient condition of the Benson proper efficiency for a nonconvex vector optimization problem with equality constraints.

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However, to the best of our knowledge, there is no paper to study necessary and sufficient conditions of the Benson proper efficiency for a vector optimization problem with general cone constraints without any convexity assumption.

Based on the optimization theories in image space, Castellani and Giannessi [3] established a unify scheme, which was so called Image Space Analysis (for short, ISA), to deal with different optimization models, which can be expressed as the impossibility of a parametric system. The ISA method is a powerful tool, and several aspect of theories in optimization are developed in recent years, such as duality, existence of solutions, optimality conditions, penalty methods and regularity conditions; see [3, 5, 6, 9, 10, 16–19]. Separation plays a crucial role in the ISA. After the disjunction of two suitable sets for the given problem has been constructed in the image space, different separation approaches are applied to the two sets, such as linear separation [6, 9, 10, 18], nonlinear or conic separation [5, 10, 16, 17, 19]. In general, authors only considered the (weakly) efficient solutions of a constrained vector optimization problem in terms of ISA, see [9, 16, 19], but the researches on properly efficient solutions by ISA are limited. That's due to the difficulty in converting the proper efficiency of the constrained vector optimization problem to the impossibility of a parametric system in the image space.

In this paper, we aim at studying the proper efficiency for the nonconvex vector optimization problem with cone constraints via ISA approach. First, we characterize the Benson proper efficiency in the image space, and show the relations among the Benson proper efficiency, image regularity condition and regular separation. Then, we discuss the generalized saddle-point and image regularity condition for the nonconvex vector optimization. Finally, we use a nonconvex separation theorem to establish the generalized saddle-point criterion without any convexity of the given problem.

The rest of the paper is organized as follows. In Section 2, we recall some concepts and the basic separation about ISA. In Section 3, we discuss the proper efficiency in image space. In Section 4, we investigate the generalized saddle-point and image regularity condition in nonconvex vector optimization.

2 Preliminaries

In this section, we recall some notations and definitions, which will be used in the sequel. Let $\mathbb{R}^\ell, \mathbb{R}^m, \mathbb{R}^n$ be Euclidean spaces. The closure, the topological interior and boundary of a set $M \subseteq \mathbb{R}^\ell$ are denoted by clM, $intM$ and bdM, respectively. Let $cone(M) := \{\lambda a : a \in M, \lambda \in \mathbb{R}_+\}$ denote the cone set produced by M, and $d(v, M) = \inf\{\|v - a\| : a \in M\}$ denote the distance from a point $v \in \mathbb{R}^\ell$ to M. Let $C \subseteq \mathbb{R}^\ell$ and $D \subseteq \mathbb{R}^m$ be nonempty, closed, convex and pointed cones with $intC \neq \emptyset$ and $intD \neq \emptyset$.

Definition 2.1. The function $f : \mathbb{R}^n \to \mathbb{R}$ is C-monotonically increasing iff

$$f(u_2) \leq f(u_1), \; \forall\, u_1 - u_2 \in C.$$

Definition 2.2. The function $f : \mathbb{R}^n \to \mathbb{R}$ is positive homogeneous iff

$$f(\alpha u) = \alpha f(u), \; \forall\, u \in \mathbb{R}^n, \; \alpha \geq 0.$$

Let $y = (y_1, ..., y_\ell) \in \mathbb{R}^\ell$. $\|y\|_1 = \sum_{i=1}^{\ell} |y_i|$, $\|y\|_2 = (y_1^2 + ... + y_\ell^2)^{1/2}$ and $\|y\|_\infty = \max\{|y_1|, ..., |y_\ell|\}$ denote the l_1, l_2, and l_∞ norms of y, respectively. The unit sphere and unit ball of $\mathbb{R}^\ell$ are denoted by $U = \{y \in \mathbb{R}^\ell : \; \|y\| = 1\}$ and $B = \{y \in \mathbb{R}^\ell : \; \|y\| \leq 1\}$, respectively. $C_U = C \cap U = \{y \in C : \; \|y\| = 1\}$ denotes the norm-base of the cone C.

Let $C^* = \{y^* \in \mathbb{R}^\ell : \langle y^*, y\rangle \geq 0, \forall y \in C\}$ and $C^\sharp = \{y^* \in \mathbb{R}^\ell : \langle y^*, y\rangle > 0, \forall y \in C\backslash\{0\}\}$ be the dual cone and strict positive dual cone of C, respectively.

The following cones called augmented dual cones of C were introduced in [7, 13, 14].

$$C^{a*} = \{(y^*, \alpha) \in C^* \times \mathbb{R}_+ : \ \langle y^*, y\rangle - \alpha\|y\| \geq 0 \text{ for all } y \in C\},$$
$$C^{a\sharp} = \{(y^*, \alpha) \in C^\sharp \times \mathbb{R}_+ : \ \langle y^*, y\rangle - \alpha\|y\| > 0 \text{ for all } y \in C\backslash\{0\}\ \}.$$

Definition 2.3 ([14])**.** Let C and K be closed cones in $\mathbb{R}^\ell$ with $intK \neq \emptyset$. A cone K is called a conic neighborhood of the cone C if $C\backslash\{0\} \subset intK$. For a positive real number ϵ, a cone $C_\epsilon = cone(C_U + \epsilon B)$ is called ϵ-conic neighborhood of C.

Theorem 2.4 ([14])**.** *Let $C = \mathbb{R}^\ell_+$. Then, for every $\epsilon \in (0, 1)$, there exists a pair $(y^*, \alpha) \in C^{a\sharp}$ such that*

$$-C\backslash\{0\} \subset int(S(y^*, \alpha)) \subset -C_\epsilon,$$

where $S(y^*, \alpha) = \{y \in \mathbb{R}^\ell : \langle y^*, y\rangle + \alpha\|y\|_1 \leq 0\}$ is a closed convex pointed cone, and

$$int(S(y^*, \alpha)) = \{y \in \mathbb{R}^\ell : \langle y^*, y\rangle + \alpha\|y\|_1 < 0\}.$$

In this paper, we consider the following vector optimization problem with cone constraints:

$$(P)\ \ \min{}_C\ f(x)$$
$$s.t.\ \ g(x) \in D,\ x \in X,$$

where $X \subset \mathbb{R}^n$ is a nonempty subset, $f : \mathbb{R}^n \to \mathbb{R}^\ell$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are vector-valued functions. As usual, we denote by $S = \{x \in X : g(x) \in D\}$ the feasible set of (P).

Definition 2.5. (i) An element $\bar{x} \in S$ is called an efficient solution of (P) if $(f(\bar{x}) - f(S)) \cap C = \{0\}$.

(ii) An element $\bar{x} \in S$ is called a weakly efficient solution of (P) if $(f(\bar{x}) - f(S)) \cap intC = \emptyset$.

(iii) An element $\bar{x} \in S$ is called a Benson properly efficient solution of (P) if $clcone(f(\bar{x}) - f(S) - C) \cap C = \{0\}$.

Next, we recall the main features of the ISA for the problem (P) with respect to the efficient solution. Take arbitrary $\bar{x} \in X$. We consider the mapping $A_{\bar{x}} : X \to \mathbb{R}^\ell \times \mathbb{R}^m$ with

$$A_{\bar{x}}(x) := (f(\bar{x}) - f(x), g(x)),\ \ \forall x \in X$$

and the sets

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \ u = f(\bar{x}) - f(x), v = g(x), x \in X\},$$
$$\mathcal{H} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \ u \in C\backslash\{0\}, v \in D\},\ \mathcal{E}_{\bar{x}} = \mathcal{K}_{\bar{x}} - cl\mathcal{H}.$$

The sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{E}_{\bar{x}}$ are called the image and extend image of (P), respectively. Obviously, $\bar{x} \in S$ is an efficient solution of (P), iff the generalized system

$$A_{\bar{x}}(x) \in \mathcal{H},\ \ x \in X, \tag{2.1}$$

has no solutions, or, equivalently

$$\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset. \tag{2.2}$$

Remark 2.6. Let $\mathcal{H}_u := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \ u \in C\backslash\{0\}, v = 0\}$. Since $\mathcal{H} + cl\mathcal{H} = \mathcal{H}$, it follows that (2.2) is equivalent to

$$\mathcal{E}_{\bar{x}} \cap \mathcal{H} = \emptyset, \tag{2.3}$$

or, equivalently,

$$\mathcal{E}_{\bar{x}} \cap \mathcal{H}_u = \emptyset. \tag{2.4}$$

Consider a real-valued function $w : \mathbb{R}^{\ell+m} \times \Pi \to \mathbb{R}$, where Π is a set of parameters to be specified case by case. Let $lev_{>0}w(\cdot;\pi) = \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \ w(u,v;\pi) > 0\}$ and $lev_{\geq 0}w(\cdot;\pi) = \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \ w(u,v;\pi) \geq 0\}$ denote the positive level set and nonnegative level set of w, respectively. Then, we recall the following concept of regular weak separation functions.

Definition 2.7 ([10])**.** The class of all the functions $w : \mathbb{R}^{\ell+m} \times \Pi \to \mathbb{R}$ such that

$$\bigcap_{\pi \in \Pi} lev_{>0}w(\cdot;\pi) = \mathcal{H}$$

is called the class of regular weak separation functions and denoted by $\mathcal{W}_R(\Pi)$.

Definition 2.8. The sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are said to admit regular separation with respect to $w \in \mathcal{W}_R(\Pi)$ and $\bar{\pi} \in \Pi$ iff

$$w(u,v;\bar{\pi}) \leq 0, \ \forall \ (u,v) \in \mathcal{K}_{\bar{x}}.$$

Obviously, if $w(\cdot;\bar{\pi})$ is $cl\mathcal{H}$-monotonically increasing, then the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are regular separation with respect to $w \in \mathcal{W}_R(\Pi)$ and $\bar{\pi} \in \Pi$ iff

$$w(u,v;\bar{\pi}) \leq 0, \ \forall \ (u,v) \in \mathcal{E}_{\bar{x}}.$$

3 Proper Efficiency in Image Space

Proper efficiency is an important optimality notion in vector optimization. However, people can't find a suitable mapping $A_{\bar{x}}$ such that solving the properly efficient solution of (P) can be expressed by the impossibility of a generalized parametric system (2.1). This leads to the few studies on the proper efficiency by ISA method.

On the other hand, it is worth noting that all the definitions of different solutions only need the information of $x \in S$, and the characterizations of efficient solution in (2.1) and (2.2) contain extra information. So, we introduce two sets as follows,

$$\mathcal{K}_1 = \mathcal{K}_{\bar{x}} \cap (\mathbb{R}^\ell \times D) \ \text{ and } \ \mathcal{E}_1 = \mathcal{E}_{\bar{x}} \cap (\mathbb{R}^\ell \times D).$$

Obviously, $\mathcal{K}_1 \cap \mathcal{H} = \emptyset$, $\mathcal{E}_1 \cap \mathcal{H} = \emptyset$, (2.2) and (2.3) are equivalent.

Then, we can analyse the proper efficiency by using $\mathcal{E}_1$ in the following.

Theorem 3.1. *Suppose that $\bar{x} \in S$. Then,*

$$cl(cone(\mathcal{E}_1)) \cap \mathcal{H}_u = \emptyset, \tag{3.1}$$

iff $\bar{x}$ is a Benson properly efficient solution of (P).

Proof. "$\Longleftarrow$". Assume that $\bar{x}$ is a Benson properly efficient solution of (P). Note that $\mathcal{E}_{\bar{x}} \cap (\mathbb{R}^\ell \times D) \subset (f(\bar{x}) - f(S) - C) \times D$, which implies that

$$P_{\mathbb{R}^\ell}(cl(cone(\mathcal{E}_1))) \subset clcone(f(\bar{x}) - f(S) - C),$$

where $P_{\mathbb{R}^\ell}(\cdot)$ denotes the projection from $\mathbb{R}^\ell \times \mathbb{R}^m$ to $\mathbb{R}^\ell$.

Thus, $clcone(f(\bar{x}) - f(S) - C) \cap C\backslash\{0\} = \emptyset$ implies (3.1).

"$\Longrightarrow$". By the contrary, suppose that $\bar{x}$ is not a Benson properly efficient solution of (P), which means that $clcone(f(\bar{x}) - f(S) - C) \cap C\backslash\{0\} \neq \emptyset$. Let $u \in clcone(f(\bar{x}) - f(S) - C) \cap C\backslash\{0\}$. Then there exist sequences $x_n \in S$, $\lambda_n > 0$ and $c_n \in C$ with $n \in \mathbb{N}$ such that

$$u = \lim_{n\to+\infty} \lambda_n(f(\bar{x}) - f(x_n) - c_n).$$

It follows from $x_n \in S$ that $g(x_n) \in D$, for all $n \in \mathbb{N}$. Take $d_n = g(x_n)$, $v_n = \lambda_n(g(x_n) - d_n) = 0$, and $u_n = \lambda_n(f(\bar{x}) - f(x_n) - c_n)$, for all $n \in \mathbb{N}$. Then we have that $(u_n, v_n) \in cone(\mathcal{E}_1)$ and

$$(u, 0) = \lim_{n\to+\infty}(u_n, v_n) = \lim_{n\to+\infty} \lambda_n(f(\bar{x}) - f(x_n) - c_n, g(x_n) - d_n),$$

which implies $(u, 0) \in clcone(\mathcal{E}_1)$. Together with $(u, 0) \in C\backslash\{0\} \times \{0\} = \mathcal{H}_u$, we obtain

$$cl(cone(\mathcal{E}_1)) \cap \mathcal{H}_u \neq \emptyset,$$

which contradicts (3.1). The proof is complete. □

Theorem 3.1 provides a characterization of the Benson proper efficiency for the constrained vector optimization problem in the image space. However, the disadvantage of $\mathcal{E}_1$ is obvious. Since $\mathcal{E}_1$ loses the information of $g(x) \notin D$, we can't use it to establish the saddle-point criterion. On the other hand, combining with $\mathcal{E}_1 \subset \mathcal{E}_{\bar{x}}$, we immediately get the following result.

Theorem 3.2. *Suppose that $\bar{x} \in S$. If*

$$cl(cone(\mathcal{E}_{\bar{x}})) \cap \mathcal{H}_u = \emptyset, \tag{3.2}$$

then, $\bar{x}$ is a Benson properly efficient solution of (P).

Remark 3.3. The condition (3.2) is called *image regularity condition* in some literatures([5, 16–19]), which is used in conic separation, penalty methods. When (P) reduces to a scalar optimization problem, and under some convexity assumptions, the condition (3.2) is equivalent to the existence of saddle-point. Obviously, $cl(cone(\mathcal{E}_{\bar{x}})) \cap \mathcal{H} = \emptyset$ and (3.2) are equivalent since $\mathcal{H} + cl\mathcal{H} = \mathcal{H}$.

In general, the converse of Theorem 3.2 doesn't hold, and we illustrate it in the following example.

Example 3.1. Let $\ell = 2, m = n = 1$, $X = \mathbb{R}$, $C = \mathbb{R}^2_+$, and $D = \mathbb{R}_+$. The mappings f and g are defined as follows,

$$f(x) := (x + 1, 2x), \quad g(x) := \begin{cases} x^{\frac{1}{2}} & \text{if } x \geq 0, \\ -x^2 & \text{otherwise.} \end{cases}$$

Obviously, $S = \mathbb{R}_+$, and $\bar{x} = 0$ is a Benson properly efficient solution of (P). Moreover, $f(\bar{x}) = (1, 0)$, and

$$\mathcal{E}_{\bar{x}} = \{(-u, -2u, -u^2) : \ u \le 0\} \cup \{(-u, -2u, u^{\frac{1}{2}}) : \ u \ge 0\} - \mathbb{R}^3_+,$$
$$\mathcal{H} = (\mathbb{R}^2_+ \backslash \{0\}) \times \mathbb{R}_+, \ \mathcal{H}_u = (\mathbb{R}^2_+ \backslash \{0\}) \times \{0\}.$$

Then, $\mathcal{E}_{\bar{x}} \cap \mathcal{H}_u = \emptyset$. On the other hand, $cl(cone(\mathcal{E}_{\bar{x}})) = \mathbb{R}^2 \times (-\mathbb{R}_+) \cup (-\mathbb{R}^2_+) \times \mathbb{R}_+$, and

$$cl(cone(\mathcal{E}_{\bar{x}})) \cap \mathcal{H}_u = (\mathbb{R}^2_+ \backslash \{0\}) \times \{0\} = \mathcal{H}_u,$$

which means that the converse of Theorem 3.2 fails.

In the following, we give some assumptions to guarantee the image regularity condition.

Proposition 3.4. *Let $C = \mathbb{R}^\ell_+$. Assume that $\bar{x}$ is a Benson properly efficient solution of (P). Suppose that there exists $\bar{r} > 0$ such that* $\max_{1 \le i \le \ell}\{f_i(\bar{x}) - f_i(x)\} \le \bar{r} d(g(x), D)$, *$\forall x \notin S$. Then the image regularity condition* (3.2) *holds.*

Proof. By the contrary, assume that (3.2) is not fulfilled. Then there exists $u \in C \backslash \{0\}$ such that $(u, 0) \in cl(cone(\mathcal{E}_{\bar{x}}))$. Since $\bar{x}$ is a Benson properly efficient solution of (P), it follows from Theorem 3.1 that there exist sequences $x_n \notin S$, $\lambda_n > 0$, $c_n \in C$ and $d_n \in D$ with $n \in \mathbb{N}$ such that

$$u = \lim_{n \to +\infty} \lambda_n (f(\bar{x}) - f(x_n) - c_n) \ \text{ and } \ v = \lim_{n \to +\infty} \lambda_n (g(x_n) - d_n) = 0. \tag{3.3}$$

Again by $u = (u_1, ..., u_\ell) \in \mathbb{R}^\ell_+ \backslash \{0\}$, then there exists $1 \le i \le \ell$ such that $u_i > 0$. Therefore,

$$\lim_{n \to +\infty} \frac{\|g(x_n) - d_n\|}{f_i(\bar{x}) - f_i(x_n) - (c_n)_i} = 0. \tag{3.4}$$

Together with $u_i > 0$ and the first equation of (3.3), we get that $f_i(\bar{x}) - f_i(x_n) - (c_n)_i > 0$, for sufficiently large n. Moreover,

$$0 \le \frac{d(g(x_n), D)}{f_i(\bar{x}) - f_i(x_n) - (c_n)_i} \le \frac{\|g(x_n) - d_n\|}{f_i(\bar{x}) - f_i(x_n) - (c_n)_i}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\lim_{n \to +\infty} \frac{d(g(x_n), D)}{f_i(\bar{x}) - f_i(x_n) - (c_n)_i} = 0,$$

so that, for every $r > 0$, there exists $n \in \mathbb{N}$ that

$$d(g(x_n), D) < \frac{1}{r}(f_i(\bar{x}) - f_i(x_n) - (c_n)_i) \le \frac{1}{r}(f_i(\bar{x}) - f_i(x_n)),$$

which is a contradiction to the assumption of this proposition. The proof is complete. □

Remark 3.5. If $\ell = 1$, then the assumptions of Proposition 3.4 reduce to that there exists $\bar{r} > 0$ such that $f(\bar{x}) - f(x) \le \bar{r} d(g(x), D)$, $\forall x \in X$, which mean that the exact penalty function exists. It is worth noting that the existence of exact penalty functions is equivalent to the image regularity condition (3.2) for the scalar optimization problem, see [17].

By the regular separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, we can also obtain a sufficient condition for image regularity condition immediately.

Theorem 3.6. *Suppose that function $w(\cdot;\pi)$ is positive homogeneous and $cl\mathcal{H}$-monotonically increasing, for $\pi \in \Pi$ and $w \in \mathcal{W}_R(\Pi)$. If the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are regular separation with respect to w and $\bar{\pi} \in \Pi$, then the image regularity condition* (3.2) *holds.*

Proof. The proof is easy, we omit it. □

In particular, we give a special class of regular separation functions as follows:

$$\overline{w}(u,v;\theta,\lambda) := \omega_1(u,\theta) + \omega_2(v,\lambda),\ u \in \mathbb{R}^\ell,\ v \in \mathbb{R}^m,\ (\theta,\lambda) \in \Pi_1 \times \Pi_2, \tag{3.6}$$

where $\omega_1 : \mathbb{R}^\ell \times \Pi_1 \longrightarrow \mathbb{R}$ and $\omega_2 : \mathbb{R}^m \times \Pi_2 \longrightarrow \mathbb{R}$ fulfill the following conditions

$$\bigcap_{\theta\in\Pi_1} lev_{>0}\, \omega_1(\cdot,\theta) = C\backslash\{0\}, \tag{3.7}$$

$$\bigcap_{\lambda\in\Pi_2} lev_{\geq 0}\, \omega_2(\cdot,\lambda) = D, \tag{3.8}$$

$$\forall\, \lambda \in \Pi_2,\ \forall\, t \geq 0,\ \exists \lambda_t \in \Pi_2 \text{ s.t. } t\omega_2(v,\lambda) = \omega_2(v,\lambda_t),\ \forall\, v \in \mathbb{R}^m. \tag{3.9}$$

It is easy to verify that $\overline{w} \in \mathcal{W}_R(\Pi_1 \times \Pi_2)$ if the functions (3.6) satisfy assumptions (3.7)-(3.9). Under suitable assumptions, we also give a sufficient condition for Benson proper efficiency.

Theorem 3.7. *Assume that the function* (3.6) *fulfills conditions* (3.7)-(3.9)*. Suppose that $\omega_1(\cdot,\theta)$ and $\omega_2(\cdot,\lambda)$ are C- and D-monotonicity increasing, respectively, and $\omega_1(\cdot,\theta)$ is positive homogeneous, for $\theta \in \Pi_1$ and $\lambda \in \Pi_2$. Moreover, $\bar{x} \in S$. If the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are regular separation with respect to $\overline{w}$ and $(\bar{\theta},\bar{\lambda}) \in \Pi_1 \times \Pi_2$, then $\bar{x}$ is a Benson properly efficient solution of (P).*

Proof. It follows from the monotonicity of ω_1 and ω_2 that $\overline{w}(\cdot;\theta,\lambda)$ is $cl\mathcal{H}$-monotonically increasing. Together with the regular separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, we have

$$\overline{w}(u,v;\bar{\theta},\bar{\lambda}) = \omega_1(u,\bar{\theta}) + \omega_2(v,\bar{\lambda}) \leq 0,\ \forall\, (u,v) \in \mathcal{E}_{\bar{x}}.$$

By (3.8), then for every $(u,v) \in \mathcal{E}_{\bar{x}}$ with $v \in D$, we get

$$\omega_1(u,\bar{\theta}) \leq 0,$$

which implies that

$$\omega_1(f(\bar{x}) - f(x),\bar{\theta}) \leq 0,\ \forall\, x \in S.$$

Again by the monotonicity and positive homogeneity of ω_1, one has

$$\omega_1(u,\bar{\theta}) \leq 0,\ \forall\, u \in clcone(f(\bar{x}) - f(S) - C). \tag{3.10}$$

It follows from (3.7) and (3.10) that $clcone(f(\bar{x}) - f(S) - C) \cap C = \{0\}$, thus $\bar{x}$ is a Benson properly efficient solution of (P). This completes the proof. □

4 Generalized Saddle-Point and Image Regularity Condition for the Nonconvex Vector Optimization

In [4], the Benson proper efficiency was characterized by the linear Lagrangian saddle-point under generalized convexity assumptions and Slater condition. However, the classic saddle-point criterion doesn't exist for nonconvex case in general, so nonlinear Lagrangian functions are introduced. In [7, 13, 14], the authors defined a conic scalarization function and used it to characterize the properly minimal element of a set (in the sense of Benson or Henig). Motivated by these results, we define a nonlinear separation function by

$$\underline{w}(u,v;\mu,\alpha,\lambda,\beta) := \langle \mu, u\rangle - \alpha\|u\|_Y + \langle \lambda, v\rangle - \beta\|v\|_Z, \tag{4.1}$$

where $(\mu,\alpha,\lambda,\beta) \in \Pi = C^{a\sharp} \times D^{a*}$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ denote the norms in $Y = \mathbb{R}^\ell$ and $Z = \mathbb{R}^m$, respectively.

Lemma 4.1. *Let* $\Pi = C^{a\sharp} \times D^{a*}$. *Then,* $\underline{w} \in \mathcal{W}_R(\Pi)$, *i.e.,*

$$\bigcap_{(\mu,\alpha,\lambda)\in\Pi} lev_{>0}\underline{w}(\cdot;\mu,\alpha,\lambda,\beta) = \mathcal{H}.$$

Proof. For every $(\mu,\alpha,\lambda,\beta) \in \Pi = C^{a\sharp} \times D^{a*}$, we have

$$\underline{w}(u,v;\mu,\alpha,\lambda,\beta) = \langle \mu, u\rangle - \alpha\|u\|_Y + \langle \lambda, v\rangle - \beta\|v\|_Z > 0,\ \forall (u,v) \in \mathcal{H},$$

which leads to

$$\bigcap_{(\mu,\alpha,\lambda,\beta)\in\Pi} lev_{>0}\underline{w}(\cdot;\mu,\alpha,\lambda,\beta) \supseteq \mathcal{H}.$$

Next, we prove the following inclusion

$$\bigcap_{(\mu,\alpha,\lambda,\beta)\in\Pi} lev_{>0}\underline{w}(\cdot;\mu,\alpha,\lambda,\beta) \subseteq \mathcal{H}. \tag{4.2}$$

By the contrary, assume that (4.2) is false. Then there exists $(u_0, v_0) \notin \mathcal{H}$ such that

$$\underline{w}(u_0,v_0;\mu,\alpha,\lambda,\beta) > 0,\ \forall (\mu,\alpha,\lambda,\beta) \in \Pi. \tag{4.3}$$

Since $(u_0, v_0) \notin \mathcal{H}$, we consider the following three cases:

Case 1. If $u_0 = 0, v_0 \in Z$, then we set $(\lambda_0, \beta_0) = (0, 0)$. For any $(\mu_0, \alpha_0) \in C^{a\sharp}$, we get

$$\underline{w}(u_0,v_0;\mu_0,\alpha_0,\lambda_0,\beta_0) = \langle \mu_0, u_0\rangle - \alpha_0\|u_0\|_Y + \langle \lambda_0, v_0\rangle - \beta_0\|v_0\|_Z = 0,$$

which contradicts (4.3).

Case 2. If $u_0 \notin C, v_0 \in Z$, then there exists $\mu_0 \in C^\sharp$ such that $\langle \mu_0, u_0\rangle \le 0$. For $(\mu_0,\alpha_0,\lambda_0,\beta_0) = (\mu_0, 0, 0, 0) \in C^{a\sharp} \times D^*$, we have

$$\underline{w}(u_0,v_0;\mu_0,\alpha_0,\lambda_0) = \langle \mu_0, u_0\rangle - \alpha_0\|u_0\|_Y + \langle \lambda_0, v_0\rangle - \beta_0\|v_0\|_Z \le 0,$$

which contradicts (4.3).

Case 3. If $u_0 \in C\backslash\{0\}, v_0 \notin D$, then there exists $\lambda_0 \in D^*$ such that $\langle \lambda_0, v_0\rangle < 0$. Let $(\mu_0, \alpha_0) \in C^{a\sharp}$, $\beta_0 = 0$ and $\lambda_n = n\lambda_0 \in D^*$. Then for sufficiently large $n \in \mathbb{N}$, we can obtain

$$\underline{w}(u_0,v_0;\mu_0,\alpha_0,\lambda_n,\beta_0) = \langle \mu_0, u_0\rangle - \alpha_0\|u_0\|_Y + n\langle \lambda_0, v_0\rangle - \beta_0\|v_0\|_Z < 0,$$

which contradicts (4.3).

The proof is complete. □

Take $p \in Y$. For separation function $\underline{w}$ in (4.1), we get a generalized Lagrangian function $L : X \times \Pi \to \mathbb{R}$ as follows:

$$\begin{aligned} L(x,\mu,\alpha,\lambda,\beta) :&= \underline{w}(p,0;\mu,\alpha,\lambda,\beta) - \underline{w}(p-f(x),g(x);\mu,\alpha,\lambda,\beta) \\ &= \langle \mu, f(x)\rangle + \alpha\|p-f(x)\|_Y - \alpha\|p\|_Y - \langle \lambda, g(x)\rangle + \beta\|g(x)\|_Z, \\ &\qquad\qquad \forall (x,\mu,\alpha,\lambda,\beta) \in X \times \Pi. \end{aligned}$$

It is worth mentioning that the choice of p is arbitrary in L.

Definition 4.2. (P) is said to satisfy the generalized saddle-point criterion of L at $\bar{x} \in X$ for some $(\bar{\mu},\bar{\alpha}) \in C^{a\sharp}$, if there exists $(\bar{\lambda},\bar{\beta}) \in D^{a*}$ such that $(\bar{x},\bar{\lambda},\bar{\beta})$ is a saddle-point of the generalized Lagrangian function $L(x,\bar{\mu},\bar{\alpha},\lambda,\beta)$, that is,

$$L(\bar{x},\bar{\mu},\bar{\alpha},\lambda,\beta) \le L(\bar{x},\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) \le L(x,\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}),\ \forall (x,\lambda,\beta) \in X \times D^{a*}.$$

Proposition 4.3. *Let $p = f(\bar{x})$. Then, the following statements are equivalent.*

(i) *For $\bar{x} \in X$ and $(\bar{\mu},\bar{\alpha}) \in C^{a\sharp}$, there exists $(\bar{\lambda},\bar{\beta}) \in D^{a*}$ such that $(\bar{x},\bar{\lambda},\bar{\beta})$ is a saddle-point of the generalized Lagrangian function $L(x,\bar{\mu},\bar{\alpha},\lambda,\beta)$,*

(ii) *$\bar{x} \in S$ and $\underline{w}(u,v;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) \le 0,\ \forall (u,v) \in \mathcal{K}_{\bar{x}}$.*

Proof. The proof is easy, we omit it. □

In the following, we show the relationships between the generalized saddle-point criterion of L and image regularity condition (3.2).

Theorem 4.4. *For $\bar{x} \in X$ and $(\bar{\mu},\bar{\alpha}) \in C^{a\sharp}$, there exists $(\bar{\lambda},\bar{\beta}) \in D^{a*}$ such that $(\bar{x},\bar{\lambda},\bar{\beta})$ is a saddle-point of the generalized Lagrangian function $L(x,\bar{\mu},\bar{\alpha},\lambda,\beta)$, then image regularity condition* (3.2) *holds and $\bar{x} \in S$. Moreover, if $(\bar{\lambda},\bar{\beta}) \in D^{a\sharp}$, then*

$$cl(cone(\mathcal{E}_{\bar{x}})) \cap cl\mathcal{H} = \{(0,0)\}. \tag{4.4}$$

Proof. Suppose that $(\bar{x},\bar{\lambda},\bar{\beta})$ is a saddle-point of L, , that is

$$\begin{aligned} &\langle \bar{\mu}, f(\bar{x})\rangle + \bar{\alpha}\|p-f(\bar{x})\|_Y - \bar{\alpha}\|p\|_Y - \langle \lambda, g(\bar{x})\rangle + \beta\|g(\bar{x})\|_Z \\ &\le \langle \bar{\mu}, f(\bar{x})\rangle + \bar{\alpha}\|p-f(\bar{x})\|_Y - \bar{\alpha}\|p\|_Y - \langle \bar{\lambda}, g(\bar{x})\rangle + \bar{\beta}\|g(\bar{x})\|_Z \\ &\le \langle \bar{\mu}, f(x)\rangle + \bar{\alpha}\|p-f(x)\|_Y - \bar{\alpha}\|p\|_Y - \langle \bar{\lambda}, g(x)\rangle + \bar{\beta}\|g(x)\|_Z,\ \forall (x,\lambda,\beta) \in X \times D^{a*}. \end{aligned} \tag{4.5}$$

It follows from the first inequality of (4.5) that $g(\bar{x}) \in D$ and $\langle \bar{\lambda}, g(\bar{x})\rangle - \bar{\beta}\|g(\bar{x})\|_Z = 0$. Together with the second inequality of (4.5), we get

$$\begin{aligned} 0 &\le \langle \bar{\mu}, f(x)-f(\bar{x})\rangle + \bar{\alpha}\|p-f(x)\|_Y - \bar{\alpha}\|p-f(\bar{x})\|_Y - \langle \bar{\lambda}, g(x)\rangle + \bar{\beta}\|g(x)\|_Z \\ &\le \langle \bar{\mu}, f(x)-f(\bar{x})\rangle + \bar{\alpha}\|f(x)-f(\bar{x})\|_Y - \langle \bar{\lambda}, g(x)\rangle + \bar{\beta}\|g(x)\|_Z,\ \forall x \in X. \end{aligned}$$

Since $\langle \bar{\mu}, \cdot\rangle + \bar{\alpha}\|\cdot\|_Y$ is C-monotonically increasing and $\langle \bar{\lambda}, \cdot\rangle + \bar{\beta}\|\cdot\|_Z$ is D-monotonically increasing ([13], Theorem 3.5), we have

$$\begin{aligned} &\langle \bar{\mu}, f(x)-f(\bar{x})+c\rangle + \bar{\alpha}\|f(x)-f(\bar{x})+c\|_Y + \langle \bar{\lambda}, -g(x)+d\rangle + \bar{\beta}\|-g(x)+d\|_Z \ge 0 \\ &\qquad\qquad \forall x \in X,\ (c,d) \in C \times D, \end{aligned}$$

which implies that

$$\underline{w}(u,v;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) = \langle\bar{\mu},u\rangle - \bar{\alpha}\|u\|_Y + \langle\bar{\lambda},v\rangle - \bar{\beta}\|v\|_Z \le 0,\ \forall(u,v)\in\mathcal{E}_{\bar{x}}.$$

On the other hand, $\underline{w}(\cdot,\cdot;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta})$ is positive homogeneous, we deduce that

$$\underline{w}(u,v;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) = \langle\bar{\mu},u\rangle - \bar{\alpha}\|u\|_Y + \langle\bar{\lambda},v\rangle - \bar{\beta}\|v\|_Z \le 0,\ \forall(u,v)\in cl(cone(\mathcal{E}_{\bar{x}})). \tag{4.6}$$

Moreover, $(\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta})\in\Pi = C^{a\sharp}\times D^{a*}$, we have

$$\underline{w}(u,v;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) = \langle\mu,u\rangle - \alpha\|u\|_Y + \langle\lambda,v\rangle - \beta\|v\|_Z > 0,\ \forall(u,v)\in\mathcal{H}. \tag{4.7}$$

It follows from (4.6) and (4.7) that $cl(cone(\mathcal{E}_{\bar{x}}))\cap\mathcal{H}=\emptyset$, then the image regularity condition (3.2) holds.

Next, we prove the second conclusion. Since $(\bar{\mu},\bar{\alpha})\in C^{a\sharp}$ and $(\bar{\lambda},\bar{\beta})\in D^{a\sharp}$, we get

$$\underline{w}(u,v;\bar{\mu},\bar{\alpha},\bar{\lambda},\bar{\beta}) = \langle\mu,u\rangle - \alpha\|u\|_Y + \langle\lambda,v\rangle - \beta\|v\|_Z > 0,\ \forall(u,v)\in cl\mathcal{H}\backslash\{(0,0)\},$$

together with (4.6), we obtain (4.4). The proof is complete. □

Combining with Theorem 3.2, we immediately obtain the following corollary.

Corollary 4.5. *If there exists $(\bar{\mu},\bar{\alpha})\in C^{a\sharp}$, such that (P) satisfies the generalized saddle-point criterion of L at $\bar{x}$ for $(\bar{\mu},\bar{\alpha})$, then $\bar{x}$ is the Benson properly efficient solution of (P).*

Next, we give an example to explain Corollary 4.5 and Theorem 4.4.

Example 4.1. Let $\ell=2, m=n=1$, $X=\mathbb{R}$, $C=\mathbb{R}^2_+$, and $D=\mathbb{R}_+$. Let $\|\cdot\|=\|\cdot\|_1$ be the norm in $\mathbb{R}^\ell$ and $\mathbb{R}^m$. The mappings f and g are defined as follows,

$$f(x) := \begin{cases} (x,-\frac{1}{2}x) & \text{if } x\le 0, \\ (x,-2x) & \text{otherwise,} \end{cases} \quad g(x):=x+2.$$

Obviously, f is nonconvex and the classic Lagrange saddle-point doesn't exist. Note that $C^{a\sharp}=\{(\mu,\alpha)\in\mathbb{R}^3_+ :\ \mu_1>\alpha,\mu_2>\alpha\}$ and $D^{a*}=\{(\lambda,\beta)\in\mathbb{R}^2_+ :\ \lambda\ge\beta\}$. Take $\bar{x}=0$, $p=0$, $(\bar{\mu},\bar{\alpha})=(2,2,1)$ and $(\bar{\lambda},\bar{\beta})=(2,2)$. Then, we can verify that $(\bar{x},\bar{\lambda},\bar{\beta})=(0,2,2)$ is a generalized saddle-point of $L(x,\bar{\mu},\bar{\alpha},\lambda,\beta)$. Moreover, $S=\{x\in\mathbb{R}: x\ge -2\}$ and

$$clcone(f(\bar{x})-f(S)-C)=\{(u,-2u):u\le 0\}\cup\{(u,-\frac{1}{2}u):u>0\}-\mathbb{R}^2_+.$$

Then, $clcone(f(\bar{x})-f(S)-C)\cap C=\{(0,0)\}$, which means that $\bar{x}=0$ is the Benson properly efficient solution of (P). On the other hand,

$$\begin{aligned}\mathcal{E}_{\bar{x}} &= \{(-u,\frac{1}{2}u,u+2):\ u\le 0\}\cup\{(-u,-2u,u+2):\ u\ge 0\}-\mathbb{R}^3_+,\\ \mathcal{H}_u &= (\mathbb{R}^2_+\backslash\{0\})\times\{0\}.\end{aligned}$$

Then, $cl(cone(\mathcal{E}_{\bar{x}}))\cap\mathcal{H}_u=\emptyset$.

In general, the image regularity condition (3.2) is not sufficient to guarantee the existence of a generalized saddle point for L. To this aim, we need the condition (4.4). Now, using the nonconvex separation theorem for conic separation, we establish the existence of a generalized saddle point for L.

Theorem 4.6. *Let $C = \mathbb{R}^\ell_+$ and $D = \mathbb{R}^m_+$. Suppose that the norms in $Y = \mathbb{R}^\ell$, $Z = \mathbb{R}^m$, and $Y \times Z = \mathbb{R}^{\ell+m}$ are all l_1 norm. Let $\bar{x} \in S$. If (4.4) holds, then there exist two pairs $(\bar{\mu}, \bar{\alpha}) \in C^{a\sharp}$ and $(\bar{\lambda}, \bar{\alpha}) \in D^{a\sharp}$ such that $(\bar{x}, \bar{\lambda}, \bar{\alpha})$ is a saddle-point of the generalized Lagrangian function $L(x, \bar{\mu}, \bar{\alpha}, \lambda, \beta)$.*

Proof. It follows from Theorem 5.2 in [13] and (4.4) that there exists a closed convex cone Q with $cl\mathcal{H}\backslash\{(0,0)\} \subset intQ$ such that

$$cl(cone(\mathcal{E}_{\bar{x}})) \cap Q = \{(0,0)\}. \tag{4.8}$$

Let $\epsilon > 0$ be sufficiently small such that $(cl\mathcal{H})_\epsilon \subset Q$. Then, by Theorem 2.4, there exists a pair $(\bar{\mu}, \bar{\lambda}, \bar{\alpha}) \in (cl\mathcal{H})^{a\sharp}$ such that

$$-cl\mathcal{H}\backslash\{(0,0)\} \subset int\{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \langle(\bar{\mu}, \bar{\lambda}), (u,v)\rangle + \bar{\alpha}\|(u,v)\|_1 \leq 0\} \subset -(cl\mathcal{H})_\epsilon \subset -Q.$$

It follows from these inclusions and (4.8) that

$$\langle(\bar{\mu}, \bar{\lambda}), (u,v)\rangle + \bar{\alpha}\|(u,v)\|_1 \geq 0 \; for \; all \; (u,v) \in -cl(cone(\mathcal{E}_{\bar{x}})),$$

which implies

$$\langle\bar{\mu}, f(x) - f(\bar{x})\rangle + \bar{\alpha}\|f(x) - f(\bar{x})\|_1 - \langle\bar{\lambda}, g(x)\rangle + \bar{\alpha}\|g(x)\|_1 \geq 0, \; \forall x \in X,$$

then, we have

$$\langle\bar{\mu}, f(x)\rangle + \bar{\alpha}\|f(x) - f(\bar{x})\|_1 - \langle\bar{\lambda}, g(x)\rangle + \bar{\alpha}\|g(x)\|_1 \geq \langle\bar{\mu}, f(\bar{x})\rangle, \; \forall x \in X. \tag{4.9}$$

On the other hand, it is easy to verify that $(cl\mathcal{H})^{a\sharp} = \{(\mu, \lambda, \alpha) \in (cl\mathcal{H})^\sharp \times \mathbb{R}_+ : (\mu, \alpha) \in C^{a\sharp}, \; (\lambda, \alpha) \in D^{a\sharp}\}$. Therefore, $(\bar{\mu}, \bar{\alpha}) \in C^{a\sharp}$ and $(\bar{\lambda}, \bar{\alpha}) \in D^{a\sharp} \subset D^{a*}$.

Letting $x = \bar{x}$ in (4.9), we get $-\langle\bar{\lambda}, g(\bar{x})\rangle + \bar{\alpha}\|g(\bar{x})\|_1 \geq 0$. Since $g(\bar{x}) \in D$ and $(\bar{\lambda}, \bar{\alpha}) \in D^{a*}$, one has $-\langle\bar{\lambda}, g(\bar{x})\rangle + \bar{\alpha}\|g(\bar{x})\|_1 \leq 0$. Thus,

$$-\langle\bar{\lambda}, g(\bar{x})\rangle + \bar{\alpha}\|g(\bar{x})\|_1 = 0. \tag{4.10}$$

Moreover, again by $g(\bar{x}) \in D$, we have

$$\langle\lambda, g(\bar{x})\rangle - \beta\|g(\bar{x})\|_1 \geq 0, \; \forall(\lambda, \beta) \in D^{a*}. \tag{4.11}$$

It follows from (4.9), (4.10) and (4.11) that $(\bar{x}, \bar{\lambda}, \bar{\alpha})$ is a saddle-point of the generalized Lagrangian function $L(x, \bar{\mu}, \bar{\alpha}, \lambda, \beta)$.

The proof is complete. □

In [7], Gasimov also obtained the Benson proper efficiency by means of the saddle-point criterion for a nonlinear Lagrangian function. He only considered a vector optimization problem with equality constraints. However, our problem is more general. Taking $D = 0$ and $p = 0$ in L, then L reduces to the generalized Lagrangian function in [7]. Thus, the characterization of the Benson proper efficiency in terms of Lagrangian function in [7] is a special case of our results. Moreover, we give a sufficient condition for the existence of the generalized saddle-point criterion.

5 Conclusions

In this paper, we discuss the Benson proper efficiency for a nonconvex vector optimization problem with cone constraints by ISA method. We have studied the relationships among the Benson properly efficient solutions, image regularity condition and regular separation. Furthermore, we introduce a nonlinear Lagrangian function, and investigate the generalized saddle-point and image regularity condition for the nonconvex vector optimization problem. Finally, we employ a nonconvex separation theorem to obtain the existence of the generalized saddle-point criterion.

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